Acta Math Vietnam (2015) 40:179–195
DOI 10.1007/s40306-015-0114-3

Graded Annihilators and Uniformly F-Compatible Ideals

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Dedicated to Ngo Viet Trung, on the occasion of his sixtieth birthday

Received: 25 June 2014 / Accepted: 4 September 2014 / Published online: 25 February 2015

Abstract Let R be a commutative (Noetherian) local ring of prime characteristic p that is F-pure. This paper is concerned with comparison of three finite sets of radical ideals of R, one of which is only defined in the case when R is F-finite (that is, is finitely generated when viewed as a module over itself via the Frobenius homomorphism). Two of the aforementioned three sets have links to tight closure, via test ideals. Among the aims of the paper are a proof that two of the sets are equal, and a proposal for a generalization of I. M. Aberbach's and F. Enescu's splitting prime.

Keywords Commutative Noetherian local ring · Prime characteristic · Frobenius homomorphism · Tight closure · Test element · Excellent ring · Frobenius skew polynomial ring · Graded annihilator · F-pure ring · Uniformly F-compatible ideal

Mathematics Subject Classification (2010) Primary 13A35 · 16S36 · 13E05 · 13F40 · Secondary 13J10

1 Introduction

Throughout the paper, let $(R, \mathfrak{m})$ be a commutative (Noetherian) local ring of prime characteristic p having maximal ideal $\mathfrak{m}$. In recent years, the study of R-modules with a Frobenius action has assisted in the development of the theory of tight closure over R. An R-module with a Frobenius action can be viewed as a left module over the Frobenius skew polynomial ring over R, and such left modules will play a central role in this paper.

The Frobenius skew polynomial ring over R is described as follows. Throughout, $f : R \longrightarrow R$ denotes the Frobenius ring homomorphism, for which $f(r) = r^p$ for all $r \in R$.

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The *Frobenius skew polynomial ring over* R is the skew polynomial ring $R[x, f]$ associated to R and f in the indeterminate x; as a left R-module, $R[x, f]$ is freely generated by $(x^i)_{i\geq 0}$, and so consists of all polynomials $\sum_{i=0}^{n} r_i x^i$, where $n \geq 0$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule $xr = f(r)x = r^p x$ for all $r \in R$.

We can think of $R[x, f]$ as a positively-graded ring $R[x, f] = \bigoplus_{n=0}^{\infty} R[x, f]_n$, where $R[x, f]_n = Rx^n$ for $n \geq 0$. The *graded annihilator* of a left $R[x, f]$-module H is the largest graded two-sided ideal of $R[x, f]$ that annihilates H; it is denoted by $\text{gr-ann}_{R[x,f]} H$.

Let G be a left $R[x, f]$-module that is x-torsion-free in the sense that $xg = 0$ for $g \in G$, only when $g = 0$. Then $\text{gr-ann}_{R[x,f]} G = \mathfrak{b}R[x, f]$, where $\mathfrak{b} = (0 :_R G)$ is a radical ideal. See [11, Lemma 1.9]. We shall use $\mathcal{I}(G)$ (or $\mathcal{I}_R(G)$) to denote the set of R-annihilators of the $R[x, f]$-submodules of G; we shall refer to the members of $\mathcal{I}(G)$ as the *G-special R-ideals*. For a graded two-sided ideal $\mathfrak{B}$ of $R[x, f]$, we denote by $\text{ann}_G(\mathfrak{B})$ or $\text{ann}_G \mathfrak{B}$ the $R[x, f]$-submodule of G consisting of all elements of G that are annihilated by $\mathfrak{B}$. Also, we shall use $\mathcal{A}(G)$ to denote the set of *special annihilator submodules of* G, that is, the set of $R[x, f]$-submodules of G of the form $\text{ann}_G(\mathfrak{A})$, where $\mathfrak{A}$ is a graded two-sided ideal of $R[x, f]$. In [11, §1], the present author showed that there is a sort of 'Galois' correspondence between $\mathcal{I}(G)$ and $\mathcal{A}(G)$. In more detail, there is an order-reversing bijection, $\Delta : \mathcal{A}(G) \longrightarrow \mathcal{I}(G)$ given by

$$\Delta : N \longmapsto \left(\text{gr-ann}_{R[x,f]} N\right) \cap R = (0 :_R N).$$

The inverse bijection, $\Delta^{-1} : \mathcal{I}(G) \longrightarrow \mathcal{A}(G)$, also order-reversing, is given by

$$\Delta^{-1} : \mathfrak{b} \longmapsto \text{ann}_G\, (\mathfrak{b}R[x, f])) \,.$$

We shall be mainly concerned in this paper with the situation where R is F-pure. We remind the reader what this means. For $j \in \mathbb{N}$ (the set of positive integers) and an R-module M, let $M^{(j)}$ denote M considered as a left R-module in the natural way and as a right R-module via f^j, the jth iterate of the Frobenius ring homomorphism. Then R is *F-pure* if, for every R-module M, the natural map $M \longrightarrow R^{(1)} \otimes_R M$ (which maps $m \in M$ to $1 \otimes m$) is injective.

Note that $R^{(j)} \cong Rx^j$ as (R, R)-bimodules. Let $i \in \mathbb{N}_0$, the set of non-negative integers. When we endow Rx^i and Rx^j with their natural structures as (R, R)-bimodules (inherited from their being graded components of $R[x, f]$), there is an isomorphism of (left) R-modules $\phi : Rx^{i+j} \otimes_R M \stackrel{\cong}{\longrightarrow} Rx^i \otimes_R (Rx^j \otimes_R M)$ for which $\phi(rx^{i+j} \otimes m) = rx^i \otimes (x^j \otimes m)$ for all $r \in R$ and $m \in M$. It follows that R is F-pure if and only if the left $R[x, f]$-module $R[x, f] \otimes_R M$ is x-torsion-free for every R-module M. This means that, when R is F-pure, there is a good supply of natural x-torsion-free left $R[x, f]$-modules.

In fact, we shall use Φ (or Φ_R when it is desirable to specify which ring is being considered) to denote the functor $R[x, f] \otimes_R \bullet$ from the category of R-modules (and all R-homomorphisms) to the category of all $\mathbb{N}_0$-graded left $R[x, f]$-modules (and all homogeneous $R[x, f]$-homomorphisms). For an R-module M, we shall identify $\Phi(M)$ with $\bigoplus_{n\in\mathbb{N}_0} Rx^n \otimes_R M$, and (usually) identify its 0th component $R \otimes_R M$ with M, in the obvious ways.

Let E be the injective envelope of the simple R-module $R/\mathfrak{m}$. We shall be concerned with $\Phi(E)$, the $\mathbb{N}_0$-graded left $R[x, f]$-module $\bigoplus_{n\in\mathbb{N}_0} Rx^n \otimes_R E$. Assume now that R is F-pure. In [12, Corollary 4.11], the present author proved that the set $\mathcal{I}(\Phi(E))$ is a finite set of radical ideals of R; in [11, Theorem 3.6 and Corollary 3.7], he proved that $\mathcal{I}(\Phi(E))$ is closed under taking primary (prime in this case) components; and in [14, Corollary 2.8], he proved that the big test ideal $\widetilde{\tau}(R)$ of R (for tight closure) is equal to the smallest member of $\mathcal{I}(\Phi(E))$ that meets R°, the complement in R of the union of the minimal prime ideals of R.

Let $\mathfrak{a} \in \mathcal{I}(\Phi(E))$ (with $\mathfrak{a} \neq R$), still in the F-pure case. The special annihilator submodule $\operatorname{ann}_{\Phi(E)}(\mathfrak{a}R[x, f])$ of $\Phi(E)$ corresponding to $\mathfrak{a}$ inherits a natural structure as a graded left module over the Frobenius skew polynomial ring $(R/\mathfrak{a})[x, f]$, and its 0th component is contained in $(0 :_E \mathfrak{a})$. As $R/\mathfrak{a}$-module, the latter is isomorphic to the injective envelope of the simple $R/\mathfrak{a}$-module. Motivated by results in [14, §3] in the case where R is complete, and by work of K. Schwede in [10, §5] in the F-finite case, we say that $\mathfrak{a}$ is *fully* $\Phi(E)$-*special* if (it is $\Phi(E)$-special and) its 0th component is exactly $(0 :_E \mathfrak{a})$. The main result of this paper is that a $\Phi(E)$-special ideal of R is always fully $\Phi(E)$-special provided that R is an (F-pure) homomorphic image of an excellent regular local ring of characteristic p. When R satisfies this condition, corollaries can be drawn from that main result: we shall establish an analogue of [14, Theorem 3.1] and, in particular, show that $R/\mathfrak{a}$ is F-pure whenever $\mathfrak{a}$ is a proper $\Phi(E)$-special ideal of R.

Along the way, we shall show that, in the case where R is F-finite as well as F-pure, the set $\mathcal{I}(\Phi(E))$ of $\Phi(E)$-special ideals of R is equal to the set of *uniformly F-compatible ideals* of R, introduced by K. Schwede in [10, §3]. An ideal $\mathfrak{b}$ of R is said to be *uniformly F-compatible* if, for every $j > 0$ and every $\phi \in \operatorname{Hom}_R(R^{(j)}, R)$, we have $\phi(\mathfrak{b}^{(j)}) \subseteq \mathfrak{b}$. In [10, Corollary 5.3 and Corollary 3.3], Schwede proved that there are only finitely many uniformly F-compatible ideals of R and that they are all radical; in [10, Proposition 4.7 and Corollary 4.8], he proved that the set of uniformly F-compatible ideals is closed under taking primary (prime in this case) components; in [10, Theorem 6.3], Schwede proved that the big test ideal $\widetilde{\tau}(R)$ of R is equal to the smallest uniformly F-compatible ideal of R that meets R°; and in [10, Remark 4.4 and Proposition 4.7], he proved that there is a unique largest proper uniformly F-compatible ideal of R, and that is prime and equal to the splitting prime of R discovered and defined by I. M. Aberbach and F. Enescu [1, §3].

