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Uniqueness of Meromorphic Functions Whose Certain Differential Polynomials Share a Small Function

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Abstract In this paper, we study the uniqueness of meromorphic functions whose certain nonlinear differential polynomials share a small function with finite weight. Our result generalizes and improves the recent results due to A. Banerjee and the present first author Sarajevo (Sarajevo J. Math. **8**(20), 69–89, (2012)).

Keywords Meromorphic function · Differential polynomials · Weighted sharing · Small function

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1 Introduction, Definitions, and Results

In this paper, by meromorphic functions, we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7, 16, 17]. Let E denote any set of positive real numbers of finite linear measure not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}(r \to \infty, r \notin E)$. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$ and by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ $(r \to \infty, r \notin E)$. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function with respect to f provided that $T(r, a) = S(r, f)$.

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Let f and g be two nonconstant meromorphic functions. We say that f and g share the value a CM (counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Throughout this paper, we need the following definition

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1999, Lahiri [8] studied the uniqueness problems of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper, regarding the nonlinear differential polynomials, Lahiri asked the following question: *What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?*

Afterwards research works concerning the above question have been done by many mathematicians and continuous efforts are being put in to relax the hypothesis of the results (see [1, 3, 6, 8, 12, 13, 15]).

In 1997, Yang and Hua [15] proved the following result.

Theorem A *Let f and g be two nonconstant meromorphic functions, $n\geq 11$ an integer, and $a\in\mathbb{C}-\{0\}$. If f^nf' and g^ng' share the value a CM, then either $f=tg$ for some $(n+1)$th root of unity 1 or $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2=-a^2$.*

In 2004, Lin and Yi [13] proved the following results.

Theorem B *Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty,f)>2/(n+1)$, $n\geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f\equiv g$.*

Theorem C *Let f and g be two nonconstant meromorphic functions, $n\geq 13$ an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share the value 1 CM, then $f\equiv g$.*

A new trend in this direction is to consider the uniqueness of a meromorphic function concerning the value sharing of the kth derivatives of a linear expression of a meromorphic function. In 2010, Dyavanal [4] considers the uniqueness problem of meromorphic function related to the value sharing of two nonlinear differential polynomials. The author proved two theorems for the value sharing of differential functions in which the multiplicities of zeros and poles of f and g are taken into account. The ideas resorted by the author in [4] are no doubt novel, but there are some mistakes in the paper. For example, on page 7, in the proof of Theorem 1.2 [4], there is a serious lacuna when a counting function is being elaborated and then restricted in terms of Nevanlinna's characteristic function (for details, please see page 3 of [2]). In 2011, the author has rectified the paper and proved the following theorems.

Theorem D [5] *Let f and g be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s\geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $g=\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f=\frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, where h is a nonconstant meromorphic function.*

Theorem E [5] *Let f and g be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f \equiv g$.*

A recent increment to uniqueness theory is to consider weighted sharing instead of sharing IM or CM; this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing, which measures how close a shared value is to being shared CM or being shared IM, has been introduced by Lahiri around 2000.

Definition 1.1 [9] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g *share* (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also, we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If α is a small function of f and g, then f, g share α with weight k means that $f-\alpha$, $g-\alpha$ share the value 0 with weight k.

In 2012, using the notion of weighted value sharing, Banerjee and Sahoo [2] proved the following theorems which improve Theorems D and E.

Theorem F *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer and $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Let $[f^n(a_1 f + a_2)]^{(k)}$ and $[g^n(a_1 g + a_2)]^{(k)}$ share (b, l) where $k(\geq 1)$, $l(\geq 0)$ are integers, a_1, a_2, b are nonzero constants and one of the following conditions holds:*

(a) $l \geq 2$ *and* $n > \max\{(3k+8)/s + 1, 3 + 2/s\}$
(b) $l = 1$ *and* $n > \max\{(4k+9)/s + 3/2, 3 + 2/s\}$
(c) $l = 0$ *and* $n > \max\{(9k+14)/s + 4, 3 + 2/s\}$

Then, either $[f^n(a_1 f + a_2)]^{(k)}[g^n(a_1 g + a_2)]^{(k)} \equiv b^2$ or $f \equiv g$.

