

Conjugate Duality and Optimization over Weakly Efficient Set

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Abstract In this article, we present a conjugate duality for nonconvex optimization problems. This duality scheme is symmetric and has zero gap. As applied to a vector-maximization problem, it transforms the latter into an optimization problem over a weakly efficient set which can be solved by monotonic optimization methods.

Keywords Quasigradient duality · Vector-optimization · Weakly efficient set · Quasi-supgradient · Monotonic optimization

Mathematics Subject Classification (2010) 80C90 · 90C60 · 49N25

1 Introduction

A nonconvex duality with zero gap, called quasigradient duality, was developed by Thach in two well-known articles [9, 10]. An important feature of this duality theory is that, with a suitable definition of quasiconjugate of a function, it transforms certain classes of difficult nonconvex optimization problems into simpler ones, sometimes even into convex problems. Moreover, the corresponding duality relationship often reveals important properties useful for the theoretical and computational study of the original problem. In particular, in [12], by defining the quasiconjugate of a function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ as

$$f^*(p) = \frac{1}{\sup\{f(x) : p^T x \leq 1, x \geq 0\}} \quad \forall p \in \mathbb{R}_+^n,$$

the maximization of a Leontiev production function over a convex feasible region is converted by duality into the maximization of an increasing linear function over a convex

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region. In [13], this duality theory, extended to vector maximization problems, is used to establish important properties related to weak Pareto efficiency. More recently, in [14], quasiconjugate duality scheme is applied for studying a specific nonlinear optimization problem under resource allocation constraints. It turned out that the feasible set of the dual to this problem is a singleton (in the case of a single resource) or the set of Pareto efficient solutions of an associated vector-maximization problem (in the case of $k > 1$ resources).

So far, quasiconjugate duality has been restricted to the classes of concave scalar-maximization and concave vector-maximization problems involving concave criteria functions. However, many functions encountered in mathematical economics and other applications are not concave but only quasiconcave. Outstanding examples are the Cobb-Douglas function, generalized Cobb-Douglas function, generalized Leontiev function, constant elasticity of substitution (C.E.S) function (cf. [3]), and posynomial function (under the assumption that all goods are useful).

To overcome this limitation, we will attempt, in the first part of the present paper, to extend quasiconjugate duality to the very general class of nonconvex scalar-maximization and vector-maximization problems involving continuous quasiconcave and increasing functions. The extension will preserve the main properties earlier established in [13], especially the symmetric property and the zero gap duality property. Also, duality relationships will be developed so that they can help to characterize the (weak) Pareto efficient solutions.

It is worth noting that most of the existing duality theories for vector-maximization problems are based on applying Lagrange duality to a scalarized problem depending on the weight parameters (cf. [2, 6, 8]). In some works, the theory of set-valued maps has also been used for studying vector-optimization problems (cf. [4, 5]). However, in all these approaches, the duality is not symmetric and often requires convexity of the primal problem.

Together with the extension of quasiconjugate duality, we will also present a new application. Specifically, it will be shown that using quasiconjugate duality for vector-optimization we can convert an optimization problem over weakly efficient sets into a bilevel optimization problem. The latter in turn can be reformulated as a monotonic optimization problem amenable to efficient solution methods developed by Tuy and his coauthors in [15–17].

The paper is organized as follows. After the introduction, in Section 2, together with a review of basic results of quasiconjugate as presented in [7, 11] and [13], we will provide some new related results and introduce the concept of quasi-subgradient of a function. Next, in Section 3, we will develop quasiconjugate duality for scalar-maximization problems and in Section 4, quasiconjugate duality for vector-maximization problems. Finally, Section 5 is devoted to an application of quasiconjugate duality to optimization over weakly efficient sets.

2 Quasiconjugate of a Function

Throughout this section, we assume that $f(x)$ is a nonnegative finite-valued function on \mathbb{R}_+^n satisfying $f(x) > 0$ for all x from

$$\mathbb{R}_{++}^n := \{x \in \mathbb{R}^n \mid x_i > 0 \forall i = 1, 2, \dots, n\}.$$

Following Thach in [11], we define the quasiconjugate of f as the function $f^* : \mathbb{R}_+^n \rightarrow \mathbb{R}$ with

$$f^*(p) = \frac{1}{\sup\{f(x) : p^T x \leq 1, x \geq 0\}} \quad \forall p \in \mathbb{R}_+^n, \tag{1}$$

(as usual, we use the convention $\frac{1}{+\infty} = 0$).

Since f is continuous and $f(x) > 0$ for every $x \in \mathbb{R}^n_{++}$, we have

$$\sup\{f(x) : p^T x \leq 1, x \geq 0\} > 0$$

for every $p \geq 0$, so $f^*(p)$ is again a nonnegative finite-valued function on \mathbb{R}^n_+ satisfying $f^*(p) > 0$ for every $p \in \mathbb{R}^n_{++}$.

In [13], we have established the following property:

If f is a polyhedral concave increasing and positively homogeneous function defined on \mathbb{R}^n then so is f^ and f is the quasiconjugate of f^* on \mathbb{R}^n_+ , i.e., $(f^*)^* = f$.*

We now extend this property to a larger class of functions f .

Proposition 1 *If f is positively homogeneous, then so is f^* .*

Proof For $x^0 > 0$, we have $p^T(kx^0) = 0 \leq 1$ for all $k = 1, 2, \dots$ and $f(kx^0) = kf(x^0) \rightarrow +\infty$ as $k \rightarrow +\infty$. So,

$$f^*(0) = \frac{1}{\sup\{f(x) : x \geq 0\}} = \frac{1}{+\infty} = 0.$$

If $p \neq 0$, then for every $\theta > 0$, we have

$$\begin{aligned} \sup\{f(x) : \theta p^T x \leq 1, x \geq 0\} &= \sup\left\{f\left(\frac{1}{\theta}x'\right) : p^T x' \leq 1, x' \geq 0\right\} \\ &= \frac{1}{\theta} \sup\left\{f(x') : p^T x' \leq 1, x' \geq 0\right\}, \end{aligned}$$

and so $f^*(\theta p) = \theta f^*(p)$. □

A set $X \subset \mathbb{R}^n$ is said to be conormal if $x' \in X$ whenever $x' \geq x \in X$ (see [16]). For $\gamma \in \mathbb{R}_+$, let F_γ denote the upper levels of f :

$$F_\gamma = \{x \in \mathbb{R}^n_+ : f(x) \geq \gamma\}.$$

Lemma 1 (see [16]) *If $f(x)$ is an increasing function on \mathbb{R}^n then the set F_γ is conormal.*

Lemma 2 *Let F be a conormal set. If $q^T x \geq \gamma$ for all $x \in F$ then $q \in \mathbb{R}^n_+$.*

Proof If we had $q_i < 0$ for some $i \in \{1, 2, \dots, n\}$, then $x^0 \in F$ would imply that $x^k = (x_1^0, \dots, x_i^0 + k, \dots, x_n^0) \in F$ for all k (because $x^k \geq x^0$) while $q^T x^k \rightarrow -\infty$ as $k \rightarrow +\infty$. □

Let X be an arbitrary set in \mathbb{R}^n . A set X^* is said to be the upper conjugate of X if

$$X^* = \left\{p \in \mathbb{R}^n_+ : p^T x \geq 1 \forall x \in X\right\}.$$

Let F_γ^\natural denote the upper levels of f^* : $F_\gamma^\natural = \{p \in \mathbb{R}^n_+ : f^*(p) \geq \gamma\}$. In [13], it has been shown that when f is a polyhedral concave increasing and positively homogeneous function defined on \mathbb{R}^n then F_1^\natural and F_1 are upper conjugate of each other, i.e.,

$$F_1^\natural = \left\{p \in \mathbb{R}^n_+ : p^T x \geq 1 \forall x \in F_1\right\}, \tag{2}$$

$$F_1 = \left\{x \in \mathbb{R}^n_+ : p^T x \geq 1 \forall p \in F_1^\natural\right\}. \tag{3}$$

More generally, we have the following theorem.

