Acta Math Vietnam (2016) 41:91–102
DOI 10.1007/s40306-014-0099-3

Some Geometric Properties of Generalized Cesàro-Musielak-Orlicz Spaces Equipped with the Amemiya Norm

Atanu Manna · P. D. Srivastava

Received: 12 December 2013 / Accepted: 6 March 2014 /
Published online: 23 November 2014

Abstract A generalized Cesàro-Musielak-Orlicz sequence space $Ces_{\Phi}(q)$ endowed with the Amemiya norm is introduced. Criteria for the coordinatewise uniformly Kadec-Klee property and the uniform Opial property of the space $Ces_{\Phi}(q)$ with respect to the Amemiya norm are obtained.

Keywords Musielak-Orlicz function · Generalized Cesàro means · Amemiya norm · Coordinatewise Kadec-Klee property · Uniform Opial property

Mathematics Subject Classification (2010) 46A45 · 46A80 · 46B20 · 46B45 · 46E30

1 Introduction

The study of geometric properties of Banach spaces such as Kadec-Klee property, Opial property, and their several generalizations play very important role in metric fixed point theory. In particular, the Opial property of a Banach space has a great importance in the fixed point theory, differential equation, and integral equations. On the other hand, the Kadec-Klee property has several applications in Ergodic theory and many other branches [23].

Recently, several authors are interested in studying the geometric properties of Cesàro, Cesàro-Orlicz, and Musielak-Orlicz sequence spaces due to their several applications in various branches of mathematical analysis. Some topological properties such as order continuity, separability, completeness, and relations between norm and modular

A. Manna (✉) · P. D. Srivastava
Department of Mathematics, IIT Kharagpur, West Bengal 721302, India
e-mail: atanumanna@maths.iitkgp.ernet.in

P. D. Srivastava
e-mail: pds@maths.iitkgp.ernet.in

as well as some geometric properties such as the Fatou property, monotonicity, Kadec-Klee property, uniform Opial property, rotundity, local rotundity, etc. are discussed in [2, 3, 6, 11, 20, 21]. Khan [13, 14] has introduced the Riesz-Musielak-Orlicz sequence space and studied some geometric properties of it. Recently, Mongkolkeha and Kumam [16] studied (H)-property and uniform Opial property for the generalized Cesàro sequence space $Ces_{(p)}(q)$. Quite recently, Manna and Srivastava [15] introduced the generalized Cesàro-Musielak-Orlicz sequence space $Ces_{\Phi}(q)$, which include the well-known Cesàro [25], generalized Cesàro [21, 22], Cesàro-Orlicz [2, 20], Cesàro-Musielak-Orlicz [26] sequence spaces, etc. as in particular cases, and studied coordinatewise uniformly Kadec-Klee property and uniform Opial property for these spaces equipped with the Luxemberg norm. In this paper, we continue our study by investigating these properties in generalized Cesàro-Musielak-Orlicz sequence space with respect to the Amemiya norm.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ the set of natural numbers, of reals, and of nonnegative reals, respectively. Let $(X, \|.\|)$ be a Banach space and l^0 be the space of all real sequences $x = (x(i))_{i=1}^{\infty}$. Let $S(X)$ and $B(X)$ denote the unit sphere and closed unit ball, respectively.

Let $(E, \|.\|_E)$ be a real normed linear subspace of l^0. E is said to be a *normed sequence lattice* [12] if it satisfies the following two conditions:

(i) For any $x \in E$ and $y \in l^0$ such that $|y(k)| \leq |x(k)|$ for every $k \in \mathbb{N}$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$.
(ii) There exists a sequence $x = (x(k))_{k=1}^{\infty} \in E$ such that $x(k) > 0$ for all $k \in \mathbb{N}$.

A *normed sequence lattice* $(E, \|.\|_E)$ with complete norm is called *Banach sequence lattice* [12].

Note In many literatures, *Banach sequence lattice* E is also called *Köthe sequence space* [2, 20]. □

An element $x \in E$ is said to be *order continuous* if for any sequence $(x_l) \subset E_+$, where $x_l = (x_l(i))_{i=1}^{\infty}, l \in \mathbb{N}$ such that $|x(i)| \geq x_l(i) \searrow 0$, i.e., $x_l(i)$ decreases to zero as $l \to \infty$ for each $i \in \mathbb{N}$, implies that $\|x_l\|_E \to 0$. The set of all order-continuous elements in E is denoted by E_a. A *Banach sequence lattice* E is said to be order continuous if $E_a = E$. It is known that E is order continuous if and only if [2]

$$\|(0, 0, \ldots, x(i+1), x(i+2), \ldots)\|_E \to 0 \text{ as } i \to \infty \text{ for any } x \in E.$$

A sequence $(x_l) \subset X$ is said to be ε-separated sequence if the separation of the sequence (x_l), defined by $\text{sep}(x_l) = \inf\{\|x_l - x_m\| : l \neq m\}$ is $> \varepsilon$ for some $\varepsilon > 0$ [9].

A Banach space X is said to have the *Kadec-Klee property*, denoted by (H), if each weakly convergent sequence on the unit sphere is strongly convergent, i.e., convergent in norm [10]. A Banach space X is said to possess *coordinatewise* Kadec-Klee property, denoted by (H_c) [7], if $x \in X$ and every sequence $(x_l) \subset X$ such that

$$\|x_l\| \to \|x\| \text{ and } x_l(i) \to x(i) \text{ for each } i \in \mathbb{N}, \text{ then } \|x_l - x\| \to 0.$$

It is known that $X \in (H_c)$ implies $X \in (H)$, because the weak convergence in any Köthe sequence space X implies the coordinatewise convergence (see also [7]). A Banach space X has the *coordinatewise uniformly* Kadec-Klee property, denoted by (UKK_c) [27], if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$(x_l) \subset B(X)$, $\text{sep}(x_l) \geq \varepsilon$, $\|x_l\| \to \|x\|$ and $x_l(i) \to x(i)$ for each $i \in \mathbb{N}$, implies $\|x\| \leq 1 - \delta$.

