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Normal Reduction Numbers for Normal Surface Singularities with Application to Elliptic Singularities of Brieskorn Type

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Abstract
In this paper, we give a formula for normal reduction number of an integrally closed $\mathfrak{m}$-primary ideal of a two-dimensional normal local ring $(A, \mathfrak{m})$ in terms of the geometric genus $p_g(A)$ of A. Also, we compute the normal reduction number of the maximal ideal of Brieskorn hypersurfaces. As an application, we give a short proof of a classification of Brieskorn hypersurfaces having elliptic singularities.

Keywords Normal reduction number · Geometric genus · Hypersurface of Brieskorn type

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1 Introduction

For a Noetherian local ring $(A, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal I, let $\overline{I}$ denote the integral closure, that is, $z \in \overline{I}$ if and only if $z^n + c_1 z^{n-1} + \cdots + c_n = 0$ for some $n \geq 1$ and $c_i \in I^i$ $(i = 1, \ldots, n)$.

For a given Noetherian local ring $(A, \mathfrak{m})$ and an integrally closed $\mathfrak{m}$-primary ideal I (i.e., $\overline{I} = I$) with minimal reduction Q, we are interested in the question:

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Question What is the minimal number r such that $\overline{I^r} \subset Q$ for every $\mathfrak{m}$-primary ideal I of A and its minimal reduction Q?

One example of this direction is the Briançon-Skoda Theorem saying;
If $(A, \mathfrak{m})$ is a d-dimensional rational singularity (characteristic 0) or an F-rational ring (characteristic $p > 0$), then $\overline{I^d} \subset Q$ and d is the minimal possible number in this case (cf. [3, 9]).

The aim of our paper is to answer this question in the case of normal two-dimensional local rings using resolution of singularities.

In what follows, we always assume that $(A, \mathfrak{m})$ is an excellent two-dimensional normal local domain. For any $\mathfrak{m}$-primary integrally closed ideal $I \subset A$ (e.g., the maximal ideal $\mathfrak{m}$) and its minimal reduction Q of I, we define two normal reduction numbers as follows:

$$\begin{aligned} \mathrm{nr}(I) &= \min\{n \in \mathbb{Z}_{\geq 0} \mid \overline{I^{n+1}} = Q\overline{I^n}\}, \\ \bar{\mathrm{r}}(I) &= \min\{n \in \mathbb{Z}_{\geq 0} \mid \overline{I^{N+1}} = Q\overline{I^N} \text{ for every } N \geq n\}. \end{aligned}$$

These are analogues of the reduction number $r_Q(I)$ of an ideal $I \subset A$. But in general, $r_Q(I)$ is not independent of the choice of a minimal reduction Q. On the other hand, $\mathrm{nr}(I) = \bar{\mathrm{r}}(I)$ is *not* known in general.

Also, we define the following:

$$\begin{aligned} \mathrm{nr}(A) &= \max\{\mathrm{nr}(I) \mid I \text{ is an } \mathfrak{m}\text{-primary integrally closed ideal of } A\}, \\ \bar{\mathrm{r}}(A) &= \max\{\bar{\mathrm{r}}(I) \mid I \text{ is an } \mathfrak{m}\text{-primary integrally closed ideal of } A\}. \end{aligned}$$

These invariants of A characterize "good" singularities.

Example 1.1 (See [8] for (1), [12] for (2)) Suppose that A is not regular.

(1) A is a rational singularity ($p_g(A) = 0$) if and only if $\mathrm{nr}(A) = \bar{\mathrm{r}}(A) = 1$.
(2) If A is an elliptic singularity, then $\bar{\mathrm{r}}(A) = 2$, where we say that A is an elliptic singularity if the arithmetic genus of the fundamental cycle on any resolution of A is 1.

One of the main aims is to compare these invariants with geometric invariants (e.g., geometric genus $p_g(A)$). In [13], we have shown that $\mathrm{nr}(A) \leq p_g(A) + 1$. But actually, it turns out that we have a much better bound (see Theorem 2.9).

Theorem 1.2 *If $(A, \mathfrak{m})$ is a normal two-dimensional local ring, then* $p_g(A) \geq \binom{\mathrm{nr}(A)}{2}$.

On the other hand, sometimes we have $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m})$. For example, if $A = K[[x, y, z]]/(f)$, where f is a homogeneous polynomial of degree $d \geq 2$ with isolated singularity, it is easy to see $\mathrm{nr}(\mathfrak{m}) = d - 1$. If $d \leq 4$, we can see by Theorem 1.2 that $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = d - 1$. We do not have an answer yet if $d = 5$.

Question 1.3 If A is a homogeneous hypersurface singularity of degree d, then $\mathrm{nr}(A) = d - 1$?

To have examples for this theory, we compute $\mathrm{nr}(\mathfrak{m})$ of Brieskorn hypersurface singularities, that is, two-dimensional normal local domains

$$A = K[[x, y, z]]/(x^a + y^b + z^c),$$

where K is an algebraically closed field of any characteristic and $2 \leq a \leq b \leq c$.

Note that our approach in this paper will be extended to the case of Brieskorn complete intersection singularity (see [11]).

We can get an explicit value of $\mathrm{nr}(\mathfrak{m})$ in this case.

Theorem 3.1 *Let A be a Brieskorn hypersurface singularity as above. Put $\mathfrak{m} = (x, y, z)A$ and $Q = (y, z)A$. Then*

$$\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = \left\lfloor \frac{(a-1)b}{a} \right\rfloor.$$

Moreover, if we put $n_k = \lfloor \frac{kb}{a} \rfloor$ for each $k \geq 0$, then

$$\overline{\mathfrak{m}^n} = Q^n + xQ^{n-n_1} + x^2 Q^{n-n_2} + \cdots + x^{a-1} Q^{n-n_{a-1}}.$$

As an application of the theorem, we can show that the Rees algebra $\mathcal{R}(\mathfrak{m})$ is normal if and only if $\bar{\mathrm{r}}(\mathfrak{m}) = a - 1$ (see Corollary 3.7). Moreover, we can determine $\ell_A(\mathfrak{m}^{n+1}/Q\mathfrak{m}^n)$ for every $n \geq 0$ and $q(\mathfrak{m}) = \ell_A(H^1(X, \mathcal{O}_X(-M)))$, where $X \to \mathrm{Spec} A$ denotes the resolution of singularity of $\mathrm{Spec} A$ and M denotes the maximal ideal cycle on X.

