Acta Mathematica Vietnamica (2019) 44:905–914
https://doi.org/10.1007/s40306-018-00306-1

On the Annihilator Submodules and the Annihilator Essential Graph

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Received: 20 November 2017 / Revised: 12 May 2018 / Accepted: 28 June 2018 /
Published online: 12 January 2019

Abstract
Let R be a commutative ring and let M be an R-module. For $a \in R$, $\mathrm{Ann}_M(a) = \{m \in M : am = 0\}$ is said to be an annihilator submodule of M. In this paper, we study the property of being prime or essential for annihilator submodules of M. Also, we introduce the annihilator essential graph of equivalence classes of zero divisors of M, $AE_R(M)$, which is constructed from classes of zero divisors, determined by annihilator submodules of M and distinct vertices $[a]$ and $[b]$ are adjacent whenever $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M. Among other things, we determine when $AE_R(M)$ is a connected graph, a star graph, or a complete graph. We compare the clique number of $AE_R(M)$ and the cardinal of $\mathrm{m} - \mathrm{Ass}_R(M)$.

Keywords Annihilator submodule · Annihilator essential graph · Zero divisor graph

Mathematics Subject Classification (2010) 13A15 · 05C99

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity and all modules are unitary. Let M be an R-module. A proper submodule P of M is said to be prime if $rm \in P$ for $r \in R$ and $m \in M$, implies that $m \in P$ or $r \in \mathrm{Ann}_R(M/P) = \{r \in R : rM \subseteq P\}$. Let $\mathrm{Spec}_R(M)$ denote the set of prime submodules of M. For $a \in R$

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we call $\mathrm{Ann}_M(a) = \{m \in M : am = 0\}$ the annihilator submodule of a in M. Let $\mathrm{m}-\mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(a), \text{ for some } 0 \neq a \in R\}$. The properties of prime submodules and $\mathrm{m}-\mathrm{Ass}_R(M)$ are studied in [8, 9] and [4]. By [8, Proposition 3.2], any maximal element of $\{\mathrm{Ann}_M(a) : a \notin \mathrm{Ann}_R(M)\}$ is a prime submodule of M. Thus, $\mathrm{m}-\mathrm{Ass}_R(M)$ is a non-empty set, when M is a Noetherian R-module. In Section 2, we study some properties of the elements of $\mathrm{m}-\mathrm{Ass}_R(M)$. In particular, we show that $\mathrm{Ann}_M(a) = \{m \in M \mid rm \in \mathrm{Ann}_R(aM)M \text{ for some } r \notin \mathrm{Ann}_R(aM)\}$ whenever $\mathrm{Ann}_M(a)$ is a prime submodule of M and $a \notin r(\mathrm{Ann}_R(M))$. Also, we compare $\mathrm{m}-\mathrm{Ass}_R(M)$ and the set of associated prime ideals of R, $\mathrm{Ass}_R(R)$, and we show that:

$$\mathrm{m}-\mathrm{Ass}_R(M) = \{\mathrm{Ann}_M(a) \mid \mathrm{Ann}_R(a) \in \mathrm{Ass}_R(R)\},$$

where M is either a free or a faithful multiplication R-module.

There are many studies of various graphs associated to rings or modules (see for instance [3, 5, 6, 10]). A submodule N of M is called an essential submodule if it has a non-zero intersection with any other non-zero submodule of M. In the third section, we investigate the property of being essential for an annihilator submodule, $\mathrm{Ann}_M(a)$, in two cases, $a \in r(\mathrm{Ann}_R(M)) = \{r \in R : r^t M = 0 \text{ for some positive integer } t\}$ or $a \notin r(\mathrm{Ann}_R(M))$. We prove that, if $\mathrm{Ann}_M(a)$, $\mathrm{Ann}_M(b) \in \mathrm{m}-\mathrm{Ass}_R(M)$, then $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M. By relying on this fact, we introduce the annihilator essential graph of equivalence classes of zero divisors of M, $AE_R(M)$, which is constructed from classes of zero divisors, determined by annihilator submodules and distinct vertices $[a]$ and $[b]$ are adjacent whenever $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M. Among other things, we determine when $AE_R(M)$ is a connected graph, a star graph, or a complete graph. An aspect of $AE_R(M)$ is the connection to elements of $\mathrm{m}-\mathrm{Ass}_R(M)$. We compare the clique number of $AE_R(M)$ and the cardinal number of $\mathrm{m}-\mathrm{Ass}_R(M)$ under the additional assumption $r(\mathrm{Ann}_R(M)) = 0$ or $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M) \neq 0$.

The zero-divisor graph determined by equivalence classes, $\Gamma_E(R)$, was introduced in [10], and further studied in [2, 7, 11]. We shall compare $\Gamma_E(R)$ and $AE_R(R)$ to determine some properties of the ring R.

Let Γ be a (undirected) graph. We say that Γ is *connected* if there is a path between any two distinct vertices. For vertices x and y of Γ, we define $\mathrm{d}(x, y)$ to be the length of a shortest path between x and y, if there is no path, then $\mathrm{d}(x, y) = \infty$. The *diameter* of Γ is

$$\mathrm{diam}(\Gamma) = \sup\{\mathrm{d}(x, y) \mid x \text{ and } y \text{ are vertices of } \Gamma\}.$$

The *girth* of Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ ($\mathrm{gr}(\Gamma) = \infty$ if Γ contains no cycle). A graph Γ is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices is denoted by K_n (we allow n to be an infinite cardinal). The *clique number*, $\omega(\Gamma)$, is the greatest integer $n > 1$ such that $K_n \subseteq \Gamma$, and $\omega(\Gamma) = \infty$ if $K_n \subseteq \Gamma$ for all integers $n \geq 1$.