The possibility $[f^n(a_1 f + a_2)]^{(k)}[g^n(a_1 g + a_2)]^{(k)} \equiv b^2$ does not arise for $k = 1$.

Theorem G *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $[f^n(a_1 f^2 + a_2 f + a_3)]^{(k)}$ and $[g^n(a_1 g^2 + a_2 g + a_3)]^{(k)}$ share (b, l) where $k(\geq 1)$, $l(\geq 0)$ are integers, a_1, a_2, a_3, b are nonzero constants and one of the following conditions holds:*

(a) $l \geq 2$ *and* $n > \max\{(3k+8)/s + 2, 4 + 4/s\}$
(b) $l = 1$ *and* $n > \max\{(4k+9)/s + 3, 4 + 4/s\}$
(c) $l = 0$ *and* $n > \max\{(9k+14)/s + 8, 4 + 4/s\}$

Then, either $[f^n(a_1f^2+a_2f+a_3)]^{(k)}[g^n(a_1g^2+a_2g+a_3)]^{(k)}\equiv b^2$ *or* $f\equiv g$ *or* f, g *satisfies the algebraic equation* $R(f,g)=0$, *where*

$$R(x,y)=x^n(a_1x^2+a_2x+a_3)-y^n(a_1y^2+a_2y+a_3).$$

The possibility $[f^n(a_1f^2+a_2f+a_3)]^{(k)}[g^n(a_1g^2+a_2g+a_3)]^{(k)}\equiv b^2$ *does not arise when* $k=1$.

This paper is motivated by the following questions.

Question 1.1 Can one deduce a generalized result in which Theorems F and G will be included?

Question 1.2 What can be said if the sharing value b in Theorems F and G is replaced by a small function of f and g?

We will concentrate our attention to the above questions and provide an affirmative answer in this direction. We now state our main result.

Theorem 1.1 *Let* n *be a positive integer. Let* f *and* g *be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer and* $\Theta(\infty,f)+\Theta(\infty,g)>4/n$. *Let* $P(z)=a_mz^m+\cdots+a_1z+a_0$, *where* m *is a positive integer and* $a_0(\neq 0), a_1,\ldots,a_m(\neq 0)$ *are complex constants. Suppose that* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share* (α,l) *where* $k(\geq 1)$, $l(\geq 0)$ *are integers and* $\alpha(\not\equiv 0,\infty)$ *is a small function of* f *and* g *and one of the following conditions holds:*

(a) $l\geq 2$ *and* $n>\max\{(3k+8)/s+m, m+2+2m/s\}$
(b) $l=1$ *and* $n>\max\{(4k+9)/s+3m/2, m+2+2m/s\}$
(c) $l=0$ *and* $n>\max\{(9k+14)/s+4m, m+2+2m/s\}$

Then, either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv\alpha^2$ *or* $f=tg$ *for a constant* t *such that* $t^d=1$, *where* $d=\gcd\{n+m,\ldots,n+m-i,\ldots,n+1,n\}$, $a_{m-i}\neq 0$ *for some* $i=0,1,\ldots,m$ *or* f *and* g *satisfy the algebraic equation* $R(f,g)=0$, *where*

$$R(f,g)=f^nP(f)-g^nP(g).$$

The possibility $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv\alpha^2$ *does not occur for* $k=1$.

Remark 1.1 Taking $m=1$ and $m=2$ in Theorem 1.1, it reduces to Theorems F and G, respectively. Since Theorems F and G are the special cases of Theorem 1.1, it improves and generalizes Theorems F and G.

Though the standard definitions and notations of the value distribution theory are available in [7], we give the following definitions and notations used in this paper.

Definition 1.2 [10] Let $a\in\mathbb{C}\cup\{\infty\}$. Denote by $N(r,a;f\mid=1)$ the counting function of simple a points of f. For a positive integer p, denote by $N(r,a;f\mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. Denote by $\overline{N}(r,a;f\mid\leq p)$ the corresponding reduced counting function.

