Acta Math Vietnam (2015) 40:37–60
DOI 10.1007/s40306-014-0096-6

The Chern Numbers and Euler Characteristics of Modules

L. Ghezzi · S. Goto · J. Hong · K. Ozeki · T. T. Phuong · W. V. Vasconcelos

Dedicated to Professors N.V. Trung and G. Valla for their groundbreaking contributions to the theory of Hilbert functions

Received: 1 January 2014 / Revised: 1 April 2014 / Accepted: 28 April 2014 /
Published online: 15 November 2014

Abstract The set of the first Hilbert coefficients of parameter ideals relative to a module—its Chern coefficients—over a local Noetherian ring codes for considerable information about its structure–noteworthy properties such as that of Cohen-Macaulayness, Buchsbaumness, and of having finitely generated local cohomology. The authors have previously studied the ring case. By developing a robust setting to treat these coefficients for unmixed rings and modules, the case of modules is analyzed in a more transparent manner. Another

L. Ghezzi
Department of Mathematics, New York City College of Technology-Cuny, 300 Jay Street, Brooklyn, NY 11201, USA
e-mail: lghezzi@citytech.cuny.edu

S. Goto (✉) · K. Ozeki
Department of Mathematics, School of Science and Technology, Meiji University
1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
e-mail: goto@math.meiji.ac.jp

K. Ozeki
e-mail: kozeki@math.meiji.ac.jp

J. Hong
Department of Mathematics, Southern Connecticut State University, 501 Crescent Street, New Haven, CT 06515-1533, USA
e-mail: hongj2@southernct.edu

T. T. Phuong
Department of Information Technology and Applied Mathematics, Ton Duc Thang University, 98 Ngo Tat To Street, Ward 19, Binh Thanh District, Ho Chi Minh City, Vietnam
e-mail: sugarphuong@gmail.com

W. V. Vasconcelos
Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA
e-mail: vasconce@math.rutgers.edu

series of integers arise from partial Euler characteristics and are shown to carry similar properties of the module. The technology of homological degree theory is also introduced in order to derive bounds for these two sets of numbers.

Keywords Hilbert function · Hilbert coefficient · Parameter ideal · Local cohomology · Euler characteristic · Cohen-Macaulay module · Vasconcelos module · Generalized Cohen-Macaulay module · Buchsbaum module

Mathematics Subject Classification (2010) 13H10 · 13H15 · 13A30

1 Introduction

Let $\mathbf{R}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let I be an $\mathfrak{m}$-primary ideal. There is a great deal of interest on the set of I-good filtrations of $\mathbf{R}$. More concretely, on the set of multiplicative, decreasing filtrations

$$\mathcal{A} = \{I_n \ \mid \ I_0 = \mathbf{R},\ I_{n+1} = II_n, n \gg 0\}$$

of $\mathbf{R}$ ideals which are integral over the I-adic filtration, conveniently coded in the corresponding Rees algebra and its associated graded ring

$$\mathcal{R}(\mathcal{A}) = \sum_{n\geq 0} I_n t^n, \quad \mathrm{gr}_{\mathcal{A}}(\mathbf{R}) = \sum_{n\geq 0} I_n/I_{n+1}.$$

Our focus here is on a set of filtrations both broader and more narrowly defined. Let M be a finitely generated $\mathbf{R}$-module. The Hilbert polynomial of the associated graded module

$$\mathrm{gr}_I(M) = \bigoplus_{n\geq 0} I^n M/I^{n+1}M,$$

more precisely the values of the length $\lambda(M/I^{n+1}M)$ of $M/I^{n+1}M$ for large n can be assembled as

$$P_M(n) = \sum_{i=0}^{r} (-1)^i e_i(I, M)\binom{n+r-i}{r-i},$$

where $r = \dim_{\mathbf{R}} M > 0$. In most of our discussion, either I or M is fixed, and by simplicity, we set $e_i(I, M) = e_i(M)$ or $e_i(I, M) = e_i(I)$ accordingly. Occasionally, the first Hilbert coefficient $e_1(I, M)$ is referred to as the *Chern coefficient* of I relative to M ([32]).

The authors have examined ([8, 9, 13, 32]) how the values of $e_1(Q, \mathbf{R})$ codes for structural information about the ring $\mathbf{R}$ itself. More explicitly, one defines the set

$$\Lambda(M) = \{e_1(Q, M) \ \mid \ Q \text{ is a parameter ideal for } M\}$$

and examines what its structure expresses about M. In case $M = \mathbf{R}$, this set was analyzed for the following extremal properties:

(a) $0 \in \Lambda(\mathbf{R})$.
(b) $\Lambda(\mathbf{R})$ contains a single element.
(c) $\Lambda(\mathbf{R})$ is bounded.

The task of determining the elements of $\Lambda(M)$ has turned out to be rather daunting. More amenable has been the approach to obtain specialized bounds using cohomological techniques. An unresolved issue has been to describe the character of the set $\Lambda(M)$, in particular the role of its extrema and the gap structure of the set itself.

The other invariant of the module M in our investigation is the following. Let $Q = (x_1, x_2, \ldots, x_r)$ be a parameter ideal for M. We denote by $\mathrm{H}_i(Q; M)$ $(i \in \mathbb{Z})$ the ith homology module of the Koszul complex $\mathrm{K}_\bullet(Q; M)$ generated by the system $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of parameters of M. We put

$$\chi_1(Q; M) = \sum_{i \geq 1} (-1)^{i-1} \lambda_{\mathbf{R}}(\mathrm{H}_i(Q; M))$$

and call it the first *Euler characteristic* of M relative to Q; hence,

$$\chi_1(Q; M) = \lambda_{\mathbf{R}}(M/QM) - e_0(Q, M)$$

by a classical result of Serre (see [1, 26]).

In analogy to $\Lambda(M)$, one defines the set

$$\Xi(M) = \{\chi_1(Q; M) \mid Q \text{ is a parameter ideal for } M\}$$

and examines again what its structure expresses about M. Most of the properties of this set can be assembled from a diverse literature, particularly from [26, Appendix II]. The outcome is a listing that mirrors, step-by-step, all the properties of the set $\Lambda(M)$ that we study.

We shall now describe more precisely our results. Section 2 starts with a review of some elementary computation rules for $e_1(Q, M)$ under hyperplane sections, more properly modulo superficial elements. Since part of our goal is to extend to modules our previous results on rings, given the ubiquity of the unmixedness hypothesis, we develop a fresh setting to treat the module case. It made for more transparent proofs. These are carried out in Sections 3–5.

Section 6 introduces homological degree techniques to obtain special bounds for the set $\Lambda(M)$. The treatment here is more general and sharper than in [32]. Thus, in Corollary 6.7, it is proved that the set

$$\Lambda_Q(M) = \{e_1(\mathfrak{q}, M) : \mathfrak{q} \text{ is a parameter ideal for } M \text{ with the same integral closure as that of } Q\}$$

is finite. In Section 7, we treat the sets $\Xi(M)$ and $\Xi_Q(M)$, focusing on the properties that have analogs in $\Lambda(M)$ (see Table 1). In particular, we prove that Euler characteristics can be uniformly bounded by homological degrees (Theorem 7.2). We also consider the numerical function, which we call the *Hilbert characteristic* of M with respect to $Q = (\mathbf{x})$:

$$\mathbf{h}(\mathbf{x}; M) = \sum_{i=0}^{r} (-1)^i e_i(Q, M).$$

If the system $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of parameters of M forms a d–sequence for M, $\mathbf{h}(\mathbf{x}; M)$ has some properties of a homological degree. They are enough to bound the Betti numbers $\beta_i(M)$ in terms of $\beta_i(\mathbf{R}/\mathfrak{m})$, a well-known property of cohomological degrees. Finally, in Section 8, we recast in the context of Buchsbaum-Rim coefficients several questions treated in this paper.

A street view of our results for the convenience of the reader is given in the following table. Let $\mathbf{R}$ be a Noetherian local ring with infinite residue class field and M, a finitely generated $\mathbf{R}$-module with $r = \dim_{\mathbf{R}} M \geq 2$. Let $\mathcal{P}(M)$ be the collection of systems

Table 1 Properties of a finitely generated module M carried by the values of either function

$\mathrm{X}_{\mathbf{f}}(M) \subseteq [0, \infty)$	M	[19]	[26, Appendix II]
$0 \in \mathrm{X}_{\mathbf{f}}(M)$	M Cohen-Macaulay	Theorem 3.1*	[26, Appendix II]
$\lvert\mathrm{X}_{\mathbf{f}}(M)\rvert < \infty$	M generalized Cohen-Macaulay	Theorem 4.5*	[6]
$\lvert\mathrm{X}_{\mathbf{f}}(M)\rvert = 1$	M Buchsbaum	Theorem 5.4*	[28]
$\lvert\{\mathbf{f}(\mathbf{x}) \mid \bar{Q} = \overline{(\mathbf{x})}\}\rvert < \infty$	Q	Corollary 6.7	Corollary 7.3

An adorned reference [XY]* requires that the module M be unmixed. The third and fourth columns refer to the functions $\mathbf{f}_1(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$, respectively

$\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of parameters of M. In [8] and in this paper, the authors study multiplicity derived numerical functions

$$\mathbf{f} : \mathcal{P}(M) \longrightarrow \mathbb{N}$$

on emphasis on the nature of its range

$$\mathrm{X}_{\mathbf{f}}(M) = \{\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{P}(M)\}.$$

For the two functions $e_1(\mathbf{x}, M)$ and $\chi_1(\mathbf{x}; M)$, more properly $\mathbf{f}_1(\mathbf{x}) = -e_1(\mathbf{x}, M)$ and $\mathbf{f}_2(\mathbf{x}) = \chi_1(\mathbf{x}; M)$, respectively, identical assertions about the character of $\mathrm{X}_{\mathbf{f}}(M)$ are expressed in the above grid.

2 Preliminaries

Throughout this section, let $\mathbf{R}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated $\mathbf{R}$–module. For basic terminology and properties of Noetherian rings and Cohen–Macaulay rings and modules, we make use of [3] and [20]. For convenience of exposition, we treat briefly the role of hyperplane sections in Hilbert functions and examine unmixed modules. We add further clarifications when we define homological degrees.

2.1 Hyperplane Sections and Hilbert Polynomials

We need rules to compute these coefficients. Typically, they involve the so-called *superficial elements* or *filter regular elements*. We keep the terminology of *generic hyperplane section*, even when dealing with Samuel's multiplicity with respect to an $\mathfrak{m}$–primary ideal I and its Hilbert coefficients $e_i(M) = e_i(I, M)$. Hopefully, this usage will not lead to undue confusion. We say that $h \in I$ is a *parameter* for M if $\dim_{\mathbf{R}} M/hM < \dim_{\mathbf{R}} M$.

