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On the Global Attractor for a Semilinear Strongly Degenerate Parabolic Equation

Mai Xuan Thao[1]

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Abstract We prove the existence of a global attractor in $\mathcal{S}_0^2(\Omega) \cap L^{2p-2}(\Omega)$ for a semilinear strongly degenerate parabolic equation in a bounded domain with the homogeneous Dirichlet boundary condition, in which the nonlinearity satisfies a polynomial type condition of arbitrary order and the external force belongs to $L^2(\Omega)$. This global attractor is then shown to have a finite fractal dimension in $L^2(\Omega)$. We also study the existence and exponential stability of the unique stationary solution to the problem.

Keywords Strongly degenerate · Global attractor · Asymptotic a priori estimate method · Fractal dimension · Stationary solution

Mathematics Subject Classification (2010) 35B41 · 35D30 · 35K65

1 Introduction

In this paper, we consider the following semilinear strongly degenerate parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - P_{\alpha,\beta}u + f(u) = g(X), \ X = (x, y, z) \in \Omega, t > 0, \\ u(X, t) = 0, \ X \in \partial\Omega, t > 0, \\ u(X, 0) = u_0(X), \ X \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$ with smooth boundary $\partial\Omega$, $P_{\alpha,\beta}$ is a strongly degenerate operator of the form

$$P_{\alpha,\beta}u = \Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u, \ \alpha, \beta \geq 0,$$

✉ Mai Xuan Thao
mxthao7@gmail.com

[1] Department of Natural Sciences, Hong Duc University,
307 Le Lai, Dong Son, Thanh Hoa City, Vietnam

$u_0 \in L^2(\Omega)$ is given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that for all $s \in \mathbb{R}$,

$$C_1|s|^p - C_0 \le f(s)s \le C_2|s|^p + C_0, \tag{2}$$
$$f'(s) \ge -C_3 \tag{3}$$

for some $p \ge 2$, where C_0, C_1, C_2, C_3 are positive constants;

(G) $g \in L^2(\Omega)$.

Under the above conditions, following the approach used in [2], Thuy and Tri [15] proved the existence and uniqueness of weak solutions to problem (1), and they also proved the existence of a compact global attractor in $L^2(\Omega)$ for the continuous semigroup $S(t)$ generated by weak solutions to (1). This result was then improved in [13] by showing the existence of a global attractor in a more regular space, namely in the space $\mathcal{S}_0^1(\Omega) \cap L^p(\Omega)$. For other results on the existence and long-time behavior of solutions to semilinear parabolic equations involving this strongly degenerate operator, we refer the reader to some recent works of Anh and Tuyet [3, 4], and Anh [1]. In this paper, we will continue studying some properties, namely the regularity and fractal dimension estimates, of the global attractor obtained in [15].

The first aim of the present paper is to prove the existence of a global attractor in the space $\mathcal{S}_0^2(\Omega) \cap L^{2p-2}(\Omega)$ for the semigroup $S(t)$, that is, we study the regularity of the global attractor obtained in [15]. As is known, the existence of a global attractor in $L^2(\Omega)$ is obtained by showing the existence of a bounded absorbing set in $\mathcal{S}_0^1(\Omega) \cap L^p(\Omega)$ and using the compactness of the embedding $\mathcal{S}_0^1(\Omega) \hookrightarrow L^2(\Omega)$. However, when proving the existence of global attractors in $L^{2p-2}(\Omega)$ and $\mathcal{S}_0^2(\Omega) \cap L^{2p-2}(\Omega)$, we cannot use embedding results because under the conditions of the problem, the solutions only belong to the space $\mathcal{S}_0^2(\Omega) \cap L^{2p-2}(\Omega)$. To overcome this difficulty, we exploit the asymptotic a priori estimate method introduced in [8, 16]. The regularity result obtained here seems to be optimal because under the considered conditions, the stationary solutions, which belong to the attractor, in general cannot belong to a function space smaller than $\mathcal{S}_0^2(\Omega) \cap L^{2p-2}(\Omega)$. In particular, this result improves the previous results in [13, 15]

The second aim of the paper is to show that the global attractor has a finite fractal dimension in $L^2(\Omega)$. To do this, we will use the method introduced by Ladyzhenskaya.

The third aim of the paper is to study the existence and stability of weak stationary solutions to problem (1). In particular, we will show that if $\lambda_1 > C_3$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $-P_{\alpha,\beta}$, then the global attractor has a very simple form $\mathcal{A} = \{u^*\}$, where u^* is the unique weak stationary solution of problem (1).

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some concepts and results on function spaces and global attractors which we will use. In Section 3, we prove the existence of global attractors in various spaces by using the asymptotic a priori estimate method. The fractal dimension of the global attractor is estimated in Section 4. In the last section, we prove the existence, uniqueness, and exponential stability of a weak stationary solution to problem (1). It is worthy noticing that, in particular, the regularity and dimension estimate results obtained in this paper extend and improve some existing ones in [13, 15] and corresponding results for non-degenerate semilinear parabolic equations in [5, 9, 10, 12, 16].

2 Preliminaries

2.1 Function Spaces and Operator

To study problem (1), we use the weighted Sobolev space $\mathcal{S}_0^1(\Omega)$ defined as the completion $C_0^\infty(\Omega)$ in the norm

$$\|u\|^2_{\mathcal{S}_0^1(\Omega)} := \int_\Omega \left(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2\right) dX.$$

This is a Hilbert space with respect to the following scalar product

$$(u, v)_{\mathcal{S}_0^1(\Omega)} = \int_\Omega \left(\nabla_x \cdot u \nabla_x v + \nabla_y u \cdot \nabla_y v + |x|^{2\alpha}|y|^{2\beta}\nabla_z u \cdot \nabla_z v\right) dX.$$

We also use the space $\mathcal{S}_0^2(\Omega)$ defined as the completion $C_0^\infty(\Omega)$ in the norm

$$\|u\|^2_{\mathcal{S}_0^2(\Omega)} := \int_\Omega |P_{\alpha,\beta} u|^2 dX.$$

We recall some embedding results in [14], see also [7] for more general results related to the function space $\mathcal{S}_0^1(\Omega)$.