Analogously, we can define $N(r,a;f\mid\geq p)$ and $\overline{N}(r,a;f\mid\geq p)$.

Definition 1.3 [9] Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then,

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq k).$$

Clearly, $N_1(r, a; f) = \overline{N}(r, a; f)$.

2 Lemmas

Let F and G be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1 [14] *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \cdots + a_1 f(z) + a_0,$$

where $a_n (\neq 0), a_{n-1}, \ldots, a_1, a_0$ are complex numbers. Then,

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

Lemma 2.2 [18] *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then,*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.1}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 2.3 *(see*[17, p. 38]*) Suppose that f is a nonconstant meromorphic function in the complex plane and k is a positive integer. Then,*

$$T(r, f) \leq \overline{N}(r, \infty; f) + N(r, 0; f) + N(r, 1; f^{(k)}) - N(r, 0; f^{(k+1)}) + S(r, f).$$

Lemma 2.4 [9] *Let f and g be two nonconstant meromorphic functions sharing* $(1, 2)$. *Then, one of the following cases holds:*

(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$
(ii) $f = g$
(iii) $fg = 1$

Lemma 2.5 [1] *Let F and G be two nonconstant meromorphic functions sharing* $(1, 1)$ *and* $H \not\equiv 0$. *Then,*

$$\begin{aligned} T(r, F) \leq\ & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ & + \frac{1}{2}\overline{N}(r, 0; F) + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 2.6 [1] *Let F and G be two nonconstant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$. Then,*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G).$$

Lemma 2.7 *Let f and g be two nonconstant meromorphic functions whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $P(z)$ be defined as in Theorem 1.1 and n, m be two positive integers. If $(n+m-2)p > 2m(1+1/s)$ then*

$$[f^nP(f)]'[g^nP(g)]' \not\equiv \alpha,$$

where α is a small function with f and g and p is the number of distinct roots of $P(z)=0$.

Proof Suppose on the contrary that

$$[f^nP(f)]'[g^nP(g)]' = \alpha.$$

Then,

$$f^{n-1}Q(f)f'g^{n-1}Q(g)g' = \alpha, \tag{2.3}$$

where $Q(z) = b_mz^m + b_{m-1}z^{m-1} + \cdots + b_1z + b_0$, where $b_j = (n+j)a_j$, $j = 0,1,\ldots,m$. We write $Q(z)$ as

$$Q(z) = b_m(z-d_1)^{l_1}(z-d_2)^{l_2}\ldots(z-d_i)^{l_i}\ldots(z-d_p)^{l_p},$$

where $\sum_{i=1}^{p} l_i = m$, $1 \leq p \leq m$; $d_i \neq d_j$, $i \neq j$, $1 \leq i,j \leq p$; d_i's are nonzero constants and l_i's are positive integers, $i = 1,2,\ldots,p$. Let z_0 $(\alpha(z_0) \neq 0,\infty)$ be a zero of f with multiplicity $p_0(\geq s)$. Then, z_0 is a pole of g with multiplicity $q_0(\geq s)$, say. From (2.3), we obtain

$$np_0 - 1 = (n+m)q_0 + 1,$$

i.e.,

$$mq_0 + 2 = n(p_0 - q_0). \tag{2.4}$$

From (2.4), we get $q_0 \geq \frac{n-2}{m}$ and so we have $p_0 \geq \frac{n+m-2}{m}$. Let z_1 $(\alpha(z_1) \neq 0,\infty)$ be a zero of $Q(f)$ with multiplicity p_1 and be a zero of $f - d_i$ of order q_i for $i = 1,2,\ldots,p$. Then, $p_1 = l_iq_i$ for some $i = 1,2,\ldots,p$. Hence, z_1 is a pole of g with multiplicity q, say. So from (2.3), we get

$$q_il_i + q_i - 1 = (n+m)q + 1 \geq (n+m)s + 1$$

i.e., $q_i \geq \frac{(n+m)s+2}{l_i+1}$ for $i = 1,2,\ldots,p$.