It is well known that the property (UKK$_c$) implies property (H_c).

A Banach space X is said to have the *Opial property* [24] if for every weakly null sequence $(x_l) \subset X$ and every nonzero $x \in X$, we have

$$\liminf_{l\to\infty} \|x_l\| < \liminf_{l\to\infty} \|x_l + x\|.$$

A Banach space X is said to have the *uniform* Opial property [24] if for each $\varepsilon > 0$ there exists $\mu > 0$ such that for any weakly null sequence (x_l) in $S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$, the following inequality holds:

$$1 + \mu \leq \liminf_{l\to\infty} \|x_l + x\|.$$

In any Banach space X the Opial property is important because it ensures that X has the weak fixed point property [8]. Opial in [18] has shown that the space $L_p[0, 2\pi](p \neq 2, 1 < p < \infty)$ does not have this property but the Lebesgue sequence space $l_p(1 < p < \infty)$ has.

For a real vector space X, a functional $\varrho : X \to [0, \infty]$ is called a modular, if for arbitrary $x, y \in X$, the following conditions hold:

(i) $\varrho(x) = 0$ if and only if $x = 0$,
(ii) $\varrho(-x) = \varrho(x)$,
(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$ for $\alpha, \beta \geq 0, \alpha + \beta = 1$.

If instead of (iii), there holds

(iii)′ $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$ for $\alpha, \beta \geq 0, \alpha + \beta = 1$, then the modular ϱ is called convex.

For any modular ϱ on X, the modular space generated by the modular ϱ is denoted by X_ϱ and is defined as

$$X_\varrho = \{x \in X : \ \varrho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\}.$$

It is shown by Orlicz [19] that the modular space X_ϱ is equivalent to the set $X_\varrho^* = \{x \in X : \ \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}$ in the case of convex modular ϱ.

A sequence (x_n) of elements of X_ϱ is called modular convergent to $x \in X_\varrho$ if there exists a $\lambda > 0$ such that $\varrho(\lambda(x_n - x)) \to$ as $n \to \infty$.

In particular, in the case that ϱ is a convex modular, X_ϱ becomes a normed linear space with the norm $\|.\|_\varrho$ induced by the convex modular ϱ defined by

$$\|x\|_\varrho^L = \inf\left\{r > 0 : \varrho\left(\frac{x}{r}\right) \leq 1\right\} \text{ for } x \in X_\varrho.$$

In the case of convex modular ϱ, it is shown in [17] that the functional

$$\|x\|_\varrho^A = \inf_{k>0} \frac{1}{k}\{1 + \varrho(kx)\}$$

defines a norm on X_ϱ and the relation $\|x\|_\varrho^L \leq \|x\|_\varrho^A \leq 2\|x\|_\varrho^L$ holds for every $x \in X_\varrho$. The norms $\|x\|_\varrho^L$ and $\|x\|_\varrho^A$ defined on the modular space X_ϱ are called the Luxemberg norm and the Amemiya norm, respectively (see [17]).

A map $\varphi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if it is an even, convex, left continuous on $[0, \infty)$, $\varphi(0) = 0$, not identically zero, and $\varphi(u) \to \infty$ as $u \to \infty$. A sequence $\Phi = (\varphi_n)_{n=1}^{\infty}$ of Orlicz functions φ_n is called a Musielak-Orlicz function [17]. A Musielak-Orlicz function $\Phi = (\varphi_n)_{n=1}^{\infty}$ is said to satisfy condition (∞_1) if for each $n \in \mathbb{N}$, we have

$$(\infty_1): \quad \lim_{u \to +\infty} \frac{\varphi_n(u)}{u} = +\infty.$$

For any Musielak-Orlicz function Φ, the complementary function $\Psi = (\psi_n)$ of Φ is defined in the sense of Young as

$$\psi_n(u) = \sup_{v \geq 0}\{\, |u|v - \varphi_n(v)\} \quad \text{for all } u \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Given any Musielak-Orlicz function Φ and $x = (x(n))_{n=1}^{\infty} \in l^0$, a convex modular $I_\Phi : l^0 \to [0, \infty]$ is defined by

$$I_\Phi(x) = \sum_{n=1}^{\infty} \varphi_n\left(|x(n)|\right)$$

and the linear space $l_\Phi = \{x \in l^0 : I_\Phi(rx) < \infty \text{ for some } r > 0\}$ is called a Musielak-Orlicz sequence space. The sequence spaces $(l_\Phi, \|x\|_{I_\Phi}^L)$ and $(l_\Phi, \|x\|_{I_\Phi}^A)$ are Banach spaces. The set of all $k > 0$ such that $\|x\|_{I_\Phi}^A = \frac{1}{k}(1 + I_\Phi(kx))$ is attained for a fixed $x \in l_\Phi^A$ is denoted by $K(x)$. Moreover, it is known that for any $x \in l_\Phi^A$, there exists a $k > 0$ such that $\|x\|_{I_\Phi}^A = \frac{1}{k}(1 + I_\Phi(kx))$ whenever $\frac{\varphi_n(u)}{u} \to \infty$ as $u \to \infty$ for each $n \in \mathbb{N}$ (see [5]). For the details about Musielak-Orlicz sequence spaces and their geometric properties, we refer to [1, 3, 11, 17].