In the last section, we discuss Brieskorn hypersurfaces with elliptic singularities. In fact, the first author proved that if A is an elliptic singularity then $\mathrm{nr}(A) = 2$. In particular, if A is an elliptic singularity then $\mathrm{nr}(\mathfrak{m}) \leq 2$. If, in addition, A is a Brieskorn hypersurface singularity $A = K[[x, y, z]]/(x^a + y^b + z^c)$, then our theorem shows that $\lfloor (a-1)b/a \rfloor \leq 2$. Using this fact, we can classify all Brieskorn hypersurfaces with elliptic singularity (see Theorem 4.4).

We are interested to know if $\mathrm{nr}(A)$ characterizes elliptic singularities or not. Namely, the question is equivalent to say, if A is not rational or elliptic, then does there exist I such that $\mathrm{nr}(I) \geq 3$? We can find such an ideal for all non-elliptic Brieskorn hypersurface singularity except $(a, b, c) = (3, 4, 6)$ or $(3, 4, 7)$.

2 Normal Reduction Numbers and Geometric Genus

Throughout this paper, let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain. In another word, A is a local domain with a resolution of singularities $f : X \to \mathrm{Spec}(A)$. For a coherent $\mathcal{O}_X$-Module $\mathcal{F}$, we denote by $h^i(\mathcal{F})$ the length $\ell_A(H^i(\mathcal{F}))$.

We define the *geometric genus* of A by the following:

$$p_g(A) = h^1(\mathcal{O}_X),$$

which is independent of the choice of resolution of singularities. When $p_g(A) = 0$, A is called a *rational singularity*.

Let $I \subset A$ be an $\mathfrak{m}$-primary integrally closed ideal. Then, there exist a resolution of singularity $X \to \mathrm{Spec} A$ and an anti-nef cycle Z on X so that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = H^0(\mathcal{O}_X(-Z))$. Then, we say that I is *represented* by Z on X and write $I = I_Z$. Then, $I_{nZ} = \overline{I^n}$ for every integer $n \geq 1$.

In what follows, let $A, X, I = I_Z$ be as above.

The authors have studied p_g-ideals in [13–15]. So, we first recall the notion of p_g-ideals in terms of $q(kI)$.

Definition 2.1 Put $q(0I) = h^1(\mathcal{O}_{\mathcal{X}})$, $q(I) := h^1(\mathcal{O}_X(-Z))$ and $q(nI) = q(\overline{I^n})$ for every integer $n \geq 1$.

Theorem 2.2 [13] *The following statements hold.*

(1) $0 \le q(I) \le p_g(A)$.
(2) $q(kI) \ge q((k+1)I)$ *for every integer* $k \ge 1$.
(3) $q(nI) = q((n+1)I) = q((n+2)I) = \cdots$ *for some integer* $n \ge 0$.

Definition 2.3 [13] The ideal I is called the p_g*-ideal* if $q(I) = p_g(A)$.

Example 2.4 Any two-dimensional excellent normal local domain over an algebraically closed field admits a p_g-ideal. Moreover, if A is a rational singularity, then every $\mathfrak{m}$-primary integrally closed ideal is a p_g-ideal.

2.1 Upper Bound on Normal Reduction Numbers

Let Q be a minimal reduction of I. Then, there exists a nonnegative integer r such that $\overline{I^{r+1}} = Q\overline{I^r}$. This is independent of the choice of a minimal reduction Q of I (see, e.g., [5, Theorem 4.5]). So we can define the following notion.

Definition 2.5 (Normal reduction number) Put

$$\begin{aligned} \mathrm{nr}(I) &= \min\{n \in \mathbb{Z}_{\ge 0} \mid \overline{I^{n+1}} = Q\overline{I^n}\}, \\ \bar{\mathrm{r}}(I) &= \min\{n \in \mathbb{Z}_{\ge 0} \mid \overline{I^{N+1}} = Q\overline{I^N} \text{ for every } N \ge n\}. \end{aligned}$$

We call them the *normal reduction numbers* of I. We also define

$$\begin{aligned} \mathrm{nr}(A) &= \max\{\mathrm{nr}(I) \mid I \text{ is a } \mathfrak{m}\text{-primary integrally closed ideal of } A\}, \\ \bar{\mathrm{r}}(A) &= \max\{\bar{\mathrm{r}}(I) \mid I \text{ is a } \mathfrak{m}\text{-primary integrally closed ideal of } A\}, \end{aligned}$$

which are called the *normal reduction numbers* of A.

Our study on normal reduction numbers is motivated by the following observation: For an $\mathfrak{m}$-primary ideal I in a two-dimensional excellent normal local domain A, I is a p_g-ideal if and only if $\bar{\mathrm{r}}(I) = 1$.

By definition, $\mathrm{nr}(I) \le \bar{\mathrm{r}}(I)$ holds in general. In the next section, we show that $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m})$ holds true for any Brieskorn hypersurface $A = K[[x, y, z]]/(x^a + y^b + z^c)$. But it seems to be open whether equality always holds for other integrally closed $\mathfrak{m}$-primary ideals.

Question 2.6 When does $\mathrm{nr}(I) = \bar{\mathrm{r}}(I)$ hold?

In order to state the main result in this section, we recall the following lemma, which gives a relationship between $\mathrm{nr}(I)$ and $q(kI)$.

