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On a Polyconvolution with a Weight Function for Fourier Cosine and Laplace Transforms

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Abstract In this paper, we introduce the generalized convolution with a weight function for the Hartley and Fourier cosine transforms. Several algebraic properties and applications of this generalized convolution to solving a class of integral equations of Toeplitz plus Hankel type and a class of systems of integral equations are presented.

Keywords Laplace transform · Fourier cosine transform · Convolution · Generalized convolution · Polyconvolution · Integral equations

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1 Introduction

Convolutions and generalized convolutions for many different integral transforms have interesting applications in several contexts of science and mathematics ([2, 3, 5, 7–10, 12, 16–18]). In 1997, Kakichev ([4]) proposed a general definition of polyconvolution for $n+1$ arbitrary integral transforms $T, T_1, T_2, \ldots, T_n$ with the weight function $\gamma(x)$ of functions $f_1, f_2, \ldots, f_n$ for which the factorization property holds

$$T\left[\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)\right](y) = \gamma(y)\,(T_1 f_1)(y)\,(T_2 f_2)(y)\ldots(T_n f_n)(y)\,.$$

An application of this notion to three integral transforms as Fourier, Fourier cosine, Fourier sine, or Hartley and types of Fourier transforms has been presented ([6, 11]). The generalized convolution generated by the Fourier cosine transform and the Laplace transform has been studied in [13–15]. Following these authors, in this paper, we construct and

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study a new polyconvolution with a weight function for a bunch of integral transforms: Fourier cosine and Laplace transforms.

We note that from the above factorization equality, the general definition of convolutions has the form

$$\overset{\gamma}{*}(f_1, f_2, ..., f_n)(x) = T^{-1}\left[\gamma(\cdot)(T_1 f_1)(\cdot)(T_2 f_2)(\cdot)...(T_n f_n)(\cdot)\right](x)$$

with T^{-1} being the inverse operator of T. Although it looks quite simple, it is not easy to have an explicit form of convolutions when applied to concrete integral transforms. Furthermore, to obtain explicit formulas for convolutions of different integral transforms, one should answer the question in which function spaces the convolutions live and which properties they own. We will approach these goals for a new polyconvolution with a weight function for two Fourier cosine transforms and one Laplace transform. As a by-product, we will apply this new notion to solving some non-standard integral equations and systems of integral equations. We note that for such systems of integral equations, a representation of their solution in a closed form is an interesting and open problem [2, 7].

The paper is organized as follows. In Section 2, we recall some known convolutions and generalized convolutions. In Section 3, we define a new polyconvolution with a weight function $\gamma(y) = e^{-y}$ of three functions for Fourier cosine and Laplace transforms and prove the existence of this polyconvolution in certain function spaces as well as the factorization equality and algebraic properties of this polyconvolution operator. In Section 4, the boundedness property of the polyconvolution operator is considered. In Section 5, we study an integral transform related to this polyconvolution. Finally, in Section 6 with the help of the new polyconvolution, we study a class of Toeplitz plus Hankel integral equations and some systems of integral equations and prove that they can be solved in a closed form.

2 Preliminaries

In this section, we recall some known convolutions and generalized convolutions. The convolution of two functions f and g in $L_1(\mathbb{R})$ for the Fourier integral transform is well known [8] as

$$\left(f \underset{F}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-y)\, g(y)\, dy, \quad x \in \mathbb{R}, \tag{1}$$

for which the following factorization property holds:

$$F\left(f \underset{F}{*} g\right)(y) = (Ff)(y)\,(Fg)(y), \quad y \in \mathbb{R}, \tag{2}$$

where the Fourier integral transform is defined by

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-iyx} dx, \quad y \in \mathbb{R}.$$

The convolution of two functions f and g for the Laplace transform is that of the form [8]

$$\left(f \underset{L}{*} g\right)(x) = \int_0^x f(x-y)\, g(y)\, dy, \quad x > 0, \tag{3}$$

which satisfies the factorization identity

$$L\left(f \underset{L}{*} g\right)(y) = (Lf)(y)\,(Lg)(y), \quad y > 0. \tag{4}$$

Here L denotes the Laplace integral transform [8]

$$(Lf)(y) = \int_0^{\infty} f(x) e^{-yx} dx, \quad y > 0.$$

The convolution of two functions f and g in $L_1(\mathbb{R}_+)$ for the Fourier cosine transform is defined by [16]

$$\left(f \underset{F_c}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \left[g(x+y) + g(|x-y|)\right] dy, \quad x > 0, \tag{5}$$

with the factorization equality

$$F_c\left(f \underset{F_c}{*} g\right)(y) = (F_c f)(y)(F_c g)(y), \quad y > 0. \tag{6}$$

Here, the Fourier cosine integral transform is defined by [16]

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos(xy)\, dx.$$

The generalized convolution for the Fourier sine and Fourier cosine transforms is defined by [8]

$$\left(f \underset{1}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y) \left[g(|x-y|) - g(x+y)\right] dy, \quad x > 0, \tag{7}$$

for the which the factorization equality holds:

$$F_s\left(f \underset{1}{*} g\right)(y) = (F_s f)(y)(F_c g)(y), \quad y > 0, \ f, g \in L_1(\mathbb{R}_+), \tag{8}$$

where F_s denotes the Fourier sine integral transform [8]