Theorem 1 *If f is a continuous quasiconcave strictly increasing function on \mathbb{R}^n_+ then $F_{\frac{1}{\gamma}}^{\natural}$ and F_{γ} are upper conjugate of each other for any $\gamma > 0$ such that $F_{\gamma} \neq \emptyset$.*

Proof For $p \in F_{\frac{1}{\gamma}}^{\natural}$, we have $p \geq 0$ and

$$\begin{aligned} f^*(p) &\geq \frac{1}{\gamma} \\ \Leftrightarrow \frac{1}{\sup\{f(x) : p^T x \leq 1, x \geq 0\}} &\geq \frac{1}{\gamma} \\ \Leftrightarrow f(x) &\leq \gamma \quad \forall x \geq 0 \text{ such that } p^T x \leq 1 \\ \Leftrightarrow p^T x &> 1 \quad \forall x \geq 0 \text{ such that } f(x) > \gamma. \end{aligned} \tag{4}$$

Also, it is easily seen that $p^T x \geq 1$ for every $x \geq 0$ satisfying $f(x) = \gamma$. Indeed, let $\bar{x} \geq 0$ satisfy $f(\bar{x}) = 1$. Since f is positive on \mathbb{R}^n_{++} and strictly increasing on \mathbb{R}^n_+ , $\{x \in \mathbb{R}^n_+ \mid f(x) > \gamma\} \neq \emptyset$. Take any sequence $\{x^l\} \subset \{x \in \mathbb{R}^n_+ \mid f(x) > \gamma\}$ such that $x^l \rightarrow \bar{x}$. By (4), $p^T x^l > 1$ for all l , so letting $l \rightarrow +\infty$ we get $p^T \bar{x} \geq 1$. Thus, $p^T x \geq 1$ for every $x \geq 0$ such that $f(x) \geq \gamma$, i.e., $p^T x \geq 1$ for every $x \in F_{\gamma}$. Hence,

$$F_{\frac{1}{\gamma}}^{\natural} \subset \left\{ p \in \mathbb{R}^n_+ : p^T x \geq 1 \quad \forall x \in F_{\gamma} \right\}.$$

Conversely, for any $p \geq 0$, we can write

$$\begin{aligned} p^T x &\geq 1 \quad \forall x \in F_{\gamma} \\ \Leftrightarrow p^T x &\geq 1 \quad \forall x \geq 0 \text{ such that } f(x) \geq \gamma \\ \Leftrightarrow f(x) &< \gamma \quad \forall x \geq 0 \text{ such that } p^T x < 1 \\ \Rightarrow f(x) &\leq \gamma \quad \forall x \geq 0 \text{ such that } p^T x \leq 1 \\ \Leftrightarrow \sup\{f(x) : p^T x \leq 1, x \geq 0\} &\leq \gamma \\ \Leftrightarrow \frac{1}{\sup\{f(x) : p^T x \leq 1, x \geq 0\}} &\geq \frac{1}{\gamma} \\ \Leftrightarrow f^*(p) &\geq \frac{1}{\gamma} \\ \Leftrightarrow p &\in F_{\frac{1}{\gamma}}^{\natural}. \end{aligned}$$

Hence, $F_{\frac{1}{\gamma}}^{\natural}$ is the upper conjugate of F_{γ} .

Next, we show that conversely F_{γ} is the upper conjugate of $F_{\frac{1}{\gamma}}^{\natural}$. Set

$$A = \left\{ x \in \mathbb{R}^n_+ : p^T x \geq 1 \quad \forall p \in F_{\frac{1}{\gamma}}^{\natural} \right\}.$$

Since $F_{\frac{1}{\gamma}}^{\natural}$ is the upper conjugate of F_{γ} , it is obvious that $F_{\gamma} \subseteq A$. Suppose that $\bar{x} \geq 0$ and $\bar{x} \notin F_{\gamma}$. Since f is continuous quasiconcave and increasing, F_{γ} is a closed convex and conormal set. So F_{γ} does not intersect the line segment $[0, \bar{x}]$. By the separation theorem, there are $q \in \mathbb{R}^n \setminus \{0\}$ and number $\alpha \in \mathbb{R}$ such that

$$\begin{aligned} q^T x &\geq \alpha \quad \forall x \in F_{\gamma}, \\ q^T x &< \alpha \quad \forall x \in [0, \bar{x}]. \end{aligned} \tag{5}$$

$$\tag{6}$$

From (5), we have $q \geq 0$ by Lemma 2, and from (6) it follows that $\alpha > 0$. Setting $p = \frac{1}{\alpha}q$, we can then write

$$p^T x \geq 1 \quad \forall x \in F_\gamma, \tag{7}$$

$$p^T \bar{x} < 1. \tag{8}$$

From (7), it follows that $p \in F_{\frac{1}{\gamma}}^\natural$. This together with (8) implies that $\bar{x} \notin A$. □

The above theorem exhibits a polarity correspondence between the level sets of the function f and those of its quasiconjugate. It is easily seen that the function f can be defined via its level sets, namely $f(x) = \sup\{\gamma > 0 \mid x \in F_\gamma\}$ (as usual, we use the convention $\sup \emptyset = 0$). Similarly, we have $f^*(p) = \sup\{\gamma > 0 \mid p \in F_\gamma^\natural\}$. Hence, $f^*(p) = \sup\{\gamma > 0 \mid p \in (F_{\frac{1}{\gamma}})^\ast\}$ for any continuous quasiconcave and strictly increasing function f .

We now examine an important case when the equality $(f^*)^\ast = f$ extends to nonconcave functions f .

Proposition 2 *Let f be a Cobb-Douglas function on \mathbb{R}_+^n , that is*

$$f(x) = \prod_{i=1}^n x_i^{\alpha_i},$$

where $\alpha_i > 0$ for every $i = 1, 2, \dots, n$. Then, f^\ast is also a Cobb-Douglas function on \mathbb{R}_+^n and $(f^\ast)^\ast = f$.

Proof If $p = (p_1, p_2, \dots, p_n)$ and $p_i = 0$ for some $i = 1, 2, \dots, n$ then $p^T x^k \leq 1$ for every $x^k = (\bar{x}_1, \bar{x}_2, \dots, k\bar{x}_i, \dots, \bar{x}_n)$, where $\bar{x} > 0$ satisfies $p^T \bar{x} \leq 1$. Since $f(\bar{x}) > 0$, $f(x^k) = k^{\alpha_i} f(\bar{x}) \rightarrow +\infty$ as $k \rightarrow +\infty$ it follows that $f^\ast(p) = 0$. If $p > 0$ then $\sup\{f(x) \mid p^T x \leq 1, x \geq 0\} = \alpha^\alpha \prod_{i=1}^n (\frac{\alpha_i}{p_i})^{\alpha_i}$, $\alpha = \sum_{i=1}^n \alpha_i$ (see [7, Example 4.9]). Hence, $f^\ast(p) = (\frac{1}{\alpha})^\alpha \prod_{i=1}^n (\frac{p_i}{\alpha_i})^{\alpha_i}$ for every $p \in \mathbb{R}_+^n$. By an argument similar to the above, we can show that $(f^\ast)^\ast = f$. □

Note that Cobb-Douglas production functions are quasiconcave but may not be concave (for example for $\alpha_i = 2$ for all i). This raises the question: What is the largest class of functions f for which we eliminate the gap between f and $(f^\ast)^\ast$?