2 Annihilators Which Are Prime Submodules

Let R be a commutative ring and M be an R-module. In this section, we investigate the primeness of annihilator submodules of M.

Theorem 1 *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. Then, there exists $a \in r(\mathrm{Ann}_R(M))$ such that $\mathrm{Ann}_M(a)$ is a prime submodule of M.*

Proof Assume that $a \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$. If $\mathrm{Ann}_M(a)$ is a maximal element of $X = \{\mathrm{Ann}_M(b) : b \notin \mathrm{Ann}_R(M)\}$, then [8, Proposition 3.2] shows that $\mathrm{Ann}_M(a)$ is a prime submodule of M and we are done. Otherwise, there exists $b \in R$ such that $\mathrm{Ann}_M(b)$ is a maximal element of X and $\mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(b)$. We show that $b \in r(\mathrm{Ann}_R(M))$. By assumption $a \in r(\mathrm{Ann}_R(M))$, so that there is an integer t such that $0 = a^t M \subseteq \mathrm{Ann}_M(b)$. Thus, $abM = 0$, since $\mathrm{Ann}_M(b)$ is a prime submodule. Hence, $b \in \mathrm{Ann}_R(aM) \subseteq \mathrm{Ann}_R(bM)$ and so $b^2M = 0$ which implies that $b \in r(\mathrm{Ann}_R(M))$. □

The following example shows that $\mathrm{Ann}_M(a)$ can be a prime submodule of M but $a \notin r(\mathrm{Ann}_R(M))$.

Example 1 Let $\mathbb{Z}_{p^2q}$ be the ring of integers modulo p^2q for some prime integers p, q. Then, $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_{p^2q}) = p^2q\mathbb{Z}$, $r(\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_{p^2q})) = pq\mathbb{Z}$ and $\mathrm{Ann}_{\mathbb{Z}_{p^2q}}(p^2) = q\mathbb{Z}_{p^2q}$ is a prime submodule of $\mathbb{Z}_{p^2q}$ while $p^2 \notin r(\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_{p^2q}))$.

Lemma 1 *If $a \in R$, then* $\mathrm{Ann}_R(M/\mathrm{Ann}_M(a)) = \mathrm{Ann}_R(aM)$.

Proof If $a \in \mathrm{Ann}_R(M)$, there is nothing to prove. Thus, assume that $a \notin \mathrm{Ann}_R(M)$ and $r \in \mathrm{Ann}_R(M/\mathrm{Ann}_M(a))$. Then, $rM \subseteq \mathrm{Ann}_M(a)$ and so $arM = 0$. Hence, $r \in \mathrm{Ann}_R(aM)$. The converse is similar. □

Theorem 2 *Let $\mathrm{Ann}_M(a)$ be a prime submodule of M and $a \notin r(\mathrm{Ann}_R(M))$. Then, $\mathrm{Ann}_M(a) = \{m \in M : rm \in \mathrm{Ann}_R(aM)M$ for some $r \in R \setminus \mathrm{Ann}_R(aM)\}$ and it is a minimal prime submodule of M.*

Proof By assumption and Lemma 1, $\mathrm{Ann}_R(M/\mathrm{Ann}_M(a)) = \mathrm{Ann}_R(aM) = \mathfrak{p}$ is a prime ideal of R. Let $H := \{m \in M : rm \in \mathfrak{p}M$ for some $r \notin \mathfrak{p}\}$ and $m \in H$. Then, there exists $s \in R \setminus \mathfrak{p}$ such that $sm \in \mathfrak{p}M$. This implies that $sm = \sum_{i=1}^{k} s_i m_i$, where $s_i \in \mathfrak{p}$. Thus, $sam = \sum_{i=1}^{k} s_i a m_i = 0$ and so $sm \in \mathrm{Ann}_M(a)$. Hence, by assumption and $s \notin \mathfrak{p}$, it follows that $m \in \mathrm{Ann}_M(a)$. Therefore, $H \subseteq \mathrm{Ann}_M(a)$. Let $m \in \mathrm{Ann}_M(a)$. Then, $am = 0 \in \mathfrak{p}M$. If $a \notin \mathfrak{p}$, we are done. Otherwise, $a^2M = 0$ and so $a \in r(\mathrm{Ann}_R(M))$, contrary to assumption. Thus, $m \in H$.

Assume that P is a prime submodule of M and $P \subseteq \mathrm{Ann}_M(a)$. Let $m \in \mathrm{Ann}_M(a)$. Then, $am = 0 \in P$ which implies that $a \in \mathrm{Ann}_R(M/P)$ or $m \in P$. If $aM \subseteq P \subseteq \mathrm{Ann}_M(a)$, then $a^2M = 0$ and so $a \in r(\mathrm{Ann}_R(M))$; it is a contradiction. Hence, $m \in P$ and $P = \mathrm{Ann}_M(a)$ which implies that $\mathrm{Ann}_M(a)$ is a minimal prime submodule of M. □

Corollary 1 *Let $\mathrm{Ann}_R(M) = \mathfrak{p}$ be a prime ideal of R and $a \notin \mathfrak{p}$. Then, the following statements are true:*
(i) $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(M)$.
(ii) *If $\mathrm{Ann}_M(a)$ is a prime submodule of M, then $\mathrm{Ann}_M(a) = \{m \in M : rm = 0$ for some $r \in R \setminus \mathfrak{p}\} = \cup_{b \notin \mathfrak{p}} \mathrm{Ann}_M(b)$.*
(iii) $|\mathrm{m} - \mathrm{Ass}_R(M)| \leq 1$.