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \sin(yx)\, dx.$$

The generalized convolution for the Fourier cosine and Fourier sine transforms is defined by [12]

$$\left(f \underset{2}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(y) \big[g(x+y) + \operatorname{sign}(y-x)\, g(|y-x|)\big] dy, \quad x > 0, \tag{9}$$

which satisfies the factorization identity

$$F_c\left(f \underset{2}{*} g\right)(y) = (F_s f)(y)(F_c g)(y), \quad y > 0. \tag{10}$$

The convolution with the weight function $\gamma_1(y) = \cos y$ for the Fourier cosine transforms is defined by [9]

$$\begin{aligned}(f \overset{\gamma_1}{\underset{F_c}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y) \Big[& g(|x+y+1)|) + g(|x-y+1)|) + g(|x+y-11)|) \\ & + g(|x-y-1)|) \Big] dy, \quad x > 0, \end{aligned} \tag{11}$$

which satisfies the factorization identity

$$F_c\left(f \overset{\gamma_1}{\underset{F_c}{*}} g\right)(y) = \cos y (F_c f)(y)(F_c g)(y), \quad y > 0. \tag{12}$$

3 The Polyconvolution with the Weight Function $\gamma(y) = e^{-y}$ for Fourier Cosine and Laplace Transforms

Definition 3.1 The polyconvolution with the weight function $\gamma(y) = e^{-y}$ of three functions f, g, h for Fourier cosine and Laplace transforms is defined by

$$\overset{\gamma}{*}(f,g,h)(x) = \frac{1}{\pi\sqrt{2\pi}} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} f(u)\,g(v)\,h(y)\Bigg[\frac{y+1}{(y+1)^2+(x+u+v)^2} + \frac{y+1}{(y+1)^2+(x+u-v)^2} + \frac{y+1}{(y+1)^2+(x-u+v)^2} + \frac{y+1}{(y+1)^2+(x-u-v)^2}\Bigg] du\,dv\,dy, \quad x>0. \tag{13}$$

Theorem 3.2 *If the functions f, g, h are given in $L_1(\mathbb{R}_+)$, then the polyconvolution (13) belongs to $L_1(\mathbb{R}_+)$ and satisfies the factorization identity*

$$F_c\left[\overset{\gamma}{*}(f,g,h)\right](y) = e^{-y}(F_c f)(y)(F_c g)(y)(Lh)(y), \quad y>0. \tag{14}$$

Moreover, when $f, g, h \in L_2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$, the Parseval type identity holds:

$$\overset{\gamma}{*}(f,g,h)(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}(F_c f)(y)(F_c g)(y)(Lh)(y)\,e^{-y}\cos(xy)\,dy. \tag{15}$$

Proof We have

$$\begin{aligned}
&\int_0^{+\infty}\Bigg|\frac{y+1}{(y+1)^2+(x+u+v)^2} + \frac{y+1}{(y+1)^2+(x+u-v)^2} + \frac{y+1}{(y+1)^2+(x-u+v)^2} + \frac{y+1}{(y+1)^2+(x-u-v)^2}\Bigg| dx \\
&\le \int_{u+v}^{+\infty}\frac{y+1}{(y+1)^2+t^2}dt + \int_{u-v}^{+\infty}\frac{y+1}{(y+1)^2+t^2}dt + \int_{-u+v}^{+\infty}\frac{y+1}{(y+1)^2+t^2}dt + \int_{-u-v}^{+\infty}\frac{y+1}{(y+1)^2+t^2}dt \\
&= 2\pi.
\end{aligned} \tag{16}$$

From (13) and (16), we have

$$\left|\overset{\gamma}{*}(f,g,h)(x)\right| \le \sqrt{\frac{2}{\pi}}\int_0^{\infty}|f(u)|\,du\int_0^{\infty}|g(v)|\,dv\int_0^{\infty}|h(y)|\,dy = \sqrt{\frac{2}{\pi}}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\mathbb{R}_+)}.$$

Therefore,

$$\|\overset{\gamma}{*}(f,g,h)\|_{L_1(\mathbb{R}_+)} \le \sqrt{\frac{2}{\pi}}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\mathbb{R}_+)} < \infty.$$

Thus

$$\overset{\gamma}{*}(f,g,h) \in L_1(\mathbb{R}_+). \tag{17}$$

On the other hand, using (13) and the formula $\int_0^\infty e^{-sx}\cos xydx = \frac{s}{s^2+y^2},\ s > 0$, we obtain

$$
\begin{aligned}
&\overset{\gamma}{*}(f,g,h)(x)\\
&=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos utdu\sqrt{\frac{2}{\pi}}\int_0^\infty g(v)\cos tvdv\int_0^\infty h(y)e^{-ty}dy\right]e^{-t}\cos xtdt\\
&=\sqrt{\frac{2}{\pi}}\int_0^\infty (F_cf)(t)(F_cg)(t)(Lh)(t)e^{-t}\cos xtdt.
\end{aligned}
$$

Thus the Parseval type identity (15) holds. Combining with (17), we get the factorization identity (14). The theorem is proved. □

Proposition 3.3 *Let $f,g,h,l\in L_1(\mathbb{R}_+)$. Then the polyconvolution (13) satisfies the following equalities*

(a) $\overset{\gamma}{*}(f,g,h)=\overset{\gamma}{*}(g,f,h)$,

(b) $\overset{\gamma}{*}\left[\left(f\underset{F_c}{*}g\right),l,h\right]=\overset{\gamma}{*}\left[\left(f\underset{F_c}{*}l\right),g,h\right]$,

(c) $\overset{\gamma}{*}\left[\left(f\underset{F_c}{*}g\right),l,h\right]=\overset{\gamma}{*}\left[\left(l\underset{F_c}{*}g\right),f,h\right]$.

