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# $L_p$-Regularity for the Cauchy-Dirichlet Problem for Parabolic Equations in Convex Polyhedral Domains

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**Abstract** In this paper, we study the regularity of the solution of the initial-boundary value problem with Dirichlet boundary conditions for second-order divergence parabolic equations in a domain of polyhedral type. We establish several results on the regularity of the solution in weighted $L_p$-Sobolev spaces.

**Keywords** Parabolic equation · Polyhedral domains · Regularity of solutions

**Mathematical Subject Classification (2010)** 35B65 · 35D30 · 35K20

## 1 Introduction

The $L_p$-theory of second-order parabolic equations has been studied widely under various regularity assumptions on the coefficients and the domains. Let us mention some works related to this topic. For the case of continuous leading coefficients and smooth domains, the $W_p^{2,1}$-solvability has been known for a long time, see, for example, [7]. In [2], Bramanti and Cerutti established the $W_p^{2,1}$-solvability of the Cauchy-Dirichlet problem for second-order

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parabolic equations with VMO coefficients in domains being of the $C^{1,1}$ class. For the case of nonsmooth domains, let us mention the works [3] and [1] in which the unique existence of weak solutions in $W^1_p$-Sobolev spaces was established. In [8], coercive estimates for strong solutions to the Dirichlet problem and to the Neumann problem for the heat operator in a dihedral angle were obtained.

In this paper, we are concerned with the regularity of the weak solution obtained in [1] in the case the domain is of polyhedral type. For elliptic boundary value problems in domains of this type, many results on the solvability and the regularity in weighted $L_p$-Sobolev spaces are known (see the monograph [6] and the references therein). Basing on the regularity results for elliptic boundary value problems in [6] together with the solvability result for the Cauchy-Dirichlet problem for parabolic equations in [1], we will establish several results on the regularity of the weak solution in weighted $L_p$-Sobolev spaces. Our method has similarities with [4] in which the authors considered only the case of weighted Sobolev spaces with the $L_2$-norms.

Our paper is organized as follows. In Section 2, we introduce some notations and preliminaries. Section 3 is devoted to studying the regularity in time of the weak solution. This is an intermediate step to investigate the global regularity of the solution in Section 4.

## 2 Notations and Preliminaries

Let $G$ be a bounded convex domain of polyhedral type in $\mathbb{R}^3$, i.e., the following conditions hold (see [6, Chapter 4]):

(i) the boundary $\partial G$ consists of smooth (of class $C^\infty$) open two-dimensional manifolds $\Gamma_j$ (the faces of $G$), $j = 1, \dots, N$, smooth curves $M_k$ (the edges), $k = 1, \dots, d$, and vertices $x^{(1)}, \dots, x^{(d')}$.
(ii) for every $\xi \in M_k$ there exists a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a $C^\infty$ mapping) $\kappa_\xi$ which maps $G \cap \mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron and $B_1$ is the unit ball.
(iii) for every vertex $x^{(j)}$ there exists a diffeomorphism $\kappa_j$ mapping $G \cap \mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a cone with edges and vertex at the origin.

Let $T$ be a positive real number. Set $Q = G \times (0, T)$ and $S = \partial G \times [0, T]$. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{N}^n$, set $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $\partial^\alpha = \partial^\alpha_x = \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$. For a function $u = u(x, t)$ defined on $Q$, we write $u_{t^k}$ instead of $\frac{\partial^k u}{\partial t^k}$ for each $k \in \mathbb{N}$.

In this paper, the letter $p$ stands for some real number, $1 < p < \infty$, and $q$ denotes its conjugate exponent, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Let $l$ be a nonnegative integer. By $W^l_p(G)$, we denote the usual Sobolev space of functions defined in $G$ with the norm

$$\|u\|_{W^l_p(G)} = \left( \int_G \sum_{|\alpha| \leqslant l} |\partial^\alpha_x u|^p dx \right)^{\frac{1}{p}}.$$

We denote the distance from $x$ to the edge $M_k$ by $r_k(x)$, the distance from $x$ to the corner $x^{(j)}$ by $\rho_j(x)$. Furthermore, we denote by $X_j$ the set of all indices $k$ such that the vertex $x^{(j)}$ is an end point of the edge $M_k$. Let $\mathcal{U}_1, \dots, \mathcal{U}_{d'}$ be domains in $\mathbb{R}^3$ such that

$$\mathcal{U}_1 \cup \ldots \cup \mathcal{U}_{d'} \supset \overline{G}, x^{(i)} \notin \overline{\mathcal{U}}_j \text{ if } i \neq j, \text{ and } \overline{\mathcal{U}}_j \cap \overline{M}_k = \varnothing \text{ if } k \notin X_j.$$

We define $V^l_{p,\beta,\delta}(G)$ as the weighted Sobolev space with the norm

$$\|u\|_{V^l_{p,\beta,\delta}(G)} = \left( \sum_{j=1}^{d'} \int_{G\cap U_j} \sum_{|\alpha|\leqslant l} \rho_j^{p(\beta_j - l + |\alpha|)} \prod_{k\in X_j} (\frac{r_k}{\rho_j})^{p(\delta_k - l + |\alpha|)} |\partial_x^\alpha u|^p dx \right)^{\frac{1}{p}},$$

where $\beta = (\beta_1, \ldots, \beta_{d'}) \in \mathbb{R}^{d'}$, $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{R}^d$, and $l$ is a nonnegative integer.