Thus, in the F-finite F-pure case, the set of uniformly F-compatible ideals of R has properties similar to some properties of $\mathcal{I}(\Phi(E))$. Are the two sets the same? We shall, during the course of the paper, show that the answer is 'yes'. It should be emphasized, however, that Schwede only defined uniformly F-compatible ideals in the F-finite case, whereas the majority of this paper is devoted to the study of fully $\Phi(E)$-special ideals in the (F-pure but) not necessarily F-finite case.

We shall use the notation of this Introduction throughout the remainder of the paper. In particular, R will denote a local ring of prime characteristic p having maximal ideal $\mathfrak{m}$. We shall sometimes use the notation $(R, \mathfrak{m})$ just to remind the reader that R is local. The completion of R will be denoted by $\widehat{R}$. We shall only assume that R is reduced, or F-pure, or F-finite, when there is an explicit statement to that effect; also E will continue to denote $E_R(R/\mathfrak{m})$. We continue to use $\mathbb{N}$, respectively $\mathbb{N}_0$, to denote the set of all positive, respectively non-negative, integers.

For $j \in \mathbb{N}_0$, the jth component of an $\mathbb{N}_0$-graded left $R[x, f]$-module G will be denoted by G_j.

2 Fully $\Phi(E)$-Special Ideals

We remind the reader that we usually identify the 0th component of $\Phi(E) = \bigoplus_{n \in \mathbb{N}_0} Rx^n \otimes_R E$ with E in the obvious natural way. For an ideal $\mathfrak{a}$ of R, we have, with this convention, that the 0th component of $\operatorname{ann}_{\Phi(E)}(\mathfrak{a}R[x, f])$ is contained in $(0 :_E \mathfrak{a})$.

Lemma 2.1 *Assume that* $(R, \mathfrak{m})$ *is* F*-pure; let* $\mathfrak{a}$ *be an ideal of* R*. Then the* 0*th component* $(\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0$ *of* $\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f])$ *contains* $(0 :_E \mathfrak{a})$ *if and only if* $\mathfrak{a}$ *is* $\Phi(E)$*-special and* $(\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0 = (0 :_E \mathfrak{a})$.

Proof Only the implication '⇒' needs proof.

Assume that $(0 :_E \mathfrak{a}) \subseteq (\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0$. Since $\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f])$ is an $R[x, f]$-submodule of $\Phi(E)$, it follows that $\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f])$ contains the image J of the map

$$\Phi((0 :_E \mathfrak{a})) = R[x, f] \otimes_R (0 :_E \mathfrak{a}) \longrightarrow R[x, f] \otimes_R E = \Phi(E)$$

induced by inclusion. Let $\mathfrak{b}$ be the radical ideal of R for which $\text{gr-ann}_{R[x,f]} J = \mathfrak{b}R[x, f]$, so that $\mathfrak{b} = (0 :_R J)$. As $J \subseteq \mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f])$, we must have $\mathfrak{a} \subseteq \mathfrak{b}$. Furthermore, $\mathfrak{b}$ annihilates $(0 :_E \mathfrak{a}) \cong \mathrm{Hom}_R(R/\mathfrak{a}, E)$, and since an R-module and its Matlis dual have the same annihilator, we also have $\mathfrak{b} \subseteq \mathfrak{a}$. Thus $\mathfrak{a} = \mathfrak{b}$ is the R-annihilator of an $R[x, f]$-submodule of $\Phi(E)$, and so $\mathfrak{a} \in \mathcal{I}(\Phi(E))$.

Finally, note that an $e \in (\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0$ must be annihilated by $\mathfrak{a}$, and so lies in $(0 :_E \mathfrak{a})$. □

Definition 2.2 Assume that $(R, \mathfrak{m})$ is F-pure; let $\mathfrak{a}$ be an ideal of R. We say that $\mathfrak{a}$ is *fully* $\Phi(E)$*-special* if the equivalent conditions of Lemma 2.1 are satisfied.

Thus $\mathfrak{a}$ is fully $\Phi(E)$-special if and only if $(0 :_E \mathfrak{a}) \subseteq (\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0$, and then $\mathfrak{a}$ is $\Phi(E)$-special and we have the equality $(0 :_E \mathfrak{a}) = (\mathrm{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0$.

To facilitate the presentation of some examples of $\Phi(E)$-special ideals that are fully $\Phi(E)$-special, we review next the theory of S-tight closure, where S is a multiplicatively closed subset of R. This theory was developed in [14]. The special case of the theory in which $S = R^\circ$ is the 'classical' tight closure theory of M. Hochster and C. Huneke [2].

Reminders 2.3 Let H be a left $R[x, f]$-module and let S be a multiplicatively closed subset of R.

(i) We define the *internal* S*-tight closure of zero in* H, denoted by $\Delta^S(H)$, to be the $R[x, f]$-submodule of H given by

$$\Delta^S(H) = \left\{h \in H : \text{ there exists } s \in S \text{ with } sx^n h = 0 \text{ for all } n \gg 0\right\}.$$

When M is an R-module and we take the graded left $R[x, f]$-module $\Phi(M) = R[x, f] \otimes_R M$ for H, the $R[x, f]$-submodule $\Delta^S(\Phi(M))$ of $\Phi(M)$ is graded, and we refer to its 0th component as the *S-tight closure of* 0 *in* M, or the *tight closure with respect to* S *of* 0 *in* M, and denote it by $0^{*,S}_M$. See [14, §1].

(ii) By [14, Example 1.3(ii)], we have, for an R-module M,

$$\Delta^S(R[x, f] \otimes_R M) = 0^{*,S}_M \oplus 0^{*,S}_{Rx\otimes_R M} \oplus \cdots \oplus 0^{*,S}_{Rx^n\otimes_R M} \oplus \cdots .$$

(iii) Recall that an *S-test element for* R is an element $s \in S$ such that, for every R-module M and every $j \in \mathbb{N}_0$, the element sx^j annihilates $1 \otimes m \in (\Phi(M))_0$ for every $m \in 0^{*,S}_M$. The ideal of R generated by all the S-test elements for R is called the *S-test ideal of* R, and denoted by $\tau^S(R)$.

Reminders 2.4 Suppose that $(R, \mathfrak{m})$ is F-pure. Let S be a multiplicatively closed subset of R. Recall that the set $\mathcal{I}(\Phi(E))$ of $\Phi(E)$-special R-ideals is finite; let $\mathfrak{b}^{S,\Phi(E)}$ denote the intersection of all the minimal members of the set

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \cap \mathcal{I}(\Phi(E)) : \mathfrak{p} \cap S \neq \emptyset\}.$$

Thus $\mathfrak{b}^{S,\Phi(E)}$ is the smallest member of $\mathcal{I}(\Phi(E))$ that meets S.

(i) By [14, Theorem 2.6], the set $S \cap \mathfrak{b}^{S,\Phi(E)}$ is (non-empty and) equal to the set of S-test elements for R.
(ii) Thus there exists an S-test element for R.
(iii) Furthermore, $\Delta^S(\Phi(E)) = \operatorname{ann}_{\Phi(E)}(\mathfrak{b}^{S,\Phi(E)} R[x, f])$ and $(0 :_R \Delta^S(\Phi(E))) = \mathfrak{b}^{S,\Phi(E)}$, by [14, Proposition 1.5].
(iv) By [14, Proposition 2.10(v)], we have $\mathfrak{b}^{S,\Phi(E)} = (0 :_R 0_E^{*,S})$.

Lemma 2.5 (Sharp [14, Corollary 2.8]) *Suppose that $(R, \mathfrak{m})$ is F-pure. Let S be the complement in R of the union of finitely many prime ideals.*

Then the S-test ideal $\tau^S(R)$ is equal to $\mathfrak{b}^{S,\Phi(E)}$, the smallest member of the finite set $\mathcal{I}(\Phi(E))$ that meets S.

We shall also use the following result from [14].

Theorem 2.6 (Sharp [14, Theorem 2.12]) *Suppose that $(R, \mathfrak{m})$ is F-pure. Let $\mathfrak{a} \in \mathcal{I}(\Phi(E))$. Then there exists a multiplicatively closed subset S of R such that $\mathfrak{a}$ is the S-test ideal of R. Moreover, S can be taken to be the complement in R of the union of finitely many prime ideals.*

We are now able to give examples of fully $\Phi(E)$-special ideals because the next result shows that, when $(R, \mathfrak{m})$ is complete and F-pure, a $\Phi(E)$-special ideal of R is automatically fully $\Phi(E)$-special.

Proposition 2.7 *Suppose that $(R, \mathfrak{m})$ is complete and F-pure. Then every $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special.*

Proof Let $\mathfrak{a}$ be a $\Phi(E)$-special ideal of R. If $\mathfrak{a} = R$, then

$$(0 :_E \mathfrak{a}) = 0 \subseteq \operatorname{ann}_{\Phi(E)}(\mathfrak{a}R[x, f])$$

and $\mathfrak{a}$ is fully $\Phi(E)$-special. We therefore assume that $\mathfrak{a}$ is proper.