Lemma 2.7 *For any integer* $n \ge 1$, *we have*

$$2 \cdot q(nI) + \ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = q((n+1)I) + q((n-1)I).$$

Proof Assume $Q = (a, b)$ and consider the exact sequence as follows:

$$0 \to \mathcal{O}_X((n-1)Z) \to \mathcal{O}_X(-Z)(-nZ)^{\oplus 2} \to \mathcal{O}_X(-(n+1)Z) \to 0,$$

where the map $\mathcal{O}_X(-nZ)^{\oplus 2} \to \mathcal{O}_X(-(n+1)Z)$ is defined by $(x, y) \mapsto ax + by$ as in Lemma 4.3 of [15]. By taking the cohomology long exact sequence, we have the following exact sequence:

$$\begin{aligned} &\to H^0(\mathcal{O}_X(-nZ))^{\oplus 2} \xrightarrow{\varphi} H^0(\mathcal{O}_X(-(n+1)Z)) \\ \to H^1(\mathcal{O}_X(-(n-1)Z)) \to\ &H^1(\mathcal{O}_X(-nZ))^{\oplus 2} \to H^1(\mathcal{O}_X(-(n+1)Z)) \to 0. \end{aligned}$$

Since $\mathrm{Coker}(\varphi) \cong \overline{I^{n+1}}/Q\overline{I^n}$, we obtain the required assertion. □

The lemma gives another description of $\mathrm{nr}(I)$ in terms of $q(kI)$:

$$\mathrm{nr}(I) = \min\{n \in \mathbb{Z}_{\ge 1} \mid q((n-1)I), q(nI), q((n+1)I) \text{ forms an arithmetic sequence}\}.$$

In particular,

$$\mathrm{nr}(I) \le \min\{n \in \mathbb{Z}_{\ge 0} \mid q((n-1)I) = q(nI) = q((n+1)I) = \cdots\} = \bar{\mathrm{r}}(I).$$

If the following question has an affirmative answer for I, then $\mathrm{nr}(I) = \bar{\mathrm{r}}(I)$ holds true.

Question 2.8 When is $\ell_A(\overline{I^{n+1}}/Q\overline{I^n})$ a non-increasing function of n?

The main result in this section is the following theorem, which refines an inequality $\mathrm{nr}(I) \le p_g(A) + 1$ (see [14, Lemma 3.1]).

Theorem 2.9 *For any $\mathfrak{m}$-primary integrally closed ideal $I \subset A$, we have*

$$p_g(A) \ge \binom{r}{2} + q(rI),$$

where $r = \mathrm{nr}(I)$. In particular, $p_g(A) \ge \binom{\mathrm{nr}(A)}{2}$.

Proof Suppose $\mathrm{nr}(I) = r$. Then, since $\overline{I^{k+1}} \ne Q\,\overline{I^k}$ for every $k = 1, 2, \ldots, r-1$ and $\overline{I^{r+1}} = Q\,\overline{I^r}$, we have

$$\begin{aligned} q((r-1)I) - q(rI) &= q(rI) - q((r+1)I), \\ q((r-2)I) - q((r-1)I) &\ge q((r-1)I) - q(rI) + 1, \\ &\vdots \\ p_g(A) - q(I) &\ge q(I) - q(2I) + 1. \end{aligned}$$

Thus, if we put $a_k = q((r-k)I)$ for $k = 0, 1, \ldots, r$, then we get

$$a_k - a_{k-1} \ge a_{k-1} - a_{k-2} + 1 \ge \cdots \ge \{a_1 - a_0\} + (k-1) \ge k-1.$$

Hence,

$$p_g(A) = a_r = \sum_{k=1}^{r}(a_k - a_{k-1}) + a_0 \ge \sum_{k=1}^{r}(k-1) + a_0 = \frac{r(r-1)}{2} + q(rI),$$

as required.

The last assertion immediately follows from the definition of $\mathrm{nr}(A)$. □

The above theorem gives a best possible bound (see also the next section).

Example 2.10 If $p_g(A) < \binom{\mathrm{nr}(J)+1}{2}$ for some $\mathfrak{m}$-primary integrally closed ideal $J \subset A$, then $\mathrm{nr}(A) = \mathrm{nr}(J)$.

Proof Suppose $\mathrm{nr}(A) \neq \mathrm{nr}(J)$. Then, $\mathrm{nr}(A) \geq \mathrm{nr}(J) + 1$. By assumption and the theorem, we have

$$\binom{\mathrm{nr}(A)}{2} \leq p_g(A) < \binom{\mathrm{nr}(J)+1}{2} \leq \binom{\mathrm{nr}(A)}{2}.$$

This is a contradiction. Therefore, $\mathrm{nr}(A) = \mathrm{nr}(J)$. □

3 Normal Reduction Numbers of the Maximal Ideal of Brieskorn Hypersurfaces

Let K be a field of any characteristic, and let a, b, c be integers with $2 \leq a \leq b \leq c$. Then, a hypersurface singularity

$$A = K[[x, y, z]]/(x^a + y^b + z^c), \quad \mathfrak{m} = (x, y, z)A$$

is called a *Brieskorn hypersurface singularity* if A is normal.

3.1 Normal Reduction Number of the Maximal Ideal

The main purpose in this section to give a formula for the reduction number of the maximal ideal $\mathfrak{m}$ in a hypersurface of Brieskorn type: $A = K[[x, y, z]]/(x^a + y^b + z^c)$. Namely, we prove the following theorem.