The following proposition provides a necessary condition for a function f to belong to this class .

Proposition 3 *The function f^\ast , quasiconjugate of f , is quasiconcave and increasing on \mathbb{R}_+^n .*

Proof Let $p, p' \in \mathbb{R}_+^n$. For each $t \in [0; 1]$, we have

$$\{x \geq 0 : (tp + (1-t)p')^T x \leq 1\} \subset \{x \geq 0 : p^T x \leq 1\} \cup \{x \geq 0 : (p')^T x \leq 1\}.$$

It follows that

$$\begin{aligned} 0 &< \sup\{f(x) : (tp + (1-t)p')^T x \leq 1, x \geq 0\} \\ &\leq \max\{\sup\{f(x) : p^T x \leq 1, x \geq 0\}; \sup\{f(x) : (p')^T x \leq 1, x \geq 0\}\}. \end{aligned}$$

This occurs if and only if $f^\ast(tp + (1-t)p') \geq \min\{f^\ast(p); f^\ast(p')\}$. So, f^\ast is quasiconcave.

Furthermore, let $0 \leq p \leq p'$. Clearly, $\{x \in \mathbb{R}_+^n : (p')^T x \leq 1\} \subset \{x \in \mathbb{R}_+^n : p^T x \leq 1\}$, and so $\sup\{f(x) : p^T x \leq 1, x \geq 0\} \geq \sup\{f(x) : (p')^T x \leq 1, x \geq 0\} > 0$. This implies that $f^*(p) \leq f^*(p')$, proving that f^* is increasing. \square

Next, we give a sufficient condition for f to be also quasiconjugate of f^* .

Theorem 2 *If f is a continuous quasiconcave and increasing function on \mathbb{R}_+^n , then f is also the quasiconjugate of f^* on \mathbb{R}_+^n , i.e.,*

$$f(x) = \frac{1}{\sup\{f^*(p) : p^T x \leq 1, p \geq 0\}} \quad \forall x \in \mathbb{R}_+^n.$$

Proof Set

$$\bar{f}(x) = \frac{1}{\sup\{f^*(p) : p^T x \leq 1, p \geq 0\}}.$$

For $\gamma > 0$, let F_γ, \bar{F}_γ denote the upper levels of f and f^* respectively, i.e.,

$$F_\gamma = \{x \in \mathbb{R}_+^n : f(x) \geq \gamma\}, \bar{F}_\gamma = \{x \in \mathbb{R}_+^n : \bar{f}(x) \geq \gamma\}.$$

To prove the equality $f(x) = \bar{f}(x)$ for every $x \in \mathbb{R}_+^n$, it suffices to show that $F_\gamma = \bar{F}_\gamma$ for any $\gamma > 0$. Let $\bar{x} \in F_\gamma$. If $\bar{x} \notin \bar{F}_\gamma$, then we have $\bar{f}(\bar{x}) < \gamma$, or

$$\frac{1}{\sup\{f^*(p) : p^T \bar{x} \leq 1, p \geq 0\}} < \gamma,$$

hence, there exists $\bar{p} \geq 0, \bar{p}^T \bar{x} \leq 1$ such that

$$\begin{aligned} f^*(\bar{p}) &> \frac{1}{\gamma} \\ \Leftrightarrow \frac{1}{\sup\{f(x) : \bar{p}^T x \leq 1, x \geq 0\}} &> \frac{1}{\gamma} \\ \Leftrightarrow \sup\{f(x) : \bar{p}^T x \leq 1, x \geq 0\} &< \gamma \\ \Leftrightarrow f(x) < \gamma \quad \forall x \geq 0 \text{ such that } \bar{p}^T x &\leq 1. \end{aligned}$$

It follows that $f(\bar{x}) < \gamma$. This conflicts with the fact $\bar{x} \in F_\gamma$. Hence, $F_\gamma \subset \bar{F}_\gamma$.

Conversely, let $\bar{x} \in \bar{F}_\gamma$. We can write

$$\begin{aligned} \bar{f}(\bar{x}) &\geq \gamma \\ \Leftrightarrow \frac{1}{\sup\{f^*(p) : p^T \bar{x} \leq 1, p \geq 0\}} &\geq \gamma \\ \Leftrightarrow f^*(p) \leq \frac{1}{\gamma} \quad \forall p \geq 0 \text{ such that } p^T \bar{x} &\leq 1 \\ \Leftrightarrow \frac{1}{\sup\{f(x) : p^T x \leq 1, x \geq 0\}} \leq \frac{1}{\gamma} \quad \forall p \geq 0 \text{ such that } p^T \bar{x} &\leq 1 \\ \Leftrightarrow \sup\{f(x) : p^T x \leq 1, x \geq 0\} \geq \gamma \quad \forall p \geq 0 \text{ such that } p^T \bar{x} &\leq 1. \end{aligned} \tag{9}$$

For $\tilde{p} > 0$ such that $\tilde{p}^T \bar{x} \leq 1$, the set $\{x \in \mathbb{R}_+^n : \tilde{p}^T x \leq 1\}$ is compact. Since f is continuous, by (9) there exists a vector $\tilde{x} \geq 0, \tilde{p}^T \tilde{x} \leq 1$ such that $f(\tilde{x}) \geq \gamma$. This implies that $\tilde{x} \in F_\gamma$. Hence, F_γ is nonempty.

Suppose now that $\bar{x} \notin F_\gamma$. The function f is quasiconcave continuous and increasing on \mathbb{R}_+^n , so F_γ is a closed convex and conormal set. Since F_γ does not intersect the line segment

$[0, \bar{x}]$, by the strong separation theorem, there exist a vector $q \neq 0$ and real number α such that

$$q^T x > \alpha \quad \forall x \in F_\gamma; \tag{10}$$

$$q^T x < \alpha \quad \forall x \in [0; \bar{x}]. \tag{11}$$

From (10), we have $q \geq 0$ by Lemma 2. Furthermore, from (11), it follows that $\alpha > 0$ and there exists $t > 0$ sufficiently small such that $(q + te)^T \bar{x} \leq \alpha$, where $e = (1, 1, \dots, 1)$. Setting $\bar{p} = \frac{1}{\alpha}(q + te)$, we have $\bar{p} > 0$ and

$$\bar{p}^T x \geq \frac{1}{\alpha} p^T x > 1 \quad \forall x \in F_\gamma; \tag{12}$$

$$\bar{p}^T \bar{x} \leq 1. \tag{13}$$

Consequently,

$$\bar{p}^T x > 1 \quad \forall x \geq 0 \text{ such that } f(x) \geq \gamma \tag{14}$$

$$\Leftrightarrow f(x) < \gamma \quad \forall x \geq 0 \text{ such that } \bar{p}^T x \leq 1. \tag{15}$$

From (9) and (13), it follows that

$$\sup\{f(x) : \bar{p}^T x \leq 1, x \geq 0\} \geq \gamma.$$

Since $\{x \in \mathbb{R}_+^n : \bar{p}^T x \leq 1\}$ is a compact set and f is continuous, there exists a vector $\hat{x} \geq 0, \bar{p}^T \hat{x} \leq 1$ such that $f(\hat{x}) \geq \gamma$. This conflicts with (15). Hence, $F_\gamma = \bar{F}_\gamma$. □

Remark 1 The property $f = (\bar{f}^*)^*$ has been established earlier in [13] for polyhedral concave increasing positively homogeneous functions. Theorem 2 thus extends this property to the substantially larger class of continuous quasiconcave increasing nonnegative finite-valued functions on \mathbb{R}_+^n and positive on \mathbb{R}_{++}^n . The latter class includes many important production or utility functions in mathematical economics, such as:

– Generalized Cobb-Douglas function

$$f(x) = \prod_{j=1}^k (f_j(x))^{\alpha_j}, \quad \alpha_j > 0,$$

where $f_j(x), j = 1, 2, \dots, k$ are positive concave and increasing on \mathbb{R}_+^n . This function is quasiconcave on \mathbb{R}_+^n (see [3]).

