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Convergence, Scalarization and Continuity of Minimal Solutions in Set Optimization

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Abstract

The paper deals with the study of two different aspects of stability in the given space as well as the image space, where the solution concepts are based on a partial order relation on the family of bounded subsets of a real normed linear space. The first aspect of stability deals with the topological set convergence of families of solution sets of perturbed problems in the image space and Painlevé–Kuratowski set convergence of solution sets of the perturbed problems in the given space. The convergence in the given space is also established in terms of solution sets of scalarized perturbed problems. The second aspect of stability deals with semicontinuity of the solution set maps of parametric perturbed problems in both the spaces.

Keywords Topological convergence · Painlevé–Kuratowski convergence · Upper semicontinuity · Lower semicontinuity · Stability · Scalarization

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1 Introduction

Stability theory plays a crucial role in set optimization in order to examine the behavior of solution sets of the perturbed problems obtained by perturbing the feasible

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set, objective function or ordering cone. Various stability aspects have been studied in literature such as essential stability (see [1, 2]), continuity of solution set maps (see [3–7]) and convergence of solution sets (see [8–12]).

In set optimization, Dhingra and Lalitha [13] and Han et al. [9] studied the convergence aspect of stability in the given space in terms of Painlevé–Kuratowski sense. Further, Gutiérrez et al. [8] and Karuna and Lalitha [10, 14] established the convergence results in the image space in terms of external and internal stability without using an appropriate notion of convergence in an ordered power set. Recently, Geoffroy [15] defined a topology on the family of lower bounded sets and introduced topological convergence for a sequence of families of sets. By perturbing both the feasible set and the objective function, he established the topological convergence for families of minimal and relaxed minimal solution sets.

The real life problems are influenced by many other parameters also, which led to study of another stability aspect, namely continuity of solution set maps in parametric set optimization. In literature [3–5, 16, 17], the continuity aspect has been explored by employing certain continuity and convexity assumptions. In [4], using the upper set order relation, Xu and Li established the continuity of u-lower level maps to study the continuity of efficient solution set map, whereas using lower set order relation, Chen et al. [16] established the continuity of strict lower level maps to provide sufficient conditions for the continuity of strict minimal solution set map. Besides, Zhang and Huang [18] examined the continuity results using topological convergence introduced by Geoffroy [15].

A well-established practice to study stability aspects is through associated perturbed scalarized problems. In this direction, Liu et al. [19] investigated the continuity aspect by employing linear and nonlinear scalarization techniques. To the best of our knowledge, the stability aspect in terms of convergence of solution sets of scalarized problems in set optimization has not been investigated so far.

Recently, Karaman et al. [20] introduced new partial order relations on the family of bounded sets by using Minkowski difference. They considered scalar problems based on generalized Gerstewitz function to characterize efficient and weak efficient solutions and discussed optimality conditions. By using a strict order relation given in [15], Geoffroy provided a topolgy on the family of lower bounded subsets of a real normed linear space. Motivated by them, in order to define a topology on the whole family of nonempty subsets of a real normed linear space rather than the family of lower bounded subsets, we consider the strict order relation given in [20]. Further, we discuss a notion of corresponding topological convergence for sequence of families of sets. In light of this topology, we study the convergence aspect and continuity aspect for set optimization problem in the image space.

We derive upper and lower set order convergence of the families of weak minimal and minimal solution sets, respectively, in the image space under the perturbations of the feasible set and objective function. Further, in the given space, we establish the lower convergence for efficient solution sets in Painlevé–Kuratowski sense without imposing any strict quasiconvexity assumption, which have been used earlier in the literature (see [10, 21]) to obtain the similar convergence results. We also establish the upper Painlevé–Kuratowski for a weaker notion of weak efficient solutions. Besides, we provide complete characterization of strict efficient and weak efficient solutions in terms of the scalar problem considered in [20]. We further derive the upper and lower Painlevé–Kuratowski convergence of solution sets of perturbed scalar problems to solution sets of the original set optimization problem.

The study is further extended to the semicontinuity of solution set maps both in the given space and image space. For this purpose, we introduce notions of upper and lower semicontinuity for minimal solution set maps and examine the semicontinuity results in the image space by imposing certain conditions such as domination property, compactness and topological convergence. We derive the upper semicontinuity of weak efficient solution set map. Analogous to lower Painlevé–Kuratowski convergence for efficient solution sets, we establish the lower semicontinuity for efficient solution set map by relaxing the assumption of strict convexity. To the best of our knowledge, the convergence aspect in set optimization using scalarization technique and continuity aspect in set optimization in the image space are being studied for the first time in this paper.

