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# The Einstein-Vlasov-Scalar Field System with Gowdy or $T^2$ Symmetry in Contracting Direction

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**Abstract**
We prove in the case of cosmological models for the Einstein-Vlasov-scalar field system with Gowdy symmetry, that the solutions exist globally in the past. The sources of the equations are generated by a distribution function and a scalar field, subject to the Vlasov and the wave equations respectively. The result is generalized for the case of $T^2$ symmetry. Using previous results, we deduce geodesic completeness.

**Keywords** Einstein · Vlasov · Scalar field · Gowdy symmetry · $T^2$ symmetry · Hyperbolic differential equations · Global existence · Geodesic completeness

**Mathematics Subject Classification (2010)** 83C20 · 83C22 · 34B05

## 1 Introduction

The question of global existence solutions of Einstein equations with matter or not is very important in general relativity. Here, for long, the practice has been to study existence of solutions under symmetry assumptions. Spacetimes with Gowdy or $T^2$ symmetry have received much attention by different authors for many years, see [1, 3, 8, 11, 15] and the references therein.

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In this paper, we prove past global existence, with respect to a geometrically defined time, for Einstein equations coupled to the Vlasov and non-linear wave equations. In the surface symmetry case with the Vlasov and linear scalar field, we have shown in [12] (without any restriction on the data) and [13] global existence in the past time direction. The symmetry assumption here is the Gowdy and $T^2$ one, and thereby we extend respectively Andréasson's result in [1] and Smulevici's one in [11] for the Vlasov case. Andréasson's result was the first result to provide a global foliation in the contracting and expanding direction of a spacetime containing both matter and gravitational waves. In that paper, Andréasson generalized in the two time directions (contracting and expanding) the Moncrief result for the vacuum case. His method of proof is inspired by a result in [3] for vacuum spacetimes admitting a $T^2$ isometry group acting on $T^3$ space-like surfaces. Gowdy spacetimes are a special case of these spacetimes. Andréasson proves that $T^3 \times \mathbb{R}$ spacetimes with Gowdy symmetry admit global foliations by conformal coordinates in the contracting direction.

We extend the result in [1] by introducing a scalar field. It can be seen as a step towards certain questions of physical interest. In recent years, cosmological models with accelerated expansion have become a very active research topic in response to astronomical observations [10]. The easiest way to obtain models with accelerated expansion is to introduce a positive cosmological constant (see [2] and the references therein). A more sophisticated and generalized way is to introduce a scalar field with non-vanishing potential. This is our case.

There are three types of time coordinates which have been studied in the inhomogeneous Einstein-Vlasov system case: constant mean curvature, areal, and conformal. A constant mean curvature time coordinate $t$ is one where each hypersurface of constant time has constant mean curvature and on each hypersurface of this kind the value of $t$ is the mean curvature of that slice. In the case of areal coordinates, the time coordinate is a function of the area of the surfaces of symmetry. In some papers of the relevant literature, it is taken to be proportional to the area. In the case of conformal coordinates, the metric is conformally flat on the manifold $Q$ which is the quotient of spacetime by the symmetry group. $Q$ is a two-dimensional Lorentzian manifold.

Now consider the past maximal globally hyperbolic development of data on an initial hypersurface with Gowdy or $T^2$ symmetry (which includes plane symmetry). Using conformal coordinates and geometrical arguments as in [1, 3], we prove that along any past inextendible timelike curve, the time coordinate $R = t$, which is the area of the symmetry orbits, tends to a constant value $R_0 = 0$, independent of the choice of the curve. This was also proved in the case of $T^2$-symmetric spacetimes with only the Vlasov matter by Weaver in [15] with results generalized later in some direction by Smulevici in [11]. Conformal coordinates have advantages for the simplification of estimations of solutions. By a long chain of geometrical arguments as in [3], one deduces the global foliations in the past time direction of the spacetime by areal coordinates system.

Now, we recall the formulation of the Einstein-Vlasov-scalar field system. The spacetime is a four-dimensional manifold $M$, with local coordinates $(x^\lambda) = (t, x^i)$ on which $x^0 = t$ denotes the time and $(x^i)$ the space coordinates. Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On $M$, a Lorentzian metric $g$ is given with signature $(-, +, +, +)$. We consider a self-gravitating collisionless gas and restrict ourselves to the case where all particles have the same rest mass $m \geq 0$, and move forward in time. We denote by $(p^\lambda)$ the momenta of the particles. The conservation of the quantity $g_{\lambda\beta} p^\lambda p^\beta$ requires that the phase space of the particle is the seven-dimensional submanifold

$$PM = \{g_{\lambda\beta} p^\lambda p^\beta = -m^2; \quad p^0 > 0\}$$

of $TM$ which is coordinatized by $(t, x^i, p^i)$. The energy-momentum tensor is given by

$$T_{\lambda\beta} = -\int_{\mathbb{R}^3} f p_\lambda p_\beta \mid g \mid^{1/2} \frac{dp^1 dp^2 dp^3}{p_0} + \nabla_\lambda \phi \nabla_\beta \phi - \left[\frac{1}{2}\nabla_\nu \phi \nabla^\nu \phi + V(\phi)\right] g_{\lambda\beta} \quad (1.1)$$

where the distribution function of the particles is a non-negative real-valued function denoted by $f$ and defined on $PM$, $p_\lambda = g_{\lambda\beta} p^\beta$, $|g|$ denotes the modulus of determinant of the metric $g$, a scalar field $\phi$ is a real-valued $C^\infty$ function on $M$, and $V \in C^\infty(\mathbb{R}^+)$ is a function such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$ (see [9]).

The Einstein field equations

$$R_{\lambda\beta} - \frac{1}{2} R g_{\lambda\beta} = 8\pi T_{\lambda\beta} \quad (1.2)$$

should be coupled to the Vlasov equation (matter equation for $f$) and to the wave equation (matter equation for $\phi$), which are respectively

$$\partial_t f + \frac{p^i}{p^0} \partial_{x^i} f - \frac{1}{p^0} \Gamma^i_{\beta\gamma} p^\beta p^\gamma \partial_{p^i} f = 0 \quad (1.3)$$

$$\nabla^\lambda \nabla_\lambda \phi - V'(\phi) = 0. \quad (1.4)$$

As a consequence, the energy-momentum tensor is divergence-free since the Bianchi identities imply that $\nabla^\lambda G_{\lambda\beta} = 0$ where $G_{\lambda\beta} = R_{\lambda\beta} - \frac{1}{2} R g_{\lambda\beta}$ and the contribution of $f$ to the energy-momentum tensor is divergence-free [4].

The outline of the paper follows in the large that of [1]. In Section 2, we present the equations in Gowdy symmetry with conformal coordinates. The main theorem is formulated in Section 3. The proof of the theorem is based on a series of estimations using a light-cone argument. These are important to obtain bounds on the field components, the matter terms, and their derivatives. In Section 4, we extend the previous results for the $T^2$ -symmetry case.

## 2 The Gowdy Case

We refer to [1, 7], or [8] for details on the notion of Gowdy symmetry. There are several choices of spacetime manifolds compatible with Gowdy symmetry. Here, we restrict our attention to the $T^3$ case. Spacetimes admitting a $T^2$ isometry group acting on $T^3$ space-like surfaces are more general than the Gowdy spacetimes: both families admit two commuting killing vectors, but in the Gowdy case the additional condition is that the twists are zero. The dynamics of the matter is governed by the Vlasov and the non-linear wave equations. The Vlasov equation models a collisionless system of particles which follow the geodesics of spacetime. We now consider a solution of the Einstein-Vlasov-scalar field system where all unknowns are invariant under this symmetry. We write the system in conformal coordinates. The circumstances under which coordinates of this type exist are discussed in [2] and the references therein. In such coordinates, the metric $g$ takes the form

$$ds^2 = -e^{2(\mu-U)} dt^2 + e^{2(\mu-U)} d\theta^2 + e^{2U}(dx + Ady)^2 + R^2 e^{-2U} dy^2, \quad (2.1)$$

where $\mu$, $U$, $R$, and $A$ are unknown real functions of $t$ and $\theta$ variables. $R$ is periodic in $\theta$ with period 1. Here, the timelike coordinate $t$ locally labels spatial hypersurfaces of the spacetime. The scalar field is a function of $t$ and $\theta$. Using the results of [1], the complete Einstein-Vlasov-scalar field system can be written in the following form:

*The Einstein-matter constraint equations:*

$$U_t^2 + U_\theta^2 + \frac{e^{4U}}{4R^2}\left(A_t^2 + A_\theta^2\right) + \frac{R_{\theta\theta}}{R} - \frac{\mu_t R_t}{R} - \frac{\mu_\theta R_\theta}{R} = -e^{2(\mu-U)}\rho, \tag{2.2}$$

$$2U_t U_\theta + \frac{e^{4U}}{2R^2}A_t A_\theta - \frac{R_{t\theta}}{R} - \frac{\mu_t R_\theta}{R} - \frac{\mu_\theta R_t}{R} = e^{2(\mu-U)}J_1. \tag{2.3}$$

*The Einstein-matter evolution equations:*

$$U_{tt} - U_{\theta\theta} = \frac{U_\theta R_\theta}{R} - \frac{U_t R_t}{R} + \frac{e^{4U}}{2R^2}\left(A_t^2 - A_\theta^2\right) + \frac{1}{2}e^{2(\mu-U)}(\rho - P_1 + P_2 - P_3), \tag{2.4}$$

$$A_{tt} - A_{\theta\theta} = \frac{A_t R_t}{R} - \frac{A_\theta R_\theta}{R} + 4(A_\theta U_\theta - A_t U_t) + 2Re^{2\mu-4U}S_{23}, \tag{2.5}$$

$$R_{tt} - R_{\theta\theta} = Re^{2(\mu-U)}(\rho - P_1), \tag{2.6}$$

$$\mu_{tt} - \mu_{\theta\theta} = U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - e^{2(\mu-U)}P_3 - \frac{A^2}{R^2}e^{2(\mu+U)}P_2 - \frac{2A}{R}e^{2\mu}S_{23} \tag{2.7}$$

$$\phi_{tt} - \phi_{\theta\theta} = -\frac{R_t}{R}\phi_t + \frac{R_\theta}{R}\phi_\theta - e^{2(\mu-U)}V'(\phi). \tag{2.8}$$

*The Vlasov equation:*

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{v^1}{v^0}\frac{\partial f}{\partial \theta} - \Bigg[(\mu_\theta - U_\theta)v^0 + (\mu_t - U_t)v^1 - \frac{e^{2U}A_\theta}{R}\frac{v^2 v^3}{v^0} \\
&\quad + \frac{U_\theta}{v^0}((v^3)^2 - (v^2)^2) - \frac{R_\theta}{R}\frac{(v^3)^2}{v^0}\Bigg]\frac{\partial f}{\partial v^1} - \left[U_\theta\frac{v^1 v^2}{v^0} + U_t v^2\right]\frac{\partial f}{\partial v^2} \\
&\quad - \left[\left(\frac{R_t}{R} - U_t\right)v^3 - \left(U_\theta - \frac{R_\theta}{R}\right)\frac{v^1 v^3}{v^0} + \frac{e^{2U}v^2}{R}\left(A_t + A_\theta\frac{v^1}{v^0}\right)\right]\frac{\partial f}{\partial v^3} = 0.
\end{aligned} \tag{2.9}$$

Since all the particles have proper mass $m$, the variables $v^\alpha$ are related to the canonical momentum variables $p^\alpha$ by the relations

$$v^0 = e^{\mu-U}p^0, \quad v^1 = e^{\mu-U}p^1, \quad v^2 = e^U p^2 + Ae^U p^3, \quad v^3 = Re^{-U}p^3,$$

so that

$$v^0 = \sqrt{m^2 + (v^1)^2 + (v^2)^2 + (v^3)^2}.$$

*Remark 2.1* If $m > 0$ then $p^0 > 0$ and $v^0 > 0$.
**Case** $m = 0$: let $I$ be the maximal existence interval of solution $s \longmapsto (x^\alpha(s), p^\alpha(s))$ of the geodesic equations

$$\frac{dx^\alpha}{ds} = p^\alpha; \quad \frac{dp^\alpha}{ds} = -\Gamma^\alpha_{\beta\lambda}p^\beta p^\lambda. \tag{2.10}$$

Take $s_0 \in I$ such that $p_0^0 = \sqrt{g^{00}(s_0, \theta(s_0))}\sqrt{g_{ij}(s_0, \theta(s_0))p_0^i p_0^j} > 0$. Then since

$$\tilde{p} = (p^1, p^2, p^3) \mapsto p^0 = \sqrt{g^{00}(s_0, \theta(s_0))}\sqrt{g_{ij}(s_0, \theta(s_0))p^i p^j}$$

is continuous on $\tilde{p}_0 = (p_0^1, p_0^2, p_0^3)$, we can find a neighborhood $W$ of $\tilde{p}_0$ such that

$$(\tilde{p} \in W) \Rightarrow \left(p^0 > \frac{p_0^0}{2} > 0\right).$$

Finally, either $m > 0$ or $m = 0$, we have $p^0 > 0$, i.e., $v^0 > 0$.

The matter terms are then defined by

$$\rho(t,\theta) = -g^{00}T_{00}(t,\theta) = \int_{\mathbb{R}^3} f(t,\theta,v)v^0 dv + \frac{1}{2}e^{-2(\mu-U)}(\phi_t^2+\phi_\theta^2) + V(\phi),$$
$$J_1(t,\theta) = -g^{11}T_{01}(t,\theta) = \int_{\mathbb{R}^3} f(t,\theta,v)v^1 dv - e^{-2(\mu-U)}\phi_t\phi_\theta,$$
$$P_1(t,\theta) = g^{11}T_{11}(t,\theta) = \int_{\mathbb{R}^3} f(t,\theta,v)\frac{(v^1)^2}{v^0}dv + \frac{1}{2}e^{-2(\mu-U)}(\phi_t^2+\phi_\theta^2) - V(\phi),$$
$$P_2(t,\theta) = e^{-2U}T_{22}(t,\theta) = \int_{\mathbb{R}^3} f(t,\theta,v)\frac{(v^2)^2}{v^0}dv + \frac{1}{2}e^{-2(\mu-U)}(\phi_t^2-\phi_\theta^2) - V(\phi),$$
$$P_3(t,\theta) = \int_{\mathbb{R}^3} f(t,\theta,v)\frac{(v^3)^2}{v^0}dv + \frac{1}{2}e^{-2(\mu-U)}(\phi_t^2-\phi_\theta^2) - V(\phi),$$
$$S_{23}(t,\theta) = \int_{\mathbb{R}^3} f(t,\theta,v)\frac{v^2v^3}{v^0}dv,$$

and

$$T_{33}(t,\theta) = A^2e^{2U}P_2 + R^2e^{-2U}P_3 + 2ARS_{23}.$$

We prescribe initial data at time $t = t_0 > 0$:

$$f(t_0,\theta,v) = f_0(\theta,v), \quad \mu(t_0,\theta) = \mu_0(\theta), \quad \mu_t(t_0,\theta) = \mu_1(\theta),$$
$$R(t_0,\theta) = R_0(\theta), \quad R_t(t_0,\theta) = R_1(\theta), \quad U(t_0,\theta) = U_0(\theta), \quad U_t(t_0,\theta) = U_1(\theta),$$
$$A(t_0,\theta) = A_0(\theta), \quad A_t(t_0,\theta) = A_1(\theta), \quad \phi(t_0,\theta) = \phi_0(\theta), \quad \phi_t(t_0,\theta) = \phi_1(\theta)$$

## 3 The Main Theorem

We begin this important section by specifying the regularity properties and the local in time existence result which we require.

**Definition 3.1** Let $I \subset ]0,\infty[$ be an interval and $(t,\theta) \in I \times \mathbb{R}$.
a) $f \in C^\infty(I \times \mathbb{R}^2)$ is regular if $f(t,\theta+1,v) = f(t,\theta,v)$ for $(t,\theta,v) \in I \times \mathbb{R}^2$, $f \geq 0$ and supp$f(t,\theta,.)$ is compact uniformly in $\theta$ and locally uniformly in $t$.
b) $\mu$ (or $R, U, A$) $\in C^2(I \times \mathbb{R})$ is regular, if $\mu(t,\theta+1) = \mu(t,\theta)$.
c) $\rho$ (or $P_k, J$) $\in C^1(I \times \mathbb{R})$ $(k = 1,2,3)$ is regular, if $\rho(t,\theta+1) = \rho(t,\theta)$.
d) $\phi \in C^2(I \times \mathbb{R})$ is regular, if $\phi(t,\theta+1) = \phi(t,\theta)$.