– Generalized Leontief function

$$f(x) = \left(\min \left\{ \frac{x_j}{c_j} \mid j = 1, 2, \dots, n \right\} \right)^\alpha,$$

where $c_j > 0, j = 1, 2, \dots, n, \alpha > 0$. This function is quasiconcave and is concave if and only if $\alpha \leq 1$ (see [3]).

– Constant elasticity of substitution (C.E.S.) function

$$f(x) = \left(a_1 x_1^\beta + a_2 x_2^\beta + \dots + a_n x_n^\beta \right)^{\frac{1}{\beta}},$$

where $a_j > 0, i = 1, 2, \dots, n, 0 < \beta \leq 1$. This function is quasiconcave on \mathbb{R}_+^n (see [3]).

– Posynomial function

$$f(x) = \sum_{j=1}^m c_j \prod_{i=1}^n (x_i)^{a_{ij}} \text{ with } c_j > 0 \text{ and } a_{ij} \geq 0.$$

This function is quasiconcave on \mathbb{R}_+^n .

It is easy to check that f^* is a continuous function on \mathbb{R}_+^n if f is polyhedral concave increasing, positively homogeneous, or if f is a Cobb-Douglas function on \mathbb{R}_+^n . More generally, the following proposition is true.

Proposition 4 *If $f(x)$ is a continuous function on \mathbb{R}_+^n , then f^* is upper semi-continuous on \mathbb{R}_+^n and lower semi-continuous on \mathbb{R}_{++}^n .*

Proof We define the point-to-set map $C : \mathbb{R}_+^n \rightrightarrows \mathbb{R}_+^n$ by

$$C(p) = \{x \geq 0 : p^T x - 1 \leq 0\}.$$

For any $\bar{p} \in \mathbb{R}_+^n$, since $p^T x - 1$ is a continuous function on $\mathbb{R}_+^n \times \bar{p}$ and \mathbb{R}_+^n is closed, C is usc at \bar{p} (cf. [8, Proposition 2.2.1]). It is easy to verify that the function $p^T x - 1$ is continuous on $C(\bar{p}) \times \bar{p}$ and convex in x for each fixed $p \in \mathbb{R}_+^n$. This together with the convexity of \mathbb{R}_+^n implies that the map C is lsc at \bar{p} (cf. [8, Proposition 2.2.2]). Hence, C is continuous on \mathbb{R}_+^n . Furthermore, from the continuity of f , it follows that the function

$$p \mapsto \sup\{f(x) : p^T x \leq 1, x \geq 0\}$$

is lsc at any $p \geq 0$ (cf. [8, Theorem 4.1.1]). Noting that f^* is finite-valued, this implies that the function $f^*(p)$ is usc at any $p \geq 0$.

We now show that $f^*(p)$ is lsc on \mathbb{R}_{++}^n . For any $\bar{p} > 0$, then $C(p)$ is uniformly compact near \bar{p} , i.e., there is a neighborhood U of \bar{p} such that the closure of the set $\cup_{p \in U} C(p)$ is compact. Indeed, since $\bar{p} > 0$, there is a closed ball B with center \bar{p} such that $p > 0$ for every $p \in B$. If $\text{cl}(\cup_{p \in B} C(p))$ were not compact, there would exist a sequence $\{x^n\}$ in $\text{cl}(\cup_{p \in B} C(p))$ satisfying $\|x^n\| \rightarrow +\infty$, so, for every $p \in B$, $p^T x^n \rightarrow +\infty$ as $n \rightarrow +\infty$, contradicting the fact that $p^T x^n \leq 1$ for some $p \in B$. Hence, $C(p)$ is uniformly compact near \bar{p} . This together with the continuity of f implies that the function

$$p \mapsto \sup\{f(x) : p^T x \leq 1, x \geq 0\}$$

is usc at \bar{p} (cf. [8, Theorem 4.1.1]). Noting that f^* is finite-valued, the lower semicontinuity of $f^*(p)$ at $\bar{p} > 0$ follows. □

Next, we introduce the concept of quasi-supdifferential for quasiconcave functions. Using this concept, we will obtain in the next section an optimality condition in the form of a generalized KKT for a quasiconcave maximization problem.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be an arbitrary function. In [10], with the quasiconjugate of $f(x)$ defined by

$$f^H(p) = \begin{cases} -\inf\{f(x) : p^T x \geq 1\} & \text{if } p \neq 0, \\ -\sup\{f(x) : x \in \mathbb{R}^n\} & \text{if } p = 0, \end{cases} \tag{16}$$

a vector p is called a quasi-subgradient of f at x if

$$p^T x = 1 \text{ and } f^H(p) + f(x) \leq 0.$$

In a similar way, with $f^*(p)$ now defined by (1), we call a vector $p \in \mathbb{R}_+^n$ a *quasi-supgradient* of f at x if

$$p^T x = 1 \text{ and } f(x) f^*(p) \geq 1.$$

The *quasi-supdifferential* of f at x , denoted by $\partial^* f(x)$, is then defined to be the set of all quasi-supgradients of f at x . If $\partial^* f(x)$ is nonempty, f is said to be quasi-supdifferentiable at x .

An immediate consequence of the definition of quasiconjugate is the following:

$$f(x) f^*(p) \leq 1 \quad \forall (x, p) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \text{ s.t. } p^T x \leq 1. \tag{17}$$

So $x \in \partial^* f(x)$ is equivalent to saying that

$$p^T x = 1 \text{ and } f(x) f^*(p) = 1.$$

Theorem 3 *If f is quasiconcave continuous strictly increasing on \mathbb{R}_+^n , then $\partial^* f(x)$ is a nonempty convex set for any $x > 0$. Moreover,*

$$p \in \partial^* f(x) \Leftrightarrow x \in \partial^* f^*(p).$$

Proof Since f is quasiconcave continuous and increasing on \mathbb{R}_+^n , $F_{f(x)}$ is closed convex and conormal. Suppose there is an open ball B of center x such that $B \subset F_{f(x)}$. Then, there exists $\hat{x} \in B$ such that $\hat{x} < x$. Since f is strictly increasing this implies that $f(\hat{x}) < f(x)$, conflicting with $\hat{x} \in F_{f(x)}$. Hence, x is a boundary point of $F_{f(x)}$. Consequently, there exist a vector $q \neq 0$ and number $\alpha \in \mathbb{R}$ such that

$$q^T z \geq \alpha \quad \forall z \in F_{f(x)}, \tag{18}$$

$$q^T x = \alpha. \tag{19}$$

From (18), by Lemma 2, we have $q \geq 0$ and from (19), it follows that $\alpha > 0$. Setting $p = \frac{1}{\alpha} q$, we have $p \geq 0$ and

$$p^T z \geq 1 \quad \forall z \in F_{f(x)}, \tag{20}$$

$$p^T x = 1. \tag{21}$$

From (20), we have, by Theorem 1, $p \in F_{\frac{1}{f(x)}}$, hence $f^*(p) \geq \frac{1}{f(x)}$, i.e., $f^*(p) f(x) \geq 1$. This together with (21) implies $p \in \partial^* f(x)$. The convexity of $\partial^* f(x)$ then follows from the definition of quasi-supgradient.

By Theorem 2, we have $(f^*)^* = f$. So, $p \in \partial^* f(x)$ is equivalent to $x \in \partial^* f^*(p)$. \square

The above theorem shows that quasiconjugate duality under consideration fits into the quasigradient duality scheme originally introduced in [9, 10].