The rest of the paper is organized as follows. In Sect. 2, we study topological convergence and its properties with respect to preference relations considered in [20]. Section 3 deals with the study of convergence of solution sets in the given space as well as the image space. We establish convergence of solution sets in the given space using a scalar technique in Sect. 4. In Sect. 5, we investigate the continuity of solution set maps in both the image space and the given space. Finally, we give some concluding remarks in Sect. 6.

2 Preliminaries

Let X be a real normed linear space and $B(x, \delta)$ denote the open ball with centre at $x \in X$ and radius $\delta > 0$. Let Y be a real normed linear space and K be a closed convex pointed cone in Y with nonempty interior, denoted by $\text{int}K$. Let $\mathcal{P}^0(Y)$ denote the family of all nonempty subsets of Y and $\mathcal{B}^0(Y)$ denote the family of all nonempty bounded subsets of Y. An element $y \in Y$ is said to be a lower bound of a set $A \in \mathcal{P}^0(Y)$ if $A \subseteq y + K$. We denote the set of all lower bounded sets in $\mathcal{P}^0(Y)$ by $\mathcal{LB}(Y)$. For $A, B \in \mathcal{P}^0(Y)$, we first recall the following notion of the Minkowski difference from [22] defined as

$$A \dot{-} B := \{y \in Y : y + B \subseteq A\}.$$

We next consider two order relations introduced by Karaman et al. [20]. For $A, B \in \mathcal{P}^0(Y)$,

$$A \preceq^m B \text{ if and only if } (A \dot{-} B) \cap (-K) \neq \emptyset$$

and

$$A \prec^m B \text{ if and only if } (A \dot{-} B) \cap (-\text{int}K) \neq \emptyset.$$

From [20, Corollary 2], we observe that $\preceq^m$ is a partial order relation on $\mathcal{B}^0(Y)$. It can also be seen easily that the relation $\prec^m$ is transitive and irreflexive.

We now consider some subsets of $\mathcal{P}^0(Y)$ with respect to $\prec^m$ relation. Similar notions have been considered in [15, Definition 2.5] for the family of sets $\mathcal{LB}(Y)$.

Definition 1 Let $A, B \in \mathcal{P}^0(Y)$.

(a) If $A \prec^m B$, then the open interval $(A; B)$ is defined as

$$(A; B) := \{C \in \mathcal{P}^0(Y) : A \prec^m C \prec^m B\}.$$

The set of all open intervals in $\mathcal{P}^0(Y)$ is denoted by $\mathcal{I}$ and clearly $\mathcal{I} \subseteq \mathcal{P}(\mathcal{P}^0(Y))$.

(b) The set A^+ is defined as $A^+ := \{C \in \mathcal{P}^0(Y) : A \prec^m C\}$.

(c) The set A^- is defined as $A^- := \{C \in \mathcal{P}^0(Y) : C \prec^m A\}$.

***Remark* 1** Since $A \prec^m A+k$ for any $k \in \text{int}K$, it follows that $A+k \in A^+$ and hence, $A^+ \neq \varnothing$. Similarly, $A^- \neq \varnothing$ as $A - k \in A^-$ for any $k \in \text{int}K$.

The following lemma shows that every open interval between two sets is nonempty.

Lemma 1 *Let $A, B \in \mathcal{P}^0(Y)$ be such that $A \prec^m B$ then $(A; B) \neq \varnothing$.*

Proof Let $-k \in (A \dot{-} B) \cap (-\text{int}K)$. Thus, $B - k \subseteq A$, that is, for any $\lambda \in (0, 1)$, $B - \lambda k - (1 - \lambda)k \subseteq A$. Thus, $-(1 - \lambda)k \in A \dot{-} (B - \lambda k)$ and so $A \prec^m B - \lambda k$. Also $B - \lambda k \prec^m B$. Hence, $B - \lambda k \in (A; B)$ which implies that $(A; B) \neq \varnothing$.

Similarly, it is easy to see that $A + \lambda k$ also belongs to $(A; B)$ for any $\lambda \in (0, 1)$ and $-k \in (A \dot{-} B) \cap (-\text{int}K)$.

Geoffroy [15] considered a topology on the family of lower bounded sets, whereas we define a similar topology on the collection of family of nonempty subsets of Y, generated by $\mathcal{I}$. We denote it by τ. Clearly $\mathcal{I}$ is a sub-base for the topology τ, that is, the collection of all finite intersection of elements of $\mathcal{I}$ is a base for τ.