**Theorem 3.2** *Given initial data for* (1.1)–(1.4)*, there is a maximal globally hyperbolic development* $(M, g, f, \phi)$ *of the data which is unique up to isometry.*

The proof is as in [5, 6]. This important result (which is a local existence in time solution for an associated hyperbolic system) will be used in this paper. However, it does not yield any conclusion concerning global existence in time directions. Our attention is concentrated only on the existence of global solutions in the past time direction for any initial data. This is an extension of Choquet's result for the hyperbolic system.

*Remark 3.3* $(M, g, f, \phi)$ is unique up to isometry means that if $(M', g', f', \phi')$ is another maximal globally hyperbolic development, then there is a diffeomorphism $\varphi : M \to M'$

such that $\varphi^* g' = g$, $\varphi^* f' = f$, $\varphi^* \phi' = \phi$ and $\varphi \circ i = i'$, where $i$ and $i'$ are the embeddings of $T^3$ into $M$ and $M'$ respectively.

**Theorem 3.4** *Let $(T^3, \mu_0, \mu_1, A_0, A_1, U_0, U_1, f_0, \phi_0, \phi_1)$ be regular initial data that satisfy the constraints (2.2)–(2.3) where the metric is coordinatized as in (2.1). Let $(g, f, \phi)$ be the local regular solution that corresponds to the initial data and $]T, t_0]$ be the maximal interval of existence. There exists a globally hyperbolic spacetime $(M, g, f, \phi)$ such that*
(i) $M = [0, t_0[\times T^3$,
(ii) $(g, f, \phi)$ *satisfies the Einstein-Vlasov-scalar field system in areal coordinates,*
(iii) $(M, g, f, \phi)$ *is isometrically diffeomorphic to the maximal globally hyperbolic development of the initial data $(T^3, \mu_0, \mu_1, A_0, A_1, U_0, U_1, f_0, \phi_0, \phi_1)$.*

In order to extend the local existence in time to the global existence, it is sufficient to obtain uniform bounds on the field components, the distribution function, the scalar field, and all their derivatives on a finite time interval $[t_1, t_2)$ on which the local solution exists. This means that the past maximal development of initial data in terms of conformal coordinates has $t \to -\infty$ as long as $R$ stays bounded away from zero, and using geometrical arguments as in [1], we have a proof of the theorem.

Let us introduce the null vector fields $\partial_\tau := \frac{1}{\sqrt{2}}(\partial_t - \partial_\theta)$, $\partial_\xi := \frac{1}{\sqrt{2}}(\partial_t + \partial_\theta)$. Then $F_\tau := \frac{1}{\sqrt{2}}(F_t - F_\theta)$ and $F_\xi := \frac{1}{\sqrt{2}}(F_t + F_\theta)$ for any function $F$ of variables $t$ and $\theta$.

*Step 1* Monotonicity of $R$ and bounds on its first derivatives.

After some calculation, the constraints (2.2) and (2.3) give respectively

$$\sqrt{2}\partial_\theta R_\xi = -Re^{2(\mu-U)}(\rho - J_1) - 2RU_\xi^2 - \frac{e^{4U}}{2R}A_\xi^2 + 2\mu_\xi R_\xi, \tag{3.1}$$

$$\sqrt{2}\partial_\theta R_\tau = Re^{2(\mu-U)}(\rho + J_1) + 2RU_\tau^2 + \frac{e^{4U}}{2R}A_\tau^2 - 2\mu_\tau R_\tau. \tag{3.2}$$

Since $|J_1| \leq \rho$ and $R > 0$, it follows from (3.1) and (3.2) that

$$\partial_\theta R_\xi < \sqrt{2}\mu_\xi R_\xi, \tag{3.3}$$

$$\partial_\theta R_\tau > -\sqrt{2}\mu_\tau R_\tau. \tag{3.4}$$

If $R_\xi(t, \theta) = 0$ for some $t \in ]t_1, t_0]$ and $\theta \in \mathbb{R}$, then by the periodicity of $R$ with respect to $\theta$ and Gronwall's lemma applied on (3.3), $0 = R_\xi(t, \theta + 1) < R_\xi(t, \theta)\exp(\int_\theta^{\theta+1} \sqrt{2}\mu_\xi) = 0$, a contradiction. Thus, $R_\xi \neq 0$ on $]t_1, t_0] \times S^1$.

Similarly, (3.4) yields the same assertion for $R_\tau$. This implies that the quantities $R_\xi$, $R_\tau$ have each a definite sign. It follows that the quantity $g^{\alpha\beta}\partial_{x^\alpha} R \partial_{x^\beta} R = g^{00} R_t^2 + g^{11} R_\theta^2 = -\frac{1}{2}e^{2(\mu-U)} R_\xi R_\tau$ does not change sign (it is strictly positive or strictly negative). Since $R$ is periodic and continuous in $\theta$, there must exist points where $R_\theta = 0$, hence the quantity above is negative everywhere ($g^{00} R_t^2 < 0$). Thus, $\nabla R$ is timelike. This means that $R_t$ is non zero everywhere. Our choice of time corresponds to contracting $T^2$ orbits so that $R_t > 0$ and $|R_\theta| < R_t$.

Next, we show that $R_t$ and $|R_\theta|$ are bounded into the past. Using (2.6) and the fact that $\rho \geq p_1$,

$$\partial_\tau R_\xi = \frac{R}{2} e^{2(\mu-U)}(\rho - P_1) \geq 0 \tag{3.5}$$

and

$$\partial_\xi R_\tau = \frac{R}{2} e^{2(\mu - U)} (\rho - P_1) \geq 0. \tag{3.6}$$

It follows that if we start at any point $(t_0, \theta_0)$ on the initial surface, we obtain

$$\frac{d}{ds} R_\xi (s, \theta + t - s) = \partial_t R_\xi - \partial_\theta R_\xi = \sqrt{2} \partial_\tau R_\xi (s, r + t - s) \geq 0.$$

After integration on $[t, t_0]$,

$$R_\xi (t, \theta) \leq R_\xi (t_0, \theta + t - t_0).$$

Similarly,

$$R_\tau (t, \theta) \leq R_\tau (t_0, \theta - t + t_0).$$

These yield

$$\sqrt{2} R_t (t, \theta) \leq R_\xi (t_0, \theta + t - t_0) + R_\tau (t_0, \theta - t + t_0) \leq \sup_{\theta \in S^1} (R_\xi + R_\tau)(t_0, \theta).$$

We conclude that $R_t$ is bounded into the past and $|R_\theta|$ is also bounded. Consequently, $R$ is uniformly bounded to the past of the initial surface.

*Step 2* Bounds on $U$, $A$, $\mu$, $\phi$, and their first derivatives.

We use the light-cone argument and Gronwall's lemma in this step. The functions involved in this case are quadratic and defined by

$$X = \frac{1}{2} R (U_t^2 + U_\theta^2) + \frac{e^{4U}}{8R} (A_t^2 + A_\theta^2) + \frac{1}{2} R (\phi_t^2 + \phi_\theta^2) + \phi^2, \tag{3.7}$$

$$Y = R U_t U_\theta + \frac{e^{4U}}{4R} A_t A_\theta + R \phi_t \phi_\theta. \tag{3.8}$$

Using (2.4), (2.5), and (2.8) we find

$$\begin{aligned} \partial_\tau (X + Y) = & -\frac{1}{2} R_\xi \left( U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2} (-A_t^2 + A_\theta^2) \right) + \left( \frac{R_\tau}{2} - \frac{R_t}{\sqrt{2}} \right) \phi_t^2 \\ & + \left( \frac{R_\tau}{2} + \frac{R_\theta}{\sqrt{2}} \right) \phi_\theta^2 + 2 \phi \phi_\tau - \frac{R}{\sqrt{2}} (\phi_t + \phi_\theta) e^{2(\mu - U)} V'(\phi) \\ & + \frac{R}{2} U_\xi e^{2(\mu - U)} (\rho - P_1 + P_2 - P_3) + \frac{e^{2U}}{2R} A_\xi e^{2(\mu - U)} S_{23}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \partial_\xi (X - Y) = & -\frac{1}{2} R_\tau \left( U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2} (-A_t^2 + A_\theta^2) \right) + \left( \frac{R_\xi}{R} - \frac{R_t}{\sqrt{2}} \right) \phi_t^2 \\ & + \left( \frac{R_\xi}{R} - \frac{R_\theta}{\sqrt{2}} \right) \phi_\theta^2 + 2 \phi \phi_\xi - \frac{R}{\sqrt{2}} (\phi_t - \phi_\theta) e^{2(\mu - U)} V'(\phi) \\ & + \frac{R}{2} U_\tau e^{2(\mu - U)} (\rho - P_1 + P_2 - P_3) + \frac{e^{2U}}{2R} A_\tau e^{2(\mu - U)} S_{23}. \end{aligned} \tag{3.10}$$