Proof (i) It is clear that $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(aM)$. To establish the reverse inclusion, let $r \in \mathrm{Ann}_R(aM)$. Then, $arM = 0$ and so $ar \in \mathrm{Ann}_R(M)$. By assumption, $\mathrm{Ann}_R(M)$ is a prime ideal of R and $a \notin \mathrm{Ann}_R(M)$, thus $r \in \mathrm{Ann}_R(M)$. Hence, $\mathrm{Ann}_R(aM) \subseteq \mathrm{Ann}_R(M)$.

(ii) It follows by (i) and Theorem 2.

(iii) It follows by (ii). □

The following lemma shows that there is a natural injective map from

$$\mathrm{Spec}_R(M) \cap \{\mathrm{Ann}_M(a) : a \notin r(\mathrm{Ann}_R(M))\} \longrightarrow \mathrm{Spec}(R) \cap \{\mathrm{Ann}_R(aM) : a \in R\}$$

given by $\mathrm{Ann}_M(a) \to \mathrm{Ann}_R(aM)$.

Lemma 2 *Let* $\mathrm{Ann}_M(a)$ *and* $\mathrm{Ann}_M(b)$ *be prime submodules of* M *and* $a, b \notin r(\mathrm{Ann}_R(M))$. *Then,* $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$ *if and only if* $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(bM)$.

Proof In view of Lemma 1, if $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$, then $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(bM)$. For the converse, assume that $m \in \mathrm{Ann}_M(a)$. Thus, $am = 0$ and $am \in \mathrm{Ann}_M(b)$. If $m \in \mathrm{Ann}_M(b)$, we are done. Otherwise, $a \in \mathrm{Ann}_R(bM) = \mathrm{Ann}_R(aM)$ which implies that $a \in r(\mathrm{Ann}_R(M))$ contrary to assumption. Thus, $m \in \mathrm{Ann}_M(b)$. □

The following result shows that the above injective map from $\mathrm{Spec}_R(M) \cap \{\mathrm{Ann}_M(a) : a \notin r(\mathrm{Ann}_R(M))\}$ to $\mathrm{Spec}(R) \cap \{\mathrm{Ann}_R(aM) : a \in R\}$, could be a bijection.

An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. Multiplication module has been studied in [1].

Theorem 3 *Let* M *be either a free or a faithful multiplication module and* $a \in R$. *Then,* $\mathrm{Ann}_M(a)$ *is a prime submodule of* M *if and only if* $\mathrm{Ann}_R(a)$ *is a prime ideal of* R. *In particular,*

$$\mathrm{m}-\mathrm{Ass}_R(M) = \{\mathrm{Ann}_M(a) : \mathrm{Ann}_R(a) \in \mathrm{Ass}_R(R)\}.$$

Proof Assume that M is a free R-module, thus $M \cong \oplus_{i\in I} R_i$ $(R_i = R)$, where I is an index set. Let $a \in R$ and $\mathrm{Ann}_R(a)$ be a prime ideal of R. It is easy to see that $\mathrm{Ann}_M(a) \cong \oplus_{i\in I}\mathrm{Ann}_R(a)$. Let $rm \in \mathrm{Ann}_M(a)$ and $r \notin \mathrm{Ann}_R(aM) = \mathrm{Ann}_R(a)$ for some $r \in R$, $m = (m_i)_{i\in I} \in M$. Thus, $rm_i \in \mathrm{Ann}_R(a)$ and so $m_i \in \mathrm{Ann}_R(a)$, for all $i \in I$. Hence, $m \in \mathrm{Ann}_M(a)$ and $\mathrm{Ann}_M(a)$ is a prime submodule of M. By the same argument, the converse follows.

By [1, Corollary 2.11], $\mathrm{Ann}_M(a)$ is a prime submodule of M if and only if $\mathrm{Ann}_R(aM)$ is prime ideal of R. On the other hand, $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(a)$ since M is faithful. Thus, $\mathrm{Ann}_M(a)$ is a prime submodule of M if and only if $\mathrm{Ann}_R(a)$ is a prime ideal of R. □

Theorem 4 *Let* M *be a projective module and* $a \in R$. *If* $\mathrm{Ann}_R(a)$ *is a prime ideal of* R, *then* $\mathrm{Ann}_M(a)$ *is a prime submodule of* M. *Furthermore,* $|\mathrm{Ass}_R(R)| \leq |\mathrm{m}-\mathrm{Ass}_R(M)|$, *whenever* M *is a faithful projective module.*

Proof By assumption, there exists a free R-module F and an R-module A such that $F \cong M \oplus A$. By assumption and Theorem 3, $\mathrm{Ann}_F(a)$ is a prime submodule of F. Let $x \in M$, $r \in R$, and $rx \in \mathrm{Ann}_M(a)$. Then, $arx = 0$ and so $ar(x, 0) = 0$. Thus, $r(x, 0) \in \mathrm{Ann}_F(a)$. Hence, $r \in \mathrm{Ann}_R(aF)$ or $(x, 0) \in \mathrm{Ann}_F(a)$. Therefore, $r \in \mathrm{Ann}_R(a(M \oplus A)) \subseteq \mathrm{Ann}_R(aM)$ or $x \in \mathrm{Ann}_M(a)$. □

3 The Annihilator Essential Graph of Zero Divisors

Recall that R is a commutative ring and M is an R-module. A submodule N of M is called an essential submodule if it has a non-zero intersection with any other non-zero submodule of M. In this section, we investigate the essentialness of the annihilator submodules of M and we introduce the annihilator essential graph of equivalence classes of zero divisors of M, $AE_R(M)$, which is constructed from classes of zero divisors, determined by annihilator submodules of M.