A Musielak-Orlicz function Φ satisfies the δ_2^0-condition, denoted by $\Phi \in \delta_2^0$, if there are positive constants a, K, a natural number m, and a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $(c_n)_{n=m}^{\infty} \in l_1$ and the inequality

$$\varphi_n(2u) \leq K\varphi_n(u) + c_n \tag{1}$$

holds for every $n \in \mathbb{N}$ and $u \in \mathbb{R}$ whenever $\varphi_n(u) \leq a$. If a Musielak-Orlicz function Φ satisfies δ_2^0-condition with $m = 1$, then Φ is said to satisfy δ_2-condition [17].

A Musielak-Orlicz function $\Phi = (\varphi_n)_{n=1}^{\infty}$ is said to vanish only at zero, which is denoted by $\Phi > 0$ if $\varphi_n(u) > 0$ for any $n \in \mathbb{N}$ and $u > 0$.

2 Class $Ces_\Phi(q)$

Let $x \in l^0$ and $\Phi = (\varphi_n)_{n=1}^{\infty}$ be a Musielak-Orlicz function. Let $q = (q_n)_{n=1}^{\infty}, q_n \geq 1\ \forall\, n \in \mathbb{N}$ be a sequence of real numbers such that $Q_n = \sum_{k=1}^{n} q_k$. The sequence space $Ces_\Phi(q)$, being studied in [15] and is defined as follows:

$$Ces_\Phi(q) = \{x \in l^0 : R^q x \in l_\Phi\} = \{x \in l^0 : \sigma_\Phi(rx) < \infty \text{ for some } r > 0\},$$

where $\sigma_\Phi(x) = I_\Phi(R^q x) = \sum_{n=1}^{\infty} \varphi_n(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x(k)|)$ and R^q is a *generalized Cesàro means* map on l^0 defined as

$$R^q x = (R^q x(n))_{n=1}^{\infty}, \text{ with } R^q x(n) = \frac{1}{Q_n}\sum_{k=1}^{n} q_k|x(k)| \text{ for each } n = 1, 2, \ldots$$

Clearly, $Ces_\Phi(q)$ is a linear space and also becomes a normed linear space under the norms $\|x\|_{\sigma_\Phi}^L = \|R^q x\|_{I_\Phi}^L$ and $\|x\|_{\sigma_\Phi}^A = \|R^q x\|_{I_\Phi}^A$ introduced with the help of the norms on l_Φ. The space $Ces_\Phi(q)$ will be called generalized Cesàro-Musielak-Orlicz sequence space. For simplifying notation, we write $\|x\|_\Phi^L$ and $\|x\|_\Phi^A$ instead of $\|x\|_{\sigma\Phi}^L$ and $\|x\|_{\sigma\Phi}^A$, respectively.

The class $Ces_\Phi(q)$ includes the following classes in particular cases:

(i) For $q_n = 1, n = 1, 2, \ldots$, the $Ces_\Phi(q)$ reduces to Cesàro-Musielak-Orlicz sequence space ces_Φ studied by Wangkeeree [26], where

$$\mathrm{ces}_\Phi = \left\{ x \in l^0 : \sum_{n=1}^{\infty} \varphi_n \left(\frac{r}{n} \sum_{k=1}^{n} |x(k)| \right) < \infty \text{ for some } r > 0 \right\}.$$

(ii) For $\varphi_n = \varphi$ for any n, the space ces_Φ becomes the well-known Cesàro-Orlicz sequence space ces_φ, studied recently by Petrot and Suantai [20], Foralewski et al. [6], and Cui et al. [2].

(iii) For $\varphi_n(x) = |x|^{p_n}$, $p_n \geq 1$ for all n, the space $Ces_\Phi(q)$ reduces to the sequence space $Ces_{(p)}(q)$ studied by Mongkolkeha et al. [16], and for $\varphi_n(x) = |x|^{p_n}$ with $p_n = p \geq 1$ for all n, then $Ces_\Phi(q)$ reduces to the sequence space $Ces_p(q)$ studied by Khan [13].

Notation For any $x \in l^0$ and $i \in \mathbb{N}$, we shall use the following notations throughout the paper:

$$\begin{aligned}
x|_i &= (x(1), x(2), x(3), \ldots, x(i), 0, 0, \ldots), \text{ called the truncation of } x \text{ at } i, \\
x|_{\mathbb{N}-i} &= (0, 0, 0, \ldots, 0, x(i+1), x(i+2), \ldots), \\
x|_I &= \{x \in l^0 : x(i) \neq 0 \text{ for all } i \in I \subseteq \mathbb{N} \text{ and } x(i) = 0 \text{ for all } i \in \mathbb{N} \setminus I\} \text{ and} \\
\operatorname{supp} x &= \{i \in \mathbb{N} : x(i) \neq 0\}.
\end{aligned}$$

For simplifying notation, we write $\mathrm{Ces}_\Phi^A(q) = (\mathrm{Ces}_\Phi(q), \|.\|_\Phi^A)$. □

3 Main results

We assume throughout that the sequence space $Ces_\Phi^A(q)$ is nontrivial, i.e., $Ces_\Phi^A(q) \neq \{0\}$. It is easy to observe that the space $Ces_\Phi^A(q)$ belongs to the class of *normed sequence lattice*.