Proposition 1 *Assume that Ω is a bounded domain in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$. Then the following embeddings hold:*

(i) $\mathcal{S}_0^1(\Omega) \hookrightarrow L^{2^*_{\alpha,\beta}}(\Omega)$ *continuously;*
(ii) $\mathcal{S}_0^1(\Omega) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in [1, 2^*_{\alpha,\beta})$,

where $2^*_{\alpha,\beta} = \frac{2N_{\alpha,\beta}}{N_{\alpha,\beta}-2}$, $N_{\alpha,\beta} = N_1 + N_2 + (\alpha + \beta + 1)N_3$.

The following result follows directly from the definitions of the spaces $\mathcal{S}_0^1(\Omega)$, $\mathcal{S}_0^2(\Omega)$ and the compactness of the embedding $\mathcal{S}_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

Proposition 2 [3] *Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$. Then the embedding $\mathcal{S}_0^2(\Omega) \hookrightarrow \mathcal{S}_0^1(\Omega)$ is compact.*

2.2 Global Attractors

We now recall some results in [16] which will be used in Section 3.

Proposition 3 *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^r(\Omega)$ and suppose that $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^r(\Omega)$. Then for any $\varepsilon > 0$ and any bounded subset $B \subset L^r(\Omega)$, there exist two positive constants $T = T(B)$ and $M = M(\varepsilon)$ such that*

$$\text{meas}(\Omega(|S(t)u_0| \geq M)) \leq \varepsilon,$$

for all $u_0 \in B$ and $t \geq T$, where $\text{meas}(e)$ *denotes the Lebesgue measure of $e \subset \Omega$ and* $\Omega(|S(t)u_0| \geq M) := \{x \in \Omega |\ |(S(t)(u_0))(x)| \geq M\}$.

Definition 1 Let X be a Banach space. The semigroup $\{S(t)\}_{t\geq 0}$ on X is called norm-to-weak continuous on X if for any $\{x_n\}_{n=1}^\infty \in X$, $x_n \to x$ and $t_n \geq 0, t_n \to t$, we have $S(t_n)x_n \rightharpoonup S(t)x$ in X.

The following result is useful for verifying that a semigroup is norm-to-weak continuous.

Proposition 4 *Let X, Y be two Banach spaces and let X^*, Y^* be their respective dual spaces. We also assume that X is a dense subspace of Y, the injection $i : X \to Y$ is continuous and its adjoint $i^* : Y^* \to X^*$ is densely injective. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X and Y, respectively, and assume furthermore that $S(t)$ is continuous or weak continuous on Y. Then $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on X if and only if $S(t)$ maps compact subsets of X into bounded subsets of X.*

To prove the existence of a global attractor in the space $L^{2p-2}(\Omega)$ for the semigroup generated by (1), we will use the following result.

Theorem 1 *Let $\{S(t)\}_{t\geq 0}$ be a norm-to-weak continuous semigroup on $L^q(\Omega)$, and be continuous or weak continuous on $L^r(\Omega)$ for some $r \leq q$, and have a global attractor in $L^r(\Omega)$. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in $L^q(\Omega)$ if and only if*

(i) *$\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^q(\Omega)$;*
(ii) *for any $\varepsilon > 0$ and any bounded subset B of $L^q(\Omega)$, there exist positive constants $M = M(\varepsilon, B)$ and $T = T(\varepsilon, B)$ such that*

$$\int_{\Omega(|S(t)u_0|\geq M)} |S(t)u_0|^q dX \leq \varepsilon,$$

for any $u_0 \in B$ and $t \geq T$.

Definition 2 The semigroup $\{S(t)\}_{t\geq 0}$ on X is called satisfying condition (C) in X if and only if for any bounded set B of X and for any $\varepsilon > 0$, there exist a positive constant t_B and a finite dimensional subspace X_1 of X, such that $\{PS(t)x|x \in B, t \geq t_B\}$ is bounded and

$$|(I - P)S(t)x| \leq \varepsilon \text{ for any } t \geq t_B \text{ and } x \in B,$$

where $P : X \to X_1$ is the canonical projector.

The following result will be used to prove the existence of a global attractor for the semigroup generated by problem (1) in the space $\mathcal{S}_0^2(\Omega)$.

Theorem 2 *Let X be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a norm-to-weak continuous semigroup on X. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in X provided that the following conditions hold:*

(i) *$\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in X*
(ii) *$\{S(t)\}_{t\geq 0}$ satisfies Condition (C) in X*

2.3 Fractal Dimensions of Global Attractors

Definition 3 Let M be a compact set in a metric space X. Then its fractal dimension is defined by

$$\dim_f M = \overline{\lim_{\varepsilon\to 0}} \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $n(M, \varepsilon)$ is the minimal number of closed balls the radius ε which cover the set M.

The following result was given in [6].

Theorem 3 *Assume that M is a compact set in a Hilbert space H. Let V be a continuous mapping in H such that $M \subset V(M)$. Assume that there exists a finite dimensional projector P in the space H such that*

$$\|P(Vu_1 - Vu_2)\|_H \leq l\|u_1 - u_2\|_H, \quad u_1, u_2 \in M,$$
$$\|(I - P)(Vu_1 - Vu_2)\|_H \leq \delta\|u_1 - u_2\|_H, \quad u_1, u_2 \in M,$$

where $\delta < 1$. We also assume that $l \geq 1 - \delta$. Then the compact set M possesses a finite fractal dimension, specifically,

$$\dim_f(M) \leq \dim P. \ln \frac{9l}{1-\delta}\left(\ln \frac{2}{1+\delta}\right)^{-1}.$$

3 Regularity of the Global Attractor

It is proved in [15] that problem (1) generates a continuous (nonlinear) semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ defined as follows

$$S(t)u_0 := u(t),$$

where $u(t)$ is the unique weak solution of the problem (1) with the initial datum u_0, and moreover, $S(t)$ has a compact global attractor $\mathcal{A}$ in $L^2(\Omega)$. We now prove that the global attractor $\mathcal{A}$ is in fact in $\mathcal{S}_0^2(\Omega) \cap L^{2p-2}(\Omega)$.