Theorem 5 *Let M be an R-module. Then, the following statements are true:*
(i) *For all $a \in R$, $aM + \mathrm{Ann}_M(a)$ is an essential submodule of M.*
(ii) *If $a \in r(\mathrm{Ann}_R(M))$, then $\mathrm{Ann}_M(a)$ is an essential submodule of M.*
(iii) *If $a \notin r(\mathrm{Ann}_R(M))$ and $\mathrm{Ann}_M(a)$ is a prime submodule of M, then $\mathrm{Ann}_M(a)$ is not an essential submodule of M.*

Proof (i) Let $a \in R$. We have to show that $aM + \mathrm{Ann}_M(a)$ is an essential submodule of M. Let N be a submodule of M. Then, $aN \subseteq aM \cap N \subseteq (aM + \mathrm{Ann}_M(a)) \cap N$. If $(aM + \mathrm{Ann}_M(a)) \cap N = 0$, then $aN = 0$ which implies that $N \subseteq \mathrm{Ann}_M(a)$. Hence, $N \subseteq (aM + \mathrm{Ann}_M(a)) \cap N$ and so $N = 0$. Therefore, $aM + \mathrm{Ann}_M(a)$ is an essential submodule of M.

(ii) By assumption, there exists an integer t, such that $a^t M = 0$. Thus, $aM \subseteq \mathrm{Ann}_M(a^{t-1})$ and so $aM + \mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(a^{t-1})$. Hence, $\mathrm{Ann}_M(a^{t-1})$ is an essential submodule of M by (i). Suppose that N is a non-zero submodule of M. Then, $\mathrm{Ann}_M(a^{t-1}) \cap N \neq 0$ and so there is $0 \neq x \in N$ such that $a^{t-1}x = 0$. Thus, $0 \neq a^i x \in N \cap \mathrm{Ann}_M(a)$, for some i with $0 \leq i \leq t-2$. Hence, $\mathrm{Ann}_M(a)$ is an essential submodule of M.

(iii) By assumption, $a \notin r(\mathrm{Ann}_R(M))$ so $aM \neq 0$. Let $am \in aM \cap \mathrm{Ann}_M(a)$. Then, by hypotheses, $m \in \mathrm{Ann}_M(a)$ which shows that $am = 0$ and so $aM \cap \mathrm{Ann}_M(a) = 0$. Thus, $\mathrm{Ann}_M(a)$ is not an essential submodule of M. □

Theorem 6 *Let M be an R-module and $a, b \in R$. Then, the following statements are true:*
(i) *If either $abM = 0$ or $\mathrm{Ann}_M(a)$, $\mathrm{Ann}_M(b)$ are prime submodules of M, then $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M.*
(ii) *If $\mathrm{Ann}_M(a)$, $\mathrm{Ann}_M(b)$ are prime submodules of M and $a, b \notin r(\mathrm{Ann}_R(M))$, then $abM = 0$.*

Proof (i) Let $a, b \in R$ and $abM = 0$. Then, $bM \subseteq \mathrm{Ann}_M(a)$ and so $bM + \mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$. Thus, by using Theorem 5 (i), the assertion follows.

Assume that $a, b \in R$ and $\mathrm{Ann}_M(a)$, $\mathrm{Ann}_M(b)$ are prime submodules of M. If either $a \in r(\mathrm{Ann}_R(M))$ or $b \in r(\mathrm{Ann}_R(M))$, then by Theorem 5 (ii), either $\mathrm{Ann}_M(a)$ or $\mathrm{Ann}_M(b)$ is an essential submodule of M and the assertion follows. Thus, assume that $a, b \notin r(\mathrm{Ann}_R(M))$. Without loss of generality, we can assume that $\mathrm{Ann}_M(a) \nsubseteq \mathrm{Ann}_M(b)$; see Theorem 2. Thus, there is $m \in M$ such that $am = 0$ and $bm \neq 0$. By $am \in \mathrm{Ann}_M(b)$, it follows that $a \in \mathrm{Ann}_R(bM)$. Hence, $abM = 0$ and the result follows by previous paragraph.

(ii) We suppose that $abM \neq 0$ and look for a contradiction. By (i), $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M. Thus, $(\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)) \cap abM \neq 0$. Hence, there are $m \in M$, $m' \in \mathrm{Ann}_M(a)$ and $m'' \in \mathrm{Ann}_M(b)$ such that $abm \neq 0$ and $abm = m' + m''$. Thus, $a^2b^2m = 0$. By assumption, $\mathrm{Ann}_M(a)$, $\mathrm{Ann}_M(b)$ are prime submodules of M, thus $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(a^2)$ and $\mathrm{Ann}_M(b) = \mathrm{Ann}_M(b^2)$. Therefore, $abm = 0$ that is a contradiction. □

Corollary 2 *Let $a, b \notin r(\mathrm{Ann}_R(M))$ and $\mathrm{Ann}_M(a)$ be a prime submodule of M. Then, $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M if and only if $\mathrm{Ann}_M(b) \nsubseteq \mathrm{Ann}_M(a)$.*

Proof If $\mathrm{Ann}_M(b) \nsubseteq \mathrm{Ann}_M(a)$, then by a similar argument to that of Theorem 6 (ii), one can show that $abM = 0$ and so $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M. Conversely, assume that $\mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(a)$. Thus, $\mathrm{Ann}_M(a)$ is an essential submodule of M and so $\mathrm{Ann}_M(a) \cap aM \neq 0$. Now, by a similar argument to that of Theorem 6 (ii), we achieve a contradiction. □

Assume $Z(M)$ denotes the set of zero divisors of M and $Z(M)^* = Z(M) \setminus \{0\}$. For $a, b \in R$, we say that $a \sim b$ if and only if $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$. As noted in [10], $\sim$ is an equivalence relation. If $[a]$ denotes the class of a, then $[0] = \mathrm{Ann}_R(M)$ and $[a] = R \setminus Z(M)$ for all $a \in R \setminus Z(M)$; the other equivalence classes form a partition of $Z(M)$.