Integrating each of the above equations along null paths starting at $(t_1, \theta)$ and ending at the initial $t_0-$surface, and adding the results we obtain

$$
\begin{aligned}
X(t_1,\theta) \;=\; & \frac{1}{2}(X+Y)(t_0,\theta-(t_0-t_1)) + \frac{1}{2}(X-Y)(t_0,\theta+(t_0-t_1)) \\
& -\frac{1}{2}\int_{t_1}^{t_0} [K_1(s,\theta-(s-t_1)) + K_2(s,\theta+(s-t_1))]\,ds \\
& -\frac{1}{2}\int_{t_1}^{t_0} \left[(U_\xi T_1)(s,\theta-(s-t_1)) + (U_\tau T_1)(s,\theta+(s-t_1))\right] ds \\
& -\frac{1}{2}\int_{t_1}^{t_0} \left[\left(\frac{e^{2U}}{2R}A_\xi T_2\right)(s,\theta-(s-t_1)) + \left(\frac{e^{2U}}{2R}A_\tau T_2\right)(s,\theta+(s-t_1))\right] ds,
\end{aligned} \tag{3.11}
$$

where

$$
\begin{aligned}
K_1 \;=\; & -\frac{1}{2}R_\tau\left(U_t^2 - U_\theta^2 + \frac{e^{4U}}{R^2}\left(-A_t^2 + A_\theta^2\right) - \phi_t^2 - \phi_\theta^2\right) \\
& -\frac{R_t}{\sqrt{2}}\phi_t^2 + \frac{R_\theta}{\sqrt{2}}\phi_\theta^2 + 2\phi\phi_\tau - \frac{R}{\sqrt{2}}(\phi_t+\phi_\theta)e^{2(\mu-U)}V'(\phi), \\
K_2 \;=\; & -\frac{1}{2}R_\xi\left(U_t^2 - U_\theta^2 + \frac{e^{4U}}{R^2}\left(-A_t^2 + A_\theta^2\right) - \phi_t^2 - \phi_\theta^2\right) \\
& -\frac{R_t}{\sqrt{2}}\phi_t^2 - \frac{R_\theta}{\sqrt{2}}\phi_\theta^2 + 2\phi\phi_\xi - \frac{R}{\sqrt{2}}(\phi_t-\phi_\theta)e^{2(\mu-U)}V'(\phi), \\
T_1 \;=\; & \frac{R}{2}e^{2(\mu-U)}(\rho - P_1 + P_2 - P_3), \\
T_2 \;=\; & e^{2(\mu-U)}S_{23}.
\end{aligned}
$$

From the expression of the matter terms and the fact that $V(\phi) > 0$, we deduce that

$$\rho - P_1 - P_2 - P_3 = \int_{\mathbb{R}^3}\frac{1}{v^0}f\,dv - e^{-2(\mu-U)}(\phi_t^2 - \phi_\theta^2) + 4V(\phi),$$

i.e.,

$$P_1 + P_2 + P_3 \le \rho + e^{-2(\mu-U)}\phi_t^2$$

and

$$P_2 + P_3 \le \rho - P_1 + e^{-2(\mu-U)}(\phi_t^2 - \phi_\theta^2) - 4V(\phi).$$

In another way

$$2|S_{23}| \le P_2 + P_3 - e^{-2(\mu-U)}(\phi_t^2 - \phi_\theta^2) + 2V(\phi),$$

therefore

$$|T_2| = e^{2(\mu-U)}|S_{23}| \le \frac{1}{2}e^{2(\mu-U)}(\rho - P_1). \tag{3.12}$$

Since $\rho - P_1 - P_2 + P_3 \ge 0$ (i.e., $P_2 - P_3 \le \rho - P_1$) and $\rho - P_1 + P_2 - P_3 \ge 0$, we deduce that

$$0 \le \rho - P_1 + P_2 - P_3 \le 2(\rho - P_1)$$

and

$$T_1 \le Re^{2(\mu-U)}(\rho - P_1). \tag{3.13}$$

In another way for $k = 2, 3$,

$$e^{2(\mu-U)}P_k \le e^{2(\mu-U)}(\rho - P_1) + \frac{1}{2}\phi_t^2. \tag{3.14}$$

Without loss of generality, choosing $V(\phi)$ such that $V(\phi) = V_0 \exp(\phi^2)$, we obtain

$$|\phi_\xi e^{2(\mu-U)} V'(\phi)| = 2|\phi_\xi \phi| e^{2(\mu-U)} V(\phi) \leq \frac{1}{2}(\phi_\xi^2 + \phi^2) e^{2(\mu-U)} (\rho - P_1)$$

and

$$|\phi_\tau e^{2(\mu-U)} V'(\phi)| = 2|\phi_\tau \phi| e^{2(\mu-U)} V(\phi) \leq \frac{1}{2}(\phi_\tau^2 + \phi^2) e^{2(\mu-U)} (\rho - P_1).$$

For any $t \in (t_1, t_0)$, Eqs. (3.5) and (3.6) give respectively after integration

$$R_\xi(t_0, \theta + t_0 - t) - R_\xi(t, \theta) = \frac{1}{2} \int_t^{t_0} [R e^{2(\mu-U)} (\rho - P_1)](s, \theta + s - t) ds, \quad (3.15)$$

$$R_\tau(t_0, \theta - t_0 + t) - R_\tau(t, \theta) = \frac{1}{2} \int_t^{t_0} [R e^{2(\mu-U)} (\rho - P_1)](s, \theta - s + t) ds. \quad (3.16)$$

It follows from Step 1 that the right hand sides of the two equalities (3.15), (3.16) are uniformly bounded. Consequently, on $(t_1, t_0)$, we obtain from (3.12), (3.13) the uniform bound of

$$\int_t^{t_0} T_1(s, \theta \pm (s - t)) ds; \quad \int_t^{t_0} |T_2|(s, \theta \pm (s - t)) ds$$

and (for some constant $C$) the estimations

$$\int_t^{t_0} \left[\frac{1}{2}(\phi_\xi^2 + \phi^2) R e^{2(\mu-U)} (\rho - P_1)\right](s, \theta + s - t) ds \leq C \sup_{[t_1, t_0]} X(t, \theta), \quad (3.17)$$

$$\int_t^{t_0} \left[\frac{1}{2}(\phi_\tau^2 + \phi^2) R e^{2(\mu-U)} (\rho - P_1)\right](s, \theta - s + t) ds \leq C \sup_{[t_1, t_0]} X(t, \theta). \quad (3.18)$$

We can deduce that

$$\int_{t_1}^{t_0} |U_\xi| T_1(s, \theta - s + t_1) ds \leq \int_{t_1}^{t_0} 2\sqrt{\frac{X}{R}} T_1(s, \theta - s + t_1) ds \leq C \sup_{[t_1, t_0]} \sqrt{X(t, \theta)}, \quad (3.19)$$

$$\int_{t_1}^{t_0} |U_\tau| T_1(s, \theta + s - t_1) ds \leq \int_{t_1}^{t_0} 2\sqrt{\frac{X}{R}} T_1(s, \theta + s - t_1) ds \leq C \sup_{[t_1, t_0]} \sqrt{X(t, \theta)}, \quad (3.20)$$

$$\int_{t_1}^{t_0} \frac{e^{2U}}{2R} |A_\xi| |T_2|(s, \theta - s + t_1) ds \leq \int_{t_1}^{t_0} 2\sqrt{\frac{X}{R}} |T_2|(s, \theta - s + t_1) ds \leq C \sup_{[t_1, t_0]} \sqrt{X(t, \theta)}, \quad (3.21)$$

$$\int_{t_1}^{t_0} \frac{e^{2U}}{2R} |A_\tau| |T_2|(s, \theta + s - t_1) ds \leq \int_{t_1}^{t_0} 2\sqrt{\frac{X}{R}} |T_2|(s, \theta + s - t_1) ds \leq C \sup_{[t_1, t_0]} \sqrt{X(t, \theta)}, \quad (3.22)$$

$$\int_{t_1}^{t_0} |K_1|(s, \theta - s + t_1) ds \leq C \int_{t_1}^{t_0} X(s, \theta) ds + C \sup_{[t_1, t_0]} \sqrt{X(t, \theta)}, \quad (3.23)$$

$$\int_{t_1}^{t_0} |K_2|(s, \theta + s - t_1) ds \leq C \int_{t_1}^{t_0} X(s, \theta) ds + C \sup_{[t_1, t_0]} \sqrt{X(t, \theta)}. \quad (3.24)$$