In the proof of the following lemmas, for the shake of brevity, we give some formal calculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [10].

3.1 Existence of a Global Attractor in $L^{2p-2}(\Omega)$

Lemma 1 *Assume that* **(F)** *and* **(G)** *hold. Then for any bounded subset B in $L^2(\Omega)$, there exists a positive constant $T = T(B)$ such that*

$$\|u_t(s)\|^2_{L^2(\Omega)} \leq \rho_1 \text{ for any } u_0 \in B \text{ and } s \geq T,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$ and ρ_1 is a positive constant independent of B.

Proof By differentiating (1) in time and denoting $v = u_t$, we get

$$v_t - P_{\alpha,\beta}v + f'(u)v = 0.$$

Multiplying the above equality by v, integrating over Ω and using (F), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)} + \|v\|^2_{\mathcal{S}_0^1(\Omega)} \leq C_3\|v\|^2_{L^2(\Omega)}. \tag{4}$$

Hence

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \leq 2C_3\|v\|^2_{L^2(\Omega)}. \tag{5}$$

It is proved in [15] the existence of a bounded absorbing set in $\mathcal{S}_0^1(\Omega) \cap L^p(\Omega)$, that is, there exist a constant R and a time $t_0(\|u_0\|_{L^2(\Omega)})$ such that

$$\|u(t)\|^2_{\mathcal{S}_0^1(\Omega)} + \|u(t)\|^p_{L^p(\Omega)} \leq R \text{ for all } t \geq t_0\left(\|u_0\|_{L^2(\Omega)}\right). \tag{6}$$

Taking the inner product of (1) with u_t, we obtain

$$\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{\mathcal{S}^1_0(\Omega)} + 2\int_\Omega F(u)dX\right) = \int_\Omega g u_t dX \le \frac{1}{2}\|g\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2(\Omega)}, \tag{7}$$

where $F(u) = \int_0^u f(\xi)d\xi$, thus

$$\|u_t\|^2_{L^2(\Omega)} + \frac{d}{dt}\left(\|u\|^2_{\mathcal{S}^1_0(\Omega)} + 2\int_\Omega F(u)dX\right) \le \|g\|^2_{L^2(\Omega)}. \tag{8}$$

Noting that from **(F)** we get

$$C_4(|u|^p - 1) \le F(u) \le C_5(|u|^p + 1). \tag{9}$$

Integrating (7) from t to $t+1$ and then using (9), we get

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} \le \|g\|^2_{L^2(\Omega)} + 2C_5|\Omega| + \|u(t)\|^2_{\mathcal{S}^1_0(\Omega)} + 2C_5\|u(t)\|^p_{L^p(\Omega)}.$$

By (6), there exist a constant C_6 which depends on $\|g\|_{L^2(\Omega)}$, C_4, C_5 and R such that

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} \le C_6, \quad \text{for all } t \ge t_0(\|u_0\|_{L^2(\Omega)}). \tag{10}$$

Combining (5) with (10), and using the uniform Gronwall inequality, we deduce that

$$\|u_t\|^2_{L^2(\Omega)} \le C\left(\|g\|^2_{L^2(\Omega)}, |\Omega|\right)$$

as t large enough. The proof is complete. □

Lemma 2 *The semigroup $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $L^{2p-2}(\Omega)$, i.e., there exists a positive constant ρ_{2p-2}, such that for any bounded subset $B \subset L^2(\Omega)$, there is a number $T = T(B \ge 0)$ such that*

$$\|u(t)\|_{L^{2p-2}(\Omega)} \le \rho_{2p-2} \ \textit{for any } t \ge T, u_0 \in B.$$

Proof Taking $|u|^{p-2}u$ as a test function, we obtain

$$\int_\Omega |u|^{p-2}u.u_t dX \ + \ \int_\Omega \left(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2|u|^{p-2}\right) dX + \int_\Omega f(u)|u|^{p-2}u dX = \int_\Omega g|u|^{p-2}u dX.$$

Hence, using (2) and the Cauchy inequality, we obtain

$$\int_\Omega \left(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2|u|^{p-2}dX\right) + C_1\int_\Omega |u|^{2p-2}dX \le C_0\int_\Omega |u|^{p-1}dX + \frac{1}{C_1}\int_\Omega |g|^2 dX + \frac{C_1}{2}\int_\Omega |u|^{2p-2}dX + \frac{1}{C_1}\int_\Omega |u_t|^2 dX.$$

Using the Cauchy inequality once again, we arrive at

$$\frac{C_1}{4}\int_\Omega |u|^{2p-2}dX \le \frac{1}{C_1}\|g\|^2_{L^2(\Omega)} + \frac{1}{C_1}\int_\Omega |u_t|^2 dX + C.$$

By Lemma 1, we have

$$\int_\Omega |u|^{2p-2}dX \le \rho_{2p-2}, \quad \text{for any } t \ge T, u_0 \in B,$$

where ρ_{2p-2} depends only on $C_0, C_1, C_2, \|g\|_{L^2(\Omega)}$. □

Choosing $Y = L^2(\Omega)$, $X = L^{2p-2}(\Omega)$, by Proposition 4, we see that the semigroup $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on $L^{2p-2}(\Omega)$. Thus, by Theorem 1, to prove the existence of a global attractor in $L^{2p-2}(\Omega)$ for the semigroup $S(t)$, we only need to prove the following

Lemma 3 *For any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist positive constants $M = M(B, \varepsilon)$ and $T = T(B, \varepsilon)$ such that*

$$\int_{\Omega(|u(t)|\geq M)} |u(t)|^{2p-2} dX \leq C\varepsilon \quad \text{for any } u_0 \in B \text{ as } t \geq T,$$

where the constant C is independent of B and ε.