It follows readily from [6, Lemma 4.1.3] that $V^l_{p,\beta,\delta}(G)$ is continuously imbedded in $V^{l'}_{p,\beta',\delta'}(G)$ provided that $l' \leqslant l, \beta_k - l \leq \beta'_k - l', \delta_j - l \leq \delta'_j - l'$ for $k = 1, \ldots, d, j = 1, \ldots, d'$.

By $\overset{\circ}{W}{}^1_p(G)$, we denote the closure of $C_0^\infty(G)$ in $W^1_p(G)$. The norm in the space $\overset{\circ}{W}{}^1_p(G)$ is the same one as in $W^1_p(G)$. We denote by $W^{-1}_p(G)$ the dual space of $\overset{\circ}{W}{}^1_q(G)$. The pairing between $W^{-1}_p(G)$ and $\overset{\circ}{W}{}^1_q(G)$ is denoted by $\langle .,. \rangle$. By identifying the dual space of $L_p(G)$ with $L_q(G)$, we have the continuous imbeddings $L_q(G) \subset W^{-1}_p(G)$ by setting

$$\langle f, v\rangle = \int_G f v dx$$

if $f \in L_q(G)$ and $v \in \overset{\circ}{W}{}^1_p(G)$.

Let $X$ be a Banach space. We denote by $L_p((0,T); X)$ the space of measurable functions $f : (0,T) \to X$ with

$$\|f\|_{L_p((0,T);X)} = \left(\int_0^T \|f(t)\|^p_X dt\right)^{\frac{1}{p}} < \infty.$$

For shortness, we set $V^{l,0}_{p,\beta,\delta}(Q) = L_p((0,T); V^l_{p,\beta,\delta}(G))$.

Finally, we introduce the Sobolev space $W^1_{p,*}(Q)$ which consists of all functions $u$ defined on $Q$ such that $u \in L_p((0,T); \overset{\circ}{W}{}^1_p(G))$ and $u_t \in L_p((0,T); W^{-1}_p(G))$ with the norm

$$\|u\|_{W^1_{p,*}(Q)} = \|u\|_{L_p((0,T);\overset{\circ}{W}{}^1_p(G))} + \|u_t\|_{L_p((0,T);W^{-1}_p(G))}.$$

In this paper, we consider the following Cauchy-Dirichlet problem for a second-order parabolic equation in divergence form

$$u_t - \operatorname{div}(A\nabla u) = f \text{ in } Q, \tag{1}$$

$$u = 0 \text{ on } S, \tag{2}$$

$$u|_{t=0} = 0 \text{ on } G, \tag{3}$$

where $A = A(x,t) = (a_{jk}(x,t))^n_{j,k=1}$ is a symmetric matrix of real bounded measurable functions defined in $\overline{Q}$ satisfying the following condition: there exists a positive constant $\mu_0$ such that

$$A(x,t)\,\xi \cdot \xi \geqslant \mu_0 |\xi|^2 \tag{4}$$

for all $\xi \in \mathbb{R}^n$ and all $(x,t) \in \overline{Q}$. Since we are paying attention to the influence of the singularity of the domain on the regularity of the solution, we assume that the coefficients $a_{jk}$ are infinitely smooth on $\overline{Q}$.

**Definition 1** Let $f \in L_p((0,T); W_p^{-1}(G))$. A function $u \in W^1_{p,*}(Q)$ is called a weak solution of the problem (1)–(3) if and only if $u(.,0) = 0$ and the equality

$$\langle u_t, v\rangle + B(t,u,v) = \langle f(t), v\rangle \tag{5}$$

holds for a.e. $t \in (0,T)$ and all $v \in \overset{\circ}{W}{}^1_q(G)$, where

$$B(t,u,v) = \int_G A(x,t)\nabla u \cdot \nabla v dx = \sum_{j,k=1}^n \int_G a_{jk}(x,t)\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_j}dx.$$

**Theorem 1** *If $f \in L_p((0,T); W_p^{-1}(G))$, then there exists a unique weak solution $u \in W^1_{p,*}(Q)$ of the problem (1)–(3) which satisfies*

$$\|u\|_{W^1_{p,*}(Q)} \leqslant C\|f\|_{L_p((0,T);W_p^{-1}(G))}, \tag{6}$$

*where $C$ is a constant independent of $f$ and $u$.*

This theorem is deduced directly from Theorem 1 in [1]. In fact, a weak solution of the problem (1)–(3) in the sense of [1] means a function $u \in \overset{\circ}{W}{}^{1,0}_p(Q) = L_p((0,T); W^1_p(G))$ satisfying

$$-\int_0^T (u, v_t)dt + \int_0^T B(t,u,v)dt = \int_0^T \langle f, v\rangle dt \tag{7}$$

for all smooth test functions $v$ in $\overline{Q}$ vanishing in a neighborhood of the lateral surface and the upper base of the cylinder $Q$, where $(u,v) = \int_G uv dx$. However, from (7), we obtain (5) with $u_t \in L_p(0,T,W_p^{-1}(G))$ and

$$\|u_t\|_{L_p(0,T,W_p^{-1}(G))} \leqslant C\left(\|u\|_{L_p(0,T,W^1_p(G))} + \|f\|_{L_p(0,T,W_p^{-1}(G))}\right),$$

where $C$ is a constant independent of $u$ and $f$, i.e., the function $u$ in fact belongs to $W^1_{p,*}(Q)$. Thus, Theorem 1 follows directly from [1, Theorem 1].