Theorem 1 *The space $Ces_\Phi^A(q) = (Ces_\Phi(q), \|.\|_\Phi^A)$ is a Banach space.*

Proof It is shown in [15, Theorem 1 (i)] that $Ces_\Phi^L(q) = (Ces_\Phi(q), \|.\|_\Phi^L)$ is a Banach space. But the norms $\|.\|_\Phi^L$ and $\|.\|_\Phi^A$ are equivalent, so the proof follows easily. □

Theorem 2 *Let $(Ces_\Phi^A(q))_a = \{x \in Ces_\Phi^A(q) : \sigma_\Phi(rx) < \infty$, for all $r > 0\}$. Then the following statements are true:*

(i) $(Ces_\Phi^A(q))_a$ *is a closed subspace of* $Ces_\Phi^A(q)$,

(ii) $(Ces_\Phi^A(q))_a \subseteq \{x \in Ces_\Phi^A(q) : \|x - x|_j\|_\Phi^A \to 0\}$,

(iii) *if Φ is a Musielak-Orlicz function satisfying the condition δ_2 then* $(Ces_\Phi^A(q))_a = Ces_\Phi^A(q)$.

Proof (i) Clearly $(Ces_\Phi^A(q))_a$ is a subspace of $Ces_\Phi^A(q)$. It is required to show that $(Ces_\Phi^A(q))_a$ is closed in $Ces_\Phi^A(q)$. For this, let $x \in \overline{(Ces_\Phi^A(q))_a}$, the closure of $(Ces_\Phi^A(q))_a$. So there exists $x_i = (x_i(k))_{k=1}^\infty \in (Ces_\Phi^A(q))_a$ for each $i \in \mathbb{N}$ such that $\|x - x_i\|_\Phi^A \to 0$ as $i \to \infty$. We prove that $x \in (Ces_\Phi^A(q))_a$. By the equivalence definition of norm and modular convergence, we have $\sigma_\Phi(r(x - x_i)) \to 0$ as $i \to \infty$ for all $r > 0$. So for all $r > 0$, there exists $J \in \mathbb{N}$ such that $\sigma_\Phi(2r(x - x_J)) < 1$. Since $x_J \in (Ces_\Phi^A(q))_a$, so we have $\sigma_\Phi(2rx_J) < \infty$ for all $r > 0$. Now, consider

$$\begin{aligned}\sum_{n=1}^\infty \varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) &\le \sum_{n=1}^\infty \varphi_n\left(\frac{r}{2Q_n}\sum_{k=1}^n (2q_k|x(k) - x_J(k)|)\right.\\ &\quad \left. + \frac{r}{2Q_n}\sum_{k=1}^n 2q_k|x_J(k)|\right)\\ &\le \frac{1}{2}\sigma_\Phi(2r(x - x_J)) + \frac{1}{2}\sigma_\Phi(2rx_J) < \infty.\end{aligned}$$

Since r is arbitrary, so we have $x \in (\mathrm{Ces}_\Phi^A(q))_a$.

(ii) Let $A = \{x \in Ces_\Phi^A(q) : \|x - x|_j\|_\Phi^A \to 0\}$, $x \in (Ces_\Phi^A(q))_a$ and $\epsilon > 0$. Since $x \in (Ces_\Phi^A(q))_a$, so there exists $j_0 \in \mathbb{N}$ such that

$$\sigma_\Phi\left(\frac{x - x|_j}{\epsilon}\right) = \sum_{n=j+1}^\infty \varphi_n\left(\frac{1}{\epsilon Q_n}\sum_{k=j+1}^n |q_k x(k)|\right) < \epsilon$$

for all $j > j_0$. Hence, by the definition of norm $\|.\|_\Phi^A$, we have

$$\|x - x|_j\|_\Phi^A \le \epsilon\left(1 + \sigma_\Phi\left(\frac{x - x|_j}{\epsilon}\right)\right) < \epsilon(1 + \epsilon)$$

for all $j > j_0$. Since ϵ is arbitrary, we have $\|x - x|_j\|_\Phi^A \to 0$ as $j \to \infty$. So $x \in A$.

(iii) We show only the inclusion $Ces_\Phi^A(q) \subset (Ces_\Phi^A(q))_a$ because the other inclusion is always true. Let $x \in Ces_\Phi^A(q)$. Then for some $t > 0$, $\sigma_\Phi(tx) < \infty$, i.e., $\sum_{n=1}^\infty \varphi_n(\frac{t}{Q_n}\sum_{k=1}^n q_k|x(k)|) < \infty$. We show that for any $r > 0$

$$\sum_{n=1}^\infty \varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^n |q_k x(k)|\right) < \infty$$

holds. If $r \in [0, t]$, t is fixed, then it follows easily because

$$\sum_{n=1}^\infty \varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) \le \sum_{n=1}^\infty \varphi_n\left(\frac{t}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) < \infty.$$

Now choose $r > t$. Since $x \in Ces_\Phi^A(q)$, i.e., for some $t > 0$, $\sigma_\Phi(tx) < \infty$, there exists a finite positive constant a such that

$$\sum_{n=1}^\infty \varphi_n\left(\frac{t}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) \le a.$$

Therefore for each $n \geq 1$, we have

$$\varphi_n\left(\frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\right) \leq a.$$

Since $\Phi = (\varphi_n)_{n=1}^{\infty}$ satisfies the δ_2-condition, so by definition there are positive constants a and K and a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $(c_n)_{n=1}^{\infty} \in l_1$ and the inequality

$$\varphi_n(2u) \leq K\varphi_n(u) + c_n$$

holds for every $n \in \mathbb{N}$ and $u \in \mathbb{R}$ whenever $\varphi_n(u) \leq a$. Let $u = \frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|$, $K > 0$ be a constant and a be chosen as above. Since $r > t$, there exists $l \in \mathbb{N}$ such that $r \leq 2^l t$. Now applying the δ_2-condition for all $n \geq 1$, we have

$$\begin{aligned}\varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\right) &\leq \varphi_n\left(\frac{2^l t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\right) + c_n \\ &\leq K^l\varphi_n\left(\frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\right) + \left(\sum_{i=0}^{l-1} K^i\right) c_n.\end{aligned}$$

Taking summation in both sides over $n \geq 1$, we obtain

$$\sum_{n=1}^{\infty}\varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\right) \leq K^l\sum_{n=1}^{\infty}\varphi_n\left(\frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\right) + \left(\sum_{i=0}^{l-1} K^i\right)\sum_{n=1}^{\infty} c_n < \infty.$$

Hence, $x \in (\mathrm{Ces}_{\Phi}^{A}(q))_a$. □

We will assume in the rest of the paper that the Musielak-Orlicz function $\Phi = (\varphi_n)_{n=1}^{\infty}$ with $\varphi_n(u) < \infty$ for each $n \in \mathbb{N}, u \in \mathbb{R}$. The following lemmas are useful to prove our result.