Suppose that z_2 $(\alpha(z_2) \neq 0,\infty)$ is a pole of f. Then, from (2.3), z_2 is either a zero of $g^{n-1}Q(g)$ or a zero of g'. Therefore,

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \sum_{i=1}^{p}\overline{N}(r,d_i;g) + \overline{N}_0(r,0;g') + \overline{N}(r,\infty;\alpha) + \overline{N}(r,0;\alpha) \\ &\quad + S(r,f) + S(r,g) \\ &\leq \left(\frac{m}{n+m-2} + \frac{m+p}{(n+m)s+2}\right)T(r,g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $gQ(g)$. We have a similar inequality for g.

Using the second fundamental theorem of Nevanlinna, we obtain

$$\begin{aligned} pT(r,f) &\le \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\sum_{i=1}^{p}\overline{N}(r,d_i;f)-\overline{N}_0(r,0;f')+S(r,f)\\ &\le \left(\frac{m}{n+m-2}+\frac{m+p}{(n+m)s+2}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;g')\\ &\quad -\overline{N}_0(r,0;f')+S(r,f)+S(r,g). \end{aligned} \tag{2.5}$$

Similarly,

$$\begin{aligned} pT(r,g) &\le \left(\frac{m}{n+m-2}+\frac{m+p}{(n+m)s+2}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;f')\\ &\quad -\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \end{aligned} \tag{2.6}$$

Adding (2.5) and (2.6), we obtain

$$\left(p-\frac{2m}{n+m-2}-\frac{2(m+p)}{(n+m)s+2}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

which contradicts the fact that $(n+m-2)p>2m(1+1/s)$. This proves the lemma. □

Note 2.1 If $P(z)=0$ has only one root of multiplicity m, then the above lemma holds for $n>m+2+2m/s$. If all the roots of $P(z)=0$ are distinct, then the lemma holds for $n>4-m+2/s$.

Lemma 2.8 *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer and let n, k be two positive integers. Let $P(z)$ be defined as in Theorem 1.1. Suppose that $F=\frac{(f^nP(f))^{(k)}}{\alpha}$ and $G=\frac{(g^nP(g))^{(k)}}{\alpha}$ where a, b are any two nonzero finite complex constants and $\alpha(\not\equiv 0,\infty)$ is a small function of f and g. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n\le(3k+3)/s+m$.*

Proof By the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,c_1;F)+S(r,F)\\ &\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F). \end{aligned} \tag{2.7}$$

By (2.1), (2.2), (2.7), and Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned} (n+m)T(r,f) &\le T(r,F)-\overline{N}(r,0;F)+N_{k+1}(r,0;f^nP(f))+S(r,f)\\ &\le \overline{N}(r,0;G)+N_{k+1}(r,0;f^nP(f))+\overline{N}(r,\infty;F)+S(r,f)\\ &\le N_{k+1}(r,0;f^nP(f))+N_{k+1}(r,0;g^nP(g))+\overline{N}(r,\infty;f)\\ &\quad +k\overline{N}(r,\infty;g)+\overline{N}(r,\infty;\alpha)+\overline{N}(r,0;\alpha)+S(r,f)+S(r,g)\\ &\le \left(\frac{k+2}{s}+m\right)T(r,f)+\left(\frac{2k+1}{s}+m\right)T(r,g)\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{2.8}$$

Similarly, we have

$$(n+m)T(r,g) \le \left(\frac{k+2}{s}+m\right)T(r,g)+\left(\frac{2k+1}{s}+m\right)T(r,f)$$
$$+S(r,f)+S(r,g). \quad (2.9)$$

Hence, from (2.8) and (2.9), we get

$$\left(n-\frac{3k+3}{s}-m\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which gives $n \le (3k+3)/s+m$. This completes the proof of the lemma. □

The following lemma can be carried out in line with the proof of Lemma 6 in [11].

Lemma 2.9 *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f)+\Theta(\infty,g) > \frac{4}{n},$$

where $n(\ge 3)$ is an integer. Then,

$$f^n(af+b)=g^n(ag+b)$$

implies $f=g$, where a, b are any two nonzero finite complex constants.