By Theorem 2.6 and [14, Corollary 2.8], there exist finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of R such that, if we set $S := R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$, then $\mathfrak{a}$ is the S-test ideal of R, that is $\mathfrak{a} = \tau^S(R) = \mathfrak{b}^{S,\Phi(E)}$, where the notation is as in 2.3(iii) and 2.4. Therefore, by 2.3(ii) and 2.4(iii),

$$\begin{aligned} 0_E^{*,S} \oplus 0_{Rx \otimes_R E}^{*,S} \oplus \cdots \oplus 0_{Rx^n \otimes_R E}^{*,S} \oplus \cdots &= \Delta^S(\Phi(E)) \\ &= \operatorname{ann}_{\Phi(E)}(\mathfrak{b}^{S,\Phi(E)} R[x, f]). \end{aligned}$$

Now, we know that $\mathfrak{b}^{S,\Phi(E)} = (0 :_R 0_E^{*,S})$, by 2.4(iv). Since R is complete, it follows from Matlis duality (see, for example, [15, p. 154]) that $0_E^{*,S} = (0 :_E \mathfrak{b}^{S,\Phi(E)})$. We have thus shown that $(0 :_E \mathfrak{a}) = R \otimes_R (0 :_E \mathfrak{a}) \subseteq (\operatorname{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0$. Thus $\mathfrak{a}$ is fully $\Phi(E)$-special. □

Next, we develop some theory for fully $\Phi(E)$-special ideals.

Lemma 2.8 *Suppose that $(R, \mathfrak{m})$ is F-pure, and let $\mathfrak{a}$ be a fully $\Phi(E)$-special ideal of R. Then $\mathfrak{a}$ is radical and every associated prime of $\mathfrak{a}$ is also fully $\Phi(E)$-special.*

Proof We can assume that $\mathfrak{a}$ is proper. Since $\mathfrak{a}$ is $\Phi(E)$-special, it must be radical. Let $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ be the minimal primary (prime in this case) decomposition of $\mathfrak{a}$, and let $i \in \{1, \ldots, t\}$.

Since $\mathfrak{a}$ is fully $\Phi(E)$-special, $(0 :_E \mathfrak{a}) \subseteq (\operatorname{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0$. Let $e \in (0 :_E \mathfrak{p}_i)$ and let $r \in \mathfrak{p}_i$. We show that rx^n annihilates the element $1 \otimes e$ of the 0th component of $\Phi(E)$. There exists

$$a \in \bigcap_{\substack{j=1 \\ j \neq i}}^{t} \mathfrak{p}_j \setminus \mathfrak{p}_i.$$

Now $(0 :_E \mathfrak{p}_i) = a(0 :_E \mathfrak{p}_i)$, because multiplication by a provides a monomorphism of $R/\mathfrak{p}_i$ into itself and E is injective. Therefore $e = ae'$ for some $e' \in (0 :_E \mathfrak{p}_i)$. Therefore $rx^n \otimes e = rx^n \otimes ae' = ra^{p^n} x^n \otimes e' = 0$ since $ra^{p^n} \in \mathfrak{a}$ and

$$(0 :_E \mathfrak{p}_i) \subseteq (0 :_E \mathfrak{a}) \subseteq \operatorname{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]).$$

Therefore $(0 :_E \mathfrak{p}_i) \subseteq (\operatorname{ann}_{\Phi(E)}(\mathfrak{p}_i R[x, f]))_0$ and $\mathfrak{p}_i$ is fully $\Phi(E)$-special. □

Proposition 2.9 *Suppose that $(R, \mathfrak{m})$ is F-pure. Let $(\mathfrak{a}_\lambda)_{\lambda \in \Lambda}$ be a non-empty family of fully $\Phi(E)$-special ideals of R. Then $\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is again fully $\Phi(E)$-special.*

Proof Set $\mathfrak{a} := \sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda$, and observe that $\mathfrak{a}R[x, f] = \sum_{\lambda \in \Lambda} (\mathfrak{a}_\lambda R[x, f])$. By assumption, we have $(0 :_E \mathfrak{a}_\lambda) \subseteq \operatorname{ann}_{\Phi(E)}(\mathfrak{a}_\lambda R[x, f])$ for all $\lambda \in \Lambda$. It follows that

$$\begin{aligned}(0 :_E \mathfrak{a}) &= \left(0 :_E \textstyle\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \bigcap_{\lambda \in \Lambda} (0 :_E \mathfrak{a}_\lambda) \\ &\subseteq \bigcap_{\lambda \in \Lambda} (\operatorname{ann}_{\Phi(E)}(\mathfrak{a}_\lambda R[x, f]))_0 \\ &= \left(\operatorname{ann}_{\Phi(E)}\left(\textstyle\sum_{\lambda \in \Lambda} (\mathfrak{a}_\lambda R[x, f])\right)\right)_0 = (\operatorname{ann}_{\Phi(E)}(\mathfrak{a}R[x, f]))_0.\end{aligned}$$

Therefore $\mathfrak{a} := \sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is fully $\Phi(E)$-special. □

Corollary 2.10 *Suppose that $(R, \mathfrak{m})$ is F-pure. Then R has a unique largest fully $\Phi(E)$-special proper ideal, and this is prime.*

Proof The zero ideal is fully $\Phi(E)$-special, and so it follows from Proposition 2.9 that the sum $\mathfrak{b}$ of all the fully $\Phi(E)$-special proper ideals of R is fully $\Phi(E)$-special (and contained in $\mathfrak{m}$), and so is the unique largest fully $\Phi(E)$-special proper ideal of R. Also $\mathfrak{b}$ must be prime, since all the associated primes of $\mathfrak{b}$ are fully $\Phi(E)$-special, by Lemma 2.8. □

In what follows, we shall have cause to pass between R and its completion. Note that if R is F-pure, then so too is $\widehat{R}$, by Hochster and Roberts [3, Corollary 6.13]. The following technical lemma will be helpful.

Lemma 2.11 (See [13, Lemma 4.3]) *There is a unique way of extending the R-module structure on $E := E_R(R/\mathfrak{m})$ to an $\widehat{R}$-module structure. Recall that, as an $\widehat{R}$-module, $E \cong E_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{m}})$.*

Since each element of $\Phi_R(E) = R[x, f] \otimes_R E$ *is annihilated by some power of* $\mathfrak{m}$*, the left* $R[x, f]$*-module structure on* $\Phi_R(E)$ *can be extended in a unique way to a left* $\widehat{R}[x, f]$*-module structure.*

The map $\beta : \Phi_R(E) = R[x, f] \otimes_R E \longrightarrow \widehat{R}[x, f] \otimes_{\widehat{R}} E = \Phi_{\widehat{R}}(E)$ *for which*

$$\beta(rx^i \otimes h) = rx^i \otimes h \quad \text{for all } r \in R,\ i \in \mathbb{N}_0 \text{ and } h \in E$$

is a homogeneous $\widehat{R}[x, f]$*-isomorphism.*

Since each element of $\Phi_R(E)$ *is annihilated by some power of* $\mathfrak{m}$*, it follows that a subset of* $\Phi_R(E)$ *is an* $R[x, f]$*-submodule if and only if it is an* $\widehat{R}[x, f]$*-submodule. Consequently,*

$$\mathcal{I}_R(\Phi_R(E)) = \left\{\mathfrak{B} \cap R : \mathfrak{B} \in \mathcal{I}_{\widehat{R}}(\Phi_{\widehat{R}}(E))\right\}.$$

Lemma 2.12 *Suppose that* $(R, \mathfrak{m})$ *is* F*-pure, and let* $\mathfrak{a}$ *be an ideal of* R*. Then* $\mathfrak{a}\widehat{R}$ *is a fully* $\Phi_{\widehat{R}}(E)$*-special ideal of* $\widehat{R}$ *if and only if* $\mathfrak{a}$ *is a fully* $\Phi_R(E)$*-special ideal of* R*.*

Proof By Lemma 2.11, when we extend the left $R[x, f]$-module structure on $\Phi_R(E)$, in the unique way possible, to a left $\widehat{R}[x, f]$-module structure, $E \cong E_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{m}})$ as $\widehat{R}$-modules and $\Phi_R(E) \cong \Phi_{\widehat{R}}(E)$ as left $\widehat{R}[x, f]$-modules. The claim therefore follows from the facts that

$$\operatorname{ann}_{\Phi_R(E)}(\mathfrak{a}R[x, f]) = \operatorname{ann}_{\Phi_R(E)}((\mathfrak{a}\widehat{R})\widehat{R}[x, f])$$

and $(0 :_E \mathfrak{a}) = (0 :_E \mathfrak{a}\widehat{R})$. □

3 The Case Where R Is an F-Pure Homomorphic Image of an Excellent Regular Local Ring of Characteristic p

The main aim of this section is to prove that, when R is an F-pure homomorphic image of an excellent regular local ring of characteristic p, every $\Phi(E)$-special ideal of R is a fully $\Phi(E)$-special ideal. This will enable us to extend some results obtained in [14, §3] about an F-pure complete local ring to an F-pure homomorphic image of an excellent regular local ring of characteristic p. We begin the section with a lemma that is derived from a result of G. Lyubeznik [5, Lemma 4.1].