Lemma 1 *Suppose $\Phi \in \delta_2$ and $\Phi > 0$. Then for any $(x_l) \subset Ces_{\Phi}^{A}(q)$, where $x_l = (x_l(i))_{i=1}^{\infty}, l \in \mathbb{N}$, $\|x_l\|_{\Phi}^{A} \to 0$ if and only if $\sigma_{\Phi}(x_l) \to 0$.*

Proof See [7, 11]. □

It is noted that, for a fixed $x \in \mathrm{Ces}_{\Phi}^{A}(q)$, the set $K(x)$ defined earlier (see Section 1) has the form $K(x) = \{k > 0 : \frac{1}{k}(1 + \sigma_{\Phi}(kx)) = \|x\|_{\Phi}^{A}\}$.

Lemma 2 *Let $x \in Ces_{\Phi}^{A}(q)$ be given and $x \neq 0$. If $K(x) = \emptyset$, then $\|x\|_{\Phi}^{A} = \sum_{n=1}^{\infty}\lambda_n R^q x(n)$, where $\lambda_n = \lim_{u\to\infty}\frac{\varphi_n(u)}{u}$ and $R^q x(n) = \frac{1}{Q_n}\sum_{i=1}^{n} q_i|x(i)|, n \in \mathbb{N}$.*

Proof Let $f(k) = \frac{1}{k}(1 + \sigma_{\Phi}(kx))$, where $\sigma_{\Phi}(x) = \sum_{n=1}^{\infty}\varphi_n(\frac{1}{Q_n}\sum_{i=1}^{n} q_i|x(i)|) = \sum_{n=1}^{\infty}\varphi_n(R^q x(n))$. Since $f(k)$ is continuous and $K(x) = \emptyset$, so we have $\|x\|_{\Phi}^{A} = \lim_{k\to\infty} f(k) = \lim_{k\to\infty}\frac{\sigma_{\Phi}(kx)}{k}$. Then $\lambda_n = \lim_{u\to\infty}\frac{\varphi_n(u)}{u}$ is finite for all $n \in \operatorname{supp} x$. If not, there exists a $n_0 \in \operatorname{supp} x$ such that

$$\|x\|_{\Phi}^{A} = \lim_{k\to\infty} \frac{\sigma_{\Phi}(kx)}{k} \geq \lim_{k\to\infty} \frac{\varphi_{n_0}(kR^q x(n_0))}{kR^q x(n_0)} R^q x(n_0) = \infty.$$

So we have

$$\|x\|_{\Phi}^{A} = \lim_{k\to\infty} \frac{\sigma_{\Phi}(kx)}{k} = \lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{\varphi_n(kR^q x(n))}{kR^q x(n)} R^q x(n) = \sum_{n=1}^{\infty} \lambda_n R^q x(n).$$

□

Lemma 3 *Let $x \in Ces_{\Phi}^{A}(q)$ be given and $x \neq 0$. If $\Phi = (\varphi_n)_{n=1}^{\infty}$ is a Musielak-Orlicz function satisfying condition (∞_1), then $K(x) \neq \emptyset$.*

Proof Suppose on contrary that $K(x) = \emptyset$. Then by Lemma 2, we obtain $\lim_{u\to\infty} \frac{\varphi_n(u)}{u} < \infty$ for each $n \in$ supp x, a contradiction to the assumption that Φ satisfies the condition (∞_1). □

Theorem 3 *The sequence space $Ces_{\Phi}^{A}(q)$ has the UKK_c-property whenever $\Phi = (\varphi_n)_{n=1}^{\infty}$ satisfying condition (∞_1), $\Phi \in \delta_2$, i.e., (1) and $\Phi > 0$.*

Proof Let $\epsilon > 0$ be given, $(x_l) \subset B(Ces_{\Phi}^{A}(q))$, $x \in Ces_{\Phi}^{A}(q)$, $\|x_l\|_{\Phi}^{A} \to \|x\|_{\Phi}^{A}$, $x_l(i) \to x(i)$ for each $i \in \mathbb{N}$ and sep$(x_l) \geq \epsilon$. We prove that $\|x\|_{\Phi}^{A} \leq 1 - \delta$. It trivially holds when $x = 0$. Let us assume that $x \neq 0$. Then using Lemma 3, we have $K(x) \neq \emptyset$, i.e., for each $x \in Ces_{\Phi}^{A}(q)$, there exists a $k_l \in \mathbb{R}_+$ such that $\|x\|_{\Phi}^{A} = \frac{1}{k_l}(1 + \sigma_{\Phi}(k_l x))$. Since $x_l \to x$ in $Ces_{\Phi}^{A}(q)$ weakly, it implies $x_l(i) \to x(i)$ for each $i \in \mathbb{N}$, so we may select a finite set $I = \{1, 2, 3, \ldots, N-1\}$ for which $x_l \to x$ uniformly. So, there exists $l_N \in \mathbb{N}$ such that

$$\|(x_l - x_m)|_I\|_{\Phi}^{A} \leq \frac{\epsilon}{2} \text{ for all } l, m \geq l_N. \tag{2}$$