## 3 The Regularity in Time

To investigate the regularity of weak solutions for initial−boundary value problems for parabolic equations in non-smooth domains, it is reasonable to study, as an intermediate step, the regularity with respect to the time variable of those solutions in Sobolev spaces in which they are attained. So, the present section is devoted to this intermediate step.

We start by proving the following auxiliary lemma.

**Lemma 1** *Assume that for each $t \in [0,T]$, $F(t,\cdot,\cdot) : \overset{\circ}{W}{}^1_p(G) \times \overset{\circ}{W}{}^1_q(G) \to \mathbb{C}$ is a bilinear map satisfying*

$$|F(t,u,v)| \leqslant C\|u\|_{W^1_p(G)}\|v\|_{W^1_q(G)} \tag{8}$$

*for all $u \in \overset{\circ}{W}{}^1_p(G)$ and $v \in \overset{\circ}{W}{}^1_q(G)$, where $C$ is a constant independent of $u$, $v$ and $t$. Assume further that $F(.,u,v)$ is measurable on $[0,T]$ for each pair $u \in \overset{\circ}{W}{}^1_p(G)$ and*

*$v \in \overset{\circ}{W}{}^1_q(G)$. Suppose that $u \in W^1_{p,*}(Q)$ satisfies $u|_{t=0} = 0$ and*

$$\langle u_t(\cdot, t), v\rangle + B(t, u(\cdot, t), v) = \int_0^t F(s, u(\cdot, s), v)ds \tag{9}$$

*for a.e. $t \in [0, T]$ and all $v \in \overset{\circ}{W}{}^1_q(G)$. Then $u \equiv 0$ on $Q$.*

*Proof* It follows from (8) that the function $g : [0, T] \to W^{-1}_p(G)$ defined by

$$\langle g(t), v\rangle = \int_0^t F(s, u(\cdot, s), v)ds,\ \ t \in [0, T],\ \ v \in \overset{\circ}{W}{}^1_q(G),$$

is a member of $L_p((0, \tau); W^{-1}_p(G))$ for each $\tau \in (0, T]$ with

$$\|g\|^p_{L_p((0,\tau);W^{-1}_p(G))} \leqslant C \int_0^\tau \int_0^t \|u(\cdot, s)\|^p_{W^1_p(G)} dsdt. \tag{10}$$

Hence, according to Theorem 1, it follows from (9) that, for each $\tau \in (0, T]$,

$$\|u\|^p_{W^1_{p,*}(Q_\tau)} \leqslant C \int_0^\tau \int_0^t \|u(\cdot, s)\|^p_{W^1_p(G)} dsdt. \tag{11}$$

Especially,

$$\|u\|^p_{L_p((0,\tau);W^1_p(G))} \leqslant C \int_0^\tau \int_0^t \|u(\cdot, s)\|^p_{W^1_p(G)} dsdt \leqslant C\tau \|u\|^p_{L_p((0,\tau);W^1_p(G))}. \tag{12}$$

Taking $\tau = \frac{1}{2C}$, it follows from (12) that $u \equiv 0$ on $[0, \frac{1}{2C}]$. Repeating these arguments leads to $u \equiv 0$ on intervals $[\frac{1}{2C}, \frac{1}{C}]$, $[\frac{1}{C}, \frac{3}{2C}]$, ... and, consequently, $u \equiv 0$ on $Q$. □

Now, we state and prove the main theorem of this section.

**Theorem 2** *Let $h$ be a nonnegative integer. Assume the function $f$ has weak derivatives with respect to $t$ up to order $h$ and the following conditions are fulfilled.*

(i) $f_{t^k} \in L_p((0, T); W^{-1}_p(G))$ *for* $k = 0, \ldots, h$,
(ii) $f_{t^k}(x, 0) = 0$ *for* $k = 0, \ldots, h - 1$.

*Then the weak solution $u$ in the space $W^1_{p,*}(Q)$ of the problem (1)–(3) has derivatives with respect to $t$ up to order $h$ with*

$$u_{t^k} \in W^1_{p,*}(Q) \text{ for } k = 0, 1, \ldots, h \tag{13}$$

*and*

$$\sum_{k=0}^h \|u_{t^k}\|_{W^1_{p,*}(Q)} \leqslant C \sum_{k=0}^h \|f_{t^k}\|_{L_p((0,T);W^{-1}_p(G))}, \tag{14}$$