Let us begin with the following.

Lemma 2.1 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring, I an $\mathfrak{m}$–primary ideal of* $\mathbf{R}$*, and M a finitely generated* $\mathbf{R}$*–module. Let $h \in I$ and suppose that* $\lambda(0 :_M h) < \infty$*. Then, we have the following.*

(a) *h is a parameter for M, if* $\dim_{\mathbf{R}} M > 0$.
(b) $\lambda(0 :_M h) \leq \lambda(\mathrm{H}^0_{\mathfrak{m}}(M/hM))$.
(c) [24, (1.5)] *If* $\dim_{\mathbf{R}} M > 1$ *and M/hM is Cohen–Macaulay, then M is Cohen–Macaulay.*

Proof Suppose that $\dim_{\mathbf{R}} M > 0$ and let $\mathfrak{p} \in \mathrm{Supp}_{\mathbf{R}} M$ with $\dim \mathbf{R}/\mathfrak{p} = \dim_{\mathbf{R}} M$. Then,

$$(0) :_{M_{\mathfrak{p}}} \frac{h}{1} = (0)$$

since $\mathfrak{p} \neq \mathfrak{m}$. As $\dim_{\mathbf{R}_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$, we get $h \notin \mathfrak{p}$. Hence, h is a parameter for M if $\dim_{\mathbf{R}} M > 0$.

We look at the exact sequence

$$0 \to (0) :_M h \to \mathrm{H}^0_{\mathfrak{m}}(M) \xrightarrow{h} \mathrm{H}^0_{\mathfrak{m}}(M) \xrightarrow{\varphi} \mathrm{H}^0_{\mathfrak{m}}(M/hM) \to \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{h} \mathrm{H}^1_{\mathfrak{m}}(M)$$
$$\to \mathrm{H}^1_{\mathfrak{m}}(M/hM) \to \cdots$$

of local cohomology modules derived from the exact sequence

$$0 \to (0) :_M h \to M \xrightarrow{h} M \to M/hM \to 0$$

of $\mathbf{R}$–modules. We then have

$$\lambda\left((0) :_M h\right) = \lambda(\mathrm{Im}\varphi) = \lambda\left(\mathrm{H}^0_{\mathfrak{m}}(M/hM)\right) - \lambda\left((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} h\right) \le \lambda\left(\mathrm{H}^0_{\mathfrak{m}}(M/hM)\right).$$

Therefore, if $\dim_{\mathbf{R}} M > 1$ and M/hM is Cohen–Macaulay, then h is M–regular and hence M is Cohen–Macaulay as well. □

We will make repeated use of [21, (22.6)] and [19, Section 3] (see also [23] and [24] for a more general version of these results).

Proposition 2.2 *Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring, I an $\mathfrak{m}$–primary ideal of $\mathbf{R}$, and M a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M > 0$. Let $h \in I$ and assume that h is superficial for M with respect to I (in particular $h \in I \setminus \mathfrak{m}I$).*

(a) *The Hilbert coefficients of M and M/hM satisfy*

$$e_i(M) = e_i(M/hM) \text{ for } 0 \le i < r-1 \text{ and}$$
$$e_{r-1}(M) = e_{r-1}(M/hM) + (-1)^r \lambda(0 :_M h).$$

(b) *Let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated $\mathbf{R}$–modules. If $t = \dim_{\mathbf{R}} A < s = \dim_{\mathbf{R}} B$, then $e_i(B) = e_i(C)$ for $0 \le i < s-t$. In particular, if $t = 0$ and $s \ge 2$, then $e_1(B) = e_1(C)$.*

(c) *If M is a module of dimension 1 and I is a parameter ideal for M, then*

$$e_1(M) = -\lambda(\mathrm{H}^0_{\mathfrak{m}}(M)).$$

(d) *If M is a module of dimension 2 and I is a parameter ideal for M, then*

$$e_1(M) = e_1(M/hM) + \lambda(0 :_M h) = -\lambda\left(\mathrm{H}^0_{\mathfrak{m}}(M/hM)\right) + \lambda(0 :_M h)$$
$$= -\lambda\left((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} h\right).$$

Proof See Proof of Lemma 2.1 for assertion (d). □

The following corollary was previously observed in [19]. By induction on $r = \dim_{\mathbf{R}} M$, it also can be achieved independently using Proposition 2.2.

Corollary 2.3 *If M is a module of positive dimension and I is a parameter ideal for M, then $e_1(I, M) \le 0$.*

2.2 Unmixed Modules

We recall the notion of unmixed local rings and modules and develop a setting to study their Hilbert coefficients.

Definition 2.4 Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then, we say that $\mathbf{R}$ is *unmixed* if $\dim \widehat{\mathbf{R}}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass}\widehat{\mathbf{R}}$, where $\widehat{\mathbf{R}}$ is the $\mathfrak{m}$–adic completion of $\mathbf{R}$. Similarly, let M be a finitely generated $\mathbf{R}$–module of dimension r. Then, we say that M is *unmixed* if $\dim \widehat{\mathbf{R}}/\mathfrak{p} = r$ for every $\mathfrak{p} \in \operatorname{Ass}_{\widehat{\mathbf{R}}}\widehat{M}$, where $\widehat{M}$ denotes the $\mathfrak{m}$–adic completion of M.

Our formulation of unmixedness is the following.

Theorem 2.5 *Let* $\mathbf{R}$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $\dim_{\mathbf{R}} M = \dim \mathbf{R}$. *Then, the following conditions are equivalent:*

(i) M *is an unmixed* $\mathbf{R}$*–module.*
(ii) *There exists a surjective homomorphism* $\mathbf{S} \to \widehat{\mathbf{R}}$ *of rings together with an embedding* $\widehat{M} \hookrightarrow \mathbf{S}^n$ *as an* $\mathbf{S}$*–module for some* $n > 0$, *where* $\mathbf{S}$ *is a Gorenstein local ring with* $\dim \mathbf{S} = \dim \mathbf{R}$.

Proof We have only to prove (i) $\Rightarrow$ (ii). We may assume $\mathbf{R}$ is complete. Thanks to Cohen's structure theorem of complete local rings, we can choose a surjective homomorphism $\mathbf{S} \to \mathbf{R}$ of rings such that $\mathbf{S}$ is a Gorenstein local ring with $\dim \mathbf{S} = \dim \mathbf{R}$. Then, because $\operatorname{Ass}_{\mathbf{S}} M \subseteq \operatorname{Ass}\mathbf{S}$ and the $\mathbf{S}_{\mathfrak{p}}$–module $M_{\mathfrak{p}}$ is reflexive for all $\mathfrak{p} \in \operatorname{Ass}_{\mathbf{S}} M$, the canonical map

$$M \to \operatorname{Hom}_{\mathbf{S}}(\operatorname{Hom}_{\mathbf{S}}(M, \mathbf{S}), \mathbf{S})$$

is injective, while we get an embedding

$$\operatorname{Hom}_{\mathbf{S}}(\operatorname{Hom}_{\mathbf{S}}(M, \mathbf{S}), \mathbf{S}) \hookrightarrow \mathbf{S}^{\mathbf{n}}$$

for some $n > 0$ because $\operatorname{Hom}_{\mathbf{S}}(M, \mathbf{S})$ is a finitely generated $\mathbf{S}$–module. Hence, the result. □

Corollary 2.6 ([11]) *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $\dim_{\mathbf{R}} M = \dim \mathbf{R} \geq 2$. *If* M *is an unmixed* $\mathbf{R}$*–module, then* $\mathrm{H}^1_{\mathfrak{m}}(M)$ *is finitely generated.*

Proof We may assume $\mathbf{R}$ is complete. We maintain the notation in Proof of Theorem 2.5 and let $\mathfrak{n}$ denote the maximal ideal of $\mathbf{S}$. Then, applying the functors $\mathrm{H}^i_{\mathfrak{n}}(*)$ to the exact sequence

$$0 \to M \to \mathbf{S}^n \to C \to 0$$

of $\mathbf{S}$–modules, we get $\mathrm{H}^1_{\mathfrak{m}}(M) \cong \mathrm{H}^0_{\mathfrak{n}}(C)$ because $\operatorname{depth}\mathbf{S} \geq 2$. Hence, $\mathrm{H}^1_{\mathfrak{m}}(M)$ is finitely generated. □

3 Vanishing of $e_1(Q, M)$

Let $\mathbf{R}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M$. Recall that a parameter ideal for M is an ideal $Q = (x_1, x_2, \ldots, x_r) \subseteq \mathfrak{m}$ in $\mathbf{R}$ with $\lambda(M/QM) < \infty$.

Theorem 3.1 *Let* $\mathbf{R}$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $\dim_{\mathbf{R}} M \geq 2$. *Suppose that* M *is unmixed and let* Q *be a parameter ideal for* M. *Then, the following conditions are equivalent:*

(i) M *is a Cohen–Macaulay* $\mathbf{R}$*–module.*
(ii) $e_1(Q, M) = 0$.

Proof We set $e_1(Q) = e_1(Q, M)$. It is enough to show that if M is not Cohen–Macaulay, then $e_1(Q) < 0$. We may assume that $\mathbf{R}$ is complete with an infinite residue field and $\dim \mathbf{R} = \dim M$.

Choose a Gorenstein local ring $(\mathbf{S}, \mathfrak{n})$ and a surjection $\mathbf{S} \to \mathbf{R}$, with $\dim \mathbf{S} = \dim \mathbf{R}$. If Q is a parameter ideal of $\mathbf{R}$, there exists a parameter ideal $\mathfrak{q}$ of $\mathbf{S}$ such that $\mathfrak{q}\mathbf{R} = Q$ ([9, Lemma 3.1]). Therefore, the associated graded module of Q relative to M is isomorphic to the associated graded module of $\mathfrak{q}$ with respect to the $\mathbf{S}$–module M:

$$\mathrm{gr}_Q(M) \simeq \mathrm{gr}_{\mathfrak{q}}(M),$$

which implies that

$$e_1(Q) = e_1(\mathfrak{q}, M),$$

where $e_1(\mathfrak{q}, M)$ denotes the first Hilbert coefficient of $\mathfrak{q}$ with respect to the $\mathbf{S}$–module M.

Consider the exact sequence of $\mathbf{S}$–modules obtained from Proposition 2.5:

$$0 \to M \to \mathbf{S}^{\mathbf{n}} \to C \to 0.$$

Let y be a superficial element for $\mathfrak{q}$ with respect to M such that y is part of a minimal generating set of $\mathfrak{q}$. We may assume that y is a nonzero divisor on M. By tensoring the exact sequence of $\mathbf{S}$–modules with $\mathbf{S}/(y)$, we get

$$0 \to T = \mathrm{Tor}_1^{\mathbf{S}}(\mathbf{S}/(y), C) \to M/yM \xrightarrow{\zeta} \mathbf{S}^{\mathbf{n}}/y\mathbf{S}^{\mathbf{n}} \to C/yC \to 0.$$

Let $M' = M/yM$ and $N = \mathrm{Im}(\zeta)$ and consider the short exact sequence:

$$0 \to T \to M' \to N \to 0.$$

Then, either $T = 0$ or T has finite length $\lambda(T) < \infty$. Note that N is an unmixed $\mathbf{S}/(y)$-module.