Proof First, we show (a). Indeed, from the factorization equality (14), we have

$$
\begin{aligned}
F_c[\overset{\gamma}{*}(f,g,h)](y) &= e^{-y}(F_cf)(y)(F_cg)(y)(Lh)(y)\\
&= e^{-y}(F_cg)(y)(F_cf)(y)(Lh)(y)\\
&= F_c[\overset{\gamma}{*}(g,f,h)](y).
\end{aligned}
$$

Thus $\overset{\gamma}{*}(f,g,h)=\overset{\gamma}{*}(g,f,h)$.

(b) Using the factorization properties (6) and (14), we can write

$$
\begin{aligned}
F_c\left\{\overset{\gamma}{*}[(f\underset{F_c}{*}g),l,h]\right\}(y) &= e^{-y}F_c(f\underset{F_c}{*}g)(y)(F_cl)(y)(Lh)(y)\\
&= e^{-y}(F_cf)(y)(F_cl)(y)(F_cg)(y)(Lh)(y)\\
&= e^{-y}F_c(f\underset{F_c}{*}l)(y)(F_cg)(y)(Lh)(y)\\
&= F_c\left\{\overset{\gamma}{*}[(f\underset{F_c}{*}l),g,h]\right\}(y),\quad y>0.
\end{aligned}
$$

So, we get $\overset{\gamma}{*}[(f\underset{F_c}{*}g),l,h]=\overset{\gamma}{*}[(f\underset{F_c}{*}l),g,h]$.

Similarly, we can prove (c). □

Theorem 3.4 (Tichmarch type theorem) *Let $f\in L_1(\mathbb{R}_+,e^{\alpha x}),\alpha>0,g,h\in L_1(\mathbb{R}_+)$. If $\overset{\gamma}{*}(f,g,h)(x)=0,\ x>0$, then either $f(x)=0,\ x>0$ or $g(x)=0,\ x>0$ or $h(x)=0,x>0$.*

Proof We have

$$\left|\frac{d^n}{dy^n}\{(\cos yx)\, f(x)\}\right| = \left|f(x)\, x^n \cos\left(yx + n\frac{\pi}{2}\right)\right|$$
$$\leq \left|e^{-\alpha x}x^n\right|\left|e^{\alpha x} f(x)\right| \leq \frac{n!}{\alpha^n}\left|e^{\alpha x} f(x)\right|.$$

Here, we used the estimation

$$0 \leq e^{-\alpha x}x^n = e^{-\alpha x}\frac{(\alpha x)^n}{n!}\frac{n!}{\alpha^n} \leq e^{-\alpha x}e^{\alpha x}\frac{n!}{\alpha^n} = \frac{n!}{\alpha^n}.$$

Since $f \in L_1(\mathbb{R}_+, e^{\alpha x})$, we get $\frac{d^n}{dy^n}[\cos yx f(x)] \in L_1(\mathbb{R}_+)$.

Since $L_1(\mathbb{R}_+, e^{\alpha x}) \subset L_1(\mathbb{R}_+)$, $(F_c f)(y)$ is analytic in $\mathbb{R}_+$. Similarly we obtain $(F_c g)$ is analytic in $\mathbb{R}_+$. On the other hand, $(Lh)(y)$ is analytic in $\mathbb{R}_+$. By using the factorization property (14) for $\overset{\gamma}{*}(f, g, h)(x) = 0$, we have

$$(F_c f)(y)(F_c g)(y)(Lh)(y) = 0, \quad y > 0.$$

It implies that, either $f(x) = 0, x > 0$ or $g(x) = 0, x > 0$ or $h(x) = 0, x > 0$.
The theorem is proved. □

4 Inequalities for the Polyconvolution

In this section, we present the norm inequalities for the polyconvolution (13) in $L_1(\mathbb{R}_+)$ and $L_p(\mathbb{R}_+, \rho)$ with $1 \leq p \leq \infty$ and ρ being a weight function. The standard norms are defined as follows

$$\|f\|_{L_1(\mathbb{R}_+)} = \int_0^\infty |f(x)|\, dx;\ \|f\|_{L_1(\mathbb{R}_+,\rho)} = \int_0^\infty |f(x)|\,\rho(x)\, dx;$$
$$\|f\|_{L_p(\mathbb{R}_+)} = \left[\int_0^\infty |f(x)|^p dx\right]^{\frac{1}{p}};\ \|f\|_{L_p(\mathbb{R}_+,\rho)} = \left[\int_0^\infty |f(x)|\,\rho(x)\, dx\right]^{\frac{1}{p}}.$$