Proof For any fixed $\varepsilon > 0$, by Proposition 3 and **(F)**, there exist $M_1 = M_1(B, \varepsilon) > 0$ and $T_1 = T_1(B, \varepsilon) > 0$, such that the following estimates are valid for any $u_0 \in B$ and $t \geq T_1$:

$$\int_{\Omega(|u(t)|\geq M_1)} |g|^2 dX < \varepsilon \text{ and meas}(\Omega|u(t)| \geq M_1) < \varepsilon,$$
$$\int_{\Omega(|u(t)|\geq M_1)} |u_t(s)|^2 dX < C\varepsilon \quad \text{for } s \geq T_1, \tag{11}$$

and $f(s) \geq 0$ for any $s \geq M_1$, $f(s) \leq 0$ for any $s \leq -M_1$. Denote $\Omega_{M_1} = \Omega(u(t) \geq M_1)$ and $\Omega_{2M_1} = \Omega(u(t) \geq 2M_1)$. Multiplying (1) by $(u - M_1)_+^{p-2}(u - M_1)_+$, where

$$(u - M_1)_+ = \begin{cases} u - M_1, & u \geq M_1 \\ 0, & u \leq M_1. \end{cases}$$

We have

$$\int_{\Omega_{M_1}} (u - M_1)_+^{p-1} u_t dX$$
$$+(p-1)\int_{\Omega_{M_1}} (u - M_1)_+^{p-2}(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2) dX$$
$$+\int_{\Omega_{M_1}} f(u)(u - M_1)_+^{p-1} dX \leq \int_{\Omega_{M_1}} |g|^2 dX \int_{\Omega_{M_1}} (u - M_1)_+^{2p-2} dX.$$

Using (11), we have

$$\int_{\Omega_{M_1}} f(u)(u - M_1)^{p-1} dX \leq C\varepsilon.$$

Therefore, we have

$$\begin{aligned}\int_{\Omega_{2M_1}} f(u)u^{p-1}\frac{1}{2^{p-1}} dX &\leq \int_{\Omega_{2M_1}} f(u)u^{p-2\left(1-\frac{M_1}{u}\right)^{p-1}} dX \\ &\leq \int_{\Omega_{M_1}} f(u)(u - M_1)^{p-1} dX \leq C\varepsilon.\end{aligned}$$

Noting that $\text{meas}(\Omega_{2M_1}) \leq \varepsilon$ and **(F)**, the above inequality implies that

$$\int_{\Omega_{2M_1}} u^{2p-2} dX \leq C\varepsilon \quad \text{as } t \geq T_1. \tag{12}$$

Now taking $|(u+M_1)_-|^{p-2}(u+M_1)_-$ as a test function, where

$$(u+M_1)_- = \begin{cases} u+M_1, & u \geq -M_1 \\ 0, & u \leq -M_1, \end{cases}$$

we have in the same fashion as above that

$$\int_{\Omega(u(t)\leq -2M_1)} |u(t)|^{2p-2} dX \leq C\varepsilon \quad \text{as } t \geq T_1. \tag{13}$$

Combining (12) and (13), we have

$$\int_{\Omega(|u(t)|\geq 2M_1)} |u(t)|^{2p-2} dX \leq C\varepsilon \quad \text{for any } u_0 \in B, t \geq T_1.$$

This completes the proof. □

Theorem 4 *Under the condition* **(F)**, **(G)**, *the semigroup* $\{S(t)\}_{t\geq 0}$ *generated by problem* (1) *has a global attractor* $\mathcal{A}_{L^{2p-2}}$ *in* $L^{2p-2}(\Omega)$, *that is,* $\mathcal{A}_{L^{2p-2}}$ *is compact, invariant in* $L^{2p-2}(\Omega)$ *and attracts every bounded set of* $L^2(\Omega)$ *in the topology of* $L^{2p-2}(\Omega)$.

3.2 Existence of a Global Attractor in $\mathcal{S}_0^2(\Omega)$

Lemma 4 *For any* $2 \leq r < \infty$ *and any bounded subset* $B \subset L^2(\Omega)$, *there exists a positive constant T, which depends on r and the* L^2*-norm of B, such that*

$$\int_\Omega |u_t(s)|^r dX \leq M \quad \text{for any } u_0 \in B, s \geq T,$$

where the positive constant M depends on r but not on B, and $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$.

Proof We prove by induction on $k (k = 0, 1, 2, ...)$ the existence of T_k, depending on k and B, such that

$$\int_\Omega |u_t(s)|^{2\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^k} dX \leq M_k \text{ for any } u_0 \in B, s \geq T_k, \tag{14}$$

and

$$\int_t^{t+1} \left(\int_\Omega |u_t(s)|^{2\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}} dX \right)^{\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}} ds \leq M_k \quad \text{for any } u_0 \in B, s \geq T_k, \tag{15}$$

where M_k depends on k but not on B.

(i) Initialization of the induction ($k = 0$): The estimate (A_0) has been proved in Lemma 1, while B_0 can be derived by integrating (4) from t to $t+1$ and using the embedding $\mathcal{S}_0^1(\Omega) \hookrightarrow L^{\frac{2N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}}(\Omega)$.