So the identity (3.11) now implies

$$\begin{aligned} \sup_\theta X(t, \theta) \leq\ & 4 \sup_\theta X(t_0, \theta) + C \sup_{[t_1, t_0] \times S^1} \sqrt{X(t, \theta)} \\ & + C \sup_{[t_1, t_0] \times S^1} X(t, \theta) + C \int_{t_1}^{t_0} \sup_\theta X(s, \theta) ds. \end{aligned} \quad (3.25)$$

Using Gronwall's lemma, we conclude as in Step 2 of [1] that $\sup_\theta X(s,\theta)$ is uniformly bounded on $(t_1, t_0)$, leading to bounds on $U$, $A$, $\phi$, $V(\phi)$, and their first derivatives. The bounds on $\mu$ and its first derivatives are obtained in a similar way since (2.7) can be written as

$$\partial_\tau \mu_\xi = U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - e^{2(\mu-U)}\left(P_3 + \frac{A^2}{R^2}e^{4U}P_2 + \frac{2A}{R}e^{2U}S_{23}\right) \quad (3.26)$$

or equivalently

$$\partial_\xi \mu_\tau = U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - e^{2(\mu-U)}\left(P_3 + \frac{A^2}{R^2}e^{4U}P_2 + \frac{2A}{R}e^{2U}S_{23}\right). \quad (3.27)$$

Using inequalities (3.12), (3.14) and the fact that the integral along null paths for the quantity $Re^{2\mu-2U}(\rho - P_1)$ is bounded to the past of the initial surface, we conclude that the integrals along the null paths for the matter terms in the right hand sides of (3.26)–(3.27) are bounded since $A, U, \phi$, and their first derivatives are bounded. We obtain that $|\mu_\xi|$ and $|\mu_\tau|$ are bounded, and therefore $\mu_t = \frac{\sqrt{2}}{2}(\mu_\tau + \mu_\xi)$, $\mu_\theta = \frac{\sqrt{2}}{2}(-\mu_\tau + \mu_\xi)$, and $\mu$ are bounded.

*Step 3* Bound on the support of the momentum.

A solution $f$ to the Vlasov equation is given by

$$f(t,\theta,v) = f_0(\Theta(t_0,t,\theta,v), V(t_0,t,\theta,v)),$$

where $\Theta$ and $V$ are solutions to the characteristic system

$$\begin{aligned}
\frac{d\Theta}{ds} &= \frac{V^1}{V^0},\\
\frac{dV^1}{ds} &= -(\mu_\theta - U_\theta)V^0 - (\mu_t - U_t)V^1 + U_\theta\frac{(V^2)^2}{V^0} - \left(U_\theta - \frac{R_\theta}{R}\right)\frac{(V^3)^2}{V^0} + \frac{A_\theta}{R}e^{2U}\frac{V^2V^3}{V^0},\\
\frac{dV^2}{ds} &= -U_tV^2 - U_\theta\frac{V^1V^2}{V^0},\\
\frac{dV^3}{ds} &= -\left(\frac{R_t}{R} - U_t\right)V^3 + \left(U_\theta - \frac{R_\theta}{R}\right)\frac{V^1V^3}{V^0} - \frac{e^{2U}}{R}\left(A_t + A_\theta\frac{V^1}{V^0}\right)V^2,
\end{aligned}$$

with $\Theta(t,t,x,v) = \theta$ and $V(t,t,x,v) = v$. Since $||f||_\infty \leq ||f_0||_\infty$, the control of

$$Q(t) := \sup\left\{|\, v\,| : \exists (s,\theta) \in [t,t_0) \times S^1 \;:\; f(t,\theta,v) \neq 0\right\}$$

and previous steps give bounds on the matter quantities $\rho$, $P_k$, $(k = 1,2,3)$ and $S_{23}$. The distribution function has compact support on the initial surface and therefore $V^k(t_0)$, $k = 1,2,3$ is bounded. Since $|V^k| \leq V^0$, $k = 1,2,3$, the Gronwall lemma applied to the characteristic system gives uniform bounds on $|V^k(t)|$ and it follows that $Q(t)$ is uniformly bounded on $(t_1, t_0]$.

*Step 4* Bounds on the second-order derivatives of $R$ and $\phi$.

From previous steps, the field components and all their first derivatives are bounded, and the matter quantities are all bounded. We deduce respectively from (2.2), (2.3), and (2.6) the uniform bound of $R_{\theta\theta}$, $R_{t\theta}$, and $R_{tt}$ as long as $R$ stays bounded away from zero.

Let $B = \partial_\theta(R\phi_\tau)$ and $D = \partial_\theta(R\phi_\xi)$, then

$$\phi_{t\theta} = \frac{\sqrt{2}}{2R}(B + D) - \frac{R_\theta}{R}\phi_t \tag{3.28}$$

and

$$\phi_{\theta\theta} = \frac{\sqrt{2}}{2R}(-B + D) - \frac{R_\theta}{R}\phi_t. \tag{3.29}$$

After some calculations and using (3.28), (3.29),

$$\begin{aligned} B_\xi \;=\; & \frac{1}{2}\left[\left(R_{\theta\theta} - \frac{R_\theta^2}{R}\right)\phi_t + \left(\frac{R_t R_\theta}{R} - R_{t\theta}\right)\phi_\theta + \frac{R_\xi}{R}B - \frac{R_\tau}{R}D\right] \\ & -\left[\left(\mu_\theta - U_\theta + \frac{R_\theta}{2}\right)V'(\phi) + \frac{1}{2}\phi_\theta V''(\phi)\right]e^{2\mu-2U} \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} D_\tau \;=\; & -\frac{1}{2}\left[\left(R_{\theta\theta} - \frac{R_\theta^2}{R}\right)\phi_t + \left(\frac{R_t R_\theta}{R} - R_{t\theta}\right)\phi_\theta + \frac{R_\xi}{R}B - \frac{R_\tau}{R}D\right] \\ & -\left[\left(\mu_\theta - U_\theta + \frac{R_\theta}{2}\right)V'(\phi) + \frac{1}{2}\phi_\theta V''(\phi)\right]e^{2\mu-2U}. \end{aligned} \tag{3.31}$$

Integrating (3.30)–(3.31) along null paths starting at $(t_1, \theta)$ and ending at the initial $t_0-$surface, we obtain

$$B(t_1,\theta) = B(t_0, \theta+t_1-t_0) + \frac{1}{2}\int_{t_1}^{t_0}\left(a-b+\frac{R_\xi}{R}B - \frac{R_\tau}{R}D\right)(t_1, \theta+s-t_1)ds \tag{3.32}$$

$$D(t_1,\theta) = D(t_0, \theta-t_1+t_0) + \frac{1}{2}\int_{t_1}^{t_0}\left(-a-b-\frac{R_\xi}{R}B + \frac{R_\tau}{R}D\right)(t_1, \theta-s+t_1)ds. \tag{3.33}$$

We take the supremum in $\theta$ of the absolute values of each equation and add the results to obtain

$$K(t) \le K(t_0) + \int_t^{t_0}\left[\sup_{\theta\in S^1}|(a+b)(s,\theta)| + \sup_{\theta\in S^1}\left(\frac{|R_\xi| + |R_\tau|}{R}\right)(s,\theta)K(s)\right]ds, \tag{3.34}$$

where $K(t) = \sup_{\theta\in S^1}(|B| + |D|)(t,\theta)$ and

$$\begin{aligned} a(t,\theta) \;&=\; \left(R_{\theta\theta} - \frac{R_\theta^2}{R}\right)\phi_t + \left(\frac{R_t R_\theta}{R} - R_{t\theta}\right)\phi_\theta, \\ b(t,\theta) \;&=\; \left[(\mu_\theta - U_\theta + \frac{R_\theta}{2})V'(\phi) + \frac{1}{2}\phi_\theta V''(\phi)\right]e^{2\mu-2U}. \end{aligned}$$

We deduce from (3.34) the Gronwall lemma and previous steps that $K(t)$ is bounded since $R$ stays bounded away from zero. This proves from (3.28)–(3.29) that $\phi_{t\theta}, \phi_{\theta\theta}$ are uniformly bounded. The uniformly bound of $\phi_{tt}$ follows from the evolution equation (2.8).