We use induction on $d = \dim M$ to show that if M is not Cohen–Macaulay, then $e_1(\mathfrak{q}, M) < 0$.

Let $d = 2$ and $\mathfrak{q} = (y, z)$. Then, $T \neq 0$ so that $\lambda(T) < \infty$. Applying the Snake Lemma to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T \cap z^n M' & \longrightarrow & z^n M' & \longrightarrow & z^n N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

we get for sufficiently large n,

$$\lambda(M'/z^n M') = \lambda(T) + \lambda(N/z^n N).$$

Computing the Hilbert polynomials, we have

$$e_1(\mathfrak{q}, M) = e_1(\mathfrak{q}/(y), M/yM) = -\lambda(T) < 0.$$

Now suppose that $d \geq 3$. From the exact sequence

$$0 \to T \to M' = M/yM \to N \to 0,$$

we have

$$e_1(\mathfrak{q}, M) = e_1(\mathfrak{q}/(y), M/yM) = e_1(\mathfrak{q}/(y), N).$$

By an induction argument, it is enough to show that N is not Cohen–Macaulay since $\dim(\mathbf{S}/(y)) = d - 1$.

Suppose that N is Cohen–Macaulay. Let $\mathfrak{n}$ be the maximal ideal of $\mathbf{S}/y\mathbf{S}$. From the exact sequence

$$0 \to T \to M' = M/yM \to N \to 0,$$

we obtain the long exact sequence:

$$0 \to \mathrm{H}^0_{\mathfrak{n}}(T) \to \mathrm{H}^0_{\mathfrak{n}}(M') \to \mathrm{H}^0_{\mathfrak{n}}(N) \to \mathrm{H}^1_{\mathfrak{n}}(T) \to \mathrm{H}^1_{\mathfrak{n}}(M') \to \mathrm{H}^1_{\mathfrak{n}}(N).$$

By the assumption that N is Cohen–Macaulay of dimension $d - 1 \geq 2$ and the fact that T is a torsion module, we get

$$0 \to T \simeq \mathrm{H}^0_{\mathfrak{n}}(M') \to 0 \to 0 \to \mathrm{H}^1_{\mathfrak{n}}(M') \to 0.$$

From the exact sequence

$$0 \to M \overset{\cdot y}{\to} M \to M' = M/yM \to 0,$$

we obtain the following exact sequence:

$$0 \to T \simeq \mathrm{H}^0_{\mathfrak{n}}(M') \to \mathrm{H}^1_{\mathfrak{n}}(M) \overset{\cdot y}{\to} \mathrm{H}^1_{\mathfrak{n}}(M) \to \mathrm{H}^1_{\mathfrak{n}}(M') = 0.$$

Since $\mathrm{H}^1_{\mathfrak{n}}(M)$ is finitely generated by Corollary 2.6 and $\mathrm{H}^1_{\mathfrak{n}}(M) = y\mathrm{H}^1_{\mathfrak{n}}(M)$, we have $\mathrm{H}^1_{\mathfrak{n}}(M) = 0$. This means that $T = 0$. Therefore,

$$0 \to T = 0 \to M/yM \simeq N \to 0.$$

Since N is Cohen–Macaulay, M/yM is Cohen–Macaulay. Since y is regular on M, M is Cohen–Macaulay, which is a contradiction. □

Example 3.2 ([32]) Let $M = \mathbf{R} = k[[x, y, z]]/(z(x, y, z))$. Then, $\mathrm{H}^0_{\mathfrak{m}}(\mathbf{R}) = (z)$ and $\mathbf{S} = \mathbf{R}/H^0_{\mathfrak{m}}(\mathbf{R}) \simeq k[[x, y]]$ is Cohen-Macaulay. If Q is a parameter ideal of $\mathbf{R}$, then $e_1(Q, \mathbf{R}) = e_1(Q\mathbf{S}, \mathbf{S}) = 0$. Hence, $e_1(Q, \mathbf{R}) = 0$, but $\mathbf{R}$ is not Cohen-Macaulay. Therefore, the unmixdness condition is necessary in Theorem 3.1.

Let us list some consequences of Theorem 3.1. Let $\mathbf{R}$ be a Noetherian local ring and M a finitely generated $\mathbf{R}$-module. We put

$$\mathrm{Assh}_{\mathbf{R}} M = \{\mathfrak{p} \in \mathrm{Ass}_{\mathbf{R}} M \mid \dim \mathbf{R}/\mathfrak{p} = \dim_{\mathbf{R}} M\}.$$

Let $(0_M) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_{\mathbf{R}} M} M(\mathfrak{p})$ be a primary decomposition of 0_M in M, where $M(\mathfrak{p})$ is a $\mathfrak{p}$–primary submodule of M for each $\mathfrak{p} \in \mathrm{Ass}_{\mathbf{R}} M$. We put

$$\mathrm{U}_M(0) = \bigcap_{\mathfrak{p} \in \mathrm{Assh}_{\mathbf{R}} M} M(\mathfrak{p})$$

and call it the *unmixed component* of M. We then have the following.

Lemma 3.3 *Let* $\mathbf{R}$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M > 0$. *Let* Q *be a parameter ideal for* M. *Let* $U = \mathrm{U}_M(0)$ *and suppose that* $U \neq (0)$. *We put* $N = M/U$. *Then, the following assertions hold.*

(a) $\dim_{\mathbf{R}} U < \dim_{\mathbf{R}} M$.

(b)

$$e_1(Q, M) = \begin{cases} e_1(Q, N) & \text{if} \quad \dim_{\mathbf{R}} U \le r-2, \\ \\ e_1(Q, N) - s_0 & \text{if} \quad \dim_{\mathbf{R}} U = r-1, \end{cases}$$

where $s_0 \ge 1$ *denotes the multiplicity of the graded* $\mathrm{gr}_Q(\mathbf{R})$*–module* $\bigoplus_{n \ge 0} U/(Q^{n+1}M \cap U)$.

(c) $e_1(Q, M) \le e_1(Q, N)$ *and the equality* $e_1(Q, M) = e_1(Q, N)$ *holds if and only if* $\dim_{\mathbf{R}} U \le r-2$.

Proof (a) This is clear since $\mathrm{U}_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \mathrm{Assh}_{\mathbf{R}} M$.

(b) We write

$$\lambda_{\mathbf{R}}\left(U/(Q^{n+1}M \cap U)\right) = s_0\binom{n+t}{t} - s_1\binom{n+t-1}{t-1} + \cdots + (-1)^t s_t$$

for $n \gg 0$ with integers $\{s_i\}_{0 \le i \le t}$, where $t = \dim_{\mathbf{R}} U$. Then, the claim follows from the exact sequence $0 \to U \to M \to N \to 0$ of $\mathbf{R}$–modules, which gives

$$\lambda_{\mathbf{R}}(M/Q^{n+1}M) = \lambda_{\mathbf{R}}(N/Q^{n+1}N) + \lambda_{\mathbf{R}}(U/(Q^{n+1}M \cap U)), \quad \forall\, n \ge 0.$$

(c) This follows from (b) and the fact that $s_0 \ge 1$.

□

Theorem 3.4 *Let* $\mathbf{R}$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \ge 2$*. Suppose that* $\mathbf{R}$ *is a homomorphic image of a Cohen–Macaulay ring. Let* $U = \mathrm{U}_M(0)$ *and let* Q *be a parameter ideal for* M*. Then, the following conditions are equivalent:*

(i) $e_1(Q, M) = 0$.

(ii) M/U *is a Cohen–Macaulay* $\mathbf{R}$*–module and* $\dim_{\mathbf{R}} U \le r-2$.

Proof It is enough to prove (i) ⇒ (ii). If $\dim_{\mathbf{R}} U = r-1$, then by (i) and Lemma 3.3–(b), we obtain $0 \ge e_1(Q, M/U) = s_0 \ge 1$, which is a contradiction. Hence, $\dim_{\mathbf{R}} U \le r-2$. This means that $0 = e_1(Q, M) = e_1(Q, M/U)$. By Theorem 3.1, M/U is Cohen–Macaulay.

□

The implication (i) ⇒ (ii) in Theorem 3.4 is not true in general without the assumption that $\mathbf{R}$ is a homomorphic image of a Cohen–Macaulay ring (see [8, Remark 2] for an example). The following corollary gives a characterization of Cohen–Macaulayness.

Corollary 3.5 *Let* $\mathbf{R}$ *be a Noetherian local ring,* M *a finitely generated* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M > 0$*, and* Q *a parameter ideal for* M*. Let* $1 \le k \le r$ *be an integer and assume that* $e_i(Q, M) = 0$ *for all* $1 \le i \le k$*. Then,*

$$\dim_{\widehat{\mathbf{R}}} \mathrm{U}_{\widehat{M}}(0) \le r-(k+1) \ \ \textit{and} \ \ \mathrm{H}_{\mathfrak{m}}^{r-j}(M) = (0)$$

for all $1 \le j \le k$*. In particular, if* $k = r$*, then* M *is a Cohen–Macaulay* $\mathbf{R}$*–module.*

Proof We may assume that $\mathbf{R}$ is complete. We put $U = \mathrm{U}_M(0)$ and $N = M/U$. Then, $e_0(Q, M) = e_0(Q, N)$ since $\dim_{\mathbf{R}} U < r$. Therefore, by Theorem 3.4, N is a Cohen–Macaulay $\mathbf{R}$–module so that we have exact sequences

$$0 \to U/Q^{n+1}U \to M/Q^{n+1}M \to N/Q^{n+1}N \to 0$$

of $\mathbf{R}$–modules for all $n \geq 0$. Hence, computing Hilbert polynomials, we get $\dim_{\mathbf{R}} U \leq r-(k+1)$. Let $1 \leq j \leq k$. Then, $\mathrm{H}_{\mathfrak{m}}^{r-j}(U) = (0)$, since $\dim_{\mathbf{R}} U < r-j$, while $\mathrm{H}_{\mathfrak{m}}^{r-j}(N) = (0)$, as N is a Cohen–Macaulay $\mathbf{R}$–module with $\dim_{\mathbf{R}} N = r$. Thus, $\mathrm{H}_{\mathfrak{m}}^{r-j}(M) = (0)$ as claimed. □

Let $\mathbf{R}$ be a Noetherian local ring and M a finitely generated $\mathbf{R}$–module. In [8], the authors examined the rings with $e_1(Q, R)$ vanishing. Here, we briefly extend this theory to modules. Let us begin with the definition.