Definition 1 The annihilator essential graph of equivalence classes of zero divisors of M, denoted $AE_R(M)$, is a graph associated to M whose vertices are the classes of elements of $Z(M)^*$, and each pair of distinct classes $[a]$ and $[b]$ are adjacent if and only if $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M.

The following remark which we include for the reader's convenience is based on the Theorems 5 and 6 and Corollary 2.

Remark 1 Let M be an R-module and let $[a]$, $[b]$ be two distinct vertices of $AE_R(M)$. Then, the following statements hold:
(i) If $a \in r(\mathrm{Ann}_R(M))$, then $[a]$ is a universal vertex of $AE_R(M)$.
(ii) If $abM = 0$, then $[a]$, $[b]$ are adjacent in $AE_R(M)$.
(iii) If $\mathrm{Ann}_M(a), \mathrm{Ann}_M(b)$ are prime submodules of M, then $[a], [b]$ are adjacent in $AE_R(M)$.
(iv) Let $a, b \notin r(\mathrm{Ann}_R(M))$ and $\mathrm{Ann}_M(a)$ be a prime submodule of M. Then, $[a]$, $[b]$ are adjacent in $AE_R(M)$ if and only if $\mathrm{Ann}_M(b) \nsubseteq \mathrm{Ann}_M(a)$.

The following example shows that $AE_R(M)$ can be an empty graph.

Example 2 Consider $M = \mathbb{Z} \times \mathbb{Z}_6$ as a $\mathbb{Z}$-module. Thus, $\mathrm{Ann}_M(2) = 0 \times 3\mathbb{Z}_6$, $\mathrm{Ann}_M(3) = 0 \times 2\mathbb{Z}_6$ and $\mathrm{Ann}_M(6) = 0 \times \mathbb{Z}_6$. Hence, $AE_{\mathbb{Z}}(M)$ is an empty graph with vertices [2, 3] and [6].

Theorem 7 *Let M be a Noetherian R-module. Then, $AE_R(M)$ is not a connected graph if and only if $r(\mathrm{Ann}_R(M)) = 0$ and there exists an element $a \in Z(M)^*$ such that $\mathrm{Ann}_M(a) \subseteq \bigcap_{P \in \mathrm{m-Ass}_R(M)} P$.*

Proof (⇒) Assume that $AE_R(M)$ is not connected. Thus, $r(\mathrm{Ann}_R(M)) = 0$, otherwise, by Remark 1 (i), $AE_R(M)$ has a universal vertex which is a contradiction. Also, Remark 1 (iii) implies that the induced subgraph of elements of $\mathrm{m} - \mathrm{Ass}_R(M)$ is complete; hence, by hypothesis, there exists $a \in Z(M)^*$ such that $\mathrm{Ann}_M(a)$ is not prime and $[a]$ is not joint with any element of $\mathrm{m}-\mathrm{Ass}_R(M)$. Therefore, $\mathrm{Ann}_M(a) \subseteq \cap_{P\in \mathrm{m}-\mathrm{Ass}_R(M)} P$ by Remark 1 (iv).

($\Leftarrow$) Let $r(\mathrm{Ann}_R(M)) = 0$ and let there be an element a in $Z(M)^*$ such that $\mathrm{Ann}_M(a) \subseteq \cap_{P\in \mathrm{m}-\mathrm{Ass}_R(M)} P$. We have to show that $[a]$ is an isolated vertex in $AE_R(M)$. By hypothesis and Remark 1 (iv), $[a]$ is not adjacent with any element of $\mathrm{m}-\mathrm{Ass}_R(M)$. On the other hand, if $\mathrm{Ann}_M(b) \notin \mathrm{m}-\mathrm{Ass}_R(M)$, then $\mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(c)$, for some $c \in R$ with $\mathrm{Ann}_M(c) \in \mathrm{m}-\mathrm{Ass}_R(M)$. Thus, $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(c)$ is not an essential submodule of M. Hence, $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is not an essential submodule of M, and so $[a]$ and $[b]$ are not adjacent. Therefore, $[a]$ is an isolated vertex and $AE_R(M)$ is not connected. □

Corollary 3 *Let M be a Noetherian R-module. Then, $AE_R(M)$ is a connected graph if and only if either $r(\mathrm{Ann}_R(M)) \neq 0$ or for each $a \in Z(M)^*$ there exists $P \in \mathrm{m}-\mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_M(a) \nsubseteq P$.*

Theorem 8 *Let M be a Noetherian R-module. If $AE_R(M)$ is a connected graph, then* $\mathrm{diam}\, AE_R(M) \leq 3$.

Proof If $r(\mathrm{Ann}_R(M)) \neq 0$, then $AE_R(M)$ has a universal vertex and therefore we have $\mathrm{diam}\, AE_R(M) \leq 2$. Now, let $r(\mathrm{Ann}_R(M)) = 0$ and $[a], [b]$ are two distinct vertices of $AE_R(M)$. By assumption, $AE_R(M)$ is connected, so there exist $a', b' \in Z(M)^*$ such that $\mathrm{Ann}_M(a')$, $\mathrm{Ann}_M(b')$ are prime submodules of M and $\mathrm{Ann}_M(a) \nsubseteq \mathrm{Ann}_M(a')$ and $\mathrm{Ann}_M(b) \nsubseteq \mathrm{Ann}_M(b')$. Thus, by Remark 1 (iv), if $[a'] = [b']$, then $[a] - [a'] - [b]$ is a path, and if $[a'] \neq [b']$, then $[a] - [a'] - [b'] - [b]$ is a path and so $\mathrm{diam}\, AE_R(M) \leq 3$. □

To see that the bound is sharp, notice that equality holds for $\mathbb{Z}_{30}$. It is easy to see that $\mathrm{diam}\, AE_{\mathbb{Z}_{30}}(\mathbb{Z}_{30}) = 3$.