From the above proof we also have

Proposition 5 *Let f be quasiconcave continuous strictly increasing on \mathbb{R}_+^n . A sufficient condition for $p \in \partial^* f(x)$ is $p^T x = 1$ and $p^T z \geq 1$ for all $z \in F_{f(x)}$.*

Let f be an arbitrary function on \mathbb{R}^n . Recall that a vector p is said to be subgradient of f at \bar{x} if $p^T (x - \bar{x}) \leq f(x) - f(\bar{x})$ for every $x \in \mathbb{R}^n$. By $\partial f(\bar{x})$ denote the set of all subgradients of f at \bar{x} .

The following theorem describes the relationships between the two concepts of subgradient and quasi-supgradient.

Theorem 4 *If f is concave continuous and strictly increasing on \mathbb{R}_+^n , then for every $\bar{x} > 0$ satisfying $f(\bar{x}) > 0$ we have*

$$-p \in \partial(-f(\bar{x})) \setminus \{0\} \Rightarrow \frac{1}{p^T \bar{x}} p \in \partial^* f(\bar{x}).$$

Proof Let $-p \in \partial(-f(\bar{x})) \setminus \{0\}$. Then

$$p^T(x - \bar{x}) \geq f(x) - f(\bar{x}) \quad \forall x \in F_{f(\bar{x})},$$

hence

$$p^T x \geq p^T \bar{x} \quad \forall x \in F_{f(\bar{x})}. \tag{22}$$

By Lemma 2, $p \geq 0$ (because $F_{f(\bar{x})}$ is conormal) and so $p^T \bar{x} > 0$. This together with (22) implies $\frac{1}{p^T \bar{x}} p^T x \geq 1$ for every $x \in F_{f(\bar{x})}$. Hence, by Proposition 5,

$$\frac{1}{p^T \bar{x}} p \in \partial^* f(\bar{x}). \quad \square$$

3 Conjugate Duality for Scalar-Maximization Problems

Let $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ be a nonnegative, quasiconcave continuous strictly increasing function, $X \subset \mathbb{R}_+^n$ a compact convex normal set with nonempty interior (A set $X \subset \mathbb{R}_+^n$ is called normal if for any two points $x, x' \in \mathbb{R}_+^n$ satisfying $x' \leq x \in X$ then $x' \in X$). Consider the optimization problem:

$$\max f(x), \text{ s.t. } x \in X. \tag{23}$$

In this section, we will develop quasiconjugate duality for this problem on the basis of an optimality condition in the form of generalized KKT condition.

If X represents a set of feasible activities and the production function is f , then this problem amounts to finding an activity vector $x \in X$ with maximal output. Since quasiconcave functions form a wide class of functions encountered in practice, the problem (23) has potential applications in various fields. In earlier papers [11, 13], a conjugate duality for (23) was developed when f is an increasing linear function on \mathbb{R}_+^n or f is a polyhedral concave increasing and positively homogeneous function defined on \mathbb{R}^n . However, many important production or utility functions in mathematical economics are only quasiconcave but not necessarily concave (see Remark 1). Our next purpose is to extend conjugate duality for problem (23) to this more general case.

Since f is continuous and X is compact, the problem (23) has a solution; moreover, its optimal value is positive.

In [10], the quasi-subdifferential was applied to provide a sufficient optimality condition for a quasiconvex minimization problem. We now use the quasi-supdifferential to develop a necessary and sufficient optimality condition for (23). For every vector $\bar{x} \in X$ denote by $N(\bar{x}, X)$, the normal cone of X at \bar{x} :

$$N(\bar{x}, X) = \{p : p^T(x - \bar{x}) \leq 0 \quad \forall x \in X\}.$$

Theorem 5 *A vector $\bar{x} \in X$ is a global optimal solution of (23) if and only if it satisfies the following generalized KKT condition*

$$0 \in \partial^* f(\bar{x}) - N(\bar{x}, X). \tag{24}$$

Proof Suppose $\bar{x} \in X$ and (24) holds. Then there exists a vector $p \in \partial^* f(\bar{x})$ such that $p \in N(\bar{x}, X)$. Since $p \in \partial^* f(\bar{x})$, we have $p^T \bar{x} = 1$ and $f^*(p) f(\bar{x}) = 1$. Since $p \in N(\bar{x}, X)$, it follows that $p^T(x - \bar{x}) \leq 0$ for every $x \in X$ or $p^T x \leq 1$ for every $x \in X$. By (17), we have

$$f^*(p) f(\bar{x}) = \max\{f^*(p) f(x) : x \in X\} = f^*(p) \max_{x \in X} f(x),$$

consequently,

$$f(\bar{x}) = \max_{x \in X} f(x),$$

so \bar{x} solves (23).

Conversely, suppose that \bar{x} solves (23). We show that $\text{int } X \cap \text{int } F_{f(\bar{x})} = \emptyset$. Suppose there is an open ball B of center x^0 such that $B \subset X \cap F_{f(\bar{x})}$. Then $f(x) = f(\bar{x})$ for every $x \in B$ and for $x^* \in B$ such that $x^* > x^0$, one would have $f(x^*) > f(x^0) = f(\bar{x})$ (because f is strictly increasing), a contradiction. Hence, $\text{int } X \cap \text{int } F_{f(\bar{x})} = \emptyset$. By the separation theorem, there are $q \in \mathbb{R}_+^n \setminus \{0\}$ and a real number α such that

$$q^T x \leq \alpha \quad \forall x \in X; \tag{25}$$

$$q^T x \geq \alpha \quad \forall x \in F_{f(\bar{x})}. \tag{26}$$

From (26) by Lemma 2, we have $q \geq 0$ (because $F_{f(\bar{x})}$ is conormal). From (25), we have $\alpha > 0$. Setting $p = \frac{1}{\alpha} q$, we can write

$$p^T x \leq 1 \quad \forall x \in X; \tag{27}$$

$$p^T x \geq 1 \quad \forall x \in F_{f(\bar{x})}. \tag{28}$$

It follows that $p^T \bar{x} = 1$. This together with (28) implies $p \in \partial^* f(\bar{x})$ by Proposition 5. From (27), it follows that $p \in N(\bar{x}, X)$. Hence, $0 \in \partial^* f(\bar{x}) - N(\bar{x}, X)$. □

Let P be the lower conjugate of X

$$P = \{p \geq 0 : p^T x \leq 1 \quad \forall x \in X\}.$$

It can easily be checked that P is also a compact convex with nonempty interior set in \mathbb{R}_+^n . Moreover, X is the lower conjugate of P (see [13]):

$$X = \{x \geq 0 : p^T x \leq 1 \quad \forall p \in P\}.$$

Following [11], we define the dual of the problem (23) to be the following problem

$$\max f^*(p), \text{ s.t. } p \in P. \tag{29}$$

By Proposition 4, f^* is usc on \mathbb{R}_+^n , so (29) has a solution. Since X and P are the conjugates of each other and the conjugate of f^* is f , the dual of the dual problem coincides with the primal problem. So the duality is symmetric.

In [13], strong duality was obtained under the assumption that f is a polyhedral concave increasing and positively homogeneous function on \mathbb{R}_+^n . We now extend strong duality to problems (23) and (29).