Definition 3.6 A finitely generated $\mathbf{R}$–module M is called a *Vasconcelos* module[1] if either $\dim_{\mathbf{R}} M = 0$ or $\dim_{\mathbf{R}} M > 0$ and $e_1(Q, M) = 0$ for some parameter ideal Q for M.

Every Cohen–Macaulay module is by definition Vasconcelos. Here is a basic characterization. We omit the proof since it is similar to those in the ring case.

Theorem 3.7 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 2$. *Let* $U = \mathrm{U}_{\widehat{M}}(0)$ *be the unmixed component of* (0) *in the* $\mathfrak{m}$*–adic completion* $\widehat{M}$ *of* M. *Then, the following conditions are equivalent:*

(i) *M is a Vasconcelos $\mathbf{R}$–module.*
(ii) $e_1(Q, M) = 0$ *for every parameter ideal Q for M.*
(iii) $\widehat{M}/\mathrm{U}_{\widehat{M}}(0)$ *is a Cohen–Macaulay* $\widehat{\mathbf{R}}$*–module and* $\dim_{\widehat{\mathbf{R}}} \mathrm{U}_{\widehat{M}}(0) \leq r-2$.
(iv) *There exists a proper* $\widehat{\mathbf{R}}$*–submodule L of $\widehat{M}$ such that $\widehat{M}/L$ is a Cohen–Macaulay* $\widehat{\mathbf{R}}$*–module with* $\dim_{\widehat{\mathbf{R}}} L \leq r-2$.

When this is the case, $\widehat{M}$ *is a Vasconcelos* $\widehat{\mathbf{R}}$*–module and* $\mathrm{H}_{\mathfrak{m}}^{r-1}(M) = (0)$.

Remark 3.8 Several properties of Vasconcelos rings such as [8, 3.5, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.15, 3.16, 3.17] can be all extended to Vasconcelos modules.

4 Generalized Cohen–Macaulayness of Modules with $\Lambda(M)$ Finite

Let $\mathbf{R}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M > 0$. In this section, we study the problem of when the set

$$\Lambda(M) = \{e_1(Q, M) \mid Q \text{ is a parameter ideal for } M\}$$

is finite. Part of the motivation comes from the fact that generalized Cohen–Macaulay modules have this property. Recall that M is said to be generalized Cohen–Macaulay if all the local cohomology modules $\{\mathrm{H}_{\mathfrak{m}}^i(M)\}_{0 \leq i < r}$ are finitely generated (see [6] where these modules originated).

Assume that M is a generalized Cohen–Macaulay $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M \geq 2$ and put

$$s = \sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M),$$

[1]The terminology is due to the other five authors.

where $h^i(M) = \lambda_{\mathbf{R}}(\mathrm{H}^i_{\mathfrak{m}}(M))$ for each $i \in \mathbb{Z}$. If Q is a parameter ideal for M, by the proof of [12, Lemma 2.4], we have that $e_1(Q, M) \geq -s$. Since $e_1(Q, M) \leq 0$ by Corollary 2.3, it follows that $\Lambda(M)$ is a finite set.

Let us establish here that if M is unmixed and $\Lambda(M)$ is finite, then M is indeed a generalized Cohen–Macaulay $\mathbf{R}$–module (Proposition 4.2).

Assume now that $\mathbf{R}$ is a homomorphic image of a Gorenstein local ring and that $\mathrm{Ass}_{\mathbf{R}} M = \mathrm{Assh}_{\mathbf{R}} M$. Then, $\mathbf{R}$ contains a system $x_1, x_2, \ldots, x_r$ of parameters of M which forms a strong d-sequence for M, that is, the sequence $x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r}$ is a d-sequence for M for all integers $n_1, n_2, \ldots, n_r \geq 1$ (see [5, Theorem 2.6] or [17, Theorem 4.2] for the existence of such systems of parameters). For each integer $q \geq 1$, let $\Lambda_q(M)$ be the set of values $e_1(Q, M)$, where Q runs over the parameter ideals for M such that $Q \subseteq \mathfrak{m}^q$ and $Q = (x_1, x_2, \ldots, x_r)$ with $x_1, x_2, \ldots, x_r$ a d-sequence for M. We then have $\Lambda_q(M) \neq \emptyset$, $\Lambda_{q+1}(M) \subseteq \Lambda_q(M)$ for all $q \geq 1$ and $\alpha \leq 0$ for every $\alpha \in \Lambda_q(M)$ (Corollary 2.3).

The following result plays a key role in our argument. The proof which we present here is based on Theorem 2.5 and slightly different from that of the ring case.

Lemma 4.1 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and assume that* $\mathbf{R}$ *is a homomorphic image of a Gorenstein ring. Let* M *be a finitely generated* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 2$ *and* $\mathrm{Ass}_{\mathbf{R}} M = \mathrm{Assh}_{\mathbf{R}} M$. *Assume that* $\Lambda_q(M)$ *is a finite set for some integer* $q \geq 1$ *and put* $\ell = -\min \Lambda_q(M)$. *Then,* $\mathfrak{m}^{\ell} \mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for all* $i \neq r$ *and hence all the local cohomology modules* $\{\mathrm{H}^i_{\mathfrak{m}}(M)\}_{0 \leq i < r}$ *are finitely generated.*

Proof Passing to the ring $\mathbf{R}/[(0) :_{\mathbf{R}} M]$, we may assume that $\mathbf{R}$ is a Gorenstein ring with $\dim \mathbf{R} = \dim_{\mathbf{R}} M = r$. Enlarging the residue class field $\mathbf{R}/\mathfrak{m}$ of $\mathbf{R}$ if necessary, we may assume the field $\mathbf{R}/\mathfrak{m}$ is infinite. By Corollary 2.6, $\mathrm{H}^1_{\mathfrak{m}}(M)$ is finitely generated since M is unmixed.

Suppose that $r = 2$. We put $\ell' = \lambda(\mathrm{H}^1_{\mathfrak{m}}(M))$. Let $Q = (x, y) \subseteq \mathfrak{m}^q$ be a system of parameters for M such that $Q\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$ and x, y is a d-sequence for M. Then, x is superficial for M with respect to Q. Hence, by Proposition 2.2 (d), we get $e_1(Q, M) = -\lambda(\mathrm{H}^1_{\mathfrak{m}}(M)) = -\ell'$. Thus, $\ell \geq \ell'$, as $-\ell' = e_1(Q, M) \in \Lambda_q(M)$. Hence, $\mathfrak{m}^{\ell}\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$ because $\mathfrak{m}^{\ell'}\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$.

Suppose that $r \geq 3$ and that our assertion holds true for $r - 1$. We have an exact sequence

$$(\sharp) \quad 0 \to M \to \mathbf{R}^n \to C \to 0$$

of $\mathbf{R}$–modules by Theorem 2.5. Choose an $\mathbf{R}$-regular element $x \in \mathbf{R}$ so that x is superficial both for M and C with respect to $\mathfrak{m}$. Let us fix an integer $m \geq 1$. We put $y = x^m$, $N = M/yM$, and look at the exact sequence

$$0 \to (0) :_C y \to N \xrightarrow{\varphi} (\mathbf{R}/y\mathbf{R})^n \to C/yC \to 0$$

of $\mathbf{R}$–modules obtained by sequence ($\sharp$). Let $L = \mathrm{Im}\varphi$. Then, $\dim_{\mathbf{R}} L = r - 1$, $\mathrm{Ass}_{\mathbf{R}} L = \mathrm{Assh}_{\mathbf{R}} L$ and $\mathrm{H}^0_{\mathfrak{m}}(N) \cong (0) :_C y$ because L is an $\mathbf{R}$–submodule of $(\mathbf{R}/y\mathbf{R})^n$ and $\lambda((0) :_C y) < \infty$. Hence, $L \cong N/\mathrm{H}^0_{\mathfrak{m}}(N)$.

Let $q' \geq q$ be an integer such that $\mathfrak{m}^{q'} N \cap \mathrm{H}^0_{\mathfrak{m}}(N) = (0)$. Let $y_2, y_3, \ldots, y_r \in \mathfrak{m}^{q'}$ be a system of parameters for L and assume that $y_2, y_3, \ldots, y_r$ is a d-sequence for L. Then, since $(y_2, y_3, \ldots, y_r)N \cap \mathrm{H}^0_{\mathfrak{m}}(N) = (0)$, we get $y_2, y_3, \ldots, y_r$ forms a d-sequence for N also. Therefore, since y is M-regular, the sequence $y_1 = y, y_2, \ldots, y_r$ forms a d-sequence for M; whence, y_1 is superficial for M with respect to $Q = (y_1, y_2, \ldots, y_r)$. Consequently,

$$e_1((y_2, y_3, \ldots, y_r), L) = e_1((y_2, y_3, \ldots, y_r), N) = e_1(Q, M) \in \Lambda_q(M),$$

so that $\Lambda_{q'}(L) \subseteq \Lambda_q(M)$. Hence, because the set $\Lambda_{q'}(L)$ is finite, the hypothesis of induction on r yields that $\mathfrak{m}^{\ell''}\mathrm{H}^i_{\mathfrak{m}}(L) = (0)$ for all $i \neq r-1$, where $\ell'' = -\min\Lambda_{q'}(L)$. Thus, because $\ell'' \leq \ell$, $\mathfrak{m}^{\ell}\mathrm{H}^i_{\mathfrak{m}}(L) = (0)$ for all $i \neq r-1$. Hence, $\mathfrak{m}^{\ell}\mathrm{H}^i_{\mathfrak{m}}(N) = (0)$ for all $1 \leq i < r-1$ because $\mathrm{H}^i_{\mathfrak{m}}(N) \cong \mathrm{H}^i_{\mathfrak{m}}(L)$ for $i \geq 1$.

Look now at the exact sequence

$$(\sharp\sharp) \cdots \to \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{x^m} \mathrm{H}^1_{\mathfrak{m}}(M) \to \mathrm{H}^1_{\mathfrak{m}}(N) \to \cdots \to \mathrm{H}^i_{\mathfrak{m}}(N)$$
$$\to \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \xrightarrow{x^m} \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \to \cdots$$

of local cohomology modules. We then have

$$\mathfrak{m}^{\ell}\left[(0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(M)} x^m\right] = (0)$$

for all integers $1 \leq i \leq r-2$ and $m \geq 1$ since $\mathfrak{m}^{\ell}\mathrm{H}^i_{\mathfrak{m}}(N) = (0)$ for all $1 \leq i \leq r-2$. Thus, $\mathfrak{m}^{\ell}\mathrm{H}^{i+1}_{\mathfrak{m}}(M) = (0)$ because

$$\mathrm{H}^{i+1}_{\mathfrak{m}}(M) = \bigcup_{m \geq 1}\left[(0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(M)} \mathfrak{m}^m\right].$$

On the other hand, from sequence $(\sharp\sharp)$, we get the embedding $\mathrm{H}^1_{\mathfrak{m}}(M) \subseteq \mathrm{H}^1_{\mathfrak{m}}(N)$, choosing the integer $m \geq 1$ so that $x^m\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$. Hence, $\mathfrak{m}^{\ell}\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$, which completes the proof of Lemma 4.1. □

Since $\Lambda(M) = \Lambda(\widehat{M})$, passing to the completion $\widehat{M}$ of M and applying Lemma 4.1, we readily get the following.