Since sep$(x_l) \geq \epsilon$, we have $\|x_l - x_m\|_{\Phi}^{A} \geq \epsilon$ for $l \neq m$ by definition. This, together with (2), implies that $\|(x_l - x_m)|_{\mathbb{N}-I}\|_{\Phi}^{A} \geq \frac{\epsilon}{2}$ for $l \neq m$ and $l, m \geq l_N$. Hence, for each $N \in \mathbb{N}$, there exists a l_N such that $\|x_{l_N}|_{\mathbb{N}-I}\|_{\Phi}^{A} \geq \frac{\epsilon}{4}$. Without loss of generality, we may assume that $\|x_l|_{\mathbb{N}-I}\|_{\Phi}^{A} \geq \frac{\epsilon}{4}$ for all $l, N \in \mathbb{N}$. Therefore, by Lemma 1, there exists $\delta_1 \in (0, \epsilon)$ such that $\sigma_{\Phi}(x_l|_{\mathbb{N}-I}) \geq \delta_1$.

Since $x \in Ces_{\Phi}^{A}(q)$ implies that $\|x - x|_I\|_{\Phi}^{A} \to 0$ for sufficiently large N, there exists a $\frac{\delta_1}{2} > 0$ such that $\|x|_I\|_{\Phi}^{A} > \|x\|_{\Phi}^{A} - \frac{\delta_1}{2}$. Also, since $x_l(i) \to x(i)$ for each i and $\|x_l\|_{\Phi}^{A} \to \|x\|_{\Phi}^{A}$, there exists $N_0 \in \mathbb{N}$ such that

$$\|x_l|_I\|_{\Phi}^{A} > \|x\|_{\Phi}^{A} - \frac{\delta_1}{2} \text{ for } l > N_0.$$

Since $\|x_l\|_{\Phi}^{A} \leq 1$ implies $k_l \geq 1$ for all $l \in \mathbb{N}$, so by the convexity of φ_n and the inequality $\varphi_n(u + v) \geq \varphi_n(u) + \varphi_n(v)$ for all $u, v \in \mathbb{R}^+$ for each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
1 &\geq \|x_l\|_{\Phi}^{A} \\
&= \frac{1}{k_l}\left(1 + \sum_{n=1}^{N-1} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=1}^{n} q_j|x_l(j)|\right) + \sum_{n=N}^{\infty} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=1}^{n} q_j|x_l(j)|\right)\right) \\
&= \frac{1}{k_l}\left(1 + \sum_{n=1}^{N-1} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=1}^{n} q_j|x_l(j)|\right)\right) + \frac{1}{k_l}\left(\sum_{n=N}^{\infty} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=1}^{N-1} q_j|x_l(j)|\right.\right. \\
&\quad \left.\left. + \frac{k_l}{Q_n}\sum_{j=N}^{n} q_j|x_l(j)|\right)\right) \\
&\geq \frac{1}{k_l}\left(1 + \sum_{n=1}^{N-1} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=1}^{n} q_j|x_l(j)|\right)\right) + \frac{1}{k_l}\sum_{n=N}^{\infty} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=1}^{N-1} q_j|x_l(j)|\right) \\
&\quad + \frac{1}{k_l}\sum_{n=N}^{\infty} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=N}^{n} q_j|x_l(j)|\right) \\
&= \frac{1}{k_l}\left(1 + \sigma_{\Phi}\left(k_l x_l|_I\right)\right) + \frac{1}{k_l}\sum_{n=N}^{\infty} \varphi_n\left(\frac{k_l}{Q_n}\sum_{j=N}^{n} q_j|x_l(j)|\right) \\
&\geq \frac{1}{k_l}\left(1 + \sigma_{\Phi}\left(k_l x_l|_I\right)\right) + \sum_{n=N}^{\infty} \varphi_n\left(\frac{1}{Q_n}\sum_{j=N}^{n} q_j|x_l(j)|\right) \quad (\text{since } k_l \geq 1) \\
&\geq \|x_l|_I\|_{\Phi}^{A} + \sigma_{\Phi}\left(x_l|_{\mathbb{N}-I}\right) \\
&> \|x\|_{\Phi}^{A} - \frac{\delta_1}{2} + \delta_1 = \|x\|_{\Phi}^{A} + \frac{\delta_1}{2} \text{ for } l > N_0.
\end{aligned}
$$

Therefore, $\|x\|_{\Phi}^{A} \leq 1 - \frac{\delta_1}{2}$. Thus, $Ces_{\Phi}^{A}(q)$ has the coordinatewise uniform Kadec-Klee property. □

Corollary 1 (i) *If $\varphi_n = \varphi$ for all $n \in \mathbb{N}, q_n = 1$ for $n \in \mathbb{N}$ and $\varphi \in \delta_2$, then the Cesàro-Orlicz sequence space ces_{φ}^{A}* [20] *has the (UKK_c).*

(ii) *If $\varphi_n(u) = |u|^{p_n}$ for all $u \in \mathbb{R}$, $1 < p_n < \infty\ \forall n$ then $Ces_{p}^{A}(q)$ has the (UKK_c).*

Theorem 4 *Let $\Phi > 0$ be a Musielak-Orlicz function satisfying conditions (∞_1) and δ_2, i.e., (1). Then, $Ces_{\Phi}^{A}(q)$ has the uniform Opial property.*