(ii) The induction argument: Assume that (A_k) and (B_k) hold for k, and we prove that they are true for $k+1$. By differentiating (1) in time and denoting $v = u_t$, we have

$$v_t - P_{\alpha,\beta} v + f'(u)v = 0. \tag{16}$$

Multiplying (16) by$|v|^{2\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}-2}.v$ and integrating over Ω, we obtain

$$\begin{aligned}
&C\frac{d}{dt}\int_\Omega |v|^{2\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}dX\\
&+C\int_\Omega \left(|\nabla_x\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)|^2+|\nabla_y\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)|^2\right.\\
&\left.+|x|^{2\alpha}|y|^{2\beta}|\nabla_z\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)|^2\right)dX\\
\leq\ & C_3\int_\Omega |v|^{2\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}dX,
\end{aligned} \tag{17}$$

where the constant C depends on the spatial dimension $N_{\alpha,\beta}$ and k. Using (B_k) and the uniform Gronwall inequality, we infer from (17) that

$$\int_\Omega |v|^{2\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}dX \leq M_{k+1} \text{ for any } t \geq T_k, \tag{18}$$

which shows that (A_k+1) is true. For (B_k+1), we integrate (17) from t to $t+1$ and use (18) to get

$$\begin{aligned}
&\int_t^{t+1}\int_\Omega \left(\left|\nabla_x\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)\right|^2+\left|\nabla_y\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)\right|^2\right.\\
&\qquad\left.+|x|^{2\alpha}|y|^{2\beta}\left|\nabla_z\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)\right|^2\right)dXds \leq M_{k+1}.
\end{aligned} \tag{19}$$

Using the embedding $\mathcal{S}_0^1(\Omega)\hookrightarrow L^{\frac{2N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}}(\Omega)$, we have

$$\begin{aligned}
&\left(\int_\Omega |v|^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}\frac{2N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}}dX\right)^{\frac{N_{\alpha,\beta}-2+\epsilon}{N_{\alpha,\beta}}} = \|v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\|^2_{L^{\frac{2N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}}(\Omega)}\\
&\leq C\int_\Omega \left(|\nabla_x\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)|^2+|\nabla_y\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)|^2\right.\\
&\quad\left.+|x|^{2\alpha}|y|^{2\beta}|\nabla_z\left(v^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^{k+1}}\right)|^2\right).
\end{aligned} \tag{20}$$

Combining (19) and (20), we deduce (B_{k+1}) immediately. Since $\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon} > 1$, we have $r \leq 2\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}\right)^k$ provided that $k \geq \log_{\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-2+\epsilon}}\frac{r}{2}$. □

Lemma 5 *For any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist $T > 0$ and $n_\varepsilon \in N$, such that*

$$\int_\Omega |v_2|^2 dX \leq C\varepsilon \quad \textit{for any } u_0 \in B,$$

provided that $t \geq T$ *and* $m \geq n_\varepsilon$*, where* $v_2 = (I - P_m)v = (I - P_m)u_t$ *and the constant* C *is independent of* B *and* ε*.*

Proof Multiplying (16) by v_2 and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|v_2\|^2_{L^2(\Omega)} + \|v_2\|^2_{\mathcal{S}^1_0(\Omega)} \leq \int_\Omega |f'(u)v||v_2|dX.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|v_2\|^2_{L^2(\Omega)} + \lambda_m\|v_2\|^2_{L^2(\Omega)} \leq \int_\Omega |f'(u)v||v_2|dX, \tag{21}$$

where λ_m is the m^{th} eigenvalue of the operator $Au := -P_{\alpha,\beta}u$ in Ω. By (F), Lemmas 3 and 4, we have

$$\int_\Omega |f'(u)v|^2dX \leq \left(\int_\Omega |f'(u)|^{2\left(\frac{p-1}{p-2}\right)}\right)^{\frac{p-2}{p-1}}\left(\int_\Omega |v|^{2(p-1)}\right)^{\frac{1}{p-1}} \leq M_0$$

for any $u_0 \in B$ provided that $t \geq T$, where the constant M_0 is independent of B and the constant T depends only on B and p. Therefore, we infer from (21) that

$$\frac{d}{dt}\|v_2\|^2_{L^2(\Omega)} + \lambda_m\|v_2\|^2_{L^2(\Omega)} \leq C.$$

If $t \geq T$, the last inequality shows that

$$\|v_2(t)\|^2_{L^2(\Omega)} \leq \|v_2(T)\|^2_{L^2(\Omega)}e^{-\lambda_m(t-T)} + \frac{C}{\lambda_m}(1 - e^{-\lambda_m(t-T)}). \tag{22}$$

□

Lemma 6 *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $\mathcal{S}^2_0(\Omega)$*, i.e., there exists a constant* $\rho_A > 0$ *such that for any bounded subset* $B \subset L^2(\Omega)$*, there is a* $T_B > 0$ *such that*

$$\|P_{\alpha,\beta}u(t)\|_{L^2(\Omega)} \leq \rho_A \ \text{ for any } t \geq T_B, u_0 \in B.$$

Proof Taking the L^2-inner product of (1) with $-P_{\alpha,\beta}u$, we obtain

$$\|P_{\alpha,\beta}u\|^2_{L^2(\Omega)} \leq \int_\Omega u_t.P_{\alpha,\beta}udX$$
$$+\int_\Omega f'(u)(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2)dX - \int_\Omega g.P_{\alpha,\beta}udX.$$

By the Hölder inequality and assumption **(F)** we have

$$\|P_{\alpha,\beta}u\|^2_{L^2(\Omega)} \leq C\left(\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{\mathcal{S}^1_0(\Omega)} + \|g\|^2_{L^2(\Omega)}\right).$$

Hence, from Lemma 1 and the fact that $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $\mathcal{S}^1_0(\Omega)$ we have

$$\|P_{\alpha,\beta}u(t)\|_{L^2(\Omega)} \leq \rho_A$$

for t large enough. This completes the proof. □

Let $\mathcal{K}(A)$ be the Kuratowski measure of noncompactness in $L^2(\Omega)$ of the subset A defined by

$$\mathcal{K}(A) = \inf\{\delta > 0 \mid A \text{ has a finite open cover of sets of diameter} < \delta\}.$$

We have the following lemma in [16].

Lemma 7 *Assume that $f(.)$ satisfies conditions* **(F)**. *Then for any subset $A \subset L^{2p-2}(\Omega)$, if $\mathcal{K}(A) < \varepsilon$ in $L^{2p-2}(\Omega)$, then we have*

$$\mathcal{K}(f(A)) < C\varepsilon \text{ in } L^2(\Omega),$$

where $f(A) = \{f(u)|u \in A\}$ and the constant C depends on the L^{2p-2}-norm of A, the Lebesgue measure of Ω and the coefficients C_0, C_1, C_2 in **(F)**.