We now present some open sets with respect to topology τ.

Lemma 2 *Let $A, B \in \mathcal{P}^0(Y)$. The following assertions hold*:

(i) *If $A \prec^m B$, then $(A; B)$ is a τ-open set.*
(ii) *The open interval $(A - k; A + k')$ for any $k, k' \in \text{int}K$ is a τ-open neighborhood of A.*
(iii) *For any $\lambda \in (0, 1)$ and $-k \in (A \dot{-} B) \cap (-\text{int}K)$, the set $(A - \lambda k; A + \lambda k)$ is a τ-open neighborhood of A.*
(iv) *The set A^- is a τ-open set.*
(v) *The set A^+ is a τ-open set.*

Proof (i) The proof follows easily as $\mathcal{I}$ is a sub-base for topology τ.

(ii) Since $A - k \prec^m A \prec^m A + k'$ therefore $A \in (A - k; A + k')$. From (i), it follows that $(A - k; A + k')$ is a τ-open neighborhood of A.
(iii) Immediately follows from (ii).
(iv) Let $C \in A^-$. Thus, $C \prec^m A$, which implies that $(C \dot{-} A) \cap (-\text{int}K) \neq \varnothing$. Let $-k \in (C \dot{-} A) \cap (-\text{int}K)$. Clearly, $C \in (C - \lambda k, C + \lambda k)$ for any $\lambda \in (0, 1)$. Since $C \prec^m A$, we have $C + \lambda k \prec^m A$ which implies that $(C - \lambda k, C + \lambda k) \subseteq A^-$. Thus, A^- is a τ-open set.
(v) The proof follows as in (iv).

Similar to a topological convergence given in [15, Definition 2.12], we next provide a notion of τ-convergence for a sequence of sets in $\mathcal{P}^0(Y)$.

Definition 2 Let $A \in \mathcal{P}^0(Y)$ and $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}^0(Y)$. The sequence $(A_n)_{n\in\mathbb{N}}$ *τ-converges* to A (denoted by $A_n \xrightarrow{\tau} A$) if for any neighborhood $\mathcal{N}$ of A there corresponds $m \in \mathbb{N}$ such that $A_n \in \mathcal{N}$ for $n \geqslant m$.

It may be noted that since $\mathcal{I}$ is sub-base for the topology τ therefore, without loss of generality, we assume that a neighborhood of A is of the form $(W; W')$ where W, $W' \in \mathcal{P}^0(Y)$. In the next two lemmas, we show the compatibility of τ-convergence with the order relations on $\mathcal{P}^0(Y)$.

Lemma 3 *Let A, $B \in \mathcal{P}^0(Y)$. Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be two sequences in $\mathcal{P}^0(Y)$ such that $A_n \xrightarrow{\tau} A$ and $B_n \xrightarrow{\tau} B$. Then the following assertions hold*:

(i) *If $A \prec^m B$, then $A_n \prec^m B_n$ for sufficiently large n.*
(ii) *If $A_n \prec^m B_n$ for sufficiently large n, then $A - k \prec^m B$ for any $k \in \operatorname{int}K$.*

Proof (i) From Lemma 1, we have $(A; B) \neq \varnothing$. Let $W \in (A; B)$ which implies that $A \in W^-$ and $B \in W^+$. Since W^- and W^+ are τ-open sets it follows that $A_n \in W^-$ and $B_n \in W^+$ for sufficiently large n. Thus, $W \in (A_n; B_n)$ for sufficiently large n, which implies that $A_n \prec^m B_n$ for sufficiently large n.

(ii) Clearly, $A - \frac{k}{2} \prec^m A$ for any $k \in \operatorname{int}K$. Thus, $A \in (A - \frac{k}{2})^+$ which is a τ-open set. Since $A_n \xrightarrow{\tau} A$ therefore $A - \frac{k}{2} \prec^m A_n$ for sufficiently large n. Similarly, $B \in (B + \frac{k}{2})^-$ for any $k \in \operatorname{int}K$. Since $B_n \xrightarrow{\tau} B$, we have $B_n \prec^m B + \frac{k}{2}$ for sufficiently large n. Thus, $A - \frac{k}{2} \prec^m A_n \prec^m B_n \prec^m B + \frac{k}{2}$, which implies that $A - k \prec^m B$ for any $k \in \operatorname{int}K$.