Theorem 3.1 *Let $A = K[[x, y, z]]/(x^a + y^b + z^c)$ be a Brieskorn hypersurface singularity. If we put $Q = (y, z)A$ and $n_k = \lfloor \frac{kb}{a} \rfloor$ for $k = 1, 2, \ldots, a-1$, then, $\mathfrak{m} = \overline{Q}$ and we have*

(1) $\overline{\mathfrak{m}^n} = Q^n + xQ^{n-n_1} + x^2Q^{n-n_2} + \cdots + x^{a-1}Q^{n-n_{a-1}}$ *for every $n \geq 1$.*
(2) $\bar{\mathrm{r}}(\mathfrak{m}) = \mathrm{nr}(\mathfrak{m}) = n_{a-1}$. *In particular, if $\bar{\mathrm{r}}(\mathfrak{m}) \leq 2$, then, $\lfloor \frac{(a-1)b}{a} \rfloor \leq 2$.*
(3) $\overline{\mathcal{R}'(\mathfrak{m})}$ *and* $\overline{G}(\mathfrak{m})$ *are Cohen-Macaulay.*

Remark 3.2 Note $0 := n_0 \leq n_1 < n_2 < \cdots < n_{a-1}$. In particular, $n_k \geq k$ for each $k = 0, 1, \ldots, a-1$.

In the following, we use the notation in this theorem and prove it.

Lemma 3.3 *For integers k, n with $n \geq 1$ and $1 \leq n \leq a-1$, we have that $x^k \in \overline{Q^n}$ if and only if $n \leq n_k$.*

Proof Suppose $n \leq n_k$. Then,

$$(x^k)^a = (x^a)^k = (-1)^k(y^b + z^c)^k \in Q^{bk} \subset Q^{an_k} = (Q^{n_k})^a.$$

Hence, $x^k \in \overline{Q^{n_k}} \subset \overline{Q^n}$.

Next, we prove the converse. Suppose $x^k \in \overline{Q^n}$. Then, there exists a nonzero element $c \in A$ such that $c(x^k)^\ell \in Q^{n\ell}$ for all large integers ℓ. By Artin-Rees' lemma [10, Theorem 8.5], we can choose an integer $\ell_0 \geq 1$ such that $Q^\ell \cap cA = cQ^{\ell-\ell_0}$ for every $\ell \geq \ell_0$.

Now suppose that $n \geq n_k + 1$. Since $\frac{kb}{a} + \frac{1}{a} \leq n_k + 1 \leq n$, we get

$$(y^b + z^c)^{k\ell} = (-1)^k x^{ka\ell} \in Q^{na\ell} : c \subset Q^{na\ell - \ell_0} \subset Q^{(n_k+1)a\ell - \ell_0} \subset Q^{(bk+1)\ell - n_0}$$

for sufficiently large ℓ. This implies that $y^{bk\ell} \in (y^{bk\ell+1}, z)$ and this is a contradiction because y, z forms a regular sequence. Therefore, $n \leq n_k$, as required. □

Corollary 3.4 *For an integer $n \geq 1$, if we put*

$$L_n = Q^n + xQ^{n-n_1} + x^2 Q^{n-n_2} + \cdots + x^{a-1} Q^{n-n_{a-1}},$$

then $Q^n \subset L_n \subset \overline{Q^n} = \overline{\mathfrak{m}^n}$.

Proof It is enough to prove $x^k y^i z^j \in \overline{Q^n}$ if and only if $i + j \geq n - n_k$. In fact, since $Q = (y, z)$ is a parameter ideal in A, [6, Corollary 6.8.13] and Lemma 3.3 imply

$$\begin{aligned} x^k y^i z^j \in \overline{Q^n} &\Longleftrightarrow x^k y^{i-1} z^j \in \overline{Q^{n-1}} \\ &\Longleftrightarrow \cdots\cdots \\ &\Longleftrightarrow x^k \in \overline{Q^{n-i-j}} \\ &\Longleftrightarrow n - n_k \leq i + j. \end{aligned}$$

Hence, $L_n \subset \overline{Q^n}$. □

Put $d = \gcd(a, b)$, $a' = \frac{a}{d}$ and $b' = \frac{b}{d}$. If we put

$$I_n = (x^k y^i z^j \mid kb' + ia' + ja' \geq n)A$$

for every $n \geq 1$, then $\{I_n\}_{n=1,2,\ldots}$ is a filtration of A.

Lemma 3.5 *$G(\{I_n\})$ is always reduced. In particular, $\mathcal{R}'(\{I_n\})$ is a Gorenstein normal domain.*

Proof One can easily see

$$G(\{I_n\}) \cong \begin{cases} K[X, Y, Z]/(X^a + Y^b + Z^c) & \text{if } b = c \\ K[X, Y, Z]/(X^a + Y^b) & \text{if } b < c. \end{cases} \tag{3.1}$$

By assumption, $K[X, Y, Z]/(X^a + Y^b + Z^c)$ is a normal domain. If $\operatorname{char} K = 0$, then $K[X, Y, Z]/(X^a + Y^b)$ is reduced. Otherwise, we put $p = \operatorname{char} K > 0$. Since A is normal, we have that p does *not* divide $\gcd(a, b) = d$. Hence $K[X, Y]/(X^a + Y^b)$ is reduced.

As A is normal, $R = \mathcal{R}'(\{I_n\})$ is a Gorenstein normal domain because $G(\{I_n\}) \cong R/t^{-1}R$. □

Lemma 3.6 *$L_n = I_{na'}$ for every $n \geq 1$.*

Proof Since L_n and $I_{na'}$ are monomial ideals, it suffices to show that $x^k y^i z^j \in L_n$ if and only if $x^k y^i z^j \in I_{na'}$. But this is clear from the definition. □

We are now ready to prove the theorem.

Proof of Theorem 3.1 (1) Since $\mathcal{R}'(\{I_n\})$ is normal by Lemma 3.5, we have that every I_n is integrally closed. In particular, $L_n = I_{na'}$ is also integrally closed by Lemma 3.6. Therefore, $L_n = \overline{Q^n} = \overline{\mathfrak{m}^n}$ by Corollary 3.4.

(2) One can easily see that $L_{n+1} = QL_n$ if and only if $n \geq n_{a-1}$. Hence, (2) is immediately follows from (1).