*where $C$ is a constant independent of $u$ and $f$.*

*Proof* This theorem can be proved by application of Lemma 1 and by an argument analogous to that used for the proof of Theorem 3.1 in [4]. We will show by induction on $h$ that not only the assertions (13), (14) but also the following equalities hold:

$$u_{t^k}|_{t=0} = 0,\ k = 1, \dots, h, \tag{15}$$

and

$$\langle u_{t^{h+1}}, \eta\rangle + \sum_{k=0}^{h} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta) = \langle f_{t^h}, \eta\rangle \text{ for all } \eta \in \overset{\circ}{W}{}^1_q(G). \tag{16}$$

The case $h = 0$ follows from Theorem 1. Assuming now that they hold for $h - 1$, we will prove them for $h$ ($h \geqslant 1$). We consider first the following problem: find a function $v \in \overset{\circ}{W}{}^1_{p,*}(Q)$ satisfying $v|_{t=0} = 0$ and

$$\langle v_t, \eta\rangle + B(t, v, \eta) = \langle f_{t^h}, \eta\rangle - \sum_{k=0}^{h-1} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta) \tag{17}$$

for all $\eta \in \overset{\circ}{W}{}^1_q(G)$ and a.e. $t \in [0, T]$.

Let $F : [0, T] \to W_p^{-1}(G)$ be a function defined by

$$\langle F(t), \eta\rangle = \langle f_{t^h}, \eta\rangle - \sum_{k=0}^{h-1} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta), \eta \in \overset{\circ}{W}{}^1_q(G). \tag{18}$$

From the inductive hypothesis, we see that $F \in L_p((0, T); W_p^{-1}(G))$ with

$$\begin{aligned}\|F\|_{L_p((0,T);W_p^{-1}(G))} &\leqslant \|f_{t^h}\|_{L_p((0,T);W_p^{-1}(G))} + C\sum_{k=0}^{h-1} \|u_{t^k}\|_{L_p((0,T);W_p^1(G))} \\ &\leqslant C\sum_{k=0}^{h} \|f_{t^k}\|_{L_p((0,T);W_p^{-1}(G))},\end{aligned}$$

where $C$ is a constant independent of $f$. Hence, according to Theorem 1, the problem (17) has a solution $v \in \overset{\circ}{W}{}^1_{p,*}(Q)$ with

$$\|v\|_{W^1_{p,*}(Q)} \leqslant C\sum_{k=0}^{h} \|f_{t^k}\|_{L_p((0,T);W_p^{-1}(G))},$$

where $C$ is a constant independent of $f$.

We put now

$$w(x, t) = \int_0^t v(x, \tau)d\tau,\ \ x \in G, t \in [0, T].$$

Then, we have $w|_{t=0} = 0$, $w_t = v$, $w_t|_{t=0} = 0$. We rewrite (17) as follows

$$\langle w_{tt}, \overline{\eta}\rangle + B(t, w_t, \eta) = \langle f_{t^h}, \eta\rangle - \sum_{k=0}^{h-1} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta). \tag{19}$$

It is noted that

$$B(t, w_t, \eta) = \frac{\partial}{\partial t} B(t, w, \eta) - B_t(t, w, \eta),$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t} \sum_{k=0}^{h-2} \binom{h-1}{k} B_{t^{h-1-k}}(t, u_{t^k}, \eta) \\
&= \sum_{k=0}^{h-2} \binom{h-1}{k} \left( B_{t^{h-k}}(t, u_{t^k}, \eta) + B_{t^{h-1-k}}(t, u_{t^{k+1}}, \eta) \right) \\
&= B_{t^h}(t, u, \eta) + \sum_{k=1}^{h-2} \left( \binom{h-1}{k} + \binom{h-1}{k-1} \right) B_{t^{h-k}}(t, u_{t^k}, \eta) + (h-1) B_t(t, u_{t^{h-1}}, \eta) \\
&= B_{t^h}(t, u, \eta) + \sum_{k=1}^{h-2} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta) + (h-1) B_t(t, u_{t^{h-1}}, \eta) \\
&= \sum_{k=0}^{h-1} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta) - B_t(t, u_{t^{h-1}}, \eta).
\end{aligned}$$

Hence, we get from (19) that

$$\begin{aligned}
\langle w_{tt}, \eta \rangle + \frac{\partial}{\partial t} B(t, w, \eta) \ = \ & \left\langle f_{t^h}, \eta \right\rangle + B_t(t, w - u_{t^{h-1}}, \eta) \\
& - \frac{\partial}{\partial t} \sum_{k=0}^{h-2} \binom{h-1}{k} B_{t^{h-1-k}}(t, u_{t^k}, \eta). \qquad (20)
\end{aligned}$$

Now by integrating equality (20) with respect to $t$ from 0 to $t$ and using the assumption (ii) and the inductive hypothesis (15), we arrive at

$$\begin{aligned}
\langle w_t, \eta \rangle + B(t, w, \eta) \ = \ & \left\langle f_{t^{h-1}}, \eta \right\rangle + \int_0^t B_t(\tau, w - u_{t^{h-1}}, \eta) d\tau \\
& - \sum_{k=0}^{h-2} \binom{h-1}{k} B_{t^{h-1-k}}(t, u_{t^k}, \eta). \qquad (21)
\end{aligned}$$

Put $z = w - u_{t^{h-1}}$. Then $z|_{t=0} = 0$. It follows from the inductive assumption (16) with $h$ replaced by $h-1$ and (21) that

$$\langle z_t(t), \eta \rangle + B(t, z(t), \eta) = \int_0^t B_t(\tau, z(., \tau), \eta) d\tau \text{ for all } \eta \in \overset{\circ}{W}{}_q^1(G). \qquad (22)$$

Now by applying Lemma 1, we can see from (22) that $z \equiv 0$ on $Q$. This implies $u_{t^h} = w_t = v \in \overset{\circ}{W}{}^1_{p,*}(Q)$. The proof is complete. □

## 4 The Global Regularity

Firstly, let us review some notations and results on elliptic boundary value problems in domains of polyhedral type (see [6, Chapter 4]).