Lemma 3.1 *Let* $(S, \mathfrak{M})$ *be a complete regular local ring of characteristic p, and let* $\mathfrak{B}$ *be a proper, non-zero ideal of S. Denote* $E_S(S/\mathfrak{M})$ *by* E_S*, and let* $S[x, f]$ *denote the Frobenius skew polynomial ring over S. Let* $n \in \mathbb{N}$*.*

Since S is regular, $S^{(n)}$ *is faithfully flat over S, and we identify* $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ *as an S-submodule of* $Sx^n \otimes_S E_S$ *in the natural way. Let* $a_1, \ldots, a_d$ *be a regular system of parameters for S. Consider the S-isomorphism* $\delta_n : Sx^n \otimes_S E_S \xrightarrow{\cong} E_S$ *of* [11, 4.2(iii)], *for which (with the notation used in the statement of that result)*

$$\delta_n\left(bx^n \otimes \left[\frac{s}{(a_1 \ldots a_d)^j}\right]\right) = \left[\frac{bs^{p^n}}{(a_1 \ldots a_d)^{jp^n}}\right] \quad \text{for all } b, s \in S \text{ and } j \in \mathbb{N}_0.$$

The isomorphism δ_n maps

(i) $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ *onto* $(0 :_{E_S} \mathfrak{B}^{[p^n]})$, *and*
(ii) $\mathfrak{B}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{B}))$ *onto* $(0 :_{E_S} (\mathfrak{B}^{[p^n]} : \mathfrak{B}))$.

Proof (i) Use of the analogue of Lyubeznik [5, Lemma 4.1] for the functor $Sx^n \otimes_S \bullet$ shows that the Matlis dual of $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ is S-isomorphic to $Sx^n \otimes_S (S/\mathfrak{B}) \cong S/\mathfrak{B}^{[p^n]}$. Since each S-module has the same annihilator as its Matlis dual, we thus see that $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ has annihilator $\mathfrak{B}^{[p^n]}$. As S is complete, $T = (0 :_{E_S} (0 :_S T))$ for each submodule T of E_S, by Matlis duality (see, for example, [15, p. 154]). It therefore follows that

$$\delta_n(Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})) = (0 :_{E_S} \mathfrak{B}^{[p^n]}).$$

(ii) Set $N := Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$. Similar reasoning shows that

$$\delta_n(\mathfrak{B}N) = (0 :_{E_S} (0 :_S \mathfrak{B}N)) = (0 :_{E_S} ((0 :_S N) : \mathfrak{B})) = (0 :_{E_S} (\mathfrak{B}^{[p^n]} : \mathfrak{B})).$$

□

Proposition 3.2 *Suppose that $R = S/\mathfrak{A}$, where $(S, \mathfrak{M})$ is a regular local ring of characteristic p, and $\mathfrak{A}$ is a proper ideal of S. Assume also that R is F-pure. Let $\mathfrak{b}$ be a proper ideal of R; let $\mathfrak{B}$ be the unique ideal of S that contains $\mathfrak{A}$ and is such that $\mathfrak{B}/\mathfrak{A} = \mathfrak{b}$.*

Then $\mathfrak{b}$ is fully $\Phi(E)$-special if and only if $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{B}^{[p^n]} : \mathfrak{B})$ for all $n \in \mathbb{N}$.

Note In the F-finite case, this result is already known and due to K. Schwede [10, Proposition 3.11 and Lemma 5.1].

Proof If $\mathfrak{A} = 0$, then R is regular, so that its big test ideal is R itself (by [6, Theorem 8.8], for example) and the only proper $\Phi(E)$-special ideal of R is 0; also, $(0^{[p^n]} : 0) = S$, and the only proper ideal $\mathfrak{B}$ of S satisfying $(0^{[p^n]} : 0) \subseteq (\mathfrak{B}^{[p^n]} : \mathfrak{B})$ for all $n \in \mathbb{N}$ is the zero ideal. Thus, the result is true when $\mathfrak{A} = 0$; we therefore assume for the remainder of this proof that $\mathfrak{A} \neq 0$.

Note that $\widehat{R} = \widehat{S}/\mathfrak{A}\widehat{S}$ is again F-pure and that $\widehat{S}$ is an excellent complete regular local ring of characteristic p, with maximal ideal $\mathfrak{M}\widehat{S}$.

We also note that $\mathfrak{b}$ is a fully $\Phi_R(E)$-special ideal of R if and only if $\mathfrak{b}\widehat{R}$ is a fully $\Phi_{\widehat{R}}(E)$-special ideal of $\widehat{R}$, by Lemma 2.12. Furthermore, by the faithful flatness of $\widehat{S}$ over S, we have, for $n \in \mathbb{N}$,

$$((\mathfrak{A}\widehat{S})^{[p^n]} : \mathfrak{A}\widehat{S}) = (\mathfrak{A}^{[p^n]} : \mathfrak{A})\widehat{S} \subseteq (\mathfrak{B}^{[p^n]} : \mathfrak{B})\widehat{S} = ((\mathfrak{B}\widehat{S})^{[p^n]} : \mathfrak{B}\widehat{S})$$

if and only if $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{B}^{[p^n]} : \mathfrak{B})$. Therefore, we can, and do, assume henceforth in this proof that S is complete.

Let $E_S := E_S(S/\mathfrak{M})$. Now $(0 :_{E_S} \mathfrak{A}) = E := E_R(R/\mathfrak{m})$ and $(0 :_{E_S} \mathfrak{B}) = (0 :_E \mathfrak{b})$. Note that $\mathfrak{b}$ is fully $\Phi_R(E)$-special if and only if, for each $n \in \mathbb{N}$ and each $r \in \mathfrak{b}$, the element $rx^n \in Rx^n$ annihilates the R-submodule $(0 :_E \mathfrak{b})$ of the 0th component E of $\Phi_R(E)$.

Let $n \in \mathbb{N}$. There is an exact sequence of (S, S)-bimodules

$$0 \longrightarrow \mathfrak{A}Sx^n \xrightarrow{\subseteq} Sx^n \xrightarrow{\nu} Rx^n \longrightarrow 0,$$

where $\nu(sx^n) = (s + \mathfrak{A})x^n$ for all $s \in S$. The map

$$Sx^n \otimes_S (0 :_{E_S} \mathfrak{A}) \longrightarrow Rx^n \otimes_S (0 :_{E_S} \mathfrak{A}) = Rx^n \otimes_R (0 :_{E_S} \mathfrak{A}) = Rx^n \otimes_R E$$

induced by ν therefore has kernel $\mathfrak{A}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{A}))$.

It follows that $\mathfrak{b}$ is fully $\Phi_R(E)$-special if and only if, for all $n \in \mathbb{N}$, $s \in \mathfrak{B}$ and $g \in (0 :_{E_S} \mathfrak{B}) = (0 :_E \mathfrak{b})$, the element $sx^n \otimes g$ of $Sx^n \otimes_S (0 :_{E_S} \mathfrak{A})$ lies in

$$\mathfrak{A}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{A})).$$

In other words, $\mathfrak{b}$ is fully $\Phi_R(E)$-special if and only if, for all $n \in \mathbb{N}$, we have

$$\mathfrak{B}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})) \subseteq \mathfrak{A}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{A})).$$

(We are here identifying $Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})$ and $Sx^n \otimes_S (0 :_{E_S} \mathfrak{A})$ with submodules of $Sx^n \otimes_S E_S$ in the obvious ways, using the faithful flatness of $S^{(n)}$ over S.)

By [11, 4.2(iii)], we have $Sx^n \otimes_S E_S \cong E_S$. Since S is complete, each submodule T of E_S satisfies $T = (0 :_{E_S} (0 :_S T))$. Set $N := Sx^n \otimes_S E_S$. Thus

$$\mathfrak{A}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{A})) = (0 :_N (0 :_S (\mathfrak{A}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{A}))))) = (0 :_N (\mathfrak{A}^{[p^n]} : \mathfrak{A})),$$

by Lemma 3.1. Similarly, $\mathfrak{B}(Sx^n \otimes_S (0 :_{E_S} \mathfrak{B})) = (0 :_N (\mathfrak{B}^{[p^n]} : \mathfrak{B}))$. It follows that $\mathfrak{b}$ is fully $\Phi_R(E)$-special if and only if

$$(0 :_N (\mathfrak{B}^{[p^n]} : \mathfrak{B})) \subseteq (0 :_N (\mathfrak{A}^{[p^n]} : \mathfrak{A})) \quad \text{for all } n \in \mathbb{N},$$

that is (since $N \cong E_S$), if and only if $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{B}^{[p^n]} : \mathfrak{B})$ for all $n \in \mathbb{N}$. □

Theorem 3.3 *Suppose that $R = S/\mathfrak{A}$ is a homomorphic image of an excellent regular local ring $(S, \mathfrak{M})$ of characteristic p, modulo a proper ideal $\mathfrak{A}$. Assume that R is F-pure.*

Then each $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special.

Proof Once again, the claim is easy to prove if $\mathfrak{A} = 0$, and so we assume henceforth in this proof that $\mathfrak{A} \neq 0$.

Note that $\widehat{R} = \widehat{S}/\mathfrak{A}\widehat{S}$ is again F-pure and that $\widehat{S}$ is an excellent complete regular local ring of characteristic p, with maximal ideal $\mathfrak{M}\widehat{S}$.

Let $\mathfrak{b}$ be a $\Phi(E)$-special R-ideal with $\mathfrak{b} \neq R$. Then $\mathfrak{b} = \mathfrak{c} \cap R$ for some $\Phi_{\widehat{R}}(E)$-special $\widehat{R}$-ideal $\mathfrak{c}$. (We have used Lemma 2.11 here.) Let $\mathfrak{C}$ be the unique ideal of $\widehat{S}$ that contains $\mathfrak{A}\widehat{S}$ and is such that $\mathfrak{C}/\mathfrak{A}\widehat{S} = \mathfrak{c}$. By Proposition 2.7, the ideal $\mathfrak{c}$ of $\widehat{R}$ is fully $\Phi_{\widehat{R}}(E)$-special, and so, by Proposition 3.2, we have

$$(\mathfrak{A}^{[p^n]} : \mathfrak{A})\widehat{S} = ((\mathfrak{A}\widehat{S})^{[p^n]} : \mathfrak{A}\widehat{S}) \subseteq (\mathfrak{C}^{[p^n]} : \mathfrak{C}) \quad \text{for all } n \in \mathbb{N}.$$

Set $\mathfrak{C} \cap S := \mathfrak{B}$, so that $\mathfrak{B}/\mathfrak{A} = \mathfrak{b}$.