(3) $\overline{\mathcal{R}'(\mathfrak{m})}$ is Cohen-Macaulay since it is a Veronese subring of a Cohen-Macaulay ring $\mathcal{R}'(\{I_n\})$. Then $\overline{G}(\mathfrak{m}) = \overline{\mathcal{R}'(\mathfrak{m})}/t^{-1}\overline{\mathcal{R}'(\mathfrak{m})}$ is also Cohen-Macaulay by [14, Theorem 4.1]. □

Corollary 3.7 *Let* $(A, \mathfrak{m})$ *be a Brieskorn hypersurface as in Theorem 3.1. Then,*

(1) $\mathcal{R}(\mathfrak{m})$ *is normal if and only if* $\bar{\mathrm{r}}(\mathfrak{m}) = a - 1$.
(2) $\overline{\mathcal{R}(\mathfrak{m})}$ *is Cohen-Macaulay if and only if* $\bar{\mathrm{r}}(\mathfrak{m}) = 1$.
(3) $\mathfrak{m}$ *is a* p_g*-ideal if and only if* $a = 2$ *and* $\bar{\mathrm{r}}(\mathfrak{m}) = 1$.

Proof (1) Suppose $\bar{\mathrm{r}}(\mathfrak{m}) = a - 1$. Then, $n_{a-1} = a - 1$ by (1) and this implies that $n_k = k$ for each $k = 1, 2, \ldots, a - 1$. Then, one can easily see that $\overline{\mathfrak{m}^n} = (Q, x)^n = \mathfrak{m}^n$ for every $n \geq 1$. Hence, $\mathcal{R}(\mathfrak{m})$ is normal.

Conversely, if $\mathcal{R}(\mathfrak{m})$ is normal, then, $\overline{\mathfrak{m}^n} = \mathfrak{m}^n = (Q, x)^n$. Then, $n_{a-1} = a - 1$.

(2) Since $F = \{\overline{\mathfrak{m}^n}\}$ is a good $\mathfrak{m}$-adic filtration, $\overline{\mathcal{R}(\mathfrak{m})} = \mathcal{R}(F)$ is Cohen-Macaulay if and only if $G(F)$ is Cohen-Macaulay and $\bar{\mathrm{r}}(\mathfrak{m}) - 2 = a(G(F)) < 0$ by [2, Part 2, Corollary 1.2] and [4, Theorem 3.8].

(3) $\mathfrak{m}$ is a p_g-ideal if and only if $R(\mathfrak{m})$ is normal and Cohen-Macaulay. Hence, the assertion follows from (1), (2). □

3.2 $q(\mathfrak{m})$ and $\ell_A(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n})$

In the proof of Theorem 3.1, we gave a formula of the integral closure of $\mathfrak{m}^n$. As an application, we give a formula of $q(\mathfrak{m})$ for Brieskorn hypersurface singularities.

Proposition 3.8 *Let* $A = K[[x, y, z]]/(x^a + y^b + z^c)$ *be a Brieskorn hypersurface singularity. Under the same notation as in Theorem 3.1, we have*

(1) $\ell_A(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n}) = \max\left(a - \lceil \frac{a(n+1)}{b} \rceil,\ 0\right)$.

(2) $q(\mathfrak{m}) = p_g(A) - \displaystyle\sum_{k=1}^{a-1}(n_k - n_{k-1})(a - k)$.

Proof Suppose $n_k \leq n < n_{k+1}$ for some $0 \leq k \leq a-2$. Then $\overline{\mathfrak{m}^n} = Q^n + xQ^{n-n_1} + \cdots + x^k Q^{n-n_k} + (x^{k+1})$ and $x^{k+1}, x^{k+2}, \ldots, x^{a-1}$ forms a K-basis of $\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n}$ and thus $\ell_A(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n}) = a - 1 - k$. Hence,

$$\ell_A(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n}) = \begin{cases} a - 1 - k & \text{if } n_k \leq n < n_{k+1},\ k = 0, 1, \ldots, a-2; \\ 0 & \text{if } n \geq n_{a-1}. \end{cases}$$

Moreover, one can easily see $k = a - \lceil \frac{a(n+1)}{b} \rceil - 1$.

(2) Put $a_n = p_g(A) - q(n\mathfrak{m})$ and $v_n = \ell_A(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n})$ for every $n \geq 0$. Then, $a_0 = 0$ and $\{a_n\}$ is an increasing sequence and $a_{n+1} = a_n$ for sufficiently large n. By Lemma 2.7, we have

$$0 = a_{n+1} - a_n = a_n - a_{n-1} - v_n = \cdots = a_1 - a_0 - \sum_{k=1}^{n} v_k$$

for sufficiently large $n \geq 1$. Hence, (1) yields

$$p_g(A) - q(\mathfrak{m}) = a_1 = \sum_{k=1}^{n} v_k = \sum_{k=1}^{a-1} (n_k - n_{k-1})(a-k),$$

as required. □

When $a = 2$, one can obtain the following.

Example 3.9 Let $A = K[[x, y, z]]/(x^2 + y^b + z^c)$ be a Brieskorn hypersurface singularity and put $r = \lfloor \frac{b}{2} \rfloor$. Then,

(1) $q(i\mathfrak{m}) = \begin{cases} p_g(A) - i(r-1) + \binom{i}{2} & \text{if } 1 \leq i \leq r-1; \\ p_g(A) - \binom{r}{2} & \text{if } i \geq r. \end{cases}$

(2) The normal Hilbert coefficients of $\mathfrak{m}$ are given as follows:

$$\overline{e}_0(\mathfrak{m}) = 2, \quad \overline{e}_1(\mathfrak{m}) = r, \quad \overline{e}_2(\mathfrak{m}) = \binom{r}{2},$$

where

$$\ell_A(A/\overline{I^{n+1}}) = \overline{e}_0(I)\binom{n+2}{2} - \overline{e}_1(I)\binom{n+1}{1} + \overline{e}_2(I)$$

for sufficiently large n.