*Step 5* Bounds on the first-order derivatives of matter quantities and second-order derivatives of the field components.

Since the second-order derivatives of $R$ and $\phi$ are bounded, one follows the techniques developed in [1] (Step 4) to bound first-order derivatives of $f$ and second-order derivatives of $U$, $A$, and $\mu$. There is a minor change on the term $k_\theta = \partial_\theta(\rho - P_1 + P_2 - P_3)$ where $\partial_\theta V(\phi)$ appears and is bounded by previous steps.

*Step 6* Bounds on higher order derivatives and completion of the proof.

The method described above can be used for obtaining bounds on higher derivatives as well. Hence, we have uniform bounds on the functions $R$, $U$, $A$, $\mu$, $f$, $\phi$, and all their derivatives on the interval $(t, t_0]$ if $R$ stays bounded away from zero. This implies that the solution extends to $t \to -\infty$ in conformal coordinates. The proof of the theorem is complete using the geometrical arguments developed in [1, 3].

## 4 The $T^2$ Symmetry Case

We consider now a solution of the Einstein-Vlasov-scalar field system where all unknowns are invariant under the $T^2$-symmetry with the twists different from zero (cf. [11, 15]). In conformal coordinates, the metric $g$ takes the form

$$ds^2 = -e^{2(\mu-U)}dt^2 + e^{2(\mu-U)}d\theta^2 + e^{2U}[dx + Ady + (G + AH)d\theta]^2 + R^2e^{-2U}(dy + Hd\theta)^2; \tag{4.1}$$

where $\mu$, $U$, $R$, $A$, $H$, and $G$ are unknown real functions of $t$ and $\theta$ variables, periodic in $\theta$ with period 1.

Using the results of [15] or [11], the complete Einstein-Vlasov-scalar field system (1.2)–(1.4) can be written in the following form:

*The Einstein-matter constraint equations:*

$$\begin{aligned} &U_t^2 + U_\theta^2 + \tfrac{e^{4U}}{4R^2}(A_t^2 + A_\theta^2) + \tfrac{R_{\theta\theta}}{R} - \tfrac{\mu_t R_t}{R} - \tfrac{\mu_\theta R_\theta}{R} \\ &= -\tfrac{e^{-2(\mu-2U)}}{4}\Gamma^2 - \tfrac{e^{-2\mu}}{4}H_t^2 - e^{2(\mu-U)}\rho, \end{aligned} \tag{4.2}$$

$$2U_tU_\theta + \frac{e^{4U}}{2R^2}A_tA_\theta - \frac{R_{t\theta}}{R} - \frac{\mu_t R_\theta}{R} - \frac{\mu_\theta R_t}{R} = e^{2(\mu-U)}J_1. \tag{4.3}$$

*The Einstein-matter evolution equations:*

$$\begin{aligned} U_{tt} - U_{\theta\theta} = {} & \tfrac{U_\theta R_\theta}{R} - \tfrac{U_t R_t}{R} + \tfrac{e^{4U}}{2R^2}(A_t^2 - A_\theta^2) \\ & + \tfrac{e^{-2(\mu-2U)}}{2}\Gamma^2 + \tfrac{1}{2}e^{2(-\mu+2U)}(\rho - P_1 + P_2 - P_3), \end{aligned} \tag{4.4}$$

$$A_{tt} - A_{\theta\theta} = \frac{A_tR_t}{R} - \frac{A_\theta R_\theta}{R} + 4(A_\theta U_\theta - A_t U_t) + 2Re^{2\mu-4U}S_{23} + R^2e^{-2\mu}\Gamma H_t, \tag{4.5}$$

$$R_{tt} - R_{\theta\theta} = Re^{2(\mu-U)}(\rho - P_1) + \frac{e^{-2(\mu-2U)}}{2}R\Gamma^2 + \frac{e^{-2\mu}}{2}R^3H_t^2, \tag{4.6}$$

$$\begin{aligned} \mu_{tt} - \mu_{\theta\theta} = {} & U_\theta^2 - U_t^2 + \tfrac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - e^{2(\mu-U)}P_3 - \tfrac{A^2}{R^2}e^{2(\mu+U)}P_2 \\ & - \tfrac{2A}{R}e^{2\mu}S_{23} - \tfrac{e^{-2(\mu-2U)}}{4}R\Gamma^2 - \tfrac{3e^{-2\mu}}{4}R^2H_t^2. \end{aligned} \tag{4.7}$$

$$\phi_{tt} - \phi_{\theta\theta} = E(t,\theta)\phi_t + F(t,\theta)\phi_\theta - e^{2(\mu-U)}V'(\phi). \tag{4.8}$$

*The auxiliary equations:*

$$\partial_\theta(Re^{-2(\mu-2U)}\Gamma) = -2Re^{\mu}J_2 \tag{4.9}$$

$$\partial_t(Re^{-2(\mu-2U)}\Gamma) = 2Re^{\mu}S_{12} \tag{4.10}$$

$$\partial_\theta(R^3e^{-2\mu}H_t) + Re^{-2(\mu-2U)}A_\theta\Gamma = -2R^2e^{\mu-2U}J_3 \tag{4.11}$$

$$\partial_t(R^3e^{-2\mu}H_t) + Re^{-2(\mu-2U)}A_t\Gamma = 2R^2e^{\mu-2U}S_{13}. \tag{4.12}$$

*The Vlasov equation:*

$$\frac{\partial f}{\partial t} + \frac{v^1}{v^0}\frac{\partial f}{\partial \theta} - \Big[(\mu_\theta - U_\theta)v^0 + (\mu_t - U_t)v^1 - \frac{e^{2U}A_\theta}{R}\frac{v^2 v^3}{v^0} + \frac{U_\theta}{v^0}((v^3)^2 + (v^2)^2)$$
$$- \frac{R_\theta}{R}\frac{(v^3)^2}{v^0} + e^{-\mu}(e^{2\mu}\Gamma v^2 + RH_t v^3)\Big]\frac{\partial f}{\partial v^1} - \Big[U_\theta \frac{v^1 v^2}{v^0} + U_t v^2\Big]\frac{\partial f}{\partial v^2}$$
$$+ \Big[\Big(\frac{R_t}{R} - U_t\Big)v^3 + \Big(U_\theta - \frac{R_\theta}{R}\Big)\frac{v^1 v^3}{v^0} + \frac{e^{2U}v^2}{R}\Big(A_t + A_\theta\frac{v^1}{v^0}\Big)\Big]\frac{\partial f}{\partial v^3} = 0, \quad (4.13)$$

where $\Gamma = G_t + AH_t$,

$$J_k = \int_{\mathbb{R}^3} f(t,\theta,v)v^k dv, \quad k \in \{2,\ 3\},$$
$$S_{ij} = \int_{\mathbb{R}^3} f(t,\theta,v)\frac{v^i v^j}{v^0}dv, \quad i,\ j \in \{1,\ 2,\ 3\};\ i \neq j,$$
$$E(t,\theta) = -\frac{R_t}{R} + (R^2 H_t H + R_t H_t H)e^{-2\mu}$$
$$+(2U_t G^2 + 2AGHU_t + GG_t + A_t HG + AH_t G)e^{-2\mu+4U},$$

and

$$F(t,\theta) = \frac{R_\theta}{R} + (R^2 H_\theta H - RR_\theta H^2 + 2R_\theta H^2 - 5R^2H^2U_\theta)e^{-2\mu}$$
$$+(G^2U_\theta + AA_\theta H^2 + GG_\theta + 4AU_\theta HG - 2AHU_\theta + AH_\theta G$$
$$+AG_\theta H + A_\theta HG + A^2 HH_\theta + A^2H^2U_\theta)e^{-2\mu+4U}.$$

*Step 7* Monotonicity of $R$ and bounds on its first derivatives

Using the constraint (4.2) and (4.3), the extra non-negative term $\frac{e^{-2\mu+4U}}{4}R\Gamma^2 + \frac{R^2e^{-2\mu}}{4}RH_t^2$ adds to the right hand side of each relation (3.1) and (3.2). The same method as in Step 1 follows.