Theorem 6 *Let $\bar{x} \in X$ and $\bar{p} \in P$. Then \bar{x} is optimal to (23) and \bar{p} is optimal to (29) if and only if*

$$f(\bar{x}) f^*(\bar{p}) = 1. \tag{30}$$

Proof Suppose $\bar{x} \in X$ and $\bar{p} \in P$ satisfy the dual relation (30). Then

$$f(\bar{x}) f^*(\bar{p}) = \max\{f(x) f^*(p) : x \in X, p \in P\} = \max_{x \in X} f(x) \max_{p \in P} f^*(p),$$

hence,

$$f(\bar{x}) = \max_{x \in X} f(x), \quad f^*(\bar{p}) = \max_{p \in P} f^*(p).$$

Thus, \bar{x} solves (23) and \bar{p} solves (29).

Conversely, suppose \bar{x} solves (23) and \bar{p} solves (29). Since \bar{x} solves (23), $f(\bar{x}) > 0$. By Theorem 5, we have

$$0 \in \partial^* f(\bar{x}) - N(\bar{x}, X),$$

i.e., there is a vector $q \in \mathbb{R}_+^n$ such that

$$q^T \bar{x} = 1, \tag{31}$$

$$f(\bar{x})f^*(q) = 1, \tag{32}$$

$$q^T x \leq 1 \quad \forall x \in X. \tag{33}$$

From (33), it follows that $q \in P$. Since $\bar{x} \in X$, $\bar{x}^T p \leq 1$ for every $p \in P$. This together with (31) implies that $\bar{x}^T(p - q) \leq 0$ for every $p \in P$, i.e., $\bar{x} \in N(q, P)$. From (31) and (32), we have $\bar{x} \in \partial^* f^*(q)$. Thus,

$$0 \in \partial^* f^*(q) - N(q, P).$$

By Theorem 5, it follows that q is an optimal solution of the problem (29). Since \bar{p} also solves (29), we have $f^*(q) = f^*(\bar{p})$. Therefore, $f^*(\bar{p})f(\bar{x}) = 1$. □

Corollary 1 *Let $\bar{x} \in X$ and $\bar{p} \in P$. Then, \bar{x} solves (23) and \bar{p} solves (29) if and only if*

$$\bar{p} \in \partial^* f(\bar{x}) \text{ or } \bar{x} \in \partial^* f^*(\bar{p}).$$

Proof The “if” part is obvious. To prove the “only if” part, suppose that \bar{x} solves (23) and \bar{p} solves (29). By Theorem 6, we have

$$f^*(\bar{p})f(\bar{x}) = 1.$$

We show that $\bar{p}^T \bar{x} = 1$. Since \bar{x} solves (23), we have $f(\bar{x}) > 0$. By Theorem 6 again, we have

$$f^*(\bar{p}) = \frac{1}{\sup\{f(x) : \bar{p}^T x \leq 1, x \geq 0\}} = \frac{1}{f(\bar{x})}.$$

Hence,

$$\sup\{f(x) : \bar{p}^T x \leq 1, x \geq 0\} = f(\bar{x}). \tag{34}$$

Suppose on the contrary that $\bar{p}^T \bar{x} = \alpha < 1$. There exists then a real number $t > 0$ such that $\bar{p}^T(\bar{x} + te) = 1$ ($e = (1, 1, \dots, 1)$). Since f is strictly increasing, $f(\bar{x}) < f(\bar{x} + te)$. This conflicts with (34). □

4 Duality for Vector-Maximization Problems

Let \mathbb{R}^n be represented as the cartesian product of k subspaces \mathbb{R}^{n_i} $i = 1, 2, \dots, k$:

$$\mathbb{R}^n = \prod_{i=1}^k \mathbb{R}^{n_i},$$

where $n = \sum_{i=1}^k n_i$ and $n_i \geq 1$ $i = 1, 2, \dots, k$. Then for any $x \in \mathbb{R}^n$ we have $x = (x^1, x^2, \dots, x^k)$ where $x^i \in \mathbb{R}_+^{n_i}$ for every $i = 1, 2, \dots, k$.

Given a compact normal convex set $X \subset \mathbb{R}_+^n$ with $\text{int}X \neq \emptyset$ and a function $f_i : \mathbb{R}_+^{n_i} \rightarrow \mathbb{R}$ for every $i = 1, 2, \dots, k$, consider the vector-maximization problem

$$\begin{aligned} f_i(x^i) &\rightarrow \max \quad i = 1, 2, \dots, k \\ \text{s.t. } x &= (x^1, x^2, \dots, x^k) \in X. \end{aligned} \tag{35}$$

This problem arises when one has to maximize the production output or utility under a given resource constraints. In [13], the conjugate duality was developed for vector-maximization problems when the objective functions are polyhedral concave increasing positively homogeneous production functions on \mathbb{R}_+^n . We now develop the conjugate duality for the vector-maximization problem (35) when the functions $f_i(x^i)$ are quasiconcave.

Let $C \subset \mathbb{R}^n$ be a convex set. A function $f : C \rightarrow \mathbb{R}$ is said to be positively homogeneous of degree $\alpha > 0$ if for every $x \in X$ one has $f(tx) = t^\alpha f(x)$ for every $t > 0$. When $\alpha = 1$, the functions are simply called positively homogeneous. The class of functions positively homogeneous of degree $\alpha > 0$ appears frequently in Economics (see Remark 1).

Consider now problem (35) when each function $f_i : \mathbb{R}_+^{n_i} \rightarrow \mathbb{R}_+$ ($i = 1, \dots, k$) is quasiconcave continuous increasing positively homogeneous of degree $\alpha_i > 0$ and satisfies $f_i(x^i) > 0$ for every $x^i > 0$. Then it is easily checked that $(f_i)^{\frac{1}{\alpha_i}}$ is a quasiconcave continuous, nonnegative, positively homogeneous, strictly increasing function on $\mathbb{R}_+^{n_i}$ and satisfies $(f_i(x^i))^{\frac{1}{\alpha_i}} > 0$ for every $x^i > 0$. Furthermore, it is immediate that the following proposition holds true.

Proposition 6 *A vector $\bar{x} \in X$ is (weakly) Pareto efficient of (35) if and only if it is (weakly) Pareto efficient of the problem*

$$\begin{aligned} (f_i(x^i))^{\frac{1}{\alpha_i}} &\rightarrow \max \quad i = 1, 2, \dots, k \\ \text{s.t. } x &= (x^1, x^2, \dots, x^k) \in X. \end{aligned} \tag{36}$$

Under the stated assumption, the problem (35) can be replaced by problem (36). Thus, we can assume that f_i is positively homogeneous for every $i = 1, 2, \dots, k$. Let $p = (p^1, p^2, \dots, p^k) \in \mathbb{R}^n$ with $p^i \in \mathbb{R}_+^{n_i}$, $i = 1, 2, \dots, k$. Denote by P the lower conjugate of X and by f_i^* the quasiconjugate of f_i on $\mathbb{R}_+^{n_i}$. The dual of (35) is defined to be the following vector-maximization problem

$$\begin{aligned} f_i^*(p^i) &\rightarrow \max \quad i = 1, 2, \dots, k \\ \text{s.t. } p &= (p^1, p^2, \dots, p^k) \in P. \end{aligned} \tag{37}$$

Since X and P are lower conjugate of each other and f_i is quasiconjugate of f_i^* for every $i = 1, 2, \dots, k$, the above defined duality for vector-maximization problems is symmetric.

Example 1 Consider the vector-maximization problem

$$\begin{aligned} f_i(x^i) &= \prod_{j=1}^{n_i} (x_j^i)^{\alpha_j^i} \rightarrow \max \quad i = 1, 2, \dots, k \\ \text{s.t. } x &= (x^1, x^2, \dots, x^k) \in X, \end{aligned}$$

where $\sum_{j=1}^{n_i} \alpha_j^i = 1, \alpha_j^i > 0, j = 1, 2, \dots, n_i$. By Proposition 2, the dual problem is

$$\begin{aligned} \prod_{j=1}^{n_i} \left(\frac{p_j^i}{\alpha_j^i}\right)^{\alpha_j^i} &\rightarrow \max \quad i = 1, 2, \dots, k \\ \text{s.t. } p &= (p^1, p^2, \dots, p^k) \in P. \end{aligned}$$

In [13], we have established the following theorems.