Theorem 4.1 *If f, g, h belong to $L_1(\mathbb{R}_+)$, then the following inequality holds*

$$\left\|\overset{\gamma}{*}(f, g, h)\right\|_{L_1(\mathbb{R}_+)} \leq \sqrt{\frac{2}{\pi}}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\mathbb{R}_+)}. \tag{18}$$

Proof From Definition 3.1 and the proof of Theorem 3.2, we obtain

$$\left\|\overset{\gamma}{*}(f, g, h)\right\|_{L_1(\mathbb{R}_+)} = \int_0^\infty \left|\overset{\gamma}{*}(f, g, h)(x)\right| dx$$
$$\leq \sqrt{\frac{2}{\pi}}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\mathbb{R}_+)}.$$

So, we obtain (18). □

Next, we study the polyconvolution on the function space $L_s(\mathbb{R}_+, e^{-\alpha x})$ and estimate its norm.

Theorem 4.2 *Let $f \in L_p(\mathbb{R}_+)$, $g \in L_q(\mathbb{R}_+)$, $h \in L_r(R_+)$, be such that $p, q, r > 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Then the polyconvolution (13) is bounded in $L_s(\mathbb{R}_+, e^{-\alpha x})$ when $s > 1, \alpha > 0$ and the following estimation holds*

$$\left\| \overset{\gamma}{*} (f, g, h) \right\|_{L_s(\mathbb{R}_+, e^{-\alpha x})} \leq \frac{4}{\alpha\pi\sqrt{2\pi}} \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}. \tag{19}$$

Proof From the proof of Theorem 3.2, we have the following estimation

$$\left| \overset{\gamma}{*} (f, g, h) (x) \right| \leq \frac{4}{\pi\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty |f(u)| |g(v)| |h(y)| \, dxdydz, \quad x \in \mathbb{R}_+.$$

Let p_1, q_1, r_1 be the conjugate exponentials of p, q, r and

$$\begin{aligned} U(u, v, y) &= |g(v)|^{\frac{q}{p_1}} |h(y)|^{\frac{r}{p_1}} \in L_1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+), \\ V(u, v, y) &= |h(y)|^{\frac{r}{q_1}} |f(u)|^{\frac{p}{p_1}} \in L_1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+), \\ W(u, v, y) &= |f(u)|^{\frac{p}{r_1}} |g(v)|^{\frac{q}{r_1}} \in L_1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+). \end{aligned}$$

We see that $UVW = |f(u)| |g(v)| |h(y)|$.

Using Fubini's theorem, we get

$$\begin{aligned} \|U\|^{p_1}_{L_{p_1}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)} &= \int_0^\infty \int_0^\infty \left\{ |g(v)|^{\frac{q}{p_1}} |h(y)|^{\frac{r}{p_1}} \right\}^{p_1} dvdy \\ &= \int_0^\infty |g(v)|^q \left(\int_0^\infty |h(y)|^r dy \right) dv \\ &= \int_0^\infty |g(v)|^q \, \|h\|^r_{L_r(\mathbb{R}_+)} \, dv \\ &= \|g\|^q_{L_q(\mathbb{R}_+)} \, \|h\|^r_{L_r(\mathbb{R}_+)}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \|V\|^{q_1}_{L_{p_1}(\mathbb{R}_+ \times \mathbb{R}_+)} &= \|f\|^p_{L_p(\mathbb{R}_+)} \, \|h\|^r_{L_r(\mathbb{R}_+)}, \\ \|W\|^{r_1}_{L_{r_1}(\mathbb{R}_+ \times \mathbb{R}_+)} &= \|f\|^p_{L_p(\mathbb{R}_+)} \, \|g\|^q_{L_q(\mathbb{R}_+)}. \end{aligned} \tag{20}$$

Since $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, we have $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} = 1$. Using Hölder's inequality and (20), we obtain

$$\begin{aligned} \left| \overset{\gamma}{*} (f, g, h) \right| &\leq \frac{4}{\pi\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty UVW dudvdy \\ &\leq \frac{4}{\pi\sqrt{2\pi}} \left(\int_0^\infty \int_0^\infty \int_0^\infty U^{p_1} dudvdy \right)^{\frac{1}{p_1}} \left(\int_0^\infty \int_0^\infty \int_0^\infty V^{q_1} dudvdy \right)^{\frac{1}{q_1}} \\ &\quad \times \left(\int_0^\infty \int_0^\infty \int_0^\infty W^{r_1} dudvdy \right)^{\frac{1}{r_1}} \\ &= \frac{4}{\pi\sqrt{2\pi}} \|U\|_{L_{p_1}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)} \|V\|_{L_{q_1}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)} \|W\|_{L_{r_1}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)} \\ &= \frac{4}{\pi\sqrt{2\pi}} \left(\|g\|^{\frac{q}{p_1}}_{L_q(\mathbb{R}_+)} \|h\|^{\frac{r}{p_1}}_{L_r(\mathbb{R}_+)} \right) \left(\|f\|^{\frac{p}{q_1}}_{L_p(\mathbb{R}_+)} \|h\|^{\frac{r}{q_1}}_{L_r(\mathbb{R}_+)} \right) \left(\|f\|^{\frac{p}{r_1}}_{L_q(\mathbb{R}_+)} \|g\|^{\frac{q}{r_1}}_{L_q(\mathbb{R}_+)} \right) \\ &= \frac{4}{\pi\sqrt{2\pi}} \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}. \end{aligned}$$