3 Proof of the Theorem

Proof of Theorem 1.1 Let F and G be defined as in Lemma 2.8. Then, F, G are transcendental meromorphic functions that share $(1,l)$. Thus, from (2.1), we obtain

$$\begin{aligned}N_2(r,0;F) &\le N_2\left(r,0;(f^nP(f))^{(k)}\right)+S(r,f)\\ &\le T\left(r,(f^nP(f))^{(k)}\right)-(n+m)T(r,f)+N_{k+2}(r,0;f^nP(f))+S(r,f)\\ &\le T(r,F)-(n+m)T(r,f)+N_{k+2}(r,0;f^nP(f))+S(r,f). \quad (3.1)\end{aligned}$$

Again, by (2.2), we have

$$N_2(r,0;F) \le k\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+S(r,f). \quad (3.2)$$

From (3.1), we get

$$(n+m)T(r,f) \le T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f). \quad (3.3)$$

We now discuss the following three cases separately.

Case 1. Let $l \geq 2$. Suppose that (i) of Lemma 2.4 holds. Then, using (3.2), we deduce from (3.3) that

$$\begin{aligned}(n+m)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+N_{k+2}(r,0;f^nP(f))\\&\quad+S(r,f)+S(r,g)\\&\leq N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))+2\overline{N}(r,\infty;f)\\&\quad+(k+2)\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\&\leq \left(\frac{k+2}{s}+m\right)\{T(r,f)+T(r,g)\}+2\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)\\&\quad+S(r,f)+S(r,g)\\&\leq \left(\frac{k+4}{s}+m\right)T(r,f)+\left(\frac{2k+4}{s}+m\right)T(r,g)+S(r,f)+S(r,g)\\&\leq \left(\frac{3k+8}{s}+2m\right)T(r)+S(r).\end{aligned}\tag{3.4}$$

Similarly,

$$(n+m)T(r,g) \leq \left(\frac{3k+8}{s}+2m\right)T(r)+S(r).\tag{3.5}$$

From (3.4) and (3.5), we obtain

$$(n-\frac{3k+8}{s}-m)T(r)\leq S(r),$$

contradicting the fact that $n > \max\{(3k+8)/s+m, m+2+2m/s\}$. So by Lemma 2.4, either $FG=1$ or $F=G$. Let $FG=1$. Then,

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}=\alpha^2,$$

a contradiction for $k=1$ by Lemma 2.7. So we have $F=G$. That is,

$$[f^nP(f)]^{(k)}=[g^nP(g)]^{(k)}.$$

Integrating both sides, we get

$$[f^nP(f)]^{(k-1)} = [g^nP(g)]^{(k-1)}+c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1}\neq 0$, from Lemma 2.8, we obtain $n\leq \frac{3k}{s}+m$, a contradiction. Hence, $c_{k-1}=0$. Repeating k times, we obtain

$$f^nP(f)=g^nP(g).\tag{3.6}$$

If $m=1$ then by Lemma 2.9, we have $f=g$. Suppose that $m\geq 2$ and $h=\frac{f}{g}$. If h is a constant, by putting $f=gh$ in (3.6), we get

$$\begin{aligned}a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+\cdots\\+a_1g^{n+1}(h^{n+1}-1)+a_0g^n(h^n-1)=0,\end{aligned}$$

which implies $h^d=1$, where $d=(n+m,\ldots,n+m-i,\ldots,n+1,n)$, for some $i=0,1,\ldots,m$. Thus, $f=tg$ for a constant t such that $t^d=1$, $d=(n+m,\ldots,n+m-i,\ldots,n+1,n)$, for some $i=0,1,\ldots,m$.