Let $H_m = \text{span}\{e_1, e_2, \ldots, e_m\}$ in $L^2(\Omega)$, where $\{e_j\}_{j=1}^{\infty}$ are eigenvectors of the operator $Au = -P_{\alpha,\beta}u$ with the homogeneous Dirichlet boundary condition in Ω and $P_m : L^2(\Omega) \to H_m$ be the orthogonal projection. We now verify that $\{S(t)\}_{t\geq 0}$ satisfies condition (C) in $\mathcal{S}_0^2(\Omega)$.

Lemma 8 *For any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist $T = T(\varepsilon, B) \geq 0$ and $n_\varepsilon \in \mathbb{N}$, such that*

$$\int_\Omega |(I - P_m)P_{\alpha,\beta}u|^2 dX \leq \varepsilon \text{ for any } u_0 \in B,$$

provided that $t \geq T$ and $m \geq n_\varepsilon$.

Proof Denoting $u_2 = (I - P_m)u$, and multiplying (1) by $-P_{\alpha,\beta}u_2$, we have

$$\begin{aligned}&\int_\Omega |(I - P_m)P_{\alpha,\beta}u|^2 dX \\ &\leq \int_\Omega u_t P_{\alpha,\beta}u_2 dX + \int_\Omega f(u)P_{\alpha,\beta}u_2 dX - \int_\Omega g(X)P_{\alpha,\beta}u_2 dX.\end{aligned}$$

By the Cauchy inequality, we have

$$\begin{aligned}&\int_\Omega |(I - P_m)P_{\alpha,\beta}u|^2 dX \\ &\leq \frac{1}{2}\int_\Omega |(I - P_m)u_t|^2 dX + \int_\Omega |f(u)|^2 dX + \frac{1}{2}\int_\Omega |(I - P_m)g|^2 dX.\end{aligned}$$

From Lemmas 5 and 7, we have

$$\int_\Omega |(I - P_m)P_{\alpha,\beta}u|^2 dX \leq \varepsilon \quad \text{for any } u_0 \in B, t \geq T, m \geq n_\varepsilon.$$

This completes the proof. □

From Lemmas 6, 8, and Theorem 2, we obtain the following result.

Theorem 5 *Assume* **(F)** *and* **(G)** *hold. Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by problem* (1) *has a global attractor $\mathcal{A}_{S_0^2}$ in $\mathcal{S}_0^2(\Omega)$, that is, $\mathcal{A}_{S_0^2}$ is compact, invariant in $\mathcal{S}_0^2(\Omega)$ and attracts every bounded set of $L^2(\Omega)$ in the topology of $\mathcal{S}_0^2(\Omega)$.*

4 Fractal Dimension Estimates of the Global Attractor

Lemma 9 *Assume that* **(F)** *and* **(G)** *hold. Then there exist a positive integer N_0, a time T^* and a positive constant $\delta < 1$, such that for any $u_0, v_0 \in \mathcal{A}$, we have*

$$\|(I - P_{N_0})(S(T^*)u_0 - S(T^*)v_0)\|_{L^2(\Omega)}^2 \leq \delta\|u_0 - v_0\|_{L^2(\Omega)}^2,$$

and there exists a constant $l > 1-\delta$ such that

$$\|P_{N_0}(S(T^*)u_0 - S(T^*)v_0)\|^2_{L^2(\Omega)} \le l\|u_0 - v_0\|^2_{L^2(\Omega)}.$$

Proof Let u, v be two solutions of problem (1) with initial data u_0, v_0, respectively. Putting $w(t) = u(t) - v(t)$, $Q_{N_0} = I - P_{N_0}$, $w_1 = P_{N_0}w$, $w_2 = Q_{N_0}w$. Obviously, the function $w(t)$ satisfies the equation

$$\frac{dw}{dt} - P_{\alpha,\beta}w + l(t)w = 0, \tag{23}$$

where $l(t) = \int_0^1 f'(su(t) + (1-s)v(t))ds$. Multiplying (23) by w_2, we have

$$\frac{1}{2}\frac{d}{dt}\|w_2\|^2_{L^2(\Omega)} + \|w_2\|^2_{\mathcal{S}^1_0(\Omega)} + \int_\Omega l(t)|w_2|^2 dX = 0.$$

Using the facts that

$$\|w_2\|^2_{\mathcal{S}^1_0(\Omega)} \ge \lambda_{N_0}\|w_2\|^2_{L^2(\Omega)},\ l(t) \ge -C_3,$$

we have

$$\frac{d}{dt}\|w_2\|^2_{L^2(\Omega)} \le 2(C_3 - \lambda_{N_0})\|w_2\|^2_{L^2(\Omega)}.$$

Hence, by the Gronwall inequality, we obtain

$$\|w_2(t)\|^2_{L^2(\Omega)} \le e^{-2(\lambda_{N_0}-C_3)t}\|w_2(0)\|^2_{L^2(\Omega)} \le e^{-2(\lambda_{N_0}-C_3)t}\|w(0)\|^2_{L^2(\Omega)}.$$

Since $\lambda_m \to +\infty$ as $m \to +\infty$, we can choose N_0 large enough such that for a fixed T^*,

$$\|(I - P_{N_0})(S(T^*)u_0 - S(T^*)v_0)\|^2_{L^2(\Omega)} \le \delta\|u_0 - v_0\|^2_{L^2(\Omega)},$$

where $0 < \delta := e^{-2(\lambda_{N_0}-C_3)T^*} < 1$.