Theorem 7 (Weak duality theorem) *For any $x \in X$ and $p \in P$, we have*

$$\sum_{i=1}^k f_i(x^i) f_i^*(p^i) \leq 1. \tag{38}$$

Theorem 8 (Duality relation) *Let $\bar{x} \in X$ and $\bar{p} \in P$. If (\bar{x}, \bar{p}) satisfies the equality*

$$\sum_{i=1}^k f_i(\bar{x}^i) f_i^*(\bar{p}^i) = 1, \tag{39}$$

then \bar{x} is primal weakly Pareto efficient and \bar{p} is dual weakly Pareto efficient.

Next, we establish strong duality for vector-maximization problems.

Theorem 9 *If \bar{x} is primal weakly Pareto efficient, then there is $\bar{p} \in P$ such that (\bar{x}, \bar{p}) satisfies the dual relation (39). Similarly, if \bar{p} is dual weakly Pareto efficient, then there is $\bar{x} \in X$ such that (\bar{x}, \bar{p}) satisfies (39).*

Proof Suppose \bar{x} is primal weakly Pareto efficient. Set

$$\Omega_{\bar{x}} = \{z \in \mathbb{R}_+^n : f_i(z^i) \geq f_i(\bar{x}^i), i = 1, 2, \dots, k\}.$$

Since the function f_i quasiconcave continuous increasing for every $i = 1, 2, \dots, k$, $\Omega_{\bar{x}}$ is a closed convex and conormal set in \mathbb{R}_+^n . We contend that $\text{int}\Omega_{\bar{x}} \cap \text{int} X = \emptyset$. In fact, if it were not so, there would exist an open ball B of center \hat{z} such that $B \subset \Omega_{\bar{x}} \cap X$. Since $\hat{z} \in \Omega_{\bar{x}}$, we have $f_i(\hat{z}^i) \geq f_i(\bar{x}^i)$ for every $i \in \{1, 2, \dots, k\}$. Taking $\hat{z} \in B$ such that $\hat{z} > \bar{z}$ yields $f_i(\hat{z}^i) > f_i(\bar{z}^i) \geq f_i(\bar{x}^i)$ for every $i \in \{1, 2, \dots, k\}$ (because f_i is strictly increasing), conflicting with \bar{x} being weakly Pareto efficient for (35). So, $\text{int}\Omega_{\bar{x}} \cap \text{int} X = \emptyset$. By the separation theorem, there are $u \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$u^T z \leq \alpha \quad \forall z \in X, \tag{40}$$

$$u^T z \geq \alpha \quad \forall z \in \Omega_{\bar{x}}. \tag{41}$$

Since $\Omega_{\bar{x}}$ is conormal, it follows from (41) that $u \geq 0$ by virtue of Lemma 2. Also, since $\text{int}X \neq \emptyset$, it follows from (40) that $\alpha > 0$. Setting $\bar{p} = \frac{1}{\alpha}u$, we can rewrite (40) and (41) as

$$\bar{p}^T z \leq 1 \quad \forall z \in X, \tag{42}$$

$$\bar{p}^T z \geq 1 \quad \forall z \in \Omega_{\bar{x}}. \tag{43}$$

From (42), it follows that $\bar{p} \in P$. From (42) and (43), we have further $\bar{p}^T \bar{x} = 1$ and

$$\begin{aligned} \min \left\{ \bar{p}^T z : z \in \bar{\Omega} \bar{x} \right\} &= \sum_{i=1}^k \min \left\{ \bar{p}^i T z^i : z^i \in \mathbb{R}_+^{n_i}, f_i(z^i) \geq f_i(\bar{x}^i) \right\} \\ &= \bar{p}^T \bar{x} = 1. \end{aligned} \tag{44}$$

For each $i \in \{1, 2, \dots, k\}$, putting $\alpha_i = \min\{\bar{p}^i T z^i : z^i \in \mathbb{R}_+^{n_i}, f_i(z^i) \geq f_i(\bar{x}^i)\}$, we then have from (44) that

$$\alpha_i = \min \left\{ \bar{p}^i T z^i : z^i \in \mathbb{R}_+^{n_i}, f_i(z^i) \geq f_i(\bar{x}^i) \right\} = \bar{p}^i T \bar{x}^i. \tag{45}$$

$$\alpha_1 + \alpha_2 + \dots + \alpha_k = 1. \tag{46}$$

Setting $I = \{i \in \{1, 2, \dots, k\} : f_i(\bar{x}^i) > 0, \alpha_i > 0\}$ yields

$$f_i(\bar{x}^i) f_i^*(\bar{p}^i) = \alpha_i = 0 \quad \forall i \in \{1, 2, \dots, k\} \setminus I, \tag{47}$$

To see this, let $i \in \{1, 2, \dots, k\} \setminus I$. If $f_i(\bar{x}^i) = 0$ then, noting that $f_i(0) = 0$ because f_i is positively homogeneous, we get from (45):

$$\alpha_i = \min \left\{ \bar{p}^i T z^i : z^i \in \mathbb{R}_+^{n_i}, f_i(z^i) \geq 0 \right\} = \bar{p}^i T 0 = 0.$$

If $\alpha_i = 0$ and $f_i(\bar{x}^i) > 0$ then $\bar{p}^i T \bar{x}^i = 0$. This together with the homogeneity of f_i implies

$$\sup \left\{ f_i(x^i) : \bar{p}^i T x^i \leq 1, x^i \geq 0 \right\} = +\infty,$$

hence

$$f_i^*(\bar{p}^i) = \frac{1}{\sup\{f_i(x^i) : \bar{p}^i T x^i \leq 1, x^i \geq 0\}} = \frac{1}{+\infty} = 0.$$

Therefore, as has been asserted in (47),

$$f_i(\bar{x}^i) f_i^*(\bar{p}^i) = \alpha_i = 0 \quad \forall i \in \{1, 2, \dots, k\} \setminus I.$$

Now for $i \in I$, we have $f_i(\bar{x}^i) > 0$ and $\alpha_i > 0$. On the other hand, by (45),

$$\bar{p}^i T z^i \geq \alpha_i \quad \forall z^i \in \{z^i \in \mathbb{R}_+^{n_i} : f_i(z^i) \geq f_i(\bar{x}^i)\},$$

or, equivalently,

$$\frac{1}{\alpha_i} \bar{p}^i T z^i \geq 1 \quad \forall z^i \in F_{f_i(\bar{x}^i)},$$

where $F_{f_i(\bar{x}^i)}$ is an upper level set of f_i . By Proposition 5, we then get

$$\frac{1}{\alpha_i} \bar{p}^i \in \partial^* f_i(\bar{x}^i) \quad \forall i \in I,$$

consequently

$$f_i(\bar{x}^i) f_i^*(\bar{p}^i) = \alpha_i \quad \forall i \in I. \tag{48}$$

From (47), (46), and (48), we obtain

$$\sum_{i=1}^k f_i(\bar{x}^i) f_i^*(\bar{p}^i) = \sum_{i=1}^k \alpha_i = 1,$$

proving the duality relation (39).