Let $\xi$ be a point on the edge $M_k$, and let $\Gamma_{k_+}, \Gamma_{k_-}$ be the faces of $G$ adjacent to $\xi$. Then by $\mathcal{D}_\xi$ we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{k_\pm}$ tangent to $\Gamma_{k_\pm}$ at $\xi$

and the edge $M_\xi^\circ = \bar\Gamma_{k_+}^\circ \cap \bar\Gamma_{k_-}^\circ$. Let $r, \varphi$ be polar coordinates in the plane perpendicular to $M_\xi^\circ$ such that

$$\Gamma_{k_\pm}^\circ = \{x \in \mathbb{R}^3 : r > 0, \varphi = \pm\theta_\xi/2\}.$$

We define the operator $A_\xi(\lambda, t)$ as follows:

$$A_\xi(\lambda, t)U = r^{2-\lambda}\nabla(A(\xi, t)\nabla u),$$

where $u(x) = r^\lambda U(\varphi)$, $\lambda \in \mathbb{C}$. The operator $A_\xi(\lambda, t)$ realizes a continuous mapping from $W_2^2(I_\xi) \cap \overset{\circ}{W}{}_2^1(I_\xi)$ into $L_2(I_\xi)$ for every $\lambda \in \mathbb{C}$, where $I_\xi$ denotes the interval $(-\theta_\xi/2, \theta_\xi/2)$. A complex number $\lambda_0$ is called an eigenvalue of the pencil $A_\xi(\lambda, t)$ if there exists a nonzero function $U \in W_2^2(I_\xi) \cap \overset{\circ}{W}{}_2^1(I_\xi)$ such that $A_\xi(\lambda_0, t)U = 0$. We denote by $\delta_+(\xi, t)$ and $\delta_-(\xi, t)$ the greatest positive real numbers such that the strip

$$-\delta_-(\xi, t) < \mathrm{Re}\lambda < \delta_+(\xi, t)$$

is free of eigenvalues of the pencil $A_\xi(\lambda, t)$. Furthermore, we define

$$\delta_\pm^{(k)} = \inf_{\xi \in M_k, t \in [0,T]} \delta_\pm(\xi, t)$$

for $k = 1, \ldots, d$.

Let $x^{(i)}$ be a vertex of $G$, and let $J_i$ be the set of all indices $j$ such that $x^{(i)} \in \bar\Gamma_j$. By assumption, there exist a neighborhood $\mathcal{U}$ of $x^{(i)}$ and a diffeomorphism $\kappa$ mapping $G \cap \mathcal{U}$ onto $\mathcal{K}_i \cap B_1$ and $\Gamma_k \cap \mathcal{U}$ onto $\Gamma_k^\circ \cap B_1$ for $k \in J_i$, where

$$\mathcal{K}_i = \{x \in \mathbb{R}^3 : x/|x| \in \Omega_i\}$$

is a cone with vertex at the origin, $\Gamma_k^\circ = \{x \in \mathbb{R}^3 : x/|x| \in \gamma_k\}$, $\Omega_i$ is a domain of polygonal type on the unit sphere $S^2$ with the sides $\gamma_k$, and $B_1$ is the open unit ball. We introduce spherical coordinates $\rho = |x|, \omega = x/|x|$ in $\mathcal{K}_i$ and define

$$\mathfrak{U}_i(\lambda, t)U = \rho^{2-\lambda}\nabla(A(x^{(i)}, t)\nabla u)$$

where $u(x) = \rho^\lambda U(\omega)$. The operator $\mathfrak{U}_i(\lambda, t)$ realizes a continuous mapping

$$W_2^2(\Omega_i) \cap \overset{\circ}{W}{}_2^1(\Omega_i) \to L_2(\Omega_i).$$

An eigenvalue of $\mathfrak{U}_i(\lambda, t)$ is a complex number $\lambda_0$ such that $\mathfrak{U}_i(\lambda_0, t)U = 0$ for some nonzero function $U \in W_2^2(\Omega_i) \cap \overset{\circ}{W}{}_2^1(\Omega_i)$.

For the following lemma on the regularity of the solutions to elliptic boundary value problems in domains of polyhedral type, we refer to Corollary 4.1.10 and Theorem 4.1.11 of [6].