Since $\int_0^\infty e^{-\alpha x}dx = \frac{1}{\alpha},\ \alpha > 0$, we have

$$\begin{aligned}\left\|\overset{\gamma}{*}(f,g,h)\right\|^s_{L_s(\mathbb{R}_+,e^{-\alpha x})} &= \int_0^\infty e^{-\alpha x}\left|\overset{\gamma}{*}(f,g,h)(x)\right|^s dx \\ &\leq \int_0^\infty e^{-\alpha x}\frac{4}{\pi\sqrt{2\pi}}\|f\|^s_{L_p(\mathbb{R}_+)}\|g\|^s_{L_q(\mathbb{R}_+)}\|h\|^s_{L_r(\mathbb{R}_+)}dx \\ &= \frac{1}{\alpha}\frac{4}{\pi\sqrt{2\pi}}\|f\|^s_{L_p(\mathbb{R}_+)}\|g\|^s_{L_q(\mathbb{R}_+)}\|h\|^s_{L_r(\mathbb{R}_+)}.\end{aligned}$$

So, we obtain

$$\left\|\overset{\gamma}{*}(f,g,h)\right\|_{L_s(\mathbb{R}_+,e^{-\alpha x})} \leq \frac{4}{\alpha\pi\sqrt{2\pi}}\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R}_+)}.$$

Thus, we have (19). The theorem is proved. □

5 The Integral Transform Related to this Polyconvolution

Now, we study an integral transform related to the polyconvolution (13), namely the transforms of the form

$$\begin{aligned} f(x) \mapsto g(x) &= (T_{k_1,k_2}f)(x) \\ &= (1-\frac{d^2}{dx^2})[\overset{\gamma}{*}(f,k_1,k_2)](x) \\ &= \frac{1}{\pi\sqrt{2\pi}}(1-\frac{d^2}{dx^2})\int_0^\infty\int_0^\infty\int_0^\infty f(u)k_1(v)k_2(y) \\ &\times\left[\frac{y+1}{(y+1)^2+(x+u+v)^2}\right. \\ &+\frac{y+1}{(y+1)^2+(x+u-v)^2}+\frac{y+1}{(y+1)^2+(x-u+v)^2} \\ &\left.+\frac{y+1}{(y+1)^2+(x-u-v)^2}\right]dudvdy, \quad x>0. \end{aligned} \tag{21}$$

Similarly to [15], we can prove the following result.

Theorem 5.1 (Watson type theorem) *Suppose that $f, k_1,\ k_2 \in L_2(\mathbb{R}_+)\cap L_1(\mathbb{R}_+)$ are given functions. Then the condition*

$$|e^{-y}(F_ck_1)(y)(Lk_2)(y)| = \frac{1}{1+y^2}, \quad y>0 \tag{22}$$

is a necessary and sufficient condition for the operator T_{k_1,k_2} to be unitary on $L_2(\mathbb{R}_+)$. Moreover, the inverse operator of T_{k_1,k_2} takes the form

$$f(x) = (T^{-1}_{k_1,k_2}g)(x) = \left(1-\frac{d^2}{dx^2}\right)[\overset{\gamma}{*}(f,\bar{k}_1,\bar{k}_2)](x), \tag{23}$$

where $\bar{k}_1,\ \bar{k}_2$ are the complex conjugate functions of k_1, k_2 respectively. So, we obtain

$$(T^{-1}_{k_1,k_2}g)(x) = (T_{\bar{k}_1,\bar{k}_2}g)(x), \quad x>0.$$

6 Integral Equations and Systems of Equations

The polyconvolution (13) allows us to obtain the solutions for integral equations and systems of integral equations in closed form.

6.1 Consider the Following Integral Equation

$$f(x) + \frac{\lambda}{\pi\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty f(x)\, g(v)\, h(y) \left[\frac{y+1}{(y+1)^2 + (x+u+v)^2} + \frac{y+1}{(y+1)^2 + (x+u-v)^2} + \frac{y+1}{(y+1)^2 + (x-u+v)^2} + \frac{y+1}{(y+1)^2 + (x-u-v)^2} \right] dudvdy = k(x), \quad x > 0, \tag{24}$$

where λ is a complex constant; g, h, k are functions in $L_1(\mathbb{R}_+)$; $and f(x)$ is an unknown function in $L_1(\mathbb{R}_+)$.