Proposition 4.2 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and* M *a finitely generated unmixed* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 2$. *Assume that* $\Lambda(M)$ *is a finite set and put* $\ell = -\min\Lambda(M)$. *Then,* $\mathfrak{m}^{\ell}\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for every* $i \neq r$ *so that* M *is a generalized Cohen–Macaulay* $\mathbf{R}$*–module.*

We conclude this section with a characterization of $\mathbf{R}$–modules for which $\Lambda(M)$ is finite. Let us note the following with a brief proof.

Lemma 4.3 *Let* $\mathbf{R}$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 2$. *Assume that there exists an integer* $t \geq 0$ *such that* $e_1(Q, M) \geq -t$ *for every parameter ideal* Q *for* M. *Then* $\dim_{\mathbf{R}} \mathrm{U}_M(0) \leq r-2$.

Proof Let $U = \mathrm{U}_M(0)$ and $N = M/U$. Assume that $\dim_{\mathbf{R}} U = r-1$. Choose a system $x_1, x_2, \ldots, x_r$ of parameters of M such that $x_r U = (0)$. Let $\ell > t$ be an integer and put $Q = \left(x_1^{\ell}, x_2, \ldots, x_r\right)$. Then, we get exact sequences

$$0 \to U/(Q^{n+1}M \cap U) \to M/Q^{n+1}M \to N/Q^{n+1}N \to 0$$

of $\mathbf{R}$–modules for all $n \geq 0$. Let us take an integer $k \geq 0$ so that

$$Q^n M \cap U = Q^{n-k}\left(Q^k M \cap U\right)$$

for $n \geq k$ and consider $U' = Q^k M \cap U$. Let $\mathfrak{q} = \left(x_1^{\ell}, x_2, \ldots, x_{r-1}\right)$. We then have

$$Q^{n-k}U' = \mathfrak{q}^{n-k}U',$$

as $x_r U = (0)$. Hence, for all $n \geq k$

$$\lambda_{\mathbf{R}}(M/Q^{n+1}M) = \lambda_{\mathbf{R}}(N/Q^{n+1}N) + \lambda_{\mathbf{R}}(U'/\mathfrak{q}^{n-k+1}U') + \lambda_{\mathbf{R}}(U/U'),$$

which yields $-t \leq e_1(Q, M) = e_1(Q, N) - e_0(\mathfrak{q}, U')$. Hence,

$$-t \leq e_1(Q, M) = e_1(Q, N) - e_0(\mathfrak{q}, U),$$

because $e_0(\mathfrak{q}, U) = e_0(\mathfrak{q}, U')$ (remember that $\lambda(U/U') < \infty$). Therefore, since $e_1(Q, N) \leq 0$ by Corollary 2.3, we get

$$\ell \leq \ell e_0((x_1, x_2, \ldots, x_{r-1}), U) = e_0(\mathfrak{q}, U) \leq e_1(Q, N) + t \leq t,$$

which is impossible. Thus, $\dim_{\mathbf{R}} U \leq r - 2$. □

Remark 4.4 Let $\mathbf{R}$ be a Noetherian local ring and M a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M \geq 2$. Assume that $\dim_{\mathbf{R}} \mathrm{U}_M(0) \leq r - 2$. Let $\mathfrak{q}$ be a parameter ideal for $N = M/\mathrm{U}_M(0)$. Then one can find a parameter ideal Q for M with $QN = \mathfrak{q}N$ so that $e_1(\mathfrak{q}, N) = e_1(\mathfrak{q}, M)$ by Lemma 3.3. Hence, $\Lambda(M) = \Lambda(N)$.

The goal of this section is the following.

Theorem 4.5 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 2$*. Let* $U = \mathrm{U}_{\widehat{M}}(0)$ *denote the unmixed component of* (0) *in the* $\mathfrak{m}$*-adic completion* $\widehat{M}$ *of* M*. Then, the following conditions are equivalent:*

(i) $\Lambda(M)$ *is a finite set.*
(ii) $\widehat{M}/U$ *is a generalized Cohen–Macaulay* $\widehat{\mathbf{R}}$*–module and* $\dim_{\widehat{\mathbf{R}}} U \leq r - 2$.

When this is the case, one has the estimation

$$0 \geq e_1(Q, M) \geq -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(\widehat{M}/U)$$

for every parameter ideal Q *for* M*.*

Proof We may assume that $\mathbf{R}$ is complete.

(i) ⇒ (ii) Since the set $\Lambda(M)$ is finite, by Proposition 4.3, we get $\dim_{\mathbf{R}} U \leq r - 2$. By Remark 4.4, the set $\Lambda(M/U)$ is finite so that M/U is a generalized Cohen–Macaulay $\mathbf{R}$–module by Proposition 4.2.
(ii) ⇒ (i) By [12, Lemma 2.4], the set $\Lambda(M/U)$ is finite and hence the set $\Lambda(M)$ is also finite by Lemma 3.3.

See [12, Lemma 2.4] for the last assertion. □

5 Buchsbaumness of Modules Possessing Constant First Hilbert Coefficients of Parameters

Let $\mathbf{R}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M > 0$. In this section, we study the problem when $e_1(Q, M)$ is independent of the choice of parameter ideals Q for M. Part of the motivation comes from the fact that Buchsbaum modules have this property. We establish here that if $e_1(Q, M)$ is constant and M is unmixed, then M is indeed a Buchsbaum $\mathbf{R}$–module (Theorem 5.4) (see [13] for the ring case).

First of all, let us recall some definitions. A system $x_1, x_2, \ldots, x_r$ of parameters of M is said to be *standard* if it forms a d^+-sequence for M, that is, $x_1, x_2, \ldots, x_r$ forms a strong d-sequence for M in any order. Remember that M possesses a standard system of parameters if and only if M is a generalized Cohen–Macaulay $\mathbf{R}$–module ([29]).

Let Q be a parameter ideal for M. Then, we say that Q is standard if it is generated by a standard system of parameters of M. Remember that Q is standard if and only if the equality

$$\lambda_{\mathbf{R}}(M/QM) - e_0(Q, M) = \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M) := \mathbb{I}(M)$$

holds ([29, Theorem 2.1]). It is known that every system of parameters of M contained in a standard parameter ideal for M is standard ([29]).

Suppose that M is a generalized Cohen–Macaulay $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M \geq 2$ and $s = \sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$. If Q is a parameter ideal for M, then by [12, Lemma 2.4], we get $e_1(Q, M) \geq -s$, where the equality holds if Q is standard ([25, Korollar 3.2]).

We say that our $\mathbf{R}$–module M is Buchsbaum if every parameter ideal for M is standard. Hence, if M is a Buchsbaum $\mathbf{R}$-module with $r = \dim_{\mathbf{R}} M \geq 2$, then M is a generalized Cohen–Macaulay $\mathbf{R}$–module with

$$e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$$

for every parameter ideal Q (see [28] for a detailed theory of Buchsbaum rings and modules).

We begin with the following two results, whose proofs are similar to those in the ring case (see [8, Lemma 4.5] and [13, Proposition 2.3]).

Lemma 5.1 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and* M *a generalized Cohen–Macaulay* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 2$ *and* $\operatorname{depth}_{\mathbf{R}} M > 0$. *Let* Q *be a parameter ideal for* M *such that* $e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$. *Then,* $Q\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for all* $1 \leq i \leq r-1$.

For each $x \in \mathfrak{m}$, we put $\mathrm{U}_M(x) := \bigcup_{n \geq 0} [xM :_M \mathfrak{m}^n]$.

Proposition 5.2 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and* M *a generalized Cohen–Macaulay* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 3$ *and* $\operatorname{depth}_{\mathbf{R}} M > 0$. *Let* $Q = (x_1, x_2, \ldots, x_r)$ *be a parameter ideal for* M. *Assume that* $(x_1, x_r)\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$ *and that the parameter ideal* $(x_1, x_2, \ldots, x_{r-1})$ *for the generalized Cohen–Macaulay* $\mathbf{R}$*–module* $M/\mathrm{U}_M(x_r)$ *is standard. Then,* $\mathrm{U}_M(x_1) \cap QM = x_1 M$.

We then have the following, which is the key in our argument. The proof is similar to the ring case [13, Theorem 2.1], but let us note a brief proof in order to see how we use the previous results of Lemma 5.1 and Proposition 5.2.

Theorem 5.3 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and let* M *be a generalized Cohen–Macaulay* $\mathbf{R}$*–module with* $r = \dim_{\mathbf{R}} M \geq 2$ *and* $\operatorname{depth}_{\mathbf{R}} M > 0$. *Let* Q *be a parameter ideal for* M. *Then, the following conditions are equivalent:*

(i) *Q is a standard parameter ideal for M.*
(ii) $e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$.

Proof We have only to show the implication (ii) ⇒ (i). To do this, we may assume that the residue class field $\mathbf{R}/\mathfrak{m}$ of $\mathbf{R}$ is infinite. We write $Q = (x_1, x_2, \ldots, x_r)$, where each x_j is superficial for M with respect to Q. Remember that by Lemma 5.1, $Q\mathrm{H}^i_\mathfrak{m}(M) = (0)$ for all $i \neq r$. Hence, Q is standard if $r = 2$ ([29, Corollary 3.7]).