Lemma 4 *Let $A \in \mathcal{B}^0(Y)$ be a closed set and $B \in \mathcal{P}^0(Y)$. Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be two sequences in $\mathcal{P}^0(Y)$ such that $A_n \xrightarrow{\tau} A$ and $B_n \xrightarrow{\tau} B$. Then the following assertions hold*:

(i) *If $A - k \prec^m B$ for any $k \in \operatorname{int}K$, then $A \preceq^m B$.*
(ii) *If $A_n \prec^m B_n$ for sufficiently large n, then $A \preceq^m B$.*
(iii) *If $A_n \preceq^m B_n$ then $A \preceq^m B$.*

Proof (i) Let $(k_n)_{n\in\mathbb{N}} \subseteq \operatorname{int}K$ be such that $k_n \to 0$. Since $A - k_n \prec^m B$, there exists $k'_n \in \operatorname{int}K$ such that $B - k'_n \subseteq A - k_n$. As A is a bounded set, there exists a subsequence $(k'_{n_m})_{m\in\mathbb{N}}$ of $(k'_n)_{n\in\mathbb{N}}$ which converges to $k' \in K$. As A is closed, we have $B - k' \subseteq A$ and hence, $A \preceq^m B$.

(ii) The proof follows immediately from (i) and Lemma 3(ii).

(iii) The proof follows as in (ii).

Analogous to a topological convergence given in [15, Definition 3.5], we now define the following notions of convergence of sequence of families of sets with respect to topology τ.

Definition 3 Let $\mathcal{A} \in \mathcal{P}(\mathcal{P}^0(Y))$ and $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}(\mathcal{P}^0(Y))$.

(a) The sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is said to *lower set order converge* to $\mathcal{A}$, if for any $A \in \mathcal{A}$ there exists $A_n \in \mathcal{A}_n$ for all n such that $A_n \stackrel{\tau}{\to} A$.
(b) The sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is said to *upper set order converge* to $\mathcal{A}$, if any sequence $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathcal{A}_n$ has a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ such that $A_{n_k} \stackrel{\tau}{\to} A$ then $A \in \mathcal{A}$.

The lower (resp. upper) set order convergence of the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ to $\mathcal{A}$ is denoted by $\mathcal{A}_n \stackrel{ls\tau}{\to} \mathcal{A}$ (resp. $\mathcal{A}_n \stackrel{us\tau}{\to} \mathcal{A}$). The sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is said to set order converge to $\mathcal{A}$, if $\mathcal{A}_n \stackrel{ls\tau}{\to} \mathcal{A}$ and $\mathcal{A}_n \stackrel{us\tau}{\to} \mathcal{A}$, and is denoted by $\mathcal{A}_n \to \mathcal{A}$.

We now consider the notion of Painlevé–Kuratowski convergence of sets from [23]. A sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is said to converge to a nonempty subset A of X in Painlevé-Kuratowski sense (denoted by $A_n \stackrel{K}{\to} A$) if $\mathrm{Ls}(A_n) \subseteq A \subseteq \mathrm{Li}(A_n)$, where

$$\mathrm{Ls}(A_n) := \{x \in X : x = \lim_{k\to\infty} x_k, \text{ such that } x_k \in M_{n_k}, \text{ for } k \in \mathbb{N}\},$$
$$\mathrm{Li}(A_n) := \{x \in X : x = \lim_{n\to\infty} x_n, \; x_n \in A_n \text{ for sufficiently large } n\}.$$

The inclusion $\mathrm{Ls}(A_n) \subseteq A$ refers to the *upper Painlevé–Kuratowski convergence* (denoted by $A_n \stackrel{K}{\rightharpoonup} A$) and $A \subseteq \mathrm{Li}(A_n)$ refers to the *lower Painlevé–Kuratowski convergence* (denoted by $A_n \stackrel{K}{\rightharpoondown} A$).

We now consider the set optimization problem

$$\text{(P)} \qquad \text{Min } F(x)$$
$$\text{s.t. } x \in S,$$

where $F : X \rightrightarrows Y$ is a nonempty bounded set-valued map and S is a nonempty subset of X. We denote the family of image sets of F on S by $\mathcal{F}$, that is, $\mathcal{F} := \{F(x) : x \in S\}$.

We next recall the notions of m-minimal and m-weak minimal solution sets of (P) from [20].

Definition 4 [20] Let $\bar{x} \in S$. Then $F(\bar{x})$ is said to be an

(a) *m-minimal solution set* of (P) if there does not exist any $x \in S$ such that

$$F(x) \preceq^m F(\bar{x}) \text{ and } F(x) \neq F(\bar{x}).$$

(b) *m-weak minimal solution set* of (P) if there does not exist any $x \in S$ such that

$$F(x) \prec^m F(\bar{x}).$$

We denote the family of all m-minimal (resp. m-weak minimal) solution sets of (P) by m-Min($\mathcal{F}$) (resp. m-WMin($\mathcal{F}$)).