3.3 Geometric Genus

In this subsection, let us consider a graded ring

$$B = K[x, y, z]/(x^a + y^b + z^c)$$

with $\deg x = q_0 = bc$, $\deg y = q_1 = ac$ and $\deg z = q_2 = ab$. Put $\mathfrak{m} = (x, y, z)A$ and $D = abc$. In particular, the a-invariant of B is given by $a(B) = D - q_0 - q_1 - q_2$. Also, we have that $A = \widehat{B_\mathfrak{m}}$ is the completion of the local ring $B_\mathfrak{m}$. Then, we can calculate $p_g(A)$ using this formula.

Lemma 3.10 *Under the above notation, we have*

$$p_g(A) = \sum_{i=0}^{a(B)} \dim_K B_i = \sharp\{(t_0, t_1, t_2) \in \mathbb{Z}_{\geq 0}^{\oplus 3} \mid D - q_0 - q_1 - q_2 \geq q_0 t_0 + q_1 t_1 + q_2 t_2\}.$$

We can find many examples of Brieskorn hypersurfaces with $p_g(A) = p$ for a given $p \geq 1$ if $\mathrm{nr}(\mathfrak{m}) = 1, 2$.

Example 3.11 Let $p \geq 1$ be an integer.

(1) If $A = \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^{6p+1})$, then $p_g(A) = p$ and $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = 1$.
(2) If $A = \mathbb{C}[[x, y, z]]/(x^2 + y^4 + z^{4p+1})$, then $p_g(A) = p$ and $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = 2$.

Example 3.12 Let $k \geq 1$ be an integer.

(1) Put $A = \mathbb{C}[[x, y, z]]/(x^2 + y^6 + z^{10k+i})$ for $i = 0, 1, \ldots, 9$. Then, $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = 3$ and

$$p_g(A) = \begin{cases} 6k, & (\text{if } i = 0, 1, 2); \\ 6k + 1, & (\text{if } i = 3, 4, 5); \end{cases} \qquad p_g(A) = \begin{cases} 6k + 3, & (\text{if } i = 6, 7, 8); \\ 6k + 4, & (\text{if } i = 9, 10, 11). \end{cases}$$

(2) Put $A = \mathbb{C}[[x, y, z]]/(x^2 + y^7 + z^{14k+i})$ for $i = 0, 1, \ldots, 13$. Then, $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = 3$ and

$$p_g(A) = \begin{cases} 9k, & (\text{if } i = 0, 1, 2); \\ 9k + 1, & (\text{if } i = 3, 4); \\ 9k + 2, & (\text{if } i = 5); \\ 9k + 3, & (\text{if } i = 6, 7, 8); \end{cases} \qquad p_g(A) = \begin{cases} 9k + 4, & (\text{if } i = 9); \\ 9k + 5, & (\text{if } i = 10, 11); \\ 9k + 6, & (\text{if } i = 12, 13). \end{cases}$$

We discuss when $p_g(A) = \binom{\mathrm{nr}(\mathfrak{m})}{2}$ holds.

Proposition 3.13 *Let* $A = \mathbb{C}[[x, y, z]]/(x^a + y^b + z^c)$ *with* $2 \leq a \leq b \leq c$. *Then,* $p_g(A) = \binom{\mathrm{nr}(\mathfrak{m})}{2}$ *if and only if one of the following cases:*
- $(a, b, c) = (2, 2, n)$ $(n \geq 1)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = 1$ *and* $p_g(A) = 0$.
- $(a, b, c) = (2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = 1$ *and* $p_g(A) = 0$.
- $(a, b, c) = (2, 4, 4), (2, 4, 5), (2, 4, 6), (2, 4, 7)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = 2$ *and* $p_g(A) = 1$.
- $(a, b, c) = (2, 2r, 2r), (2, 2r, 2r + 1), (2, 2r, 2r + 2)$ $(r \geq 3)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = r$ *and* $p_g(A) = \binom{r}{2} \geq 3$.
- $(a, b, c) = (2, 2r+1, 2r+1), (2, 2r+1, 2r+2)$ $(r \geq 2)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = r$ *and* $p_g(A) = \binom{r}{2}$.
- $(a, b, c) = (3, 3, 3), (3, 3, 4), (3, 3, 5)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = 2$ *and* $p_g(A) = 1$.
- $(a, b, c) = (3, 3s + 1, 3s + 1)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = 2s$ *and* $p_g(A) = \binom{2s}{2}$.
- $(a, b, c) = (3, 3s + 2, 3s + 2), (3, 3s + 2, 3s + 3)$. *In this case,* $\mathrm{nr}(A) = \mathrm{nr}(\mathfrak{m}) = 2s + 1$ *and* $p_g(A) = \binom{2s+1}{2}$.

Proof We give a proof of only if part. Put $r = \mathrm{nr}(\mathfrak{m})$. By Theorem 3.1, we have $\lfloor \frac{(a-1)b}{a} \rfloor$. So, we can write $(a-1)b = ra + \varepsilon$, where ε is an integer with $0 \leq \varepsilon \leq a - 1$. Now suppose

$$abc - bc - ca - ab \geq bc\lambda_0 + ca\lambda_1 + ab\lambda_2.$$

Then,

$$(ra + \varepsilon)c - ca - ab \geq bc\lambda_0 + ca\lambda_1 + ab\lambda_2. \qquad \text{(eq.pg)}$$

Suppose $\lambda_0 = 0$. Then,

$$\left(r - 1 + \frac{\varepsilon}{a} - \lambda_1\right) \frac{c}{b} \geq \lambda_2 + 1$$

and thus $\lambda_1 < r - 1 + \frac{\varepsilon}{a}$. By Lemma 3.10 and assumption, we have

$$\begin{aligned}\binom{r}{2} = p_g(A) &\geq \sum_{k=0}^{r-1} \left\lfloor \left(r - 1 + \frac{\varepsilon}{a} - k\right) \frac{c}{b} \right\rfloor \\ &\geq \sum_{k=0}^{r-2} \left\lfloor (r - 1 - k) \frac{c}{b} \right\rfloor + \left\lfloor \frac{\varepsilon}{a} \cdot \frac{c}{b} \right\rfloor \\ &\geq \sum_{k=0}^{r-2} (r - 1 - k) + \left\lfloor \frac{\varepsilon}{a} \cdot \frac{c}{b} \right\rfloor \geq \binom{r}{2}.\end{aligned}$$