If h is not a constant, then from (3.6), we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(f,g)=f^nP(f)-g^nP(g).$$

Case 2. Let $l = 1$ and $H \not\equiv 0$. Using Lemma 2.5 and (3.2), from (3.3), we obtain

$$\begin{aligned}(n+m)T(r,f) &\le N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F)\\ &\quad + \frac{1}{2}\overline{N}(r,\infty;F) + N_{k+2}(r,0;f^nP(f)) + S(r,f) + S(r,g)\\ &\le N_{k+2}(r,0;f^nP(f)) + N_{k+2}(r,0;g^nP(g)) + \frac{1}{2}N_{k+1}(r,0;f^nP(f))\\ &\quad + \frac{k+5}{2}\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\ &\le \left(\frac{2k+5}{s} + \frac{3m}{2}\right)T(r,f) + \left(\frac{2k+4}{s} + m\right)T(r,g)\\ &\quad + S(r,f) + S(r,g)\\ &\le \left(\frac{4k+9}{s} + \frac{5m}{2}\right)T(r) + S(r). \end{aligned} \tag{3.7}$$

Similarly,

$$(n+m)T(r,g) \le \left(\frac{4k+9}{s} + \frac{5m}{2}\right)T(r) + S(r). \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$[n - \frac{4k+9}{s} - \frac{3m}{2}]T(r) \le S(r),$$

a contradiction since $n > \max\{(4k+9)/s + 3m/2, m + 2 + 2m/s\}$. We now assume that $H = 0$. That is,

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.9}$$

where $A(\neq 0)$ and B are constants. From (3.9), it is clear that F, G share the value 1 CM and so they share $(1, 2)$. Hence, $n > \max\{(3k+8)/s + m, m + 2 + 2m/s\}$. Now, we discuss the following three subcases.

Subcase (i). Let $B \neq 0$ and $A = B$. Then, from (3.9), we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.10}$$

If $B = -1$, then from (3.10), we obtain $FG = 1$, a contradiction for $k = 1$ by Lemma 2.7.

If $B \neq -1$, from (3.10), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now, from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}T(r,G) &\le \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G)\\ &\le \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G).\end{aligned}$$

Using (2.1) and (2.2), from the above inequality, we obtain

$$T(r,G) \le N_{k+1}(r,0;f^nP(f)) + k\overline{N}(r,\infty;f) + T(r,G) + N_{k+1}(r,0;g^nP(g)) \\ -(n+m)T(r,g) + \overline{N}(r,\infty;g) + S(r,g).$$

Hence,

$$(n+m)T(r,g) \le \left(\frac{2k+1}{s}+m\right)T(r,f) + \left(\frac{k+2}{s}+m\right)T(r,g) + S(r,g).$$

Thus, we obtain

$$\left(n - \frac{3k+3}{s} - m\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

a contradiction as $n > \max\{(3k+8)/s+m, m+2+2m/s\}$.

Subcase (ii). Let $B \ne 0$ and $A \ne B$. Then, from (3.9), we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r,0;F)$. Proceeding as in subcase (i), we obtain a contradiction.

Subcase (iii). Let $B = 0$ and $A \ne 0$. Then, from (3.9), we get $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \ne 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r,0;G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r,0;F)$. So by Lemma 2.8, we have $n \le \frac{3k+3}{s}+m$, a contradiction. Thus, $A=1$, and hence, $F=G$. Then, the result follows from case 1.

Case 3. Let $l=0$ and $H \not\equiv 0$. Using Lemma 2.6 and (3.2), from (3.3), we get

$$\begin{aligned}(n+m)T(r,f) &\le N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ &\quad +N_{k+2}(r,0;f^nP(f))+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\ &\le N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))+2N_{k+1}(r,0;f^nP(f))\\ &\quad +N_{k+1}(r,0;g^nP(g))+(2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)\\ &\quad +S(r,f)+S(r,g)\\ &\le \left(\frac{5k+8}{s}+3m\right)T(r,f)+\left(\frac{4k+6}{s}+2m\right)T(r,g)+S(r,f)+S(r,g)\\ &\le \left(\frac{9k+14}{s}+5m\right)T(r)+S(r). \end{aligned} \tag{3.11}$$

Similarly,

$$(n+m)T(r,g) \le \left(\frac{9k+14}{s}+5m\right)T(r)+S(r). \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$[n - \frac{9k+14}{s} - 4m]T(r) \le S(r),$$

which contradicts the facts that $n > \max\{(9k+14)/s+4m, m+2+2m/s\}$. So $H=0$, and then proceeding as in case 2, the result follows. This proves the theorem. □

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