**Lemma 2** *Let $l, l'$ be nonnegative integers, $l, l' \geqslant 1$, and let $\beta = (\beta_1, \ldots, \beta_d)$, $\beta' = (\beta'_1, \ldots, \beta'_d)$, $\delta = (\delta_1, \ldots, \delta_{d'})$, $\delta' = (\delta'_1, \ldots, \delta'_{d'})$ be tuples of real numbers. Let us fix some $t \in [0, T]$. Let $u \in V^l_{p,\beta,\delta}(G)$ be a solution of the following elliptic boundary problem*

$$div(A(x, t)\nabla u) = f \text{ in } G, \quad (23)$$

$$u = 0 \text{ on } \partial G, \quad (24)$$

*where $f \in V^{l-2}_{p,\beta,\delta}(G) \cap V^{l'-2}_{p,\beta',\delta'}(G)$. Suppose that the following conditions are satisfied.*

(i) *The closed strip between the lines* $\mathrm{Re}\lambda = l - \beta_j - 3/p$ *and* $\mathrm{Re}\lambda = l' - \beta'_j - 3/p$ *does not contain eigenvalues of the operator pencils $\mathfrak{U}_j(\lambda, t)$, $j = 1, \ldots, d'$,*

(ii) $-\delta_+^{(k)} < \delta_k - l + 2/p < \delta_-^{(k)}$ *and* $-\delta_+^{(k)} < \delta'_k - l' + 2/p < \delta_-^{(k)}$ *for* $k = 1, \dots, d$. *Then* $u \in V^{l'}_{p,\beta',\delta'}(G)$ *and*

$$\|u\|_{V^{l'}_{p,\beta',\delta'}(G)} \le C\|f\|_{V^{l'-2}_{p,\beta',\delta'}(G)} \tag{25}$$

*with a constant* $C$ *independent of* $u$ *and* $f$.

We now state the main theorem of this section.

**Theorem 3** *Let* $m$, $h$ *be nonnegative integers,* $m \geqslant 2$, *and* $\beta = (\beta_1, \dots, \beta_d), \delta = (\delta_1, \dots, \delta'_d)$ *be real tuples,* $-1 \leqslant \beta_j, \delta_k \leqslant 1$. *Let* $\ell_m$ *be the greatest integer less than* $\frac{m-1}{2}$. *Assume that the following conditions are satisfied.*

(i) $f_{t^k} \in V^{m-2,0}_{p,\beta,\delta}(Q)$ *for* $k = 0, 1, \dots, h$,
(ii) $f_{t^{h+k}} \in V^{2\ell_m-2k,0}_{p,\beta,\delta}(Q)$ *for* $k = 0, 1, \dots, \ell_m$,
(iii) $f_{t^{h+\ell_m+1}} \in L_p((0,T); W_p^{-1}(G))$,
(iv) $f_{t^k}(x, 0) = 0$ *for* $k \leqslant h + \ell_m$.

*Additionally, suppose that the closed strip between the lines* $\mathrm{Re}\lambda = 1 - 3/p$ *and* $\mathrm{Re}\lambda = m - \beta_j - 3/p$ *does not contain eigenvalues of the operator pencils* $\mathfrak{U}_j(\lambda, t)$, $j = 1, \dots, d$, $t \in [0, T]$, *and*

$$-\delta_+^{(k)} < 2/p - 1 < \delta_-^{(k)},\ -\delta_+^{(k)} < \delta_k - m + 2/p < \delta_-^{(k)},\ k = 1, \dots, d'.$$

*Let* $u \in W^1_{p,*}(Q)$ *be the weak solution of the problem* (1)–(3). *Then*

$$u_{t^k} \in V^{m,0}_{p,\beta,\delta}(Q) \text{ for } k = 0, 1, \dots, h, \tag{26}$$

*and, if* $\ell_m \geqslant 1$,

$$u_{t^{h+1+k}} \in V^{2\ell_m-2k,0}_{p,\beta,\delta}(Q) \text{ for } k = 0, \dots, \ell_m. \tag{27}$$

*Moreover, the following estimate holds*

$$\sum_{k=0}^{h} \|u_{t^k}\|_{V^{m,0}_{p,\beta,\delta}(Q)} + \sum_{k=0}^{\ell_m} \|u_{t^{h+1+k}}\|_{V^{2\ell_m-2k,0}_{p,\beta,\delta}(Q)} \leqslant C\Bigg(\sum_{k=0}^{h} \|f_{t^k}\|_{V^{m-2,0}_{p,\beta,\delta}(Q)} + \sum_{k=0}^{\ell_m} \|f_{t^{h+k}}\|_{V^{2\ell_m-2k,0}_{p,\beta,\delta}(Q)} + \|f_{t^{h+\ell_m+1}}\|_{L_p((0,T);W_p^{-1}(G))}\Bigg), \tag{28}$$

*where* $C$ *is a constant independent of* $u$ *and* $f$.