Let $n \in \mathbb{N}$ and $s \in (\mathfrak{A}^{[p^n]} : \mathfrak{A})$. Therefore, $s \in (\mathfrak{C}^{[p^n]} : \mathfrak{C})$. It follows from G. Lyubeznik and K. E. Smith [6, Lemma 6.6] that $\mathfrak{C}^{[p^n]} \cap S = (\mathfrak{C} \cap S)^{[p^n]}$. (Lyubeznik's and Smith's proof of this result uses work of N. Radu [9, Corollary 5], which, in turn, uses D. Popescu's general Néron desingularization [7, 8].) We can now deduce that

$$s(\mathfrak{C} \cap S) \subseteq s\mathfrak{C} \cap S \subseteq \mathfrak{C}^{[p^n]} \cap S = (\mathfrak{C} \cap S)^{[p^n]},$$

so that $s \in ((\mathfrak{C} \cap S)^{[p^n]} : \mathfrak{C} \cap S) = (\mathfrak{B}^{[p^n]} : \mathfrak{B})$.

We have thus shown that $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{B}^{[p^n]} : \mathfrak{B})$ for all $n \in \mathbb{N}$, so that $\mathfrak{b} = \mathfrak{B}/\mathfrak{A}$ is fully $\Phi(E)$-special by Proposition 3.2. □

In the case where R is an F-pure homomorphic image of an excellent regular local ring of characteristic p, the characterization of $\mathcal{I}(\Phi(E))$ afforded by Proposition 3.2 and Theorem 3.3 enables us to see that set behaves well under localization. As the ideals in $\mathcal{I}(\Phi(E))$ are precisely those that can be expressed as intersections of finitely many prime members of $\mathcal{I}(\Phi(E))$, it is of interest to examine the behaviour of $\mathcal{I}(\Phi(E)) \cap \mathrm{Spec}(R)$ under localization. The next proposition, which is an extension of part of [12, Proposition 2.8], is in preparation for this investigation.

Proposition 3.4 *Let S be a regular local ring of characteristic p, and let $n \in \mathbb{N}$. Let $\mathfrak{A}, \mathfrak{B}_1, \ldots, \mathfrak{B}_t, \mathfrak{C}$ be ideals of S with $0 \neq \mathfrak{A} \neq S$, and let $\mathfrak{A} = \mathfrak{Q}_1 \cap \ldots \cap \mathfrak{Q}_t$ be a minimal primary decomposition of $\mathfrak{A}$.*

(i) *We have $(\mathfrak{B}_1 \cap \cdots \cap \mathfrak{B}_t)^{[p^n]} = \mathfrak{B}_1^{[p^n]} \cap \cdots \cap \mathfrak{B}_t^{[p^n]}$.*
(ii) *If $\mathfrak{Q}$ is a $\mathfrak{P}$-primary ideal of S, then $\mathfrak{Q}^{[p^n]}$ is also $\mathfrak{P}$-primary.*
(iii) *The equation $\mathfrak{A}^{[p^n]} = \mathfrak{Q}_1^{[p^n]} \cap \cdots \cap \mathfrak{Q}_t^{[p^n]}$ provides a minimal primary decomposition of $\mathfrak{A}^{[p^n]}$.*
(iv) *We have $(\mathfrak{A} : \mathfrak{C})^{[p^n]} = (\mathfrak{A}^{[p^n]} : \mathfrak{C}^{[p^n]})$ and $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq \left((\mathfrak{A} : \mathfrak{C})^{[p^n]} : (\mathfrak{A} : \mathfrak{C})\right)$.*
(v) *If $\mathfrak{P}$ is an associated prime ideal of $\mathfrak{A}$, then $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{P}^{[p^n]} : \mathfrak{P})$.*
(vi) *Since $0 \neq \mathfrak{A} \neq S$, we have $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \neq S$. If $\mathfrak{P}_1 := \sqrt{\mathfrak{Q}_1}$ is a minimal prime ideal of $\mathfrak{A}$, then $\mathfrak{P}_1$ is a minimal prime ideal of $(\mathfrak{A}^{[p^n]} : \mathfrak{A})$ and the unique $\mathfrak{P}_1$-primary component of $(\mathfrak{A}^{[p^n]} : \mathfrak{A})$ is $(\mathfrak{Q}_1^{[p^n]} : \mathfrak{Q}_1)$.*

Proof Parts (i), (ii) and (iii) were essentially proved in [12, Proposition 2.8], while parts (iv), (v) and (vi) can be proved by obvious modifications of the arguments used to prove the corresponding parts of [12, Proposition 2.8]. □

Corollary 3.5 *Suppose that R is F-pure and a homomorphic image of an excellent regular local ring S of characteristic p modulo a proper ideal $\mathfrak{A}$. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then*

$$\mathcal{I}_{R_\mathfrak{p}}(\Phi_{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))) \cap \mathrm{Spec}(R_\mathfrak{p}) = \left\{\mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \in \mathcal{I}(\Phi(E)) \cap \mathrm{Spec}(R) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\right\}.$$

Proof Note that, by M. Hochster and J. L. Roberts [3, Lemma 6.2], the localization $R_\mathfrak{p}$ is again F-pure. The claim is easy to prove when $\mathfrak{A} = 0$, and so we assume that $\mathfrak{A} \neq 0$.

For each lower case fraktur letter that denotes an ideal of R, let the corresponding upper case fraktur letter denote the unique ideal of S that contains $\mathfrak{A}$ and has quotient modulo $\mathfrak{A}$ equal to the specified ideal of R. For example, $\mathfrak{P}$ denotes the unique ideal of S that contains $\mathfrak{A}$ and is such that $\mathfrak{P}/\mathfrak{A} = \mathfrak{p}$.

Note that $R_\mathfrak{p} \cong S_\mathfrak{P}/\mathfrak{A}S_\mathfrak{P}$ is again a homomorphic image of an excellent regular local ring $S_\mathfrak{P}$ of characteristic p. Let $\mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$.

Suppose first that $\mathfrak{q} \in \mathcal{I}(\Phi(E)) \cap \mathrm{Spec}(R)$. By Theorem 3.3, we see that $\mathfrak{q}$ is fully $\Phi(E)$-special; use of Proposition 3.2 shows that $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{Q}^{[p^n]} : \mathfrak{Q})$ for all $n \in \mathbb{N}$. Therefore

$$((\mathfrak{A}S_\mathfrak{P})^{[p^n]} : \mathfrak{A}S_\mathfrak{P}) \subseteq ((\mathfrak{Q}S_\mathfrak{P})^{[p^n]} : \mathfrak{Q}S_\mathfrak{P}) \quad \text{for all } n \in \mathbb{N}.$$

Since the standard isomorphism $S_\mathfrak{P}/\mathfrak{A}S_\mathfrak{P} \xrightarrow{\cong} R_\mathfrak{p}$ maps $\mathfrak{Q}S_\mathfrak{P}/\mathfrak{A}S_\mathfrak{P}$ onto $\mathfrak{q}R_\mathfrak{p}$, it follows from Proposition 3.2 that $\mathfrak{q}R_\mathfrak{p}$ is fully $\Phi_{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$-special.

Conversely, suppose that $\mathfrak{q}R_\mathfrak{p}$ is $\Phi_{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$-special, so that, by Theorem 3.3, it is fully $\Phi_{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$-special. By Proposition 3.2, this means that

$$((\mathfrak{A}S_\mathfrak{P})^{[p^n]} : \mathfrak{A}S_\mathfrak{P}) \subseteq ((\mathfrak{Q}S_\mathfrak{P})^{[p^n]} : \mathfrak{Q}S_\mathfrak{P}) \quad \text{for all } n \in \mathbb{N}.$$

Let ^{e} and ^{c} denote extension and contraction of ideals under the natural ring homomorphism $S \longrightarrow S_{\mathfrak{P}}$. Contract the last displayed inclusion relations back to S to see that

$$(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{A}^{[p^n]} : \mathfrak{A})^{ec} \subseteq (\mathfrak{Q}^{[p^n]} : \mathfrak{Q})^{ec} = (\mathfrak{Q}^{[p^n]} : \mathfrak{Q}) \quad \text{for all } n \in \mathbb{N}$$

because $(\mathfrak{Q}^{[p^n]} : \mathfrak{Q})$ is $\mathfrak{Q}$-primary (for all $n \in \mathbb{N}$), by Proposition 3.4(vi). It follows from Proposition 3.2 that $\mathfrak{Q}/\mathfrak{A} = \mathfrak{q}$ is fully $\Phi(E)$-special. □

We can now recover a special case of a result of Lyubeznik and Smith.

Corollary 3.6 (G. Lyubeznik and K. E. Smith [6, Theorem 7.1]) *Suppose that R is F-pure and a homomorphic image of an excellent regular local ring S of characteristic p modulo a proper ideal $\mathfrak{A}$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the big test ideal of $R_{\mathfrak{p}}$ is the extension to $R_{\mathfrak{p}}$ of the big test ideal of R. In symbols, $\tilde{\tau}(R_{\mathfrak{p}}) = \tilde{\tau}(R)R_{\mathfrak{p}}$.*

Proof The big test ideal $\tilde{\tau}(R)$ of R is equal to the intersection of the (finitely many) members of $\mathcal{I}(\Phi(E)) \cap \operatorname{Spec}(R)$ of positive height, and a similar statement holds for $R_{\mathfrak{p}}$. The claim therefore follows from Corollary 3.5. □

Some results were obtained in [14, Theorem 3.1] for an F-pure complete local ring of characteristic p. We can now use Theorem 3.3 to establish analogous results for an F-pure homomorphic image of an excellent regular local ring of characteristic p.