*Step 8* Bounds on $U$, $A$, $\mu$, and their first derivatives.

Define

$$X = \frac{1}{2}R(U_t^2 + U_\theta^2) + \frac{e^{4U}}{4R^2}(A_t^2 + A_\theta^2); \quad Y = RU_tU_\theta + \frac{e^{4U}}{4R}A_tA_\theta.$$

Following Step 2, relations (3.9) and (3.10) are respectively replaced by

$$\partial_\tau(X+Y) = -\frac{1}{2}R_\xi\left(U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2}(-A_t^2 + A_\theta^2)\right) + \frac{e^{2U}}{2R}A_\xi R^2 e^{2(\mu+U)}\Gamma H_t$$
$$-\frac{R}{2}U_\xi\left(e^{-2\mu+4U}\Gamma^2 + e^{2(\mu-U)}(\rho - P_1 + P_2 - P_3)\right) + A_\xi e^{2\mu}S_{23}.$$

and

$$\partial_\xi(X-Y) = -\frac{1}{2}R_\tau\left(U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2}(-A_t^2 + A_\theta^2)\right) + \frac{e^{2U}}{2R}A_\tau R^2 e^{2(\mu+U)}\Gamma H_t$$
$$-\frac{R}{2}U_\tau\left(e^{-2\mu+4U}\Gamma^2 + e^{2(\mu-U)}(\rho - P_1 + P_2 - P_3)\right) + A_\tau e^{2\mu}S_{23}.$$

After integration and summation, we obtain relation (3.11) with

$$\begin{aligned} K_1 &= -\frac{1}{2}R_\tau\left(U_t^2 - U_\theta^2 + \frac{e^{4U}}{R^2}(-A_t^2 + A_\theta^2)\right),\\ K_2 &= -\frac{1}{2}R_\xi\left(U_t^2 - U_\theta^2 + \frac{e^{4U}}{R^2}(-A_t^2 + A_\theta^2)\right),\\ T_1 &= \frac{R}{2}e^{2(\mu-U)}(\rho - P_1 + P_2 - P_3) + \frac{R}{2}e^{-2\mu+4U}\Gamma^2,\\ T_2 &= e^{2(\mu-U)}S_{23} + R^2e^{2(\mu-U)}\Gamma H_t; \end{aligned}$$

and

$$\begin{aligned} T_1 &\le Re^{2(\mu-U)}(\rho - P_1) + \frac{R}{2}e^{-2\mu+4U}\Gamma^2,\\ |T_2| &\le \frac{1}{2}e^{2(\mu-U)}(\rho - P_1) + \frac{R^2}{2}e^{2(\mu-U)}(\Gamma^2 + H_t^2). \end{aligned}$$

Consequently, relations (3.15) and (3.16) are respectively replaced by

$$R_\xi(t_0, \theta + t_0 - t) - R_\xi(t,\theta) = \tfrac{1}{2}\int_t^{t_0}[Re^{2(\mu-U)}(\rho - P_1) + \tfrac{Re^{-2\mu}}{2}(e^{4U}\Gamma^2 + R^2H_t^2)](s, \theta + t_0 - s)ds, \tag{4.14}$$

$$R_\tau(t_0, \theta - t_0 + t) - R_\tau(t,\theta) = \tfrac{1}{2}\int_t^{t_0}[Re^{2(\mu-U)}(\rho - P_1) + \tfrac{Re^{-2\mu}}{2}(e^{4U}\Gamma^2 + R^2H_t^2)](s, \theta - t_0 + s)ds. \tag{4.15}$$

This permits as in Step 2 to obtain bounds on $A$, $U$, and their first derivatives. The bounds of $\mu$ and its first derivatives follow from relations (3.26) or (3.27) replaced by

$$\begin{aligned} \partial_\tau\mu_\xi = {} & U_\theta^2 - U_t^2 + \tfrac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - \tfrac{e^{-2\mu+4U}}{4}R\Gamma^2 - \tfrac{3R^2e^{-2\mu}}{4}H_t^2\\ & -e^{2(\mu-U)}P_3 - \tfrac{A^2}{R^2}e^{2(\mu+U)}P_2 - \tfrac{2A}{R}e^{2\mu}S_{23}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} \partial_\xi\mu_\tau = {} & U_\theta^2 - U_t^2 + \tfrac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - \tfrac{e^{-2\mu+4U}}{4}R\Gamma^2 - \tfrac{3R^2e^{-2\mu}}{4}H_t^2\\ & -e^{2(\mu-U)}P_3 - \tfrac{A^2}{R^2}e^{2(\mu+U)}P_2 - \tfrac{2A}{R}e^{2\mu}S_{23}. \end{aligned} \tag{4.17}$$

Under the assumption $|\phi_t| \le |\phi_\theta|$, we deduce that

$$e^{2(\mu-U)}P_k \le e^{2(\mu-U)}(\rho - P_1) + \frac{1}{2}e^{2(-\mu+U)}(\phi_t^2 - \phi_\theta^2) - 3V(\phi) \le e^{2(\mu-U)}(\rho - P_1) \tag{4.18}$$

Using inequalities (3.12), (4.18) and the fact that the right hand sides of (4.14), (4.15) are bounded (Step 7), we conclude that the integral along null paths for the matter terms in the right hand sides of (4.16), (4.17) are bounded since $A$, $U$, $A_t$, $A_\theta$, $U_t$, $U_\theta$ are bounded. We obtain that $|\mu_\xi|$ and $|\mu_\tau|$ are bounded and therefore $\mu_t$ and $\mu_\theta$ are bounded.

*Step 9* Bounds on $G$, $H$, their derivatives, and the support of the momentum.

Consider the characteristic system associated to the Vlasov equation (4.13). It is analogous to the one in Step 3, except the term $-e^{-\mu}(e^{2\mu}\Gamma V^2 + RH_tV^3)$ which is added in the second hand of the second equation $\frac{dV^1}{ds}$ of the system. Since $|V^k| < V^0$, $k = 1,\ 2,\ 3$, the integration and Gronwall's lemma applied respectively to the third and fourth equations of this system give with previous Steps (7 and 8) uniform bounds of $|V^k(t)|$, $k = 2, 3$ on $(t, t_0)$. So, we can conclude that $\operatorname{Sup}\{|V^2| + |V^3| + 1 : \exists(s, r, v^1) \in [0, t] \times \mathbb{R}_+ \times \mathbb{R}$ with $f(s, r, v) \neq 0\}$ is also uniformly bounded. Now, define $Q^1(t) = \operatorname{Sup}\{|V^1| + 1 : \exists(s, r, v^2, v^3) \in$

$[0, t]\times\mathbb{R}_+\times\mathbb{R}^2$ with $f(s, r, v) \neq 0\}$. Now integrate the second equation of the characteristic system and use Steps 7 and 8 and the fact that $|V^k| < V^0$, $|V^k| < C$ to obtain

$$Q^1(t) \leq Q^1(t_0) + C \int_{t_1}^{t_0} [Q^1(s) + \sup_\theta|\Gamma|(s, \theta) + \sup_\theta|H_t|(s, \theta)]ds. \tag{4.19}$$

Add and subtract respectively auxiliary equations (4.9)–(4.10) to obtain

$$\partial_\xi(Re^{-2(\mu-2U)}\Gamma) = 2Re^\mu(S_{12} - J_2),$$
$$\partial_\tau(Re^{-2(\mu-2U)}\Gamma) = 2Re^\mu(S_{12} + J_2).$$

Integrating along null paths and previous steps give

$$\sup_\theta|\Gamma|(t, \theta) \leq \sup_\theta|\Gamma|(t_0, \theta) + C \int_{t_1}^{t_0} [1 + Q^1(s)]ds. \tag{4.20}$$

Analogously, (4.11)–(4.12) give

$$\sup_\theta|H_t|(t, \theta) \leq \sup_\theta|H_t|(t_0, \theta) + C \int_{t_1}^{t_0} [1 + Q^1(s) + \sup_\theta|\Gamma|(s, \theta)]ds. \tag{4.21}$$

Adding (4.19)–(4.21) and applying Gronwall's lemma give uniform bounds on $H_t$, $\Gamma$, and $Q^1(t)$. Then $H$, $G_t$, and $G$ are bounded. From (4.11), we deduce bounds on $H_{t\theta}$ and then on $H_\theta$. Consequently, (4.9) gives bounds on $G_{t\theta}$ and then on $G_\theta$. We deduce respectively from (4.12) and (4.10) the bounds on $H_{tt}$ and $G_{tt}$. Since $Q^1(t)$ and $V^k$ are bounded, the integral terms of the matter quantities are also bounded.