Assume that $r \geq 3$ and that our assertion holds true for $r-1$. Let $1 \leq j \leq r$ be an integer. We put $N = M/x_jM$, $\overline{M} = N/\mathrm{H}^0_\mathfrak{m}(N)$ $(= M/\mathrm{U}_M(x_j))$ and $Q_j = (x_i \mid 1 \leq i \leq r, i \neq j)$. Then, $\mathrm{H}^i_\mathfrak{m}(N) \cong \mathrm{H}^i_\mathfrak{m}(\overline{M})$ for all $i \geq 1$. On the other hand, since $x_j\mathrm{H}^i_\mathfrak{m}(M) = (0)$ for $i \neq r$ and x_j is M-regular, for each $0 \leq i \leq r-2$, we have the short exact sequence

$$0 \to \mathrm{H}^i_\mathfrak{m}(M) \to \mathrm{H}^i_\mathfrak{m}(N) \to \mathrm{H}^{i+1}_\mathfrak{m}(M) \to 0$$

of local cohomology modules. Hence, $\mathbb{I}(M) = \mathbb{I}(N)$ and

$$\begin{aligned} e_1(Q, M) = e_1(Q_j, N) &= e_1(Q_j, \overline{M}) \\ &\geq -\sum_{i=1}^{r-2} \binom{r-3}{i-1} h^i(\overline{M}) \\ &= -\sum_{i=1}^{r-2} \binom{r-3}{i-1} h^i(N) \\ &= -\sum_{i=1}^{r-2} \binom{r-3}{i-1} [h^i(M) + h^{i+1}(M)] \\ &= -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M) \\ &= e_1(Q, M), \end{aligned}$$

so that the equality

$$e_1(Q_j, \overline{M}) = -\sum_{i=1}^{r-2} \binom{r-3}{i-1} h^i(\overline{M})$$

holds for the parameter ideal Q_j for the generalized Cohen–Macaulay $\mathbf{R}$–module $\overline{M} = M/\mathrm{U}_M(x_j)$. Thus, by the hypothesis of induction on $r = \dim_\mathbf{R} M$, Q_j is a standard parameter ideal for $M/\mathrm{U}_M(x_j)$ for every $1 \leq j \leq r$. Hence, $\mathrm{U}_M(x_1) \cap QM = x_1M$ by Proposition 5.2. Thus, Q_1 is a standard parameter ideal for M/x_1M ([29, Corollary 2.3]). Therefore, Q is a standard parameter ideal for M because $\mathbb{I}(M) = \mathbb{I}(M/x_1M)$ ([29, Corollary 2.4]). □

We are now ready to prove the main result of this section.

Theorem 5.4 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and M an unmixed* $\mathbf{R}$*–module with* $r = \dim_\mathbf{R} M \geq 2$. *Then, the following conditions are equivalent:*

(i) *M is a Buchsbaum* $\mathbf{R}$*–module.*

(ii) *The first Hilbert coefficient $e_1(Q, M)$ of M is constant and independent of the choice of parameter ideals Q for M.*

When this is the case, one has the equality

$$e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$$

for every parameter ideal Q for M, where $h^i(M) = \lambda(\mathrm{H}^i_{\mathfrak{m}}(M))$ for each $1 \le i \le r-1$.

Proof (i) $\Rightarrow$ (ii) This is due to Schenzel [25].
(ii) $\Rightarrow$ (i) Since $\sharp\Lambda(M) = 1$, by Proposition 4.2, M is a generalized Cohen–Macaulay **R**–module. Hence, $\Lambda(M) = \{-\sum_{i=1}^{r-1}\binom{r-2}{i-1}h^i(M)\}$ by [25, Korollar 3.2] so that by Theorem 5.3, every parameter ideal Q for M is standard. Thus, M is, by definition, a Buchsbaum **R**–module ([27]).
See [25] for the last assertion. □

We are now in a position to conclude this section with a characterization of **R**–modules possessing $\sharp\Lambda(M) = 1$.

Theorem 5.5 *Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated **R**–module with $r = \dim_{\mathbf{R}} M \ge 2$. Let $U = \mathrm{U}_{\widehat{M}}(0)$ be the unmixed component of (0) in the $\mathfrak{m}$-adic completion $\widehat{M}$ of M. Then, the following conditions are equivalent:*

(i) $\sharp\Lambda(M) = 1$
(ii) *$\widehat{M}/U$ is a Buchsbaum $\widehat{\mathbf{R}}$–module and $\dim_{\widehat{\mathbf{R}}} U \le r-2$.*

When this is the case, one has the equality

$$e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(\widehat{M}/U)$$

for every parameter ideal Q for M.

Proof We may assume **R** is complete.

(i) $\Rightarrow$ (ii) Since $\sharp\Lambda(M) = 1$, $\dim_{\mathbf{R}} U \le r-2$ by Proposition 4.3. We get $\sharp\Lambda(M/U) = 1$ by Remark 4.4 so that by Theorem 5.4, M/U is a Buchsbaum **R**-module.
(ii) $\Rightarrow$ (i) We get by Theorem 5.4 that $\sharp\Lambda(M/U) = 1$ and hence $\sharp\Lambda(M) = 1$ by Lemma 3.3.

See Theorem 5.4 for the last assertion. □

6 Homological Degrees

In this section, we deal with the variation of the extended degree function hdeg ([7, 30]), labeled hdeg_I (see [18], [31, p. 142]). We recall the basic properties of these functions. These techniques and their relationships to $e_1(I)$ have been mentioned in [32], but the treatment here is more focused. It will lead to sharper bounds in the case of $e_1(I, M)$.

6.1 Cohomological Degrees

Let $\mathbf{R}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue class field. Let $\mathcal{M}(\mathbf{R})$ denote the category of finitely generated $\mathbf{R}$-modules and let I be an $\mathfrak{m}$–primary ideal of $\mathbf{R}$. Then, one has the following extension of the classical multiplicity.

Definition 6.1 A *cohomological degree*, or *extended multiplicity function* with respect to I, is a function

$$\operatorname{Deg}(\cdot) : \mathcal{M}(\mathbf{R}) \to \mathbb{N}$$

that satisfies the following conditions. Let $M \in \mathcal{M}(\mathbf{R})$.

(a) If $L = \Gamma_{\mathfrak{m}}(M)$ is the $\mathbf{R}$-submodule of elements of M that are annihilated by a power of the maximal ideal $\mathfrak{m}$ and $\overline{M} = M/L$, then

$$\operatorname{Deg}(M) = \operatorname{Deg}(\overline{M}) + \lambda(L).$$

(b) (Bertini's rule) If M has positive depth, then

$$\operatorname{Deg}(M) \geq \operatorname{Deg}(M/hM)$$

for every generic hyperplane section $h \in I \setminus \mathfrak{m}I$.

(c) (The calibration rule) If M is a Cohen-Macaulay $\mathbf{R}$–module, then

$$\operatorname{Deg}(M) = \deg(M),$$

where $\deg(M) = e_0(I, M)$ is the Samuel multiplicity of M with respect to I.

The existence of cohomological degrees in arbitrary dimensions was established in [30]. Let us formulate it for the case where the ring $\mathbf{R}$ is complete. The use of the more general Samuel multiplicities was introduced in [18]. When precision demands, we denote the degree and homological degree functions associated to the $\mathfrak{m}$-primary ideal I by $\deg_I$ and hdeg_I, respectively.

For the rest of this section, suppose that $\mathbf{R}$ is complete. For each finitely generated $\mathbf{R}$–module M and $j \in \mathbb{Z}$, let

$$M_j = \operatorname{Hom}_{\mathbf{R}}\left(\mathrm{H}^j_{\mathfrak{m}}(M), E\right),$$

where $E = \mathrm{E}_{\mathbf{R}}(\mathbf{R}/\mathfrak{m})$ denotes the injective envelope of the residue class field. Then, thanks to the local duality theorem, one gets $\dim_{\mathbf{R}} M_j \leq j$ for all $j \in \mathbb{Z}$.

Definition 6.2 Let M be a finitely generated $\mathbf{R}$-module with $r = \dim_{\mathbf{R}} M > 0$. Then, the *homological degree* of M is the integer

$$\operatorname{hdeg}(M) = \deg(M) + \sum_{j=0}^{r-1} \binom{r-1}{j} \cdot \operatorname{hdeg}\left(M_j\right).$$

We call attention to the fact (see [30] for details) that the notion of generic hyperplane section used for $\operatorname{hdeg}(M)$ are superficial elements not only for M and for all M_j but also for the iterated ones of these modules (there are only a finite number of them).

We will employ hdeg to derive lower bounds for $e_1(I, M)$.

6.2 Homological Torsion

There are other combinatorial expressions of the terms $\mathrm{hdeg}_I(M_j)$ that behave well under hyperplane sections.

Definition 6.3 Let M be an $\mathbf{R}$-module with $r = \dim_{\mathbf{R}} M \geq 2$. For each integer $1 \leq i \leq r-1$, we put

$$\mathbf{T}_I^{(i)}(M) = \sum_{j=1}^{r-i} \binom{r-i-1}{j-1} \cdot \mathrm{hdeg}_I(M_j).$$

Hence,

$$\mathrm{hdeg}_I(M) > \mathbf{T}_I^{(1)}(M) \geq \mathbf{T}_I^{(2)}(M) \geq \cdots \geq \mathbf{T}_I^{(r-1)}(M).$$

If M is a generalized Cohen-Macaulay $\mathbf{R}$–module, then

$$\mathbf{T}_I^{(i)}(M) = \sum_{j=1}^{r-i} \binom{r-i-1}{j-1} \lambda\left(\mathrm{H}_{\mathfrak{m}}^j(M)\right)$$

which is independent of I.

We then have the following.

Theorem 6.4 [30, Theorem 2.13] *Let M be a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M$ and let h be a generic hyperplane section. Then, $\mathbf{T}_I^{(i)}(M/hM) \leq \mathbf{T}_I^{(i)}(M)$ for all $1 \leq i \leq r-2$.*

We now turn this into a uniform bound for the first Hilbert coefficient of a module M relative to an ideal I generated by a system of parameters of M. We note that there are general bounds for all Hilbert coefficients $e_i(I, M)$ for arbitrary $\mathfrak{m}$-primary ideals I ([22]). Those developed here have a more specialized character and hold only for $e_1(I, M)$ and parameter ideals I.