*Proof* According to [6, Lemma 4.1.3], we obtain $V^0_{p,\beta,\delta}(G) \subset W_p^{-1}(G)$ for all $\delta_k, \beta_j \le 1$. Then it follows from (i), (ii), and (iii) that

$$f_{t^k} \in L_p((0,T); W_p^{-1}(G)) \text{ for } k = 0, \dots, h + \ell_m + 1.$$

Thus, by Theorem 2, we have $u_{t^k} \in W^1_{p,*}(Q)$ for $k = 0, \dots, h + \ell_m + 1$. Moreover, since $\overset{\circ}{W}{}^1_p(G) \subset V^0_{p,\beta,\delta}(G)$ for all $\delta_k, \beta_j \ge -1$, it holds that

$$u_{t^k} \in V^{0,0}_{p,\beta,\delta}(Q) \text{ for } k = 0, \dots, h + \ell_m + 1. \tag{29}$$

Now we prove the assertions of the theorem by induction on $m$. Firstly, let us consider the case $m = 2$. In this case, $\ell_m = 0$. We will use induction on $h$. From the hypothesis and (29), we have $f - u_t \in V^0_{p,\beta,\delta}(G)$, for a.e. $t \in [0, T]$ . Thus, we can apply Lemma 2 (for $l = 1, \beta_j = \delta_k = 0, l' = 2, \beta'_j = \beta_j, \delta'_k = \delta_k$) to the following problem

$$-\operatorname{div}(A(x,t)\nabla u) = f - u_t \text{ in } G, \tag{30}$$

$$u = 0 \qquad \text{on } \partial G, \tag{31}$$

to deduce that $u(\cdot, t) \in V^2_{p,\beta,\delta}(G)$, for a.e. $t \in [0, T]$, and

$$\|u\|_{V^2_{p,\beta,\delta}(G)} \le C\|f - u_t\|_{V^0_{p,\beta,\delta}(G)} \le C(\|f\|_{V^0_{p,\beta,\delta}(G)} + \|u_t\|_{\overset{\circ}{W}{}^1_p(G)}).$$

Integrating with respect to $t$ from 0 to $T$ and using Theorem 2 again, we arrive at

$$\|u\|_{V^{2,0}_{\beta,\delta}(Q)} \le C\left(\|f\|_{V^{0,0}_{\beta,\delta}(Q)} + \|f_t\|_{L_p(0,T,W_p^{-1}(G))}\right), \tag{32}$$

where $C$ stands for constants independent of $u$, $f$, and $t$. Thus, the assertions of the theorem hold for $h = 0$ (in the case of $m = 2$). Assume inductively that they are true for $h - 1$. Differentiating both sides of (30), (31) $h$ times with respect to $t$, we have

$$-\operatorname{div}(A\nabla u_{t^h}) = \hat{f} \equiv f_{t^h} - u_{t^{h+1}} + \sum_{k=0}^{h-1}\binom{h}{k}\operatorname{div}(A_{t^{h-k}}\nabla u_{t^k}) \text{ in } G, \tag{33}$$

$$u_{t^h} = 0 \text{ on } \partial G. \tag{34}$$

From the inductive assumption, we see that

$$\|\sum_{k=0}^{h-1}\binom{h}{k}\nabla(A_{t^{h-k}}\nabla u_{t^k})\|_{V^{0,0}_{\beta,\delta}(Q)} \leqslant C\sum_{k=0}^{h-1}\|u_{t^k}\|_{V^{2,0}_{\beta,\delta}(Q)} \leqslant C\sum_{k=0}^{h}\|f_{t^k}\|_{V^{0,0}_{\beta,\delta}(Q)},$$

where $C$ is a constant independent of $f$ and $u$. This fact, together with (29) and the hypothesis (i) imply that $\hat{f} \in V^{0,0}_{\beta,\delta}(Q)$. Thus, we can use the same arguments as above to get from (33), (34) that $u_{t^h} \in V^{2,0}_{\beta,\delta}(Q)$ and

$$\begin{aligned}\|u_{t^h}\|_{V^{2,0}_{\beta,\delta}(Q)} \leqslant C\|\hat{f}\|_{V^{0,0}_{\beta,\delta}(Q)} &\leqslant C\left(\sum_{k=0}^{h}\|f_{t^k}\|_{V^{0,0}_{\beta,\delta}(Q)} + \|u_{t^{h+1}}\|_{L_p((0,T);\overset{\circ}{W}{}^1_p(G))}\right)\\ &\leqslant C\left(\sum_{k=0}^{h}\|f_{t^k}\|_{V^{0,0}_{\beta,\delta}(Q)} + \|f_{t^{h+1}}\|_{L_p(0,T,W_p^{-1}(G))}\right)\end{aligned}$$

with the constants $C$ independent of $u$ and $f$. Thus, the assertions of the theorem hold for the case of $m = 2$ and $h \in \mathbb{N}$.

Now assuming that the claims of theorem are true for $m - 1$ and for arbitrary $h$, we will prove them for $m$. Firstly, we will prove (27). From (29), we see that (27) holds for $k = \ell_m$. Suppose that it holds for $k = \ell_m, \ell_m - 1, \ldots, d + 1$ $(1 \leqslant d \leqslant \ell_m - 1)$. Differentiating both sides of (30), (31) $h + d$ times with respect to $t$, we have

$$-\operatorname{div}(A\nabla u_{t^{h+d}}) = f_{t^{h+d}} - u_{t^{h+d+1}} + \sum_{k=0}^{h+d-1}\binom{h+d}{k}\operatorname{div}(A_{t^{h+d-k}}\nabla u_{t^k}) \text{ in } G, \tag{35}$$

$$u_{t^{h+d}} = 0 \text{ on } \partial G. \tag{36}$$

Notice that $f_{t^{h+d}} \in V^{2\ell_m-2d,0}_{\beta,\delta}(Q)$ by the hypothesis (ii). From the inductive assumption, we see that if $h + 1 \le k \le h + 1 + d - 1$, then $u_k = u_{t^{h+1+j}} \in V^{2\ell_{m-1}-2j,0}_{\beta,\delta}(Q) \subset V^{2\ell_m-2d+2,0}_{\beta,\delta}(Q)$ for $j = 0\ldots, d - 2$, and if $0 \le k \le h$, then $u_{t^k} \in V^{m-1,0}_{\beta,\delta}(Q) \subset$