Theorem 6.1 *Assume that* $1 + \lambda e^{-y}(F_c g)(y)(Lh)(y) \neq 0\ \forall y > 0$. *Then the integral* (24) *has a unique solution in* $L_1(\mathbb{R}_+)$ *in the form*

$$f(x) = k(x) - \left(k \underset{F_c}{*} l \right)(x),$$

where $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c l)(y) = \frac{\lambda e^{-y}(F_c g)(y)(Lh)(y)}{1 + \lambda e^{-y}(F_c g)(y)(Lh)(y)}.$$

Proof Using Definition 3.1, (24) can be rewritten in the form

$$f(x) + \lambda \left[\overset{\gamma}{*}(f, g, h) \right](x) = k(x).$$

Because f and $\overset{\gamma}{*}(f, g, h)(x)$ and k are functions in $L_1(\mathbb{R}_+)$, one can apply the factorization property (14) to get

$$(F_c f)(y) + \lambda e^{-y}(F_c f)(y)(F_c g)(y)(Lh)(y) = (F_c k)(y).$$

Thus

$$(F_c f)(y)\left[1 + \lambda e^{-y}(F_c g)(y)(Lh)(y)\right] = (F_c k)(y).$$

With the condition $1 + \lambda e^{-y}(F_c g)(y)(Lh)(y) \neq 0\ \forall y > 0$, we have

$$(F_c f)(y) = (F_c k)(y)\left[1 - \frac{\lambda e^{-y}(F_c g)(y)(Lh)(y)}{1 + \lambda e^{-y}(F_c g)(y)(Lh)(y)}\right]. \tag{25}$$

Due to the Wiener-Lévy theorem [1], there exists a function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{\lambda e^{-y}(F_c g)(y)(Lh)(y)}{1 + \lambda e^{-y}(F_c g)(y)(Lh)(y)}. \tag{26}$$

From (25), (26), and (8), we obtain

$$\begin{aligned}(F_c f)(y) &= (F_c k)(y)\left[1 - (F_c l)(y)\right] \\ &= (F_c k)(y) - F_c\left(k \underset{F_c}{*} l \right)(y).\end{aligned}$$

Therefore

$$f(x) = k(x) - \left(k \underset{F_c}{*} l\right)(x) \in L(\mathbb{R}_+).$$

The proof is complete. □

6.2 Consider the System of Two Integral Equations

$$\begin{aligned} &f(x) + \frac{\lambda_1}{\pi\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty \varphi_1(u)\, g(v)\, \varphi_2(y) \left[\frac{y+1}{(y+1)^2 + (x+u+v)^2}\right. \\ &+ \frac{y+1}{(y+1)^2 + (x+u-v)^2} + \frac{y+1}{(y+1)^2 + (x-u+v)^2} \\ &\left. + \frac{y+1}{(y+1)^2 + (x-u-v)^2}\right] dudvdy = p(x), \\ &g(x) + \frac{\lambda_2}{\sqrt{2\pi}} \int_0^\infty f(u)\left[\varphi_3(u+x) + \varphi_3(|u-x|)\right] du = q(x), \end{aligned} \tag{27}$$

where λ_1, λ_2 are complex constants, $\varphi_1, \varphi_2, \varphi_3, p, q$ are functions in $L_1(\mathbb{R}_+)$, *and* f, g are unknown functions.

Theorem 6.2 *If the following condition is satisfied*

$$1 + \lambda_1\lambda_2 F_c\left[\overset{\gamma}{*}(\varphi_1, \varphi_3, \varphi_2)\right](y) \neq 0 \quad \forall y > 0,$$

then there exists a unique solution in $L_1(\mathbb{R}_+)$ *of system* (27)*, which is defined by*

$$\begin{aligned} f(x) &= p(x) - \lambda_1\left[\overset{\gamma}{*}(\varphi_1, q, \varphi_2)\right](x) + \left(p \underset{F_c}{*} l\right)(x) - \lambda_1\left[\overset{\gamma}{*}(\varphi_1, q, \varphi_2) \underset{F_c}{*} l\right](x); \\ g(x) &= q(x) - \lambda_2\left(p \underset{F_c}{*} \varphi_3\right)(x) + \left(l \underset{F_c}{*} q\right)(x) - \lambda_2\left[l \underset{F_c}{*} \left(p \underset{F_c}{*} \varphi_3\right)\right](x). \end{aligned}$$

Proof Using Definition 3.1 and (9), the system (27) can be rewritten in the form

$$\begin{aligned} &f(x) + \lambda_1\left[\overset{\gamma}{*}(\varphi_1, g, \varphi_2)\right](x) = p(x), \\ &\lambda_2\left(f \underset{F_c}{*} \varphi_3\right)(x) + g(x) = q(x). \end{aligned} \tag{28}$$

Due to the factorization properties (6), (14) we obtain the linear system of algebraic equations

$$\begin{aligned} &(F_c f)(y) + \lambda_1 e^{-y}(F_c\varphi_1)(y)(F_c g)(y)(L\varphi_2)(y) = (F_c p)(y), \\ &\lambda_2(F_c f)(y)(F_c\varphi_3)(y) + (F_c g)(y) = (F_c g)(y). \end{aligned}$$

The inverse of the determinant of this system has the form

$$\frac{1}{\Delta} = 1 + \frac{\lambda_1\lambda_2 F_c\left[\overset{\gamma}{*}(\varphi_1, \varphi_3, \varphi_2)\right](y)}{1 - \lambda_1\lambda_2 F_c\left[\overset{\gamma}{*}(\varphi_1, \varphi_3, \varphi_2)\right](y)}.$$