Proof Take any $\epsilon > 0$ and $x \in Ces_{\Phi}^{A}(q)$ with $\|x\|_{\Phi}^{A} \geq \epsilon$. Let $(x_l) \subset S(Ces_{\Phi}^{A}(q))$ be any weakly null sequence. We show that for every $\epsilon > 0$, there is a $\mu > 0$ such that

$$
\liminf_{l\to\infty} \|x_l + x\|_{\Phi}^{A} \geq 1 + \mu
$$

for each $x \in Ces_{\Phi}^{A}(q)$. Since $\Phi \in \delta_2$ and $\Phi > 0$, so by Lemma 1, there is a $\delta \in (0, \frac{4}{5})$ independent of x such that $\sigma_{\Phi}(\frac{x}{2}) \geq \delta$. Since $\Phi \in \delta_2$ implies $Ces_{\Phi}^{A}(q) = (Ces_{\Phi}^{A}(q))_a$ by

Theorem 2 (iii), we have $\|x - x|_n\|_\Phi^A \to 0$ as $n \to \infty$ (by Theorem 2(ii)). Therefore, for a given $\delta > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\left\|x - x|_{n_0}\right\|_\Phi^A = \left\|x|_{\mathbb{N}-n_0}\right\|_\Phi^A < \frac{\delta}{8}$$

and

$$\sum_{n=n_0+1}^{\infty} \varphi_n \left(\frac{1}{2Q_n} \sum_{j=1}^{n} q_j |x(j)| \right) < \frac{\delta}{8}.$$

Since $\sigma_\Phi(\frac{x}{2}) \geq \delta$, it follows that

$$\begin{aligned} \delta &\leq \sum_{n=1}^{n_0} \varphi_n \left(\frac{1}{2Q_n} \sum_{j=1}^{n} q_j |x(j)| \right) + \sum_{n=n_0+1}^{\infty} \varphi_n \left(\frac{1}{2Q_n} \sum_{j=1}^{n} q_j |x(j)| \right) \\ &< \sum_{n=1}^{n_0} \varphi_n \left(\frac{1}{2Q_n} \sum_{j=1}^{n} q_j |x(j)| \right) + \frac{\delta}{8}. \end{aligned}$$

This gives $\sum_{n=1}^{n_0} \varphi_n(\frac{1}{2Q_n} \sum_{j=1}^{n} q_j |x(j)|) > \frac{7\delta}{8}$. Since $x_l \to 0$ weakly, it implies $x_l(i) \to 0$ as $l \to \infty$ for each i, so we have $\sigma_\Phi(x_l|_{n_0}) \to 0$ as $l \to \infty$. Hence, by Lemma 1, there exists a natural number l_0 such that $\left\|x_l|_{n_0}\right\|_\Phi^A < \frac{\delta}{8}$ for all $l \geq l_0$. This, together with $(x_l) \subset S(Ces_\Phi^A(q))$, i.e., $\|x_l\|_\Phi^A = 1$, implies that

$$\left\|x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A > 1 - \frac{\delta}{8} \quad \text{for all} \quad l \geq l_0. \tag{3}$$

Now, for all $l \geq l_0$, we have

$$\begin{aligned} \|x_l + x\|_\Phi^A &= \left\|(x_l + x)|_{n_0} + (x_l + x)|_{\mathbb{N}-n_0}\right\|_\Phi^A \\ &\geq \left\|(x_l + x)|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A - \frac{\delta}{8} \\ &\geq \left\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A - \frac{\delta}{8} - \frac{\delta}{8} = \left\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A - \frac{\delta}{4}. \end{aligned}$$

Since Φ satisfies condition (∞_1), so by Lemma 3, there exists $k_l > 0$ such that for $l \geq l_0$, we have

$$\left\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A = \frac{1}{k_l} \left(1 + \sigma_\Phi \left(k_l \left(x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right)\right)\right).$$

Now, using the fact that $\sigma_\Phi(u + v) \geq \sigma_\Phi(u) + \sigma_\Phi(v)$, whenever $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$, we have

$$\begin{aligned} \|x_l + x\|_\Phi^A &\geq \frac{1}{k_l} + \frac{1}{k_l}\sigma_\Phi \left(k_l x|_{n_0}\right) + \frac{1}{k_l}\sigma_\Phi \left(k_l x_l|_{\mathbb{N}-n_0}\right) - \frac{\delta}{4} \\ &\geq \left\|x_l|_{\mathbb{N}-n_0}\right\|_\Phi^A + \frac{1}{k_l}\sigma_\Phi \left(k_l x|_{n_0}\right) - \frac{\delta}{4}. \end{aligned} \tag{4}$$

Without loss of generality, we may assume that $k_l \geq \frac{1}{2}$ for all l because if $k_l < \frac{1}{2}$, then we have $\|x_l + x\|_\Phi^A > 2 - \frac{\delta}{4} > 1 + \delta$. Using the convexity of Φ, we have $\sigma_\Phi(k_l x|_{n_0}) \geq$

$2k_l\sigma_\Phi(\frac{1}{2}x|_{n_0})$. Now using (3), from (4), we have

$$\begin{aligned}\|x_l + x\|_\Phi^A &\geq \|x_l|_{\mathbb{N}-n_0}\|_\Phi^A + 2\sigma_\Phi\left(\frac{1}{2}x|_{n_0}\right) - \frac{\delta}{4} \\ &> \|x_l|_{\mathbb{N}-n_0}\|_\Phi^A + 2\sum_{n=1}^{n_0}\varphi_n\left(\frac{1}{2Q_n}\sum_{j=1}^{n} q_j|x(j)|\right) - \frac{\delta}{4} \\ &\geq 1 - \frac{\delta}{8} + 2.\frac{7\delta}{8} - \frac{\delta}{4} = 1 + \frac{11\delta}{8}.\end{aligned}$$

which implies that $\liminf_{n\to\infty}\|x_l + x\|_\Phi^A \geq 1 + \mu$, where μ depends upon δ. This completes the proof. □