Corollary 4 *If R is Noetherian, then $AE_R(R)$ is connected and* $\mathrm{diam}\, AE_R(R) \leq 3$.

Proof It is an immediate consequence of Theorems 7 and 8. □

Theorem 9 *Let M be a Noetherian module and let $AE_R(M)$ be a connected graph. If $AE_R(M)$ has a cycle, then* $\mathrm{gr}(AE_R(M)) \leq 4$.

Proof If $r(\mathrm{Ann}_R(M)) \neq 0$ or $|\ \mathrm{m}-\mathrm{Ass}_R(M)\ | \geq 3$, then the result follows by Remark 1 (i) and (iii). Assume that $r(\mathrm{Ann}_R(M)) = 0$ and $|\ \mathrm{m}-\mathrm{Ass}_R(M)\ | \leq 2$. Thus, we have $|\ \mathrm{m}-\mathrm{Ass}_R(M)\ | = 2$ since $AE_R(M)$ is connected. Suppose that $\mathrm{Ann}_R(a)$ and $\mathrm{Ann}_R(b)$ are two prime submodules of M and $[c]$ is an arbitrary vertex of $AE_R(M)$. Then, $[c]$ is adjacent to $[a]$ or $[b]$, by Corollary 3 and Remark 1 (iv). Thus, we have a cycle of length at most four and therefore $\mathrm{gr}(AE_R(M)) \leq 4$. □

Theorem 10 *Let M be a Noetherian R-module. Then, the following statements are true:*
(i) *If* $r(\mathrm{Ann}_R(M)) = 0$, *then* $\omega(AE_R(M)) =|\ \mathrm{m}-\mathrm{Ass}_R(M)\ |$.
(ii) *If* $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M) \neq 0$, *then* $\omega(AE_R(M)) =|\ \mathrm{m}-\mathrm{Ass}_R(M)\ | +1$.

Proof (i) Let $n := \omega(AE_R(M))$ and $k :=|\ \mathrm{m}-\mathrm{Ass}_R(M)\ |$. Then, by Remark 1 (iii), $k \leq n$. If $\mathrm{Ann}_M(a)$ is not a prime submodule, then there is an annihilator prime submodule $\mathrm{Ann}_M(b)$ such that $[a]$ and $[b]$ are not adjacent by Remark 1 (iv), where $a, b \in Z(M)^*$.

Assume that H is a maximal clique and $k < n$. So, there is a vertex of $\mathrm{m}-\mathrm{Ass}_R(M)$ which is not a vertex of H. Assume that H contains $k-t$ vertices of $\mathrm{m}-\mathrm{Ass}_R(M)$, where $t > 0$. Thus, at least $t+1$ vertices of H are adjacent to $k-t$ vertices of $\mathrm{m}-\mathrm{Ass}_R(M)$ and could not be adjacent to all other t vertices of $\mathrm{m}-\mathrm{Ass}_R(M)$. Hence, there exist at least two vertices $[c]$, $[d]$ of H that are not adjacent to a vertex $[e]$ of $\mathrm{m}-\mathrm{Ass}_R(M)$. Then, $\mathrm{Ann}_M(c)+\mathrm{Ann}_M(d) \subseteq \mathrm{Ann}_M(e)$ and $\mathrm{Ann}_M(e)$ is not essential, by Theorem 5 (iii). Thus, $[c]$, $[d]$ are not adjacent and it is a contradiction. Hence, $k = n$.

(ii) Assume that $a \in \mathrm{Ann}_R(M)$. Thus, $[a]$ is a universal vertex of $AE_R(M)$. Now, by a similar argument to that of (i), the result follows. □

Theorem 11 *Let M be a Noetherian R-module and $r(\mathrm{Ann}_R(M)) = 0$. Then, the following statements are equivalent:*
(i) *$AE_R(M)$ is a complete graph;*
(ii) *$\mathrm{Ann}_M(a)$ is a prime submodule of M, for all $a \in Z(M)^*$;*
(iii) *$AE_R(M) = K_1$ or K_2.*

Proof (i) ⇒ (ii) Assume that $a \in Z(M)^*$. If $\mathrm{Ann}_M(a)$ is a maximal element of $X = \{\mathrm{Ann}_M(b) : b \notin \mathrm{Ann}_R(M)\}$, then it is a prime submodule of M and the result follows. Otherwise, there is $b \in Z(M)^*$ such that $\mathrm{Ann}_M(b)$ is a maximal element of X and $\mathrm{Ann}_M(a) \subsetneq \mathrm{Ann}_M(b)$. Thus, $[a]$ is not adjacent to $[b]$ by Remark 1 (iv) contrary to assumption. Hence, $\mathrm{Ann}_M(a)$ is a prime submodule of M, for all $a \in Z(M)^*$.

(ii) ⇒ (i) It is obvious by Remark 1 (iii).

(i) ⇒ (iii) Assume that $a, b, c \in Z(M)^*$ and assume $[a]$, $[b]$, $[c]$ are all distinct vertices of $AE_R(M)$. Thus, in view of (i) ⇔ (ii) and Theorem 6 (ii), we have $ab = ac = bc = 0$. Hence, $a(b+c) = 0$ and so $b+c \in Z(M)^*$. Therefore, $[b+c]$ is a vertex of $AE_R(M)$. If $[b+c]$ and $[b]$ are two distinct vertices of $AE_R(M)$, then by assumption $[b+c]$ and $[b]$ are adjacent and so as above $0 = b(b+c) = b^2$ which implies that $b \in r(\mathrm{Ann}_R(M)) = 0$; this is a contradiction. Now, assume that $[b+c] = [b]$. Thus, $\mathrm{Ann}_M(b+c) = \mathrm{Ann}_M(b)$. So that, $cM \subseteq \mathrm{Ann}_M(b) = \mathrm{Ann}_M(b+c)$. Thus, $(b+c)cM = 0$ and so $c^2 = 0$. It leads to a similar contradiction. Hence, $AE_R(M)$ has at most two vertices.