Theorem 3.7 *Suppose $(R, \mathfrak{m})$ is F-pure and that every $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special. (For example, by Theorem 3.3, this would be the case if R were a homomorphic image of an excellent regular local ring of characteristic p.) Let $\mathfrak{c}$ be a proper ideal of R that is $\Phi(E)$-special. In the light of Theorem 2.6, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_w$ be prime ideals of R for which the multiplicatively closed subset $S = R \setminus \bigcup_{i=1}^{w} \mathfrak{p}_i$ of R satisfies $\mathfrak{c} = \tau^S(R)$. Set $J := \Delta^S(\Phi(E))$, a graded left $R[x, f]$-module.*

(i) *We have $J = 0_E^{*,S} \oplus 0_{Rx\otimes_R E}^{*,S} \oplus \cdots \oplus 0_{Rx^n\otimes_R E}^{*,S} \oplus \cdots = \operatorname{ann}_{\Phi(E)}(\mathfrak{c}R[x, f])$.*

(ii) *When we regard J as a graded left $(R/\mathfrak{c})[x, f]$-module in the natural way, it is x-torsion-free and has $\mathcal{I}_{R/\mathfrak{c}}(J) = \{\mathfrak{g}/\mathfrak{c} : \mathfrak{g} \in \mathcal{I}(\Phi(E)) : \mathfrak{g} \supseteq \mathfrak{c}\}$.*

(iii) *The 0th component J_0 of J is $(0 :_E \mathfrak{c})$; as $R/\mathfrak{c}$-module, this is isomorphic to $E_{R/\mathfrak{c}}((R/\mathfrak{c})/(\mathfrak{m}/\mathfrak{c}))$.*

(iv) *The ring $R/\mathfrak{c}$ is F-pure.*

(v) *We have $\mathcal{I}(\Phi_{R/\mathfrak{c}}(J_0)) \subseteq \mathcal{I}_{R/\mathfrak{c}}(J)$, so that*

$$\left\{\mathfrak{d} : \mathfrak{d} \text{ is an ideal of } R \text{ with } \mathfrak{d} \supseteq \mathfrak{c} \text{ and } \mathfrak{d}/\mathfrak{c} \in \mathcal{I}(\Phi_{R/\mathfrak{c}}(J_0))\right\} \subseteq \mathcal{I}(\Phi_R(E)).$$

Proof Since the $\Phi(E)$-special ideal $\mathfrak{c}$ is fully $\Phi(E)$-special, we have $J_0 = (0 :_E \mathfrak{c})$. Given this observation, one can now use the arguments employed in the proof of [14, Theorem 3.1] to furnish a proof of this theorem. □

The next corollary follows from Theorem 3.7 just as, in [14], Corollary 3.2 follows from Theorem 3.1.

Corollary 3.8 *Suppose that $(R, \mathfrak{m})$ is local, F-pure and that every $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special. (For example, by Theorem 3.3, this would be the case if R were a homomorphic image of an excellent regular local ring of characteristic p.) Let $\mathfrak{c}$ be a proper ideal of R that is $\Phi(E)$-special. Denote $R/\mathfrak{c}$ by $\overline{R}$, and note that $\overline{R}$ is F-pure, by*

Theorem 3.7(iv). Let T be a multiplicatively closed subset of $\overline{R}$ which is the complement in $\overline{R}$ of the union of finitely many prime ideals. The finitistic T-test ideal $\tau^{\mathrm{fg},T}(\overline{R})$ *of $\overline{R}$ is defined to be $\bigcap_L (0 :_{\overline{R}} 0_L^{*,T})$, where the intersection is taken over all finitely generated $\overline{R}$-modules L.*

(i) *If $\mathfrak{h}$ denotes the unique ideal of R that contains $\mathfrak{c}$ and is such that $\mathfrak{h}/\mathfrak{c} = \tau^{\mathrm{fg},T}(\overline{R})$, the finitistic T-test ideal of $\overline{R}$, then $\mathfrak{h} \in \mathcal{I}(\Phi(E))$.*

(ii) *In particular, if $\mathfrak{h}'$ denotes the unique ideal of R that contains $\mathfrak{c}$ and is such that $\mathfrak{h}'/\mathfrak{c} = \tau(\overline{R})$, the test ideal of $\overline{R}$, then $\mathfrak{h}' \in \mathcal{I}(\Phi(E))$.*

(iii) *If $\mathfrak{g}$ denotes the unique ideal of R that contains $\mathfrak{c}$ and is such that $\mathfrak{g}/\mathfrak{c} = \tau^T(\overline{R})$, the T-test ideal of $\overline{R}$, then $\mathfrak{g} \in \mathcal{I}(\Phi(E))$.*

(iv) *In particular, if $\mathfrak{g}'$ denotes the unique ideal of R that contains $\mathfrak{c}$ and is such that $\mathfrak{g}'/\mathfrak{c} = \widetilde{\tau}(\overline{R})$, the big test ideal of $\overline{R}$, then $\mathfrak{g}' \in \mathcal{I}(\Phi(E))$.*

Proof Straightforward modifications of the arguments given in the proof of [14, Corollary 3.2] will provide a proof for this. □

Lemma 3.9 *Assume that $(R, \mathfrak{m})$ is local, F-pure and a homomorphic image of an excellent regular local ring of characteristic p.*

(i) *There is a strictly ascending chain $0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t \subset \tau_{t+1} = R$ of radical ideals of R such that, for each $i = 0, \ldots, t$, the reduced local ring R/τ_i is F-pure and its test ideal is τ_{i+1}/τ_i. We call this the* test ideal chain of *R. All of $\tau_0 = 0, \tau_1, \cdots, \tau_t$, and all their associated primes, belong to $\mathcal{I}(\Phi(E))$.*

(ii) *There is a strictly ascending chain $0 = \widetilde{\tau}_0 \subset \widetilde{\tau}_1 \subset \cdots \subset \widetilde{\tau}_w \subset \widetilde{\tau}_{w+1} = R$ of radical ideals in $\mathcal{I}(\Phi(E))$ such that, for each $i = 0, \ldots, w$, the reduced local ring $R/\widetilde{\tau}_i$ is F-pure and its big test ideal is $\widetilde{\tau}_{i+1}/\widetilde{\tau}_i$. We call this the* big test ideal chain of *R. All of $\widetilde{\tau}_0 = 0, \widetilde{\tau}_1, \cdots, \widetilde{\tau}_w$, and all their associated primes, belong to $\mathcal{I}(\Phi(E))$.*

Note In the case when R is an (F-pure) homomorphic image of an F-finite regular local ring, part (i) of this result is known and due to Janet Cowden Vassilev [16, §3].

Proof (i) Set $\tau_1 := \tau(R)$, and note that $\tau(R) \in \mathcal{I}(\Phi(E))$. If $\tau_1 \neq R$, apply Theorem 3.7 with the choice $\mathfrak{c} = \tau(R) = \tau_1$. That shows that R/τ_1 is F-pure. Now argue by induction on $\dim R$, noting that R/τ_1 is a homomorphic image of an excellent regular local ring of characteristic p. Use Theorem 3.7(v) to show that all of $\tau_0, \tau_1, \ldots, \tau_t$ belong to $\mathcal{I}(\Phi(E))$.

(ii) This is proved similarly. □

4 The F-Finite Case

In the F-finite case, the results above have strong connections with work of K. Schwede in [10], and the purpose of this section is to explore some of those connections. The introduction contains a description of certain properties of the set of all uniformly F-compatible ideals in an F-finite, F-pure local ring R, and some of these are similar to properties of the set of all fully $\Phi(E)$-special ideals of R: we shall show in this section that, in this special case, an ideal of R is uniformly F-compatible if and only if it is $\Phi(E)$-special, and that this is the case if and only if it is fully $\Phi(E)$-special.

Definition 4.1 Suppose that R is F-finite, let $\mathfrak{b}$ be an ideal of R. Then $\mathfrak{b}$ is said to be *uniformly F-compatible* if, for every $n > 0$ and every $\phi \in \mathrm{Hom}_R(R^{(n)}, R)$, we have $\phi(\mathfrak{b}^{(n)}) \subseteq \mathfrak{b}$.

Proposition 4.2 (Schwede [10, Lemma 5.1]) *Suppose that* $(R, \mathfrak{m})$ *is* F*-finite, let* $\mathfrak{b}$ *be an ideal of* R. *Then* $\mathfrak{b}$ *is uniformly* F*-compatible if and only if* $(0 :_E \mathfrak{b}) \subseteq (\mathrm{ann}_{\Phi(E)}(\mathfrak{b}R[x, f]))_0$.

Thus, when R *is* F*-finite and* F*-pure,* $\mathfrak{b}$ *is uniformly* F*-compatible if and only if it is fully* $\Phi(E)$*-special.*

Proof Let $n \in \mathbb{N}$ and $r \in R$. Multiplication by r yields an R-homomorphism of $R^{(n)}$, which, strictly speaking, we should denote by $r\mathrm{Id}_{R^{(n)}}$. Also $f^n : R \longrightarrow R^{(n)}$ is an R-homomorphism. Thus we can consider the composition of R-homomorphisms $R \xrightarrow{f^n} R^{(n)} \xrightarrow{r} R^{(n)}$.