Since the duality scheme is symmetric, by the arguments similar to the above we can show that if \bar{p} is primal weakly Pareto efficient then there is $\bar{x} \in X$ such that the duality relation (39) holds at (\bar{x}, \bar{p}) . □

Theorem 9, which is the main result of this paper, shows that the developed duality for general quasiconcave vector-maximization problems has zero gap and, moreover, that the corresponding duality relation (39) helps to characterize weakly efficient solutions of the primal and the dual problems. Since polyhedral concave nondecreasing and positively homogeneous function are special cases of quasiconcave continuous increasing nonnegative positively homogeneous functions, the above results have thus extended those earlier obtained in [13]. Specifically, strong duality theorem for vector-maximization problems which was established in [13] when the objective functions $f_i, i = 1, 2, \dots, k$, are concave and polyhedral still remains true when f_i are quasiconcave and not necessarily polyhedral. The proof of strong duality theorem in [13] requires concavity and finiteness of the objective functions f_i on \mathbb{R}^{n_i} . In the present paper, thanks to a different approach, only quasiconcavity and finiteness on $\mathbb{R}_+^{n_i}$ of the objective functions are needed.

5 Optimization over Weakly Pareto Efficient Set

In this last section, as an application of the above results, we present an approach to optimization over weakly Pareto efficient set based on bilevel programming and monotonic optimization.

Denote by X_{WE} the set of all weakly Pareto efficient of the vector-maximization problem (35) when each function $f_i : \mathbb{R}_+^{n_i} \rightarrow \mathbb{R}_+ (i = 1, \dots, k)$ is quasiconcave continuous increasing positively homogeneous and satisfies $f_i(x^i) > 0$ for every $x^i > 0$. Consider the following optimization problem over the weakly Pareto efficient set

$$\max\{q(x)|s.t.x \in X_{WE}\}, \tag{49}$$

where $q(x)$ is a continuous concave function defined on \mathbb{R}_+^n such that $q(x) > 0$ for every $x > 0$. If $q(x)$ and X represent, respectively, a profit and a set of feasible production programs, this problem amounts to finding a weakly efficient production program with maximal profit. For this type of problems, several solution methods are available (see [1, 18]).

By Theorems 7, 8, and 9, a vector $\bar{x} \in X_{WE}$ if and only if $\bar{x} \in X$ and there exists $\bar{p} \in P$ such that (\bar{x}, \bar{p}) is a optimal solution of the following problem

$$\max \left\{ \sum_{i=1}^k f_i(x^i) f_i^*(p^i) \mid x \in X, p \in P \right\}.$$

Thus, the problem (49) can be rewritten as the *bilevel programming problem*

$$\max q(x) \quad s.t. \tag{50}$$

$$x \in X, p \in P, (x, p) \text{ solves} \tag{51}$$

$$\max \left\{ \sum_{i=1}^k f_i(x^i) f_i^*(p^i) \mid x \in X, p \in P \right\}. \tag{52}$$

This problem belongs to the class studied in [16]. Using the method proposed in the latter paper for solving bilevel programs, we define

$$\theta(p) = \max \left\{ q(x) \mid \sum_{i=1}^k f_i(x^i) f_i^*(p^i) \geq 1, x \in X \right\} \quad \forall p \in \mathbb{R}_+^n$$

with the convention $\max \emptyset = 0$.

Note that the function $\sum_{i=1}^k f_i(x^i) f_i^*(p^i)$ is quasiconcave in x for fixed p , so the feasible set of the subproblem that defines $\theta(p)$ is a convex subset of X . Therefore, the value of $\theta(p)$ is obtained by solving a convex problem (maximizing a concave function over a convex compact set).

Proposition 7 $\theta(p)$ is an usc increasing function on \mathbb{R}_+^n .

Proof Since $q(x)$ and $\sum_{i=1}^k f_i(x^i) f_i^*(p^i)$ are continuous on \mathbb{R}_+^n , by an argument similar to the one in Proposition 4, we can prove that $\theta(p)$ is an usc function on \mathbb{R}_+^n . Suppose $0 \leq p \leq p'$. Since f_i, f_i^* are nonnegative and f_i^* is an increasing function on $\mathbb{R}_+^{n_i}$ for any $i = 1, 2, \dots, k$, we have

$$\sum_{i=1}^k f_i(x^i) f_i^*(p'^i) \geq \sum_{i=1}^k f_i(x^i) f_i^*(p^i) \quad \forall x \in X.$$

This implies that

$$\left\{ x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(p'^i) \geq 1 \right\} \subset \left\{ x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(p^i) \geq 1 \right\}.$$

Hence, $\theta(p') \geq \theta(p)$. □

We now prove that problem (50)–(52) is equivalent to the following monotonic optimization problem:

$$\max\{\theta(p) \mid p \in P\}. \tag{53}$$

Since P is compact and $\theta(p)$ is an usc function on \mathbb{R}_+^n , (53) has an optimal solution.

Theorem 10 The optimal values in problem (53) and problem (50)–(52) are equal, i.e., $\theta^* = q^*$. Moreover, if \bar{p} is an optimal solution of (53) then \bar{x} is an optimal solution of (50)–(52), where \bar{x} is a maximizer of the function $q(x)$ on $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) = 1\}$.

Proof Let \bar{x} be an optimal solution of (50)–(52), i.e., $q(\bar{x}) = q^*$ and there exists $\bar{p} \in P$ such that (\bar{x}, \bar{p}) is a maximizer of the function $\sum_{i=1}^k f_i(x^i) f_i^*(p^i)$ on $X \times P$, i.e., $\sum_{i=1}^k f_i(\bar{x}^i) f_i^*(\bar{p}^i) = 1$ by virtue of (38). Then,

$$\begin{aligned} q^* &= q(\bar{x}) \\ &\leq \max \left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) \geq 1, x \in X \right\} \\ &= \theta(\bar{p}) \\ &\leq \theta^*. \end{aligned}$$

This particularly implies $\theta^* > 0$. Conversely, let \bar{p} be an optimal solution of (53), i.e., $\bar{p} \in P$ and $\theta(\bar{p}) = \theta^*$. We show that $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) = 1\} \neq \emptyset$. In fact, suppose on the contrary, that $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) = 1\} = \emptyset$. By Theorem 7, we have

$$\left\{ x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) \geq 1 \right\} = \left\{ x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) = 1 \right\} = \emptyset.$$

So,

$$\begin{aligned} \theta^* &= \theta(\bar{p}) \\ &= \max \left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) \geq 1, x \in X \right\} \\ &= \max \left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) = 1, x \in X \right\} \\ &= \max \emptyset \\ &= 0 \\ &< \theta^*, \end{aligned}$$

which is a contradiction.

Let \bar{x} be a maximizer of the function $q(x)$ on the set

$$\left\{ x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) = 1 \right\}.$$

Then $x \in X_{WE}$ and

$$\begin{aligned} \theta^* &= \theta(\bar{p}) \\ &= \max \left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) \geq 1, x \in X \right\} \\ &= \max \left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\bar{p}^i) = 1, x \in X \right\} \\ &= q(\bar{x}) \\ &\leq q^*. \end{aligned}$$

Consequently, $\theta^* = q^*$. This particularly implies that \bar{x} is an optimal solution of the problem (50)–(52). □

6 Conclusions

In an earlier paper [13], we have studied the quasigradient duality for problems of maximizing a polyhedral concave increasing positively homogeneous function over a convex feasible set. In the present article, the quasigradient duality is developed for problems of maximizing a quasiconcave continuous increasing nonnegative positively homogeneous and finite-valued, function over a convex feasible set. As we have seen, the main results in [13] remain true for this more general class of problems. Specifically, the conjugate duality

scheme is still symmetric and a strong duality theorem is still true. We also present duality relationships that can help to characterize the (weak) Pareto efficient solutions, and to derive a sufficient condition for Pareto efficiency. As a result, an optimization problem over the weakly efficient set reduces to a bilevel optimization problem which can be solved by an monotonic optimization algorithm.

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