(iii) ⇒ (i) It is obvious. □

Theorem 12 *Let M be an R-module and $\mathrm{Ann}_R(M) \neq 0$. Then, the following statements are true:*
(i) *If $AE_R(M)$ is a star graph with more than two vertices, then $\mathrm{Ann}_R(M)$ is a prime ideal of R.*
(ii) *If M is Noetherian and $\mathrm{Ann}_R(M)$ is a prime ideal of R, then $AE_R(M)$ is a star graph.*

Proof (i) Assume that $AE_R(M)$ is a star graph with center vertex $[c]$, where $c \in \mathrm{Ann}_R(M)$. We have to show that $\mathrm{Ann}_R(M)$ is a prime ideal of R. Assume that $a, b \in R$ and $abM = 0$. If either $aM = 0$ or $bM = 0$, we are done. Otherwise, with condition $\mathrm{Ann}_M(a) \neq \mathrm{Ann}_M(b)$, $[a]$ and $[b]$ are adjacent in $AE_R(M)$, by Remark 1 (ii). So that we can assume that $[a] = [c]$ and then $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(c)$ which implies that $a \in \mathrm{Ann}_R(M)$, a contradiction. Hence, $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$. Thus, $a^2M = 0$ and so $a \in r(\mathrm{Ann}_R(M))$ and $[a]$ is the center vertex of $AE_R(M)$ by Remark 1 (i) is again a contradiction. Therefore, either $aM = 0$ or $bM = 0$ and $\mathrm{Ann}_R(M)$ is a prime ideal of R.

(ii) In view of Corollary 1, $\mid \mathrm{m}-\mathrm{Ass}_R(M) \mid = 1$. Let $\mathrm{Ann}_M(a)$ be the unique element of $\mathrm{m}-\mathrm{Ass}_R(M)$ and so the unique maximal element of $X = \{\mathrm{Ann}_M(b) : b \notin \mathrm{Ann}_R(M)\}$. Then, $\mathrm{Ann}_M(a)$ is not an essential submodule of M, by Theorem 5. Assume that $b, c \in$

$Z(M) \setminus \mathrm{Ann}_R(M)$. Thus, $\mathrm{Ann}_M(b) + \mathrm{Ann}_M(c) \subseteq \mathrm{Ann}_M(a)$ and so $\mathrm{Ann}_M(b) + \mathrm{Ann}_M(c)$ is not essential submodule of M. Hence, $[b]$ and $[c]$ are not adjacent in $AE_R(M)$. Therefore, all vertices of $AE_R(M)$ are adjacent to $[d]$ where $d \in \mathrm{Ann}_R(M)$ that implies $AE_R(M)$ is a star graph. □

The annihilator essential graph $AE_R(R)$ and the graph of equivalence classes of zero divisors $\Gamma_E(R)$ have the same vertex sets (see [10]). Moreover, if $[a], [b]$ are adjacent vertices of $\Gamma_E(R)$, then $ab = 0$, so by Remark 1 (ii), $[a], [b]$ are adjacent in $AE_R(R)$. Therefore, $\Gamma_E(R)$ is a subgraph of $AE_R(R)$. Now, the question arises as when these graphs are the same?

Theorem 13 *If R is a reduced ring, then $AE_R(R) = \Gamma_E(R)$.*

Proof Assume that $[a], [b]$ are adjacent in $AE_R(R)$. Then, $\mathrm{Ann}_R(a) + \mathrm{Ann}_R(b)$ is an essential ideal of R. If $ab \neq 0$, then $(\mathrm{Ann}_R(a) + \mathrm{Ann}_R(b)) \cap abR \neq 0$. Thus, there exist $r \in R$, $s \in \mathrm{Ann}_R(a)$ and $t \in \mathrm{Ann}_R(b)$ such that $0 \neq abr = s + t$. Hence, $a^2b^2r = 0$, and so $abr \in \mathrm{Nil}(R) = 0$ is a contradiction. Thus, $ab = 0$. The converse is true by the previous paragraph. □

Let p be a prime number and $R = \mathbb{Z}_{p^3}$. Then, $AE_R(R) = \Gamma_E(R) = K_2$, but $\mathrm{Nil}(R) \neq 0$. So that the converse of Theorem 13 is not true necessarily.

Theorem 14 *Let R be a Noetherian non-reduced ring and $AE_R(R) = \Gamma_E(R)$. Then, the following statements are true:*
(i) $Z(R) = \mathrm{Ann}_R(a)$, *for some* $a \in \mathrm{Nil}(R)$.
(ii) *If* $\mathrm{Ann}_R(a)$ *is an essential ideal of R, then* $a \in \mathrm{Nil}(R)$.
(iii) $\mathrm{Nil}(R) = r(\mathrm{Ann}_R(Z(R)))$.