Theorem 6.5 *Let M be a finitely generated $\mathbf{R}$–module with $\dim_{\mathbf{R}} M = \dim \mathbf{R} \geq 2$ and let Q be a parameter ideal of $\mathbf{R}$. Then,*

$$-e_1(Q, M) \leq \mathbf{T}_Q^{(1)}(M).$$

Proof Let $d = \dim \mathbf{R}$ and let $h \in Q \setminus \mathfrak{m}Q$ be a generic hyperplane section used for $\mathrm{hdeg}_Q(M)$. Since

$$-e_1(Q, M) = -e_1\left(Q, M/\mathrm{H}_{\mathfrak{m}}^0(M)\right) \text{ and } \mathbf{T}_Q^{(1)}\left(M/\mathrm{H}_{\mathfrak{m}}^0(M)\right) \leq \mathbf{T}_Q^{(1)}(M),$$

replacing M with $M/\mathrm{H}_{\mathfrak{m}}^0(M)$ if necessary, we may assume $\mathrm{depth}_{\mathbf{R}} M \geq 1$. We may also assume that h is superficial for M and for all M_j $(0 \leq j \leq d-1)$ with respect to Q. Hence, h is M–regular and $\lambda(M_1/hM_1) < \infty$ (remember that $\dim_{\mathbf{R}} M_1 \leq 1$). Suppose $d = 2$. Then, $\mathbf{T}_Q^{(1)}(M) = \mathrm{hdeg}_Q(M_1)$ and $-e_1(Q, M) = \lambda\left((0) :_{\mathrm{H}_{\mathfrak{m}}^1(M)} h\right)$ by Proposition 2.2 (d). On the other hand, from the exact sequence

$$0 \longrightarrow M \xrightarrow{h} M \longrightarrow M/hM \longrightarrow 0$$

of **R**–modules, we obtain the exact sequence

$$0 \to (0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} h \to \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{h} \mathrm{H}^1_{\mathfrak{m}}(M).$$

Then, taking the Matlis dual, we have an epimorphism

$$M_1/hM_1 \to \mathrm{Hom}_{\mathbf{R}}\left((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} h, E\right) \to 0$$

so that

$$\lambda\left((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} h\right) = \mathrm{hdeg}_Q\left(\mathrm{Hom}_{\mathbf{R}}\left((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} h, E\right)\right) \le \mathrm{hdeg}_Q(M_1/hM_1) \le \mathrm{hdeg}_Q(M_1)$$

by Theorem 6.4. Thus, $-e_1(Q, M) \le \mathbf{T}^{(1)}_Q(M)$. If $d \ge 3$, then we get

$$\mathbf{T}^{(1)}_{Q/(h)}(M/hM) \le \mathbf{T}^{(1)}_Q(M).$$

Hence, the result follows since $-e_1(Q, M) = -e_1(Q/(h), M/hM)$ by Proposition 2.2 (a). □

When the module is generalized Cohen-Macaulay, we recover the bound discussed at the beginning of Section 4.

Corollary 6.6 *If M is a generalized Cohen-Macaulay* **R**–*module with* $\dim_{\mathbf{R}} M \ge 1$*, then the Hilbert coefficients $e_1(Q, M)$ are bounded for all parameter ideals Q for M.*

Proof Passing to the ring $\mathbf{R}/[(0) :_{\mathbf{R}} M]$, we may assume that $\dim \mathbf{R} = \dim_{\mathbf{R}} M$ and that Q is a parameter ideal of **R**. Then, $e_1(Q, M) \le 0$ by Corollary 2.3. We get by Theorem 6.5 $-e_1(Q, M) \le \mathbf{T}^{(1)}_Q(M)$, while $\mathbf{T}^{(1)}_Q(M) = \sum_{j=1}^{d-1}\binom{d-2}{j-1}\lambda\left(\mathrm{H}^j_{\mathfrak{m}}(M)\right)$ is independent of the choice of Q. Hence, the result. □

Corollary 6.7 *Suppose that* $\dim_{\mathbf{R}} M \ge 1$. *Then, the set*

$$\{e_1(Q, M) \mid Q \text{ are parameter ideals of } M \text{ with the same integral closure}\}$$

is finite.

Proof For each parameter ideal Q of M, we get $e_1(Q, M) \le 0$, while Theorem 6.5 asserts that $e_1(Q, M) \ge -\mathbf{T}^{(1)}_Q(M)$. Hence, the result follows because $\mathbf{T}^{(1)}_Q(M)$ depends only on $\bar{Q}$, the integral closure of Q. □

7 Euler Characteristics and Hilbert Characteristics

The relationship between partial Euler characteristics and superficial elements make for a straightforward comparison with extended degree functions. Unless otherwise specified, throughout, it is assumed that **R** is a Noetherian complete local ring with infinite residue class field. We will prove that Euler characteristics can be uniformly bounded by homological degrees. The basic tool is the following observation, which is found in the proof of [3, Theorem 4.6.10 (a)].

Proposition 7.1 *Let M be a finitely generated* **R***-module with* $r = \dim_{\mathbf{R}} M \ge 2$*. Let* $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ *be a system of parameters for M and set* $\mathbf{x}' = \{x_2, \ldots, x_r\}$*. Then,*

$$\chi_1(\mathbf{x}; M) = \chi_1\left(\mathbf{x}'; M/x_1M\right) + \chi_1\left(\mathbf{x}'; 0 :_M x_1\right).$$

Theorem 7.2 *Let M be a finitely generated $\mathbf{R}$-module with $\dim_{\mathbf{R}} M = \dim \mathbf{R} = d \geq 1$. Then, for every system $\mathbf{x} = \{x_1, x_2, \ldots, x_d\}$ of parameters of $\mathbf{R}$, one has*

$$\chi_1(\mathbf{x}; M) \leq \operatorname{hdeg}_Q(M) - \deg_Q(M),$$

where $Q = (\mathbf{x})$.

Proof As $\lambda(M/QM) = \chi_1(\mathbf{x}; M) + \deg_Q(M)$, we have only to show $\lambda(M/QM) \leq \operatorname{hdeg}_Q(M)$. Let $h \in Q \setminus \mathfrak{m}Q$ be a generic hyperplane section used for $\operatorname{hdeg}_Q(M)$. Then, since $\lambda(M/QM) = \lambda((M/hM)/Q{\cdot}(M/hM))$ and $\operatorname{hdeg}_{Q/(h)}(M/hM) \leq \operatorname{hdeg}_Q(M)$, by induction on d, we may assume $d = 1$. When $d = 1$, $\chi_1(\mathbf{x}; M) = \lambda(0 :_M x_1)$, and hence $\lambda(M/QM) = \chi_1(\mathbf{x}; M) + \deg_Q(M) \leq \lambda\left(\mathrm{H}^0_{\mathfrak{m}}(M)\right) + \deg_Q(M) = \operatorname{hdeg}_Q(M)$, as wanted. □

Corollary 7.3 *Suppose that $\dim_{\mathbf{R}} M \geq 1$. Then, for every primary ideal I of M, the set*

$$\Xi_I(M) = \left\{\chi_1(\mathbf{x}, M) \mid \mathbf{x} \text{ are systems of parameters of } M \text{ with } \overline{(\mathbf{x})} = \overline{I}\right\}$$

is finite.

Proof Both $\operatorname{hdeg}_Q(M)$ and $\deg_Q(M)$ depend only on the integral closure $\overline{I} = \bar{Q}$ of $Q = (\mathbf{x})$. □

Definition 7.4 Let $\mathbf{R}$ be a Noetherian local ring and M a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M \geq 1$. For each system $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of parameters of M, the *Hilbert characteristic of M with respect to $Q = (\mathbf{x})$* is defined to be

$$\mathbf{h}(\mathbf{x}; M) = \sum_{i=0}^{r} (-1)^i e_i(Q, M).$$

The following proposition shows that the Hilbert characteristic can be characterized as a *quasi-cohomological degree* for M.

Proposition 7.5 *Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated $\mathbf{R}$–module with $r = \dim_{\mathbf{R}} M \geq 1$. Let $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ be a system of parameters of M and a d–sequence for M. Then, the Hilbert characteristic of M with respect to $\mathbf{x}$ satisfies the following.*

(a) *Suppose that x_1 is a superficial element for M and $\operatorname{depth}_{\mathbf{R}} M \geq 1$. Then,*

$$\mathbf{h}(\mathbf{x}; M) = \mathbf{h}(\mathbf{x}'; M/x_1M),$$

where $\mathbf{x}' = \{x_2, \ldots, x_r\}$.

(b) *Let $M_0 = \mathrm{H}^0_{\mathfrak{m}}(M)$ and $M' = M/M_0$. Then,*

$$\mathbf{h}(\mathbf{x}; M) = \mathbf{h}(\mathbf{x}; M') + \lambda(M_0).$$

Proof Let $Q = (\mathbf{x})$. Recall that, by [14, Proposition 3.4], we have $(-1)^r e_r(Q, M) = \lambda\left(H^0_{\mathfrak{m}}(M)\right)$.

(a) We may assume that x_1 is M–regular. By Proposition 2.2, we obtain

$$\begin{aligned}\mathbf{h}(\mathbf{x}; M) &= \sum_{i=0}^{r-1}(-1)^i e_i(Q, M) + (-1)^r e_r(Q, M)\\ &= \sum_{i=0}^{r-1}(-1)^i e_i\left(\mathbf{x}', M/x_1M\right) = \mathbf{h}\left(\mathbf{x}'; M/x_1M\right).\end{aligned}$$

(b) By applying Proposition 2.2-(b) to the exact sequence $0 \to M_0 \to M \to M' \to 0$, we get

$$e_i(Q, M) = e_i\left(Q, M'\right) \quad \text{for all } 0 \le i \le r-1.$$

Note that $(-1)^r e_r(Q, M') = \lambda(H^0_{\mathfrak{m}}(M')) = 0$. Hence,

$$\begin{aligned}\mathbf{h}(\mathbf{x}; M) = \textstyle\sum_{i=0}^{r}(-1)^i e_i(Q, M) &= \textstyle\sum_{i=0}^{r-1}(-1)^i e_i(Q, M) + \lambda(M_0)\\ &= \textstyle\sum_{i=0}^{r-1}(-1)^i e_i(Q, M') + \lambda(M_0)\\ &= \textstyle\sum_{i=0}^{r}(-1)^i e_i(Q, M') + \lambda(M_0)\\ &= \mathbf{h}(\mathbf{x}; M') + \lambda(M_0).\end{aligned}$$

□

Proposition 7.6 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and* M *a finitely generated* $\mathbf{R}$–*module with* $r = \dim_{\mathbf{R}} M \ge 1$. *Let* $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ *be a system of parameters of* M *and a d–sequence for* M. *Let* $Q = (\mathbf{x})$. *Then,*

$$\mathbf{h}(\mathbf{x}; M) = \lambda(M/QM).$$

In particular, $\mathbf{h}(\mathbf{x}; M) \ge e_0(Q, M)$ *with equality if and only if* M *is Cohen–Macaulay.*

Proof Using [10, Theorem 3.7], one can prove that

$$(-1)^i e_i(Q, M) = \chi_1\left(x_1, \ldots, x_{r-i}, x_{r-i+1}; M\right) - \chi_1\left(x_1, \ldots, x_{r-i}; M\right) \ge 0$$

for all $1 \le i \le r$. Therefore,

$$\begin{aligned}\mathbf{h}(\mathbf{x}; M) &= e_0(Q, M) + \sum_{i=1}^{r}(-1)^i e_i(Q, M)\\ &= e_0(Q, M) + \sum_{i=1}^{r}\left(\chi_1(x_1, \ldots, x_{r-i}, x_{r-i+1}; M) - \chi_1(x_1, \ldots, x_{r-i}; M)\right)\\ &= \chi_0(\mathbf{x}; M) + \chi_1(\mathbf{x}; M)\\ &= \lambda(M/QM).\end{aligned}$$

□

Corollary 7.7 *Let* $\mathbf{x}$ *be a system of parameters of* M *which is a d–sequence for* M. *Suppose that* $\mathbf{x} \in \mathfrak{m} \setminus \mathfrak{m}^2$. *Then, the Betti numbers* $\beta_i^{\mathbf{R}}(M)$ *satisfy*

$$\beta_i^{\mathbf{R}}(M) \le \lambda(M/(\mathbf{x})M) \cdot \beta_i^{\mathbf{R}}(k).$$

Proof It follows from the argument of [31, Theorem 2.94], where we use the properties of $\mathbf{h}(\mathbf{x}; M)$ in the induction part. □

Remark 7.8 Note that the condition $\mathbf{x} \in \mathfrak{m} \setminus \mathfrak{m}^2$ in Corollary 7.7 is needed in the induction argument which requires the inequality of Betti numbers $\beta_i^{\mathbf{R}/(x_1)}(k) \leq \beta_i^{\mathbf{R}}(k)$([15, Corollary 3.4.2]).