Application of the functor $\bullet \otimes_R E$ yields a composition of R-homomorphisms

$$R \otimes_R E \longrightarrow R^{(n)} \otimes_R E \xrightarrow{r} R^{(n)} \otimes_R E,$$

where the 'r' over the second arrow is an abbreviation for $r\mathrm{Id}_{R^{(n)}} \otimes_R E$. But $R^{(n)} \cong Rx^n$ as (R, R)-bimodules; furthermore, $(0 :_E \mathfrak{b}) \cong \mathrm{Hom}_R(R/\mathfrak{b}, E)$. It follows that $(0 :_E \mathfrak{b}) \subseteq (\mathrm{ann}_{\Phi(E)}(\mathfrak{b}R[x, f]))_0$ if and only if, for all $n \in \mathbb{N}$ and all $r \in \mathfrak{b}$, the composition

$$(0 :_E \mathfrak{b}) \xrightarrow{\subseteq} E \xrightarrow{\cong} R \otimes_R E \longrightarrow R^{(n)} \otimes_R E \xrightarrow{r} R^{(n)} \otimes_R E$$

(in which the second map is the natural isomorphism) is zero.

Let M be an R-module. Recall that there is an R-homomorphism

$$\xi_M : M \otimes_R E \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), E)$$

such that, for $m \in M$, $e \in E$ and $g \in \mathrm{Hom}_R(M, R)$, we have $(\xi_M(m \otimes e))(g) = g(m)e$. Furthermore, as M varies, the ξ_M constitute a natural transformation of functors; also ξ_M is an isomorphism whenever M is finitely generated. We shall use D to denote the functor $\mathrm{Hom}_R(\bullet, E)$.

Since $R^{(n)}$ is a finitely generated R-module, $(0 :_E \mathfrak{b}) \subseteq (\mathrm{ann}_{\Phi(E)}(\mathfrak{b}R[x, f]))_0$ if and only if, for all $n \in \mathbb{N}$ and all $r \in \mathfrak{b}$, the composition

$$D(R/\mathfrak{b}) \to D(R) \xrightarrow{\cong} D(\mathrm{Hom}_R(R, R)) \to D(\mathrm{Hom}_R(R^{(n)}, R)) \xrightarrow{r} D(\mathrm{Hom}_R(R^{(n)}, R))$$

is zero. (Here, the first map is induced from the natural epimorphism $R \longrightarrow R/\mathfrak{b}$, the second map is the natural isomorphism, and the sequence from the middle term rightwards is the result of application of the functor $\mathrm{Hom}_R(\mathrm{Hom}_R(\bullet, R), E)$ to the composition $R \xrightarrow{f^n} R^{(n)} \xrightarrow{r} R^{(n)}$ described at the beginning of the proof.)

Since D is a faithful functor (because E is an injective cogenerator for R), we can deduce that $(0 :_E \mathfrak{b}) \subseteq (\mathrm{ann}_{\Phi(E)}(\mathfrak{b}R[x, f]))_0$ if and only if, for all $n \in \mathbb{N}$ and all $r \in \mathfrak{b}$, the composition

$$\mathrm{Hom}_R(R^{(n)}, R) \xrightarrow{r} \mathrm{Hom}_R(R^{(n)}, R) \longrightarrow \mathrm{Hom}_R(R, R) \xrightarrow{\cong} R \longrightarrow R/\mathfrak{b}$$

is zero, that is, if and only if $\mathfrak{b}$ is uniformly F-compatible. □

Proposition 4.3 (Schwede [10]) *Suppose that* $(R, \mathfrak{m})$ *is* F*-finite, and let* $\mathfrak{a}$ *be an ideal of* R*. Note that the completion* $\widehat{R}$ *of* R *is again* F*-finite.*

(i) *If* $\mathfrak{a}$ *is a uniformly* F*-compatible ideal of* R*, then* $\mathfrak{a}\widehat{R}$ *is a uniformly* F*-compatible ideal of* $\widehat{R}$*. See Schwede* [10, Lemma 3.9].
(ii) *If* $\mathfrak{C}$ *is a uniformly* F*-compatible ideal of* $\widehat{R}$*, then* $\mathfrak{C} \cap R$ *is a uniformly* F*-compatible ideal of* R*. See Schwede* [10, Lemma 3.8].

Proof For a finitely generated R-module M, we identify $\widehat{M}$ with $M \otimes_R \widehat{R}$ in the usual way, and we note that there is a natural $\widehat{R}$-isomorphism $\psi_M : \mathrm{Hom}_R(M, R) \otimes_R \widehat{R} \xrightarrow{\cong} \mathrm{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, R \otimes_R \widehat{R})$ for which $\psi_M(g \otimes \widehat{r}) = \widehat{r}(g \otimes \mathrm{Id}_{\widehat{R}})$ for all $g \in \mathrm{Hom}_R(M, R)$ and $\widehat{r} \in \widehat{R}$. Let $n \in \mathbb{N}$. Consideration of Cauchy sequences shows that $\widehat{M^{(n)}} = \widehat{M}^{(n)}$. In particular, $\widehat{R^{(n)}} = \widehat{R}^{(n)}$ and $\widehat{\mathfrak{a}^{(n)}} = (\widehat{\mathfrak{a}})^{(n)} = (\mathfrak{a}\widehat{R})^{(n)}$.

There is an $\widehat{R}$-isomorphism $\gamma : R^{(n)} \otimes_R \widehat{R} \xrightarrow{\cong} \widehat{R}^{(n)}$ which maps $\mathfrak{a}^{(n)} \otimes_R \widehat{R}$ onto $(\mathfrak{a}\widehat{R})^{(n)}$. Also, the natural $\widehat{R}$-isomorphism $\delta : R \otimes_R \widehat{R} \xrightarrow{\cong} \widehat{R}$ maps $\mathfrak{a} \otimes_R \widehat{R}$ onto $\mathfrak{a}\widehat{R}$.

(i) Let $\theta \in \mathrm{Hom}_{\widehat{R}}(R^{(n)} \otimes_R \widehat{R}, R \otimes_R \widehat{R})$. By the above, there exist $\phi_1, \ldots, \phi_t \in \mathrm{Hom}_R(R^{(n)}, R)$ and $\widehat{r}_1, \ldots, \widehat{r}_t \in \widehat{R}$ such that $\theta = \widehat{r}_1(\phi_1 \otimes \mathrm{Id}_{\widehat{R}}) + \cdots + \widehat{r}_t(\phi_t \otimes \mathrm{Id}_{\widehat{R}})$. Since $\phi_i(\mathfrak{a}^{(n)}) \subseteq \mathfrak{a}$ for all $n \in \mathbb{N}$ and $i = 1, \ldots, t$, we see that $\theta(\mathfrak{a}^{(n)} \otimes_R \widehat{R}) \subseteq \mathfrak{a} \otimes_R \widehat{R}$ for all $n \in \mathbb{N}$. Use of the above-mentioned isomorphisms γ and δ now enables us to conclude that $\mathfrak{a}\widehat{R}$ is a uniformly F-compatible ideal of $\widehat{R}$.

(ii) Let $\phi \in \mathrm{Hom}_R(R^{(n)}, R)$, and set $\mathfrak{c} := \mathfrak{C} \cap R$. Then

$$\phi \otimes \mathrm{Id}_{\widehat{R}} \in \mathrm{Hom}_{\widehat{R}}(R^{(n)} \otimes_R \widehat{R}, R \otimes_R \widehat{R})$$

and $\delta \circ (\phi \otimes \mathrm{Id}_{\widehat{R}}) \circ \gamma^{-1}$ maps $\mathfrak{C}^{(n)}$ into $\mathfrak{C}$, and therefore maps $(\mathfrak{c}\widehat{R})^{(n)}$ into $\mathfrak{C}$. Therefore $\delta \circ (\phi \otimes \mathrm{Id}_{\widehat{R}})$ maps $\mathfrak{c}^{(n)} \otimes_R \widehat{R}$ into $\mathfrak{C}$, so that $\phi(a) \in \mathfrak{C} \cap R = \mathfrak{c}$ for all $a \in \mathfrak{c}^{(n)}$. Therefore $\mathfrak{c}$ is a uniformly F-compatible ideal of R. □

Theorem 4.4 *Suppose that* $(R, \mathfrak{m})$ *is* F*-pure and* F*-finite. Then each* $\Phi(E)$*-special ideal* $\mathfrak{a}$ *of* R *is automatically fully* $\Phi(E)$*-special.*

Proof Note that $\widehat{R}$ is also F-pure, by Hochster and Roberts [3, Corollary 6.13]. Also, $\widehat{R}$ is F-finite, because the completion of the finitely generated R-module $R^{(1)}$ is $\widehat{R}^{(1)}$.

Thus, by definition, $\mathfrak{a}$ is the R-annihilator of an $R[x, f]$-submodule of $\Phi(E)$. It follows from Lemma 2.11 that $\mathfrak{a} = \mathfrak{A} \cap R$ for some ideal $\mathfrak{A}$ of $\widehat{R}$ that is the $\widehat{R}$-annihilator of an $\widehat{R}[x, f]$-submodule of $\Phi_{\widehat{R}}(E)$. Thus $\mathfrak{A}$ is $\Phi_{\widehat{R}}(E)$-special. It follows from Proposition 2.7 that $\mathfrak{A}$ is a fully $\Phi_{\widehat{R}}(E)$-special ideal of $\widehat{R}$, and so is uniformly F-compatible, by Proposition 4.2. Therefore, by Proposition 4.3(ii), the contraction $\mathfrak{A} \cap R = \mathfrak{a}$ is a uniformly F-compatible ideal of R, and is therefore fully $\Phi(E)$-special, by Proposition 4.2 again. □

Corollary 4.5 *Suppose that* $(R, \mathfrak{m})$ *is* F*-pure and* F*-finite; let* $\mathfrak{a}$ *be an ideal of* R*. Then the following statements are equivalent:*

(i) $\mathfrak{a}$ *is uniformly* F*-compatible;*
(ii) $\mathfrak{a}$ *is* $\Phi(E)$*-special;*
(iii) $\mathfrak{a}$ *is fully* $\Phi(E)$*-special.*

Proof This is now immediate from Proposition 4.2 and Theorem 4.4. □

Question 4.6 Suppose that $(R, \mathfrak{m})$ is F-pure.