Hence, $\left(r - 1 + \frac{\varepsilon}{a} - k\right) \frac{c}{b} < r - k$ for each $k = 0, 1, \ldots, r - 1$. Moreover, if $\lambda_0 \geq 1$, then, since $\lambda_1 = \lambda_2 = 0$ does not satisfy the condition (eq.pg) by Theorem 2.9, we get the following:

$$abc - bc - ca - ab < bc, \quad \text{that is,} \quad \frac{2}{a} + \frac{1}{b} + \frac{1}{c} > 1.$$

This implies $a = 2, 3$. If $a = 2$, then $\varepsilon = 0, 1$. If $a = 3$, then $\varepsilon = 0, 1, 2$.

Now suppose $a = 3$ and $\varepsilon = 2$. Then, as $2b = 3r + 2$, we can write $r = 2s$, $b = 3s + 1$, where $s \geq 1$. Moreover, the condition holds true if and only if $(2s + \frac{2}{3} - 1)\frac{c}{3s+1} < 2s$. This means $c < 3s + 1 + \frac{3s+1}{6s-1}$. Hence, $c = 3s + 1$ because $c \geq b = 3s + 1$. Similarly, easy calculation yields the required assertion.

□

3.4 Weighted Dual Graph

In this subsection, let us explain how to construct the weighted dual graph of the minimal good resolution of singularity $X \to \operatorname{Spec} A$ for a Brieskorn hypersurface singularity $A = K[[x, y, z]]/(x^a + y^b + z^c)$. Though it is obtained in [7], we use the notation of [11] in which the first author studies complete intersection singularities of Brieskorn type. Let E be the exceptional set of $X \to \operatorname{Spec} A$ and E_0 the central curve with genus g and $E_0^2 = -c_0$. We define positive integers $a_i, \ell_i, \alpha_i, \lambda_i, \widehat{g_i}$ (i = 1,2,3), $\widehat{g}$, and ℓ as follows:

$$\begin{array}{lll} a_1 = a, & a_2 = b, & a_3 = c, \\ \ell_1 = \operatorname{lcm}(b, c), & \ell_2 = \operatorname{lcm}(a, c), & \ell_3 = \operatorname{lcm}(a, b), \\ \alpha_1 = \frac{a_1}{(a_1, \ell_1)}, & \alpha_2 = \frac{a_2}{(a_2, \ell_2)}, & \alpha_3 = \frac{a_3}{(a_3, \ell_3)}, \\ \lambda_1 = \frac{\ell_1}{(a_1, \ell_1)}, & \lambda_2 = \frac{\ell_2}{(a_2, \ell_2)}, & \lambda_3 = \frac{\ell_3}{(a_3, \ell_3)}, \\ \hat{g}_1 = (b, c), & \hat{g}_2 = (a, c), & \hat{g}_3 = (a, b). \end{array}$$

We put $\widehat{g} = \frac{abc}{\operatorname{lcm}(a,b,c)}$ and $\ell = \operatorname{lcm}(a, b, c)$, and define integers β_i by the following condition:

$$\lambda_i \beta_i + 1 \equiv 0 \pmod{\alpha_i}, \quad 0 \leq \beta_i < \alpha_i.$$

Then, E_0 has $\hat{g}_1 + \hat{g}_2 + \hat{g}_3$ branches. For each $w = 1, 2, 3$, we have $\hat{g}_w$ branches as follows:

$$B_w : \quad E_{w,1} - E_{w,2} - \cdots - E_{w,s_w},$$

where $E_{w,j}^2 = -c_{w,j}$ and

$$\frac{\alpha_w}{\beta_w} = [[c_{w,1}, c_{w,2}, \ldots, c_{w,s_w}]]$$

is a Hirzebruch-Jung continued fraction if $\alpha_w \geq 2$, we regard B_w empty if $\alpha_w = 1$. Moreover, we have

$$2g - 2 = \hat{g} - \sum_{w=1}^{3} \hat{g}_w, \qquad c_0 = \sum_{w=1}^{3} \frac{\hat{g}_w \beta_w}{\alpha_w} + \frac{\hat{g}}{\ell}.$$

For instance, if $(a, b, c) = (3, 4, 7)$, then we have the following:

$$\begin{aligned}
&a_1 = 3,\ a_2 = 4,\ a_3 = 7,\ \ell_1 = 28,\ \ell_2 = 21,\ \ell_3 = 12,\\
&\alpha_1 = 3,\ \alpha_2 = 4,\ \alpha_3 = 7,\ \lambda_1 = 28,\ \lambda_2 = 21,\ \lambda_3 = 12,\\
&\beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 4,\ \hat{g}_1 = 1,\quad \hat{g}_2 = 1,\quad \hat{g}_3 = 1,\\
&\hat{g} = 1,\quad \ell = 84.
\end{aligned}$$

Thus, $g = 0$ and $c_0 = 2$. Therefore, each irreducible component of E is a rational curve, and the weighted dual graph of E is represented as in Fig. 1, where the vertex ■ has weight -4 and other vertices • have weight -2.

See [11, 4.4] for more details.

4 Brieskorn Hypersurfaces with Elliptic Singularities

We use the notation of Section 3.4. Let Z_E denote the fundamental cycle.

We call $p_f(A) := p_a(Z_E)$ the *fundamental genus* of A. The singularity A is said to be *elliptic* if $p_f(A) = 1$. We have the following.