8 Buchsbaum-Rim Coefficients

In this section, let us note another set of related questions concerned about the vanishing and the negativity of the Buchsbaum-Rim coefficients of modules.

Let $\mathbf{R}$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim \mathbf{R} \geq 1$. The Buchsbaum-Rim multiplicity ([4]) arises in the context of an embedding

$$0 \to E \longrightarrow F = \mathbf{R}^s \longrightarrow C \to 0$$

of $\mathbf{R}$–modules, where $E \subseteq \mathfrak{m}F$ and C has finite length. Let

$$\varphi : \mathbf{R}^m \longrightarrow F = \mathbf{R}^s$$

be an $\mathbf{R}$–linear map represented by a matrix with entries in $\mathfrak{m}$ such that $\mathrm{Im}\varphi = E$. We then have a homomorphism

$$\mathcal{S}(\varphi) : \mathcal{S}(\mathbf{R}^m) \longrightarrow \mathcal{S}(\mathbf{R}^s)$$

of symmetric algebras, whose image is the Rees algebra $\mathcal{R}(E)$ of E, and whose cokernel we denote by $C(\varphi)$. Hence,

$$0 \to \mathcal{R}(E) \longrightarrow \mathcal{S}(\mathbf{R}^s) = \mathbf{R}[T_1, T_2, \ldots, T_s] \longrightarrow C(\varphi) \to 0.$$

This exact sequence (with a different notation) is studied in [4] in great detail. Of significance for us is the fact that $C(\varphi)$, with the grading induced by the homogeneous homomorphism $S(\varphi)$, has components of finite length for which we have the following. Let $E^n = [\mathcal{R}(E)]_n$ and $F^n = [\mathcal{S}(F)]_n$ for $n \geq 0$, where $F = \mathbf{R}^s$.

Theorem 8.1 *$\lambda(F^n/E^n)$ is a polynomial in n of degree $d + s - 1$ for $n \gg 0$:*

$$\lambda(F^n/E^n) = br(E)\binom{n+d+s-2}{d+s-1} - br_1(E)\binom{n+d+s-3}{d+s-2} + \textit{lower terms}.$$

This polynomial is called the Buchsbaum-Rim polynomial of E. The leading coefficient $\mathrm{br}(E)$ is the Buchsbaum-Rim multiplicity of φ; if the homomorphism φ is understood, we shall simply denote it by $\mathrm{br}(E)$. This number is determined by an Euler characteristic of the Buchsbaum-Rim complex ([4]).

Assume now the residue class field of $\mathbf{R}$ is infinite. The minimal reductions U of E are generated by $d + s - 1$ elements. We refer to U as a parameter module of F. The corresponding coefficients are $\mathrm{br}(U) = \mathrm{br}(E)$ but $\mathrm{br}_1(U) \leq \mathrm{br}_1(E)$. It is not clear what the possible values of $\mathrm{br}_1(U)$ are and, in similarity to the case of ideals, we can ask the following:

(a) $\mathrm{br}_1(U) \leq 0$?
(b) Suppose that $\mathbf{R}$ is unmixed. Then, is $\mathbf{R}$ Cohen-Macaulay if $\mathrm{br}_1(U) = 0$?
(c) Are the values of $\mathrm{br}_1(U)$ bounded?
(d) What happens in low dimensions?

As for question (a), a surprising result of Hayasaka and Hyry shows the negativity of $\mathrm{br}_1(U)$ in the following way. It gives an eminent proof of Corollary 2.3.

Theorem 8.2 ([16, Theorem 1.1]) $\lambda(F^n/U^n) \geq br(U)\binom{n+d+s-2}{d+s-1}$ *for all* $n \geq 0$. *Hence,*

$$br_1(U) \leq 0.$$

They also proved that $\mathbf{R}$ is a Cohen-Macaulay ring once $\lambda(F^n/U^n) = \mathrm{br}(U)\binom{n+d+s-2}{d+s-1}$ for some $n \geq 1$. When this is the case, one has the equality $\lambda(F^n/U^n) = \mathrm{br}(U)\binom{n+d+s-2}{d+s-1}$ for all $n \geq 0$, whence $\mathrm{br}_1(U) = 0$ ([2, Theorem 3.4]).

Note that question (c) is answered affirmatively for $s = 1$ in Corollary 6.7.

We close this paper with the following.

Conjecture 8.3 *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring with* $\dim \mathbf{R} \geq 2$ *and let* $U \subseteq \mathfrak{m}\mathbf{R}^s$ *be a parameter module of* $\mathbf{R}^s$ $(s > 0)$. *Then,* $\mathbf{R}$ *is a Cohen-Macaulay ring if and only if* $\mathbf{R}$ *is unmixed and* $br_1(U) = 0$.

Acknowledgments The first author is partially supported by a grant from the City University of New York PSC-CUNY Research Award Program-44. The second author is partially supported by Grant-in-Aid for Scientific Researches (C) in Japan (19540054). The fourth author is supported by a grant from MIMS (Meiji Institute for Advanced Study of Mathematical Sciences). The fifth author is supported by JSPS Ronpaku (Dissertation of PhD) Program. The last author is partially supported by the NSF

References

1. Auslander, M., Buchsbaum, D.: Codimension and multiplicity. Ann. Math. **68**, 625–657 (1958)
2. Brennan, J., Ulrich, B., Vasconcelos, W.V.: The Buchsbaum–Rim polynomial of a module. J. Algebra **241**, 379–392 (2001)
3. Bruns, W., Herzog, J.: Cohen-Macaulay Rings. Cambridge University Press, Cambridge (1993)
4. Buchsbaum, D., Rim, D.S.: A generalized Koszul complex II. Depth and multiplicity. Trans. Am. Math. Soc. **111**, 197–224 (1965)
5. Cuong, N.T.: p-standard system of parameters and p-standard ideals in local rings. Acta Math. Vietnam. **20**, 145–161 (1995)
6. Cuong, N.T., Schenzel, P., Trung, N.V.: Verallgemeinerte Cohen–Macaulay–Moduln. Math. Nachr. **85**, 57–73 (1978)
7. Doering, L.R., Gunston, T., Vasconcelos, W.V.: Cohomological degrees and Hilbert functions of graded modules. Am. J. Math. **120**, 493–504 (1998)
8. Ghezzi, L., Goto, S., Hong, J., Ozeki, K., Phuong, T.T., Vasconcelos, W.V.: Cohen–Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals. J. Lond. Math. Soc. **81**, 679–695 (2010)
9. Ghezzi, L., Hong, J., Vasconcelos, W.V.: The signature of the Chern coefficients of local rings. Math. Res. Lett. **16**, 279–289 (2009)
10. Goto, S., Hong, J., Vasconcelos, W.V.: The homology of parameter ideals. J. Algebra **368**, 271–299 (2012)
11. Goto, S., Nakamura, Y.: Multiplicities and tight closures of parameters. J. Algebra **244**, 302–311 (2001)
12. Goto, S., Nishida, K.: Hilbert coefficients and Buchsbaumness of associated graded rings. J. Pure Appl. Algebra **181**, 61–74 (2003)
13. Goto, S., Ozeki, K.: Buchsbaumness in local rings possessing constant first Hilbert coefficient of parameters. Nagoya Math. J. **199**, 95–105 (2010)
14. Goto, S., Ozeki, K.: Uniform bounds for Hilbert coefficients of parameters. In: Commutative Algebra and its Connections to Geometry, 97–118, Contemp. Math, 555, Am. Math. Soc., Providence, RI (2011)
15. Gulliksen, T., Levin, G.: Homology of local rings, Queen's Paper in Pure and Applied Mathematics, No. 20, Queen's University, Kingston, Ont (1969)
16. Hayasaka, F., Hyry, E.: On the Buchsbaum-Rim function of a parameter module. J. Algebra **327**, 307–315 (2011)

17. Kawasaki, T.: On Cohen–Macaulayfication of certain quasi-projective schemes. J. Math. Soc. Japan **50**, 969–991 (1998)
18. Linh, C.H.: Upper bound for the Castelnuovo-Mumford regularity of associated graded modules. Commun. Algebra **33**, 1817–1831 (2005)
19. Mandal, M., Singh, B., Verma, J.K.: On some conjectures about the Chern numbers of filtrations. J. Algebra **325**, 147–162 (2011)
20. Matsumura, H.: Commutative Algebra. Benjamin/Cummings, Reading (1980)
21. Nagata, M.: Local Rings. Interscience, New York (1962)
22. Rossi, M.E., Trung, N.V., Valla, G.: Castelnuovo-Mumford regularity and extended degree. Trans. Am. Math. Soc. **355**, 1773–1786 (2003)
23. Rossi, M.E., Valla, G.: On the Chern number of a filtration. Rend. Semin. Mat. Univ. Padova **121**, 201–222 (2009)
24. Rossi, M.E., Valla, G.: Hilbert functions of filtered modules. Lecture Notes of the Unione Matematica Italiana, vol. 9. Springer, Berlin (2010)
25. Schenzel, P.: Multiplizitäten in verallgemeinerten Cohen–Macaulay–Moduln. Math. Nachr. **88**, 295–306 (1979)
26. Serre, J.-P.: Algèbre Locale. Multiplicités. Lecture Notes in Mathematics, vol. 11. Springer, Berlin (1965)
27. Stückrad, J., Vogel, W.: Toward a theory of Buchsbaum singularities. Am. J. Math. **100**, 727–746 (1978)
28. Stückrad, J., Vogel, W.: Buchsbaum rings and applications. Springer, Berlin (1986)
29. Trung, N.V.: Toward a theory of generalized Cohen–Macaulay modules. Nagoya Math. J. **102**, 1–49 (1986)
30. Vasconcelos, W.V.: The homological degree of a module. Trans. Am. Math. Soc. **350**, 1167–1179 (1998)
31. Vasconcelos, W.V.: Integral closure. Springer Monographs in Mathematics, New York (2005)
32. Vasconcelos, W.V.: The Chern coefficients of local rings. Michigan Math. J. **57**, 725–743 (2008)