We have seen that each $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special if R is complete (by Proposition 2.7) or if R is a homomorphic image of an excellent regular local ring of characteristic p (by Theorem 3.3) or if R is F-finite (by Theorem 4.4).

Note that each complete local ring is excellent, and that each F-finite local ring of characteristic p is excellent (by E. Kunz [4, Theorem 2.5]). The above results raise the following question. If the F-pure local ring R is excellent, is it the case that every $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special?

5 A Generalization of Aberbach's and Enescu's Splitting Prime

Recall from [6, Remark 2.8 and Proposition 2.9] that G. Lyubeznik and K. E. Smith defined $(R, \mathfrak{m})$ to be *strongly F-regular* (even in the case where R is not F-finite) precisely when the zero submodule of E is tightly closed in E. See M. Hochster and C. Huneke [2, §8].

Theorem 5.1 *Suppose that $(R, \mathfrak{m})$ is F-pure and that every $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special. (For example, by Theorem 3.3, this would be the case if R were a homomorphic image of an excellent regular local ring of characteristic p; it would also be the case if R were F-finite, by Theorem 4.4.)*

- (i) *There exists a unique largest $\Phi(E)$-special proper ideal, $\mathfrak{c}$ say, of R and this is prime. Furthermore, $R/\mathfrak{c}$ is strongly F-regular.*
- (ii) *Let T be the $R[x, f]$-submodule of $\Phi(E)$ generated by $(0 :_E \mathfrak{m}) \subseteq R \otimes_R E$. Then* $\text{gr-ann}_{R[x,f]}T = \mathfrak{c}R[x, f]$.

Proof (i) By Corollary 2.10, there is a unique largest $\Phi(E)$-special proper ideal $\mathfrak{c}$ of R, and this is prime. By Corollary 3.8(iv), the big test ideal of $R/\mathfrak{c}$ is $R/\mathfrak{c}$ itself, so that $1_{R/\mathfrak{c}}$ is a big test element for $R/\mathfrak{c}$. Therefore, the zero submodule of $E_{R/\mathfrak{c}}(R/\mathfrak{m})$ is tightly closed in $E_{R/\mathfrak{c}}(R/\mathfrak{m})$, and so $R/\mathfrak{c}$ is strongly F-regular.

(ii) Note that T is the image of the $R[x, f]$-homomorphism

$$R[x, f] \otimes_R (0 :_E \mathfrak{m}) \longrightarrow R[x, f] \otimes_R E = \Phi(E)$$

induced by the inclusion map $(0 :_E \mathfrak{m}) \xrightarrow{\subseteq} E$. Let $\mathfrak{d}$ be the $\Phi(E)$-special ideal of R for which $\text{gr-ann}_{R[x,f]}T = \mathfrak{d}R[x, f]$. Since $\mathfrak{d}$ annihilates $(0 :_E \mathfrak{m})$, we see that $\mathfrak{d}$ is proper. Suppose that there exists $\mathfrak{h} \in \mathcal{I}(\Phi(E))$ such that $\mathfrak{d} \subset \mathfrak{h} \subseteq \mathfrak{m}$. (The symbol '$\subset$' is reserved to denote strict inclusion.) Thus, we have $(0 :_E \mathfrak{m}) \subseteq (0 :_E \mathfrak{h}) \subseteq (0 :_E \mathfrak{d})$. But we know that every $\Phi(E)$-special ideal of R is fully $\Phi(E)$-special, and therefore $(0 :_E \mathfrak{h}) \subseteq (\text{ann}_{\Phi(E)}(\mathfrak{h}R[x, f]))_0$. Since $\text{ann}_{\Phi(E)}(\mathfrak{h}R[x, f])$ is an $R[x, f]$-submodule of $\Phi(E)$, it follows that

$$T \subseteq \text{ann}_{\Phi(E)}(\mathfrak{h}R[x, f]) \subseteq \text{ann}_{\Phi(E)}(\mathfrak{d}R[x, f]).$$

Now take graded annihilators: in view of the bijective correspondence between the sets $\mathcal{I}(\Phi(E))$ and $\mathcal{A}(\Phi(E))$ alluded to in the Introduction, we have

$$\begin{aligned} \mathfrak{d}R[x, f] &= \text{gr-ann}_{R[x,f]}(\text{ann}_{\Phi(E)}(\mathfrak{d}R[x, f])) \\ &\subseteq \text{gr-ann}_{R[x,f]}(\text{ann}_{\Phi(E)}(\mathfrak{h}R[x, f])) = \mathfrak{h}R[x, f] \\ &\subseteq \text{gr-ann}_{R[x,f]}T = \mathfrak{d}R[x, f]. \end{aligned}$$

Hence $\mathfrak{h} = \mathfrak{d}$ and we have a contradiction.

Thus $\mathfrak{d}$ is a maximal member of the set of proper $\Phi(E)$-special ideals of R; therefore $\mathfrak{d} = \mathfrak{c}$. □

Definition 5.2 (I. M. Aberbach and F. Enescu [1, Definition 3.2]) Suppose $(R, \mathfrak{m})$ is F-finite and reduced. Let u be a generator for the socle $(0 :_E \mathfrak{m})$ of E. Aberbach and Enescu defined

$$\mathfrak{P} = \left\{r \in R : r \otimes u = 0 \text{ in } R^{(n)} \otimes_R E \text{ for all } n \gg 0\right\},$$

an ideal of R.

In [1, §3], Aberbach and Enescu showed that in the case where $(R, \mathfrak{m})$ is F-finite and F-pure, and with the notation of 5.2, the ideal $\mathfrak{P}$ is prime and is equal to the set of elements $c \in R$ for which, for all $e \in \mathbb{N}$, the R-homomorphism $\phi_{c,e} : R \longrightarrow R^{1/p^e}$ for which $\phi_{c,e}(1) = c^{1/p^e}$ does not split over R. Aberbach and Enescu call this $\mathfrak{P}$ the *splitting prime for* R. By [1, Theorem 4.8(i)], the ring $R/\mathfrak{P}$ is strongly F-regular.

Proposition 5.3 *Suppose that* $(R, \mathfrak{m})$ *is* F*-finite and* F*-pure. Let* $\mathfrak{P}$ *be Aberbach's and Enescu's splitting prime, as in 5.2. Let* $\mathfrak{q}$ *be the unique largest* $\Phi(E)$*-special proper ideal of* R*, as in Theorem 5.1. Then* $\mathfrak{P} = \mathfrak{q}$.

Proof Let u be a generator for the socle $(0 :_E \mathfrak{m})$ of E. We can write

$$\mathfrak{P} = \left\{r \in R : rx^n \otimes u = 0 \text{ in } Rx^n \otimes_R E \text{ for all } n \gg 0\right\}.$$

Now for a positive integer j and $r \in R$, if $rx^j \otimes u = 0$ in $\Phi(E)$, then

$$x(rx^{j-1} \otimes u) = r^p x^j \otimes u = 0,$$

so that $rx^{j-1} \otimes u = 0$ because the left $R[x, f]$-module $\Phi(E)$ is x-torsion-free. Therefore

$$\mathfrak{P} = \left\{r \in R : rx^n \otimes u = 0 \text{ in } Rx^n \otimes_R E \text{ for all } n \geq 0\right\}.$$

Let T be the $R[x, f]$-submodule of $\Phi(E)$ generated by $(0 :_E \mathfrak{m}) \subseteq R \otimes_R E$. We thus see that $\mathfrak{P}R[x, f] = \text{gr-ann}_{R[x,f]}T$, and this is $\mathfrak{q}R[x, f]$ by Theorem 5.1. Hence $\mathfrak{P} = \mathfrak{q}$. □

Remark 5.4 Suppose that $(R, \mathfrak{m})$ is F-pure and a homomorphic image of an excellent regular local ring S of characteristic p modulo an ideal $\mathfrak{A}$. By Theorem 5.1(i), there exists a unique largest $\Phi(E)$-special proper ideal, $\mathfrak{q}$ say, of R and this is prime. Let $\mathfrak{Q}$ be the unique ideal of S containing $\mathfrak{A}$ for which $\mathfrak{Q}/\mathfrak{A} = \mathfrak{q}$.

(i) The results of this section suggest that $\mathfrak{q}$ can be viewed as a generalization of Aberbach's and Enescu's splitting prime: for example, Proposition 5.3 shows that $\mathfrak{q}$ is that splitting prime in the case where R is, in addition, F-finite.
(ii) Note that $R/\mathfrak{q}$ is strongly F-regular (in the sense of Lyubeznik and Smith mentioned at the beginning of the section).
(iii) By Proposition 3.2, we have $(\mathfrak{A}^{[p^n]} : \mathfrak{A}) \subseteq (\mathfrak{Q}^{[p^n]} : \mathfrak{Q})$ for all $n \in \mathbb{N}$. In the special case in which S is F-finite, this result was obtained by Aberbach and Enescu [1, Proposition 4.4].

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