On the other hand, multiplying (23) by w_1, we get

$$\frac{1}{2}\frac{d}{dt}\|w_1\|^2_{L^2(\Omega)} + \|w_1\|^2_{\mathcal{S}^1_0(\Omega)} \le C_3\|w_1\|^2_{L^2(\Omega)}.$$

Hence it follows that

$$\frac{d}{dt}\|w_1\|^2_{L^2(\Omega)} \le 2(C_3 - \lambda_1)\|w_1\|^2_{L^2(\Omega)}.$$

Applying the Gronwall inequality, we have

$$\|w_1(t)\|^2_{L^2(\Omega)} \le e^{2(C_3-\lambda_1)t}\|w_1(0)\|^2_{L^2(\Omega)} \le e^{2(C_3-\lambda_1)t}\|w(0)\|^2_{L^2(\Omega)}.$$

By choosing $T^* > 0$ (small enough if $C_3 < \lambda_1$) such that $l := e^{2(C_3-\lambda_1)T^*} > 1-\delta$, we have

$$\|P_{N_0}(S(T^*)u_0 - S(T^*)v_0)\|^2_{L^2(\Omega)} \le l\|u_0 - v_0\|^2_{L^2(\Omega)}.$$

This completes the proof. □

From Lemma 9 and Theorem 3, we get the following

Theorem 6 *Assume that* **(F)** *and* **(G)** *hold. Then the global attractor $\mathcal{A}$ has a finite fractal dimension in $L^2(\Omega)$, specifically,*

$$\dim_f(\mathcal{A}) \le N_0 \ln\frac{9l}{1-\delta}\left(\ln\frac{2}{1+\delta}\right)^{-1},$$

where N_0, δ, l are given in Lemma 9.

5 Existence and Exponential Stability of Stationary Solutions

A weak stationary solution to problem (1) is an element $u^* \in \mathcal{S}_0^1(\Omega) \cap L^p(\Omega)$ such that

$$\int_\Omega (\nabla_x u^* \cdot \nabla_x v + \nabla_y u^* \cdot \nabla_y v + |x|^{2\alpha}|y|^{2\beta}\nabla_z u^* \cdot \nabla_z v)dX + \int_\Omega f(u^*)vdX = \int_\Omega gvdX \quad (24)$$

for all test functions $v \in \mathcal{S}_0^1(\Omega) \cap L^p(\Omega)$.

Theorem 7 *Assume that conditions* **(F)** $-$ **(G)** *hold. Then problem* (1) *admits at least one weak stationary solution* u^* *satisfying*

$$||u^*||^2_{\mathcal{S}_0^1(\Omega)} + 2C_1\|u^*\|^p_{L^p(\Omega)} \le 2C_0|\Omega| + \frac{1}{\lambda_1}\|g\|^2_{L^2(\Omega)}. \quad (25)$$

Moreover, if the following condition holds

$$\lambda_1 > C_3, \quad (26)$$

where $\lambda_1 > 0$ *is the first eigenvalue of the operator* $-P_{\alpha,\beta}$*,* C_3 *is the constant in* (3)*, then the weak stationary solution* u^* *of* (1) *is unique and exponentially stable.*

Proof *(i) Existence.* The estimate (25) can be obtained taking into account that in particular any weak stationary solution u^*, if it exists, should verify

$$\|u^*\|^2_{\mathcal{S}_0^1(\Omega)} + \int_\Omega f(u^*)u^* dX = \int_\Omega g u^* dX.$$

Using (2) and the Cauchy inequality, we have

$$||u^*||^2_{\mathcal{S}_0^1(\Omega)} + C_1\|u^*\|^p_{L^p(\Omega)} - C_0|\Omega| \le \frac{\lambda_1}{2}\|u^*\|^2_{L^2(\Omega)} + \frac{1}{2\lambda_1}\|g\|^2_{L^2(\Omega)}.$$

Hence, by using the inequality $\|u^*\|^2_{\mathcal{S}_0^1(\Omega)} \ge \lambda_1\|u^*\|^2_{L^2(\Omega)}$, we obtain the desired estimate (25).

For the existence, let $\{v_j\}_{j=1}^\infty$ be a basis of $\mathcal{S}_0^1(\Omega) \cap L^p(\Omega)$. For each $m \ge 1$, let us denote $V_m = \text{span}\{v_1, ..., v_m\}$ and we would like to define an approximate strong stationary solutions u^m of (1) by

$$u^m = \sum_{i=1}^m \gamma_{mi} v_i,$$

such that

$$\int_\Omega (\nabla_x u^m \cdot \nabla_x v + \nabla_y u^m \cdot \nabla_y v + |x|^{2\alpha}|y|^{2\beta}\nabla_z u^m \cdot \nabla_z v)dX + \int_\Omega f(u^m)vdX$$
$$= \int_\Omega gvdX \quad (27)$$

for all $v \in V_m$. To prove the existence of u^m, we define operators $R_m : V_m \to V_m$ by

$$((R_m u, v)) = \int_\Omega (\nabla_x u \cdot \nabla_x v + \nabla_y u \cdot \nabla_y v + |x|^{2\alpha}|y|^{2\beta}\nabla_z u \cdot \nabla_z v)dX$$
$$+ \int_\Omega f(u)vdX - \int_\Omega gvdX \quad \forall u, v \in V_m.$$

For all $u \in V_m$, using (2) and the Cauchy inequality, we have

$$\begin{aligned}((R_m u, u)) &= \|u\|^2_{\mathcal{S}^1_0(\Omega)} + \int_\Omega f(u)u dX - \int_\Omega gu dX \\ &\geq ||u||^2_{\mathcal{S}^1_0(\Omega)} + C_1\|u\|^p_{L^p(\Omega)} - C_0|\Omega| - \frac{\lambda_1}{2}\|u\|^2_{L^2(\Omega)} - \frac{1}{2\lambda_1}\|g\|^2_{L^2(\Omega)} \\ &\geq \frac{1}{2}||u||^2_{\mathcal{S}^1_0(\Omega)} - C_0|\Omega| - \frac{1}{2\lambda_1}\|g\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus, if we take

$$\beta = \left(2C_0|\Omega| + \frac{1}{\lambda_1}\|g\|^2_{L^2(\Omega)}\right)^{1/2},$$

we obtain that $((R_m u, u)) \geq 0$ for all $u \in V_m$ satisfying $||u||_{\mathcal{S}^1_0(\Omega)} = \beta$. Consequently, by a corollary of the Brouwer fixed point theorem (see [11, Chapter 2, Lemma 1.4]), for each $m \geq 1$, there exists $u_m \in V_m$ such that $R_m(u_m) = 0$, with $\|u_m\| \leq \beta$. Taking $v = u^m$ in (27) we get

$$\|u^m\|^2_{\mathcal{S}^1_0(\Omega)} + C_1\|u^m\|^p_{L^p(\Omega)} - C_0|\Omega| \leq \frac{\lambda_1}{2}\|u^*\|^2_{L^2(\Omega)} + \frac{1}{2\lambda_1}\|g\|^2_{L^2(\Omega)}.$$