$V_{\beta,\delta}^{2\ell_m-2d+2,0}(Q)$. Therefore, the right-hand side of (35) belongs to $V_{\beta,\delta}^{2\ell_m-2d,0}(Q)$. Thus, we can apply Lemma 2 to get from (35), (36) that $u_{t^{h+d}} \in V_{\beta,\delta}^{2\ell_m-2d+2,0}(Q)$, and therefore, (27) follows.

Finally, it remains to show (26). We will use again induction on $h$. If $h = 0$, from (27), we have $u_t \in V_{\beta,\delta}^{2\ell_m,0}(Q) \subset V_{\beta,\delta}^{m-2,0}(Q)$, since $m-2 \leqslant 2\ell_m$. Thus, applying Lemmas 2 to problem (30), (31) leads to $u \in V_{\beta,\delta}^{m,0}(Q)$. This means (26) holds for $h = 0$. Assume that it is true for $h-1$. To prove it for $h$, we use again (33), (34). There, the last term of $\hat{f}$ belongs to $V_{\beta,\delta}^{m-2,0}(Q)$ by the inductive assumption. On the other hand, we have $u_{t^{h+1}} \in V_{\beta,\delta}^{2\ell_m,0}(Q) \subset V_{\beta,\delta}^{m-2,0}(Q)$ by (27) and $f_{t^h} \in V_{\beta,\delta}^{m-2,0}(Q)$ by the hypothesis (i). Thus, $\hat{f} \in V_{\beta,\delta}^{m-2,0}(Q)$, and therefore, we can apply Lemma 2 to the problem (33), (34) to conclude that $u_{t^h} \in V_{\beta,\delta}^{m,0}(Q)$. So (26) holds for $h$. The estimate (28) follows from the estimates (25) and (14). The proof is complete. □

## 5 An Example

To illustrate Theorem 3, in this section, we consider as example the case of operator $\operatorname{div}(A\nabla) = \Delta$. For the following information concerning the eigenvalues of pencils $A_\xi(\lambda, t)$ and $\mathfrak{U}_j(\lambda, t)$ introduced in the previous section, we refer to [5, Chapter 2]. The eigenvalue of the operator pencil $A_\xi(\lambda, t)$ are

$$\lambda_k = k\pi/\theta_\xi, \quad k = \pm 1, \pm 2, \ldots,$$

where $\theta_\xi$ is the inner angle at the edge point $\xi$ (see [5, Section 2.1.1]). We see that $\delta_+(\xi) = \delta_-(\xi) = \pi/\theta_\xi$ are the greatest positive real numbers such that the strip

$$-\pi/\theta_\xi < \operatorname{Re}\lambda < \pi/\theta_\xi$$

is free of eigenvalues of the pencils $A_\xi(\lambda, t)$. Set

$$\theta_k = \sup_{\xi \in M_k} \theta_\xi,$$

then we get

$$\delta_\pm^{(k)} = \inf_{\xi \in M_k} \delta_\pm(\xi) = \pi/\theta_k.$$

Let $\hat{\lambda}$ be the eigenvalues of the Laplace-Beltrami operator $-\delta$ (with the Dirichlet condition) on the subdomain $\Omega_j$ of the unit sphere ($\Omega_j$ is defined in the previous section). Then the eigenvalues of the pencils $\mathfrak{U}_j(\lambda, t)$ are given by

$$\Lambda_{\pm k} = -\frac{1}{2} \pm \sqrt{\hat{\lambda} + 1/4}.$$

It is well-known that the spectrum $-\delta$ is a countable set of positive eigenvalues (see [5, Section 2.2.1]). Hence, the interval $[-1, 0]$ is free of eigenvalues of the pencils $\mathfrak{U}_j(\lambda, t)$ for all $j = 1, \ldots d'$. We denote the smallest positive eigenvalue of the $\mathfrak{U}_j(\lambda, t)$ by $\Lambda_j^+$. Then the interval $[-1-\Lambda_j^+, \Lambda_j^+]$ does not contain eigenvalues of the pencils $\mathfrak{U}_j(\lambda, t)$. Now, the conditions about the eigenvalues of pencils $A_\xi(\lambda, t)$ and $\mathfrak{U}_j(\lambda, t)$ in Theorem 3 can be written down simply as follows

$$-1 - \Lambda_j^+ < 1 - 3/p, \; m - \beta_j - 3/p < \Lambda_j^+, \; j = 1, \ldots d,$$

and

$$|2/p - 1| < \pi/\theta_k, |2/p + \delta_k - m| < \pi/\theta_k, \; k = 1, \ldots, d'.$$

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