*Step 10* Bounds on $\phi$ and its first derivatives.

Define

$$M = \frac{1}{2}R(\phi_t^2 + \phi_\theta^2) + \frac{R}{2}\phi^2; \quad N = R\phi_t\phi_\theta.$$

After some calculation and using (4.8), we obtain

$$\partial_\tau(M + N) = \frac{R_\tau}{R}(M + N) - R\phi_\xi(E\phi_t + F\phi_\theta + e^{2(\mu-U)}V'(\phi)) + R\phi_\tau\phi,$$
$$\partial_\xi(M - N) = \frac{R_\xi}{R}(M - N) - R\phi_\tau(E\phi_t + F\phi_\theta + e^{2(\mu-U)}V'(\phi)) + R\phi_\xi\phi.$$

Integrating these along null paths and adding results give

$$\begin{aligned} M(t_1, \theta) = {} & \tfrac{1}{2}(M + N)(t_0, \theta - (t_0 - t_1)) + \tfrac{1}{2}(M - N)(t_0, \theta + (t_0 - t_1)) \\ & + \textstyle\int_{t_1}^{t_0} \left[R\phi_\xi(E\phi_t + F\phi_\theta)(s, \theta - (s - t_1)) + R\phi_\tau(E\phi_t + F\phi_\theta)(s, \theta + (s - t_1))\right] ds \\ & + \textstyle\int_{t_1}^{t_0}[(R\phi_\xi e^{2(\mu-U)}V'(\phi) - R\phi_\tau\phi)(s, \theta - (s - t_1)) \\ & + (R\phi_\tau e^{2(\mu-U)}V'(\phi) - R\phi_\xi\phi)(s, \theta - (s - t_1))]ds. \end{aligned}$$

Following previous Steps 7, 8, and 9, we have

$$\sup_\theta M(t, \theta) \leq \sup_\theta M(t_0, \theta) + C \int_{t_1}^{t_0} \sup_\theta M(s, \theta)ds. \tag{4.22}$$

We deduce by Gronwall's lemma the uniform bounded of $\sup_\theta M(t, \theta)$ which leads to the bounds of $\phi$, $\phi_t$, $\phi_\theta$, and $V(\phi)$.

*Step 11* Following Steps 4, 5, and 6, ones obtains with minor changes, bounds on the first-order derivatives of matter quantities, second and higher order derivatives of field components and matter quantities.

We conclude this section by the following theorem:

**Theorem 4.1** *Let* $(T^3, \mu_0, \mu_1, A_0, A_1, U_0, U_1, G_0, G_1, H_0, H_1, f_0, \phi_0, \phi_1)$ *be regular initial data that satisfy the constraint equations* (4.2)–(4.3) *where the metric is coordinatized as in* (4.1)*. Let* $(g, f, \phi)$ *be the local regular solution that corresponds to the initial data and* $]T, t_0]$ *be the maximal interval of existence. There exists a globally hyperbolic spacetime* $(M, g, f, \phi)$ *such that*
(i) $M = [0, t_0[\times T^3$,
(ii) $(g, f, \phi)$ *satisfies the Einstein-Vlasov-scalar field system in areal coordinates,*
(iii) $(M, g, f, \phi)$ *is isometrically diffeomorphic to the maximal globally hyperbolic development of the initial data* $(T^3, \mu_0, \mu_1, A_0, A_1, U_0, U_1, G_0, G_1, H_0, H_1, f_0, \phi_0, \phi_1)$.

## 5 Geodesic Completeness

Consider a solution $s \longmapsto (x^\alpha(s), p^\alpha(s))$ of the trajectory equations (2.10) which exists on the maximal interval $I = ]s_-, s_+[$. Since particles are future pointing, we have $\frac{dt}{ds} = p^0 > 0$. We can then parameterize the trajectory by the time coordinate $t \in I$ and obtain the system:

$$\frac{dx^i}{dt} = \frac{p^i}{p^0}; \quad \frac{dp^i}{dt} = -\Gamma^i_{\beta\gamma}\frac{p^\beta p^\gamma}{p^0}, \quad i = 1, 2, 3.$$

Since supp$f$ is compact, the right hand side of the previous system is linearly bounded in $p^i$ with respect to the two directions of time (see [14] for the future direction). Thus, $t(s_\pm) = \pm\infty$. Now, due to the inequality $\frac{dt}{ds} = p^0 \leq C$, one obtains by integration

$$\int_0^s dt \leq C\int_0^s ds' \text{ i.e., } \quad t(s) - t(0) \leq Cs.$$

Thus, $s_\pm = \pm\infty$, and $I = \mathbb{R}$. We conclude that geodesics are complete for the Einstein-Vlasov-scalar field system with Gowdy or $T^2$ symmetry.

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## References

1. Andréasson, H.: Global foliations of matter spacetimes with Gowdy symmetry. Comm. Math. Phys. **206**(2), 337–365 (1999)
2. Andréasson, H., Rein, G., Rendall, A.D.: On the Einstein-Vlasov system with hyperbolic symmetry. Math. Proc. Cambridge Philos. Soc. **124**(3), 529–549 (2003)
3. Berger, B.K., Chruściel, P., Isenberg, J., Moncrief, V.: Global foliations of vacuum spacetimes with $T^2$ isometry. Ann. Phys. **260**(1), 117–148 (1997)
4. Ehlers, J. In: Israel, W. (ed.): A Survey of General Relativity Theory Astrophysics and Cosmology, pp. 1–125. Reidel, Dordrecht (1973)
5. Choquet-Bruhat, Y.: Problème de Cauchy pour le système intégro-différentiel d'Einstein-Liouville. Ann. Inst. Fourier **21**(3), 181–201 (1971)
6. Choquet-Bruhat, Y., Geroch, R.: Global aspects of the Cauchy problem in general relativity. Comm. Math. Phys. **14**, 329–335 (1969)
7. Chruściel, P.T.: On space-times with U(1)× U(1) symmetric compact Cauchy surfaces. Ann. Phys. **202**(1), 100–150 (1990)
8. Gowdy, R.H.: Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: topologies and boundary conditions. Ann. Phys. **83**, 203–241 (1974)

9. Ringström, H.: Future stability of the Einstein non-linear scalar field system. Invent. Math. **173**(1), 123–208 (2008)
10. Straumann, N.: On the cosmological constant problems and the astronomical evidence for a homogeneous energy density with negative pressure. In: Duplantier, B., Rivasseau, V. (eds.) Vaccum Energy-Renormalization, Poincaré Seminar. Birkhauser, Basel (2002)
11. Smulevici, J.: On the area of the symmetry orbits of cosmological spacetimes with toroidal or hyperbolic symmetry. Anal. PDE. **4**(2), 191–245 (2011)
12. Tegankong, D.: On the area of the symmetry orbits of the Einstein-Vlasov-scalar field system with plane and hyperbolic symmetry. J. Inform. Math. Sci. **5**(3), 131–142 (2013)
13. Tegankong, D., Rendall, A.D.: On the nature of initial singularities for solutions of the Einstein-Vlasov-scalar field system with surface symmetry. Math. Proc. Cambridge Philos. Soc. **141**(3), 547–562 (2006)
14. Tegankong, D.: The Einstein-Vlasov-scalar field system with Gowdy symmetry in the expanding direction. Classical Quantum Gravity **31**, 18pp (2014)
15. Weaver, M.: On the area of the symmetry orbits in $T^2$ symmetric spacetimes with Vlasov matter. Classical Quantum Gravity **21**(4), 1079–1097 (2004)