Corollary 2 (i) *Let $q_n = 1, n = 1, 2, \ldots$ and $\varphi_n(u) = |u|^{p_n}$ for all $u \in \mathbb{R}$, $1 < p_n < \infty\ \forall n$. Then, $\Phi \in \delta_2$ if and only if* $\limsup_{n\to\infty} p_n < \infty$. *Therefore, $ces^A_{(p)}$* [21] *has the uniform Opial property.*

(ii) *If $\varphi_n = \varphi\ \forall n$, $q_n = 1$ for $n = 1, 2, \ldots$ and $\varphi \in \delta_2$, then the Cesàro-Orlicz sequence space ces^A_φ* [20] *has the uniform Opial property.*

Acknowledgments The authors are very much grateful to the editor and the anonymous referee for their careful reading and positive comments which improved the presentation of our manuscript. The first author is thankful to CSIR, New Delhi, Govt. of India, for the financial assistance during this work.

References

1. Chen, S.T.: Geometry of Orlicz spaces. Dissertationes Math. **356**, 1–204 (1996)
2. Cui, Y., Hudzik, H., Petrot, N., Suantai, S., Szymaszkiewicz, A.: Basic topological and geometric properties of Cesàro-Orlicz spaces. Proc. Indian Acad. Sci. (Math. Sci.) **115**(4), 461–476 (2005)
3. Cui, Y., Hudzik, H.: Maluta's coefficient and Opial's properties in Musielak-Orlicz sequence spaces equipped with the Luxemberg norm. Nonlinear Anal. **35**, 475–485 (1999)
4. Cui, Y., Hudzik, H.: On the uniform Opial property in some modular sequence spaces. Func. Approx. Comment. **26**, 93–102 (1998)
5. Cui, Y., Hudzik, H., Zhu, H.: Maluta's coefficient in Musielak-Orlicz sequence spaces equipped with the Orlicz norm. Proc. Am. Math. Soc. **126**(1), 115–121 (1998)
6. Foralewski, P., Hudzik, H., Szymaszkiewicz, A.: Some remarks on Cesàro-Orlicz spaces. Math. Inequal. Appl. **13**(2), 363–386 (2010)
7. Foralewski, P., Hudzik, H.: On some geometrical and topological properties of generalized Calderón-Lozanovskiǐ sequence spaces. Houston J. Math. **25**(3), 523–542 (1999)
8. Gossez, J.P., Lami Dozo, E.: Some geometric properties related to fixed point theory for nonexpansive mappings. Pacific J. Math. **40**, 565–573 (1972)
9. Huff, R.: Banach spaces which are nearly uniformly convex. Rocky Mountain J. Math. **10**, 743–749 (1980)
10. Kadets, M.I.: The relation between some properties of convexity of the unit ball of a Banach space. Funct. Anal. Appl. **16**(3), 204–206 (1982)
11. Kaminska, A.: Uniform rotundity of Musielak-Orlicz sequence spaces. J. Approx. Theory **47**(4), 302–322 (1986)
12. Kantorovich, L.V., Akilov, G.P.: Functional analysis, 2nd edn. Pergamon and "Nauka" (1982)
13. Khan, V.A.: Some geometric properties of generalized Cesaro sequence spaces. Acta Math. Univ. Comenianae **79**(1), 1–8 (2010)
14. Khan, V.A.: On Riesz-Musielak-Orlicz sequence spaces. Numer. Funct. Anal. Optim. **28**(7-8), 883–895 (2007)

15. Manna, A., Srivastava, P.D.: Some geometric properties of generalized Cesàro-Musielak-Orlicz sequence spaces. Mathematics and Computing 2013: International Conference in Haldia, India, Chapter 19, pp. 283–296 (2014)
16. Mongkolkeha, C., Kumam, P.: On H-property and uniform Opial property of generalized Cesàro sequence spaces. J. Inequal. Appl. **2012**, 76 (2012). 9 pp
17. Musielak, J.: Orlicz spaces and modular spaces, p. 1034. Springer (1983)
18. Opial, Z.: Weak convergence of the sequence of successive approximations for non expansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
19. Orlicz, W.: A note on modular spaces I. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9**, 157–162 (1961)
20. Petrot, N., Suantai, S.: Some geometric properties in Cesàro-Orlicz spaces. Sci. Asia **31**, 173–177 (2005)
21. Petrot, N., Suantai, S.: Uniform Opial properties in generalized Cesàro sequence spaces. Nonlinear Anal. **63**, 1116–1125 (2005)
22. Petrot, N., Suantai, S.: On uniform Kadec-Klee properties and rotundity in generalized Cesàro sequence spaces. Internat. J. Math. Math. Sci. **2004**(2), 91–97 (2004)
23. Prus, S.: Geometrical background of metric fixed point theory. In: Kirk, W.A., Sims, B. (eds.) Handbook of metric fixed point theory, pp. 93–132. Kluwer Academic, Dordrecht (2001)
24. Prus, S.: Banach spaces with uniform Opial property. Nonlinear Anal. Theory Appl. **18**(8), 697–704 (1992)
25. Shiue, J.S.: Cesàro sequence spaces. Tamkang J. Math. **1**, 19–25 (1970)
26. Wangkeeree, R.: On property (k-NUC) in Cesàro-Musielak-Orlicz sequence spaces. Thai J. Math. **1**, 119–130 (2003)
27. Zhang, T.: The coordinatewise uniformly Kadec-Klee property in some Banach spaces. Siberian Math. J. **44**(2), 363–365 (2003)