Hence we deduce that

$$||u^m||^2_{\mathcal{S}^1_0(\Omega)} + \|u^m\|^p_{L^p(\Omega)} \leq C\left(1 + \|g\|^2_{L^2(\Omega)}\right). \tag{28}$$

Hence we deduce that the sequence $\{u^m\}$ is bounded in $\mathcal{S}^1_0(\Omega) \cap L^p(\Omega)$, and consequently, by the compact injection of $\mathcal{S}^1_0(\Omega)$ into $L^2(\Omega)$, we can extract a subsequence $\{u^{m'}\} \subset \{u^m\}$ that converges weakly in $\mathcal{S}^1_0(\Omega)\cap L^p(\Omega)$ and strongly in $L^2(\Omega)$ to an element $u^* \in \mathcal{S}^1_0(\Omega)\cap L^p(\Omega)$. It is now standard to take limits in (27) and obtain that u^* is a weak stationary solution of (1).

(ii) Uniqueness and exponential stability. Denote $w(t) = u(t) - u^*$, one has

$$\int_\Omega w_t v dX + \int_\Omega (\nabla_x w \cdot \nabla_x v + \nabla_y w \cdot \nabla_y v + |x|^{2\alpha}|y|^{2\beta}\nabla_z w \cdot \nabla_z v) dX + \int_\Omega (f(u) - f(u^*)) v dX = 0$$

for all test functions $v \in \mathcal{S}^1_0(\Omega) \cap L^p(\Omega)$. In particular, replacing v by $w(t)$, we have

$$\frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{\mathcal{S}^1_0(\Omega)} + \int_\Omega (f(u) - f(u^*))(u - u^*) dX = 0.$$

Using condition (3) we have

$$\|w(t)\|^2_{L^2(\Omega)} + 2(\lambda_1 - C_3)\|w(t)\|^2_{L^2(\Omega)} \leq 0.$$

By the Gronwall inequality, we arrive at

$$\|w(t)\|^2_{L^2(\Omega)} \leq \|w(0)\|^2_{L^2(\Omega)} e^{-2(\lambda_1 - C_3)t}.$$

Hence if condition (26) holds, then we get the desired conclusion. □

Remark 1 Since every stationary solution, if it exists, must lie on the global attractor, from the regularity results of the global attractor in Section 3, we deduce that the stationary solution u^* belongs to the space $\mathcal{S}^2_0(\Omega) \cap L^{2p-2}(\Omega)$, that is, we get a regularity result of the stationary solution.

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References

1. Anh, C.T.: Global attractor for a semilinear strongly degenerate parabolic equation on $\mathbb{R}^N$. Nonlinear Differ. Equ. Appl. **21**, 663–678 (2014)
2. Anh, C.T., Ke, T.D.: Existence and continuity of global attractor for a semilinear degenerate parabolic equation. Electron. J. Differ. Equ. **61**, 1–13 (2009)
3. Anh, C.T., Tuyet, L.T.: Strong solutions to a strongly degenerate semilinear parabolic equation. Vietnam J. Math. **41**, 217–232 (2013)
4. Anh, C.T., Tuyet, L.T.: On a semilinear strongly degenerate parabolic equation in an unbounded domain. J. Math. Sci. Univ. Tokyo **20**, 91–113 (2013)
5. Chepyzhov, V.V., Vishik, M.I.: Attractors for Equations of Mathematical Physics. Am. Math. Soc. Colloq. Publ. Vol. 49, Am. Math. Soc., Providence, RI (2002)
6. Chueshov, I.D.: Introduction to the Theory of Infinite-Dimensional Dissipative System. ACTA Scientific Publishing House (1999)
7. Kogoj, A.E., Lanconelli, E.: On semilinear Δ_λ-Laplace equation. Nonlinear Anal. **75**, 4637–4649 (2012)
8. Ma, Q.F., Wang, S.H., Zhong, C.K.: Necessary and sufficient conditions for the existence of global attractor for semigroups and applications. Indiana University Math. J **51**, 1541–1559 (2002)
9. Marion, M.: Attractors for reaction-diffusion equations: existence and estimate of their dimension. Appl. Anal. **25**, 101–147 (1987)
10. Robinson, J.C.: Introduction to Infinite-Dimensional Dynamical Systems. Cambridge University Press, Cambridge (2001)
11. Temam, R.: Navier-Stokes Equations: Theory and Numerical Analysis, 2nd edition. North-Holland, Amsterdam (1979)
12. Temam, R.: Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd edition. Springer-Verlag (1997)
13. Thao, M.X., Tuyet, L.T.: Global attractors for a semilinear strongly degenerate parabolic equations. J. Sciences Vinh Univ. **42**, 78–85 (2013)
14. Thuy, P.T., Tri, N.M.: Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations. Nonlinear Differ. Equ. Appl. **19**, 279–298 (2012)
15. Thuy, P.T., Tri, N.M.: Long-time behavior of solutions to semilinear parabolic equations involving strongly degenerate elliptic differential operators. Nonlinear Differ. Equ. Appl. **20**, 1213–1224 (2013)
16. Zhong, C.K., Yang, M.H., Sun, C.Y.: The existence of global attractors for the norm-to-weak continuous semigoup and application to the nonlinear reaction-diffusion equations. J. Differ. Equ. **15**, 367–399 (2006)