According to the Wiener-Lévy theorem [1], there exists a function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{\lambda_1 \lambda_2 F_c \left[\overset{\gamma}{*} (\varphi_1, \varphi_3, \varphi_2)\right](y)}{1 - \lambda_1 \lambda_2 F_c \left[\overset{\gamma}{*} (\varphi_1, \varphi_3, \varphi_2)\right](y)}, \quad \forall y > 0.$$

Hence

$$\frac{1}{\Delta} = 1 + (F_c l)(y).$$

Therefore, using (6), we have

$$(F_c f)(y) = (F_c p)(y) - \lambda_1 F_c \left[\overset{\gamma}{*} (\varphi_1, q, \varphi_2)\right](y) + F_c \left(p \underset{F_c}{*} l\right)(y)$$
$$-\lambda_1 F_c \left[\overset{\gamma}{*} (\varphi_1, q, \varphi_2) \underset{F_c}{*} l\right](y).$$

It follows that

$$f(x) = p(x) - \lambda_1 \left[\overset{\gamma}{*} (\varphi_1, q, \varphi_2)\right](x) + \left(p \underset{F_c}{*} l\right)(x) -$$
$$-\lambda_1 \left[\overset{\gamma}{*} (\varphi_1, q, \varphi_2) \underset{F_c}{*} l\right](x) \in L_1(\mathbb{R}_+).$$

Similarly, we obtain the formula for g as stated in the theorem. □

6.3 A System of Three Integral Equations

$$\frac{\lambda_1}{\pi\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty f(u)\varphi_1(v)\varphi_2(y) \left[\frac{y+1}{(y+1)^2 + (x+u+v)^2}\right.$$
$$+\frac{y+1}{(y+1)^2 + (x+u-v)^2} + \frac{y+1}{(y+1)^2 + (x-u+v)^2}$$
$$\left.+\frac{y+1}{(y+1)^2 + (x-u-v)^2}\right] dudvdy + g(x) + h(x) = p(x),$$
$$f(x) + \frac{\lambda_2}{\sqrt{2\pi}} \int_0^\infty g(u)[\varphi_3(u+x) + \varphi_3(|u-x|)]dx + h(x) = q(x),$$
$$f(x) + g(x) + \frac{\lambda_3}{2\sqrt{2\pi}} \int_0^\infty h(v) \Big[\varphi_4(x+1+v) + \varphi_4(|x+1-v|)$$
$$+\varphi_4(|x-1+v|) + \varphi_4(|x-1-v|)\Big] dv = r(x), \quad x \in \mathbb{R}. \tag{29}$$

Here, $\lambda_1, \lambda_2, \lambda_3$ are complex constants; $\varphi_1, \varphi_2, \varphi_3, \varphi_4, p, q, r$ are functions from $L_1(\mathbb{R}_+)$; and f, g, h are unknown functions.

Theorem 6.3 *Under the condition*

$$\lambda_1 \lambda_2 \lambda_3 F_c \left\{\overset{\gamma}{*} \left[\varphi_1, (\varphi_3 \underset{F_c}{\overset{\gamma_1}{*}} \varphi_4), \varphi_2\right]\right\}(y) + 2 - \lambda_2 (F_c \varphi_3)(y)$$
$$-\lambda_1 e^{-y} (F_c \varphi_1)(y)(L\varphi_2)(y) - \lambda_3 \cos y (F_c \varphi_4)(y) \neq 0 \quad \forall y \in \mathbb{R},$$

there exists a unique solution in $L_1(\mathbb{R}_+)$ *of* (29) *given by*

$$
\begin{aligned}
f(x) &= \lambda_2\Big[p \underset{F_c}{*} (\varphi_3 \overset{\gamma_1}{\underset{F_c}{*}} \varphi_4)\Big](x) + q(x) + r(x) - p(x) + \lambda_2\left\{\Big[p \underset{F_c}{*} (\varphi_3 \overset{\gamma_1}{\underset{F_c}{*}} \varphi_4)\Big] \underset{F_c}{*} l\right\}(x)\\
&\quad -\lambda_2(r \underset{F_c}{*} \varphi_3)(x) + (q \underset{F_c}{*} l)(x) + (r \underset{F_c}{*} l)(x) - \lambda_3\left\{(q \overset{\gamma}{\underset{F_c}{*}} \varphi_4) \underset{F_c}{*} l\right\}(x)\\
&\quad -\lambda_2\Big[(r \underset{F_c}{*} \varphi_3) \underset{F_c}{*} l\Big](x) - (p \underset{F_c}{*} l)(x),\\
g(x) &= \lambda_1\lambda_3\Big[\overset{\gamma}{*}(\varphi_1, (q \overset{\gamma_1}{\underset{F_c}{*}} \varphi_4), \varphi_2)\Big](x) + p(x) + r(x) - q(x) - (p \overset{\gamma_1}{\underset{F_c}{*}} \varphi_4)\\
&\quad -\lambda_1\Big[\overset{\gamma}{*}(\varphi_1, r, \varphi_2)\Big](x),\\
h(x) &= \lambda_1\left\{\Big[\overset{\gamma}{*}(\varphi_1, (\varphi_3 \underset{F_c}{*} r), \varphi_2)\Big] \underset{F_c}{*} l\right\}(x) + (p \underset{F_c}{*} l)(x) + (q \underset{F_c}{*} l)(x) - (r \underset{F_c}{*} l)(x)\\
&\quad -\lambda_2\Big[(p \underset{F_c}{*} \varphi_3) \underset{F_c}{*} l\Big](x) - \lambda_1\left\{l \underset{F_c}{*} \Big[\overset{\gamma}{*}(\varphi_1, q, \varphi_2)\Big]\right\}(x) + p(x) + q(x) - r(x)\\
&\quad -\lambda_2(p \underset{F_c}{*} \varphi_3)(x) - \lambda_1\Big[*(\varphi_1, q, \varphi_2)\Big](x).
\end{aligned} \tag{30}
$$