Theorem 4.1 [12] *If* $p_f(A) = 1$, *then* $\bar{\mathrm{r}}(A) = 2$.

Remark 4.2 By [1], $\mathrm{nr}(A) = 1$ if and only if A is rational. Therefore, $\mathrm{nr}(A) = \bar{\mathrm{r}}(A)$ if $\bar{\mathrm{r}}(A) = 2$.

It is natural to ask whether the converse of Theorem 4.1 holds or not. In the following, we classify Brieskorn hypersurface singularities with $p_f(A) = 1$ or $\bar{\mathrm{r}}(A) = 2$ as an application of results in Section 3. Before doing that, we need the following formula of $p_f(A)$ in the case of Brieskorn hypersurfaces. Put $\alpha = \alpha_1\alpha_2\alpha_3$.

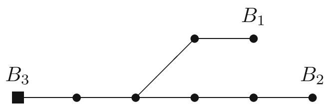

Fig. 1 The weighted dual graph of $k[[x, y, z]]/(x^3 + y^4 + z^7)$

Lemma 4.3 ([16], [7, Theorem 1.7], [11, 5.4]) *If* $\lambda_3 \le \alpha$, *then* $-Z_E^2 = \hat{g}_3 \lceil \lambda_3/\alpha_3 \rceil$ *and*

$$\begin{aligned} p_f(A) &= \frac{1}{2}\lambda_3 \left\{ \hat{g} - \frac{(2\lceil \lambda_3/\alpha_3 \rceil - 1)\hat{g}_3}{\lambda_3} - \frac{\hat{g}_1}{\alpha_1} - \frac{\hat{g}_2}{\alpha_2} \right\} + 1 \\ &= \frac{1}{2}\left(ab - a - b - (2\lceil \lambda_3/\alpha_3 \rceil - 1)(a, b)\right) + 1. \end{aligned}$$

We are now ready to state our result in the case of $p_f(A) = 1$.

Theorem 4.4 $(A, \mathfrak{m})$ *is elliptic (i.e.,* $p_f(A) = 1$*) if and only if* (a, b, c) *is one of the following.*

(1) $(2, 3, c)$, $c \ge 6$.
(2) $(2, 4, c)$, $c \ge 4$.
(3) $(2, 5, c)$, $5 \le c \le 9$.
(4) $(3, 3, c)$, $c \ge 3$.
(5) $(3, 4, c)$, $4 \le c \le 5$.

Proof If A is elliptic, then by Theorem 4.1 and Theorem 3.1, we have $\lfloor \frac{(a-1)b}{a} \rfloor \le 2$. Thus possible pairs (a, b) are as follows:

$$(2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4).$$

We know that A is rational if $(a, b, c) = (2, 2, c)$ with $2 \le c$ or $(2, 3, c)$ with $3 \le c \le 5$. We obtain the assertion by Lemma 4.3, for example, $p_f(A) = 3 - \lceil 10/c \rceil$ for $(a, b, c) = (2, 5, c)$, and $p_f(A) = 4 - \lceil 12/c \rceil$ for $(a, b, c) = (3, 4, c)$. □

We can classify Brieskorn hypersurface singularities with $\bar{r}(A) = 2$ except $(a, b, c) = (3, 4, 6)$ or $(3, 4, 7)$.

Proposition 4.5 $\bar{r}(A) = 2$ *if and only if* $p_f(A) = 1$, *except* $(a, b, c) = (3, 4, 6)$, *or* $(3, 4, 7)$.

Proof It suffices to check whether $\mathrm{nr}(A) \ge 3$ for singularities with $\bar{r}(\mathfrak{m}) = 2$ and $p_f(A) \ge 2$.

Suppose $(a, b, c) = (2, 5, c)$, $c \ge 10$. Let $Q = (y, z^2)$ and $J = \overline{Q}$. Then, $xz \notin Q$ and $(xz)^2 = (y^5 + z^c)z^2 \in Q^6 = (Q^3)^2$. Hence, $\mathrm{nr}(J) \ge 3$.

Next suppose that $(a, b, c) = (3, 4, c)$, $c \ge 8$. Let $Q = (y, z^2)$ and $J = \overline{Q}$, again. Then, $x^2z \notin Q$ and $(x^2z)^3 = (y^4 + z^c)^2z^3 \in Q^9 = (Q^3)^3$. Hence, $\mathrm{nr}(J) \ge 3$. □

Applying the result of [11], we can show that the formula for $\bar{r}(\mathfrak{m})$ for Brieskorn complete intersection singularities. Thus, the statement above can be extended to those singularities.

Remark 4.6 Suppose that $p_g(A) = 3$. It follows from Theorem 2.9 and its proof that $\mathrm{nr}(I) = 3$ if and only if $q(I) = 1$ and $q(nI) = 0$ for $n \ge 2$. In particular, $q(nI) = q(I)$ for $n \ge 2$ if $q(I) \ge 2$.

Remark 4.7 If $(a, b, c) = (3, 4, 6)$ or $(3, 4, 7)$, we have the following:

(1) $p_g(A) = 3$, $p_f(A) = 2$, $h^1(\mathcal{O}_X(-Z_E)) = 1$.

(2) There exists a point $p \in E$ such that $\mathfrak{m}\mathcal{O}_X = \mathcal{I}_p\mathcal{O}_X(-Z_E)$, where $\mathcal{I}_p \subset \mathcal{O}_X$ is the ideal sheaf of the point p; so $\mathfrak{m} = H^0(\mathcal{O}_X(-Z_E))$, but $\mathfrak{m}$ is not represented by Z_E.

Note that $H^0(\mathcal{O}_X(-nZ_E)) \neq \overline{\mathfrak{m}^n}$ for $n \geq 2$. On the other hand, $\mathcal{O}_X(-2Z_E) = \mathcal{O}_X(K_X)$ is generated by global sections. By the vanishing theorem, $h^1(\mathcal{O}_X(-nZ_E)) = 0$ for $n \geq 2$.

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