Proof (i) Assume that $0 \neq x \in \mathrm{Nil}(R)$ thus $\mathrm{Ann}_R(x)$ is an essential ideal of R, see Theorem 5 (ii), and so x is a universal vertex of $AE_R(R)$. Hence, by assumption, either $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(y)$ or $xy = 0$, for all $y \in Z(R)$. We first show that $Z(R)$ is an ideal of R. Let $a, b \in Z(R)$. It is enough to show that $a + b \in Z(R)$. If $\mathrm{Ann}_R(b) = \mathrm{Ann}_R(a)$, then we are done. Thus, assume that $\mathrm{Ann}_R(b) \neq \mathrm{Ann}_R(a)$. If $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(a)$, then there is an integer n such that $x^n a = 0$ and $x^n \neq 0$. Moreover $xb = 0$ since $\mathrm{Ann}_R(x) \neq \mathrm{Ann}_R(b)$. Hence, $x^n(a + b) = 0$ and so $a + b \in Z(R)$. If $\mathrm{Ann}_R(x) \neq \mathrm{Ann}_R(a)$ and $\mathrm{Ann}_R(x) \neq \mathrm{Ann}_R(b)$, then $xa = xb = 0$ which implies that $x(a + b) = 0$ and $a + b \in Z(R)$. Thus, $Z(R)$ is an ideal of R. We have $Z(R) = \cup_{\mathrm{Ann}_R(x) \in \mathrm{Ass}_R(R)} \mathrm{Ann}_R(x)$. Thus, by Prime Avoidance Theorem $Z(R) = \mathrm{Ann}_R(a)$, for some $a \in Z(R)$. Furthermore, by $a \in Z(R) = \mathrm{Ann}_R(a)$, it follows that $a^2 = 0$ and $a \in \mathrm{Nil}(R)$.

(ii) Let $a \in R$ and assume that $\mathrm{Ann}_R(a)$ is an essential ideal of R. If $\mathrm{Ann}_R(a) = \mathrm{Ann}_R(a^2)$, then $\mathrm{Ann}_R(a) \cap Ra = 0$. Thus, by hypotheses $Ra = 0$ and so $a = 0 \in \mathrm{Nil}(R)$. Now, assume that $\mathrm{Ann}_R(a) \neq \mathrm{Ann}_R(a^2)$. Thus, $[a], [a^2]$ are two distinct adjacent vertices of $AE(R)$. Hence, by assumption, $a^3 = 0$ and therefore $a \in \mathrm{Nil}(R)$.

(iii) Let $a \in r(\mathrm{Ann}_R(Z(R)))$. Then, $a^n Z(R) = 0$, for some integer n, and so $a^{n+1} = 0$ since $a \in Z(R)$. Hence, $a \in \mathrm{Nil}(R)$ and therefore $r(\mathrm{Ann}_R(Z(R))) \subseteq \mathrm{Nil}(R)$. Assume that $Z(R)$ is generated by $a_1, a_2, \ldots, a_n$ and $a \in \mathrm{Nil}(R)$. Then, by hypotheses and Remark 1 (i), for all $i = 1, 2, \ldots, n$, either $aa_i = 0$ or $\mathrm{Ann}_R(a_i) = \mathrm{Ann}_R(a)$. If $aa_i = 0$ for all $i = 1, 2, \ldots, n$, then $a \in \mathrm{Ann}_R(Z(R))$. Now, assume that there is j with $1 \leq j \leq n$ such that $\mathrm{Ann}_R(a_j) = \mathrm{Ann}_R(a)$. Thus, $a^k a_j = 0$ for some integer k. Thus, $a^k a_i = 0$, for all $i = 1, 2, \ldots, n$ that implies $a \in r(\mathrm{Ann}_R(Z(R)))$. So, $\mathrm{Nil}(R) \subseteq r(\mathrm{Ann}_R(Z(R)))$. □

Theorem 15 *Let R be a Noetherian ring and let $AE_R(R)$ be a graph with at least four vertices. Then, $AE_R(R)$ is not a cycle graph.*

Proof Let $AE_R(R)$ be a cycle graph with vertices $[a_1], \cdots, [a_n]$. Then, [11, Proposition 1.8] implies that $AE_R(R) \neq \Gamma_E(R)$ so we can assume that $[a_1], [a_n]$ are not adjacent in $\Gamma_E(R)$. Thus, in view of [11, Corollary 3.3], it follows that $\mathrm{Ann}_R(a_2)$, $\mathrm{Ann}_R(a_{n-1})$ are prime ideal of R. Hence, $[a_2], [a_{n-1}]$ are adjacent in $AE_R(R)$, by Remark 1 (iii). So that $n = 4$ and moreover $\Gamma_E(R)$ is the path $[a_1] - [a_2] - [a_3] - [a_4]$. Now, $a_1a_4 \in Z(R)^*$ and $\mathrm{Ann}_R(a_1a_4) \neq \mathrm{Ann}_R(a_i)$, for all i with $1 \leq i \leq 4$ since $a_1a_3 \neq 0$, $a_2a_4 \neq 0$ and $a_2^2 \neq 0$. Therefore, $AE_R(R)$ is not a cycle graph. □

Theorem 16 *If M is a faithful multiplication R-module, then $AE_R(R)$ and $AE_R(M)$ are isomorphic.*

Proof Let $a, b \in Z(M)$; we show that $\mathrm{Ann}_R(a) = \mathrm{Ann}_R(b)$ if and only if $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$. Let $\mathrm{Ann}_R(a) = \mathrm{Ann}_R(b)$. Then, $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(bM)$ since M is faithful. Thus, $\mathrm{Ann}_M(a) = \mathrm{Ann}_R(aM)M = \mathrm{Ann}_R(bM)M = \mathrm{Ann}_M(b)$ since M is multiplication. Conversely, if $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$, then $\mathrm{Ann}_R(aM) = \mathrm{Ann}_R(bM)$ by Lemma 1. So that $\mathrm{Ann}_R(a) = \mathrm{Ann}_R(b)$. Now, in view of [1, Theorem 2.13], $\mathrm{Ann}_R(a) + \mathrm{Ann}_R(b)$ is an essential ideal of R if and only if $\mathrm{Ann}_M(a) + \mathrm{Ann}_M(b)$ is an essential submodule of M. Therefore, $AE_R(R)$ and $AE_R(M)$ are isomorphic. □

Acknowledgements We would like to thank the referee for a careful reading of our article and insightful comments which saved us from several errors.

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