Proof The system (29) can be rewritten in the form

$$
\begin{aligned}
&\lambda_1\Big[\overset{\gamma}{*}(f, \varphi_1, \varphi_2)\Big](x) + g(x) + h(x) = p(x),\\
&f(x) + \lambda_2(g \underset{F_c}{*} \varphi_3)(x) + h(x) = q(x),\\
&f(x) + g(x) + \lambda_3(h \overset{\gamma_1}{\underset{F_c}{*}} \varphi_4)(x) = r(x).
\end{aligned} \tag{31}
$$

Using the factorization identities (6), (11), and (14), we obtain

$$
\begin{aligned}
&\lambda_1 e^{-y}(F_c f)(y)(F_c\varphi_1)(y)(L\varphi_2)(y) + (F_c g)(y) + (F_c h)(y) = (F_c p)(y),\\
&(F_c f)(y) + \lambda_2(F_c g)(y)(F_c\varphi_3)(y) + (F_c h)(y) = (F_c q)(y),\\
&(F_c f)(y) + (F_c g)(y) + \lambda_3 \cos y (F_c h)(y)(F_c\varphi_4)(y) = (F_c r)(y).
\end{aligned} \tag{32}
$$

The determinant of this system is

$$
\begin{aligned}
\Delta &= \lambda_1\lambda_2\lambda_3 F_c\left\{\overset{\gamma}{*}[\varphi_1, (\varphi_3 \overset{\gamma_1}{\underset{F_c}{*}} \varphi_4), \varphi_2]\right\}(y) + 2 - \lambda_3\cos y(F_c\varphi_4)(y)\\
&\quad -\lambda_2(F_c\varphi_3)(y) - \lambda_1 e^{-y}(F_c\varphi_1)(y)(L\varphi_2)(y)\\
&\neq 0.
\end{aligned}
$$

Thus,

$$\frac{1}{\Delta}$$
$$= 1 + \Bigg[\lambda_1\lambda_2\lambda_3 F_c \left\{\overset{\gamma}{*}[\varphi_1, (\varphi_3 \underset{F_c}{\overset{\gamma_1}{*}} \varphi_4), \varphi_2]\right\}(y) + 2 - \lambda_3 \cos y (F_c\varphi_4)(y) - \lambda_2(F_c\varphi_3)(y)$$
$$-\lambda_1 e^{-y}(F_c\varphi_1)(y)(L\varphi_2)(y)\Bigg]\Bigg[-1 - \lambda_1\lambda_2\lambda_3 F_c \left\{\overset{\gamma}{*}[\varphi_1, (\varphi_3 \underset{F_c}{\overset{\gamma_1}{*}} \varphi_4), \varphi_2]\right\}(y)$$
$$+\lambda_3 \cos y (F_c\varphi_4)(y) + \lambda_2(F_c\varphi_3)(y) + \lambda_1 e^{-y}(F_c\varphi_1)(y)(L\varphi_2)(y)\Bigg]^{-1}.$$

According to the Wiener-Lévy's theorem ([1]), there exists a function $l \in L_1(\mathbb{R})$ such that

$$(F_c l)(y)$$
$$= \Bigg[\lambda_1\lambda_2\lambda_3 F_c \left\{\overset{\gamma}{*}[\varphi_1, (\varphi_3 \underset{F_c}{\overset{\gamma_1}{*}} \varphi_4), \varphi_2]\right\}(y) - \lambda_3 \cos y (F_c\varphi_4)(y) - \lambda_2(F_c\varphi_3)(y)$$
$$+2 - \lambda_1 e^{-y}(F_c\varphi_1)(y)(L\varphi_2)(y)\Bigg]\Bigg[-1 - \lambda_1\lambda_2\lambda_3 F_c \left\{\overset{\gamma}{*}[\varphi_1, (\varphi_3 \underset{F_c}{\overset{\gamma_1}{*}} \varphi_4), \varphi_2]\right\}(y)$$
$$+\lambda_3 \cos y (F_c\varphi_4)(y) + \lambda_2(F_c\varphi_3)(y) + \lambda_1 e^{-y}(F_c\varphi_1)(y)(L\varphi_2)(y)\Bigg]^{-1}, \quad \forall y > 0.$$

So, we obtain

$$\frac{1}{\Delta} = 1 + (F_c l)(y).$$

From this we arrive at (30). □

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