We say that $\bar{x} \in S$ is an m-efficient (resp. m-weak efficient) solution of (P) if $F(\bar{x})$ is an m-minimal (resp. m-weak minimal) solution set of (P). The set m-Eff($\mathcal{F}$) (resp. m-WEff($\mathcal{F}$)) refers to the set of m-efficient (resp. m-weak efficient) solutions of (P).

Similar to Definition 2.7 in [24], we now give the following notion of m-strict efficient solutions of (P).

Definition 5 A point $\bar{x} \in S$ is said to be an *m-strict efficient solution* of (P) if for any $x \in S$,

$$F(x) \preceq^m F(\bar{x}) \Rightarrow x = \bar{x}.$$

We denote the set of m-strict efficient solutions of (P) by m-SEff($\mathcal{F}$). It can be easily seen that m-SEff($\mathcal{F}$) $\subseteq$ m-Eff($\mathcal{F}$) $\subseteq$ m-WEff($\mathcal{F}$). However, the reverse inclusion may not hold as can be seen from [25, Example 2.1].

We now recall a strict lower set order relation on $\mathcal{LB}(Y)$ from [15] and denote it by $\prec^{sl}$. For $A, B \in \mathcal{LB}(Y)$,

$$A \prec^{sl} B \text{ if and only if there exists } k \in K \setminus \{0\} \text{ such that } B \subseteq A + k + K.$$

We next recall the notion of relaxed minimal solution from [15, Definition 3.11]. A point $\bar{x} \in S$ is a relaxed minimal solution of (P) if there does not exists any $x \in S$ such that $F(x) \prec^{sl} F(\bar{x})$. We observe that every relaxed minimal solution of (P) is an m-weak efficient solution of (P). However, the converse need not be true as can be seen from the following example.

***Example* 1** Let $X = \mathbb{R}$, $S = [-1, 1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \{(x, 0)\}, & \text{if } x < 0, \\ \{(x, x)\}, & \text{if } x \geqslant 0. \end{cases}$$

For $\bar{x} = 0$, we observe that $\bar{x} \in m$-WEff($\mathcal{F}$), but $\bar{x}$ is not a relaxed minimal solution of (P).

Analogous to Definition 4.3 in [26], we next have the following notion of domination property which is used in the sequel.

Definition 6 Let S be a nonempty subset of X. Then a family of sets $\mathcal{F}$ is said to satisfy *m-domination property* on S if for each $F(x) \in \mathcal{F}$, there exists $F(\bar{x}) \in m$-Min($\mathcal{F}$) such that $F(\bar{x}) \preceq^m F(x)$.

We now recall the notions of the upper and lower continuity for a set-valued map from [23], which we refer to as semicontinuity notions.

Definition 7 [23, Definition 3.1.1] The map F is said to be *upper (resp.lower) semicontinuous* at $\bar{x} \in S$ if for any open set $V \subseteq Y$ with $F(\bar{x}) \subseteq V$ (resp. $F(\bar{x}) \cap V \neq \emptyset$) there is a neighborhood U of $\bar{x}$ such that $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$) for all $x \in U \cap S$.

The following proposition provides sequential characterization of the semicontinuity notions.

Proposition 1 *Let $\bar{x} \in S$.*

(i) [23, Proposition 3.1.6] *The map F is lower semicontinuous at $\bar{x}$ if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq S$ with $x_n \to \bar{x}$ and any $\bar{y} \in F(\bar{x})$, there exists $y_n \in F(x_n)$ such that $y_n \to \bar{y}$.*

(ii) [23, Proposition 3.1.9] *If $F(\bar{x})$ is compact, then F is upper semicontinuous at $\bar{x}$ if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq S$ with $x_n \to \bar{x}$ and $y_n \in F(x_n)$, there exist $\bar{y} \in F(\bar{x})$ and a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to \bar{y}$.*

3 Convergence of Families of Solution Sets

In literature (see [9, 10, 13]), the convergence aspect of stability has been established in the given space. By virtue of topological convergence for sequence of families of sets, we now study the stability of minimal solution sets both in the given space and image space.

Here we consider the perturbation of both the objective map and the feasible set. A family of the perturbed problems (P_n) is defined as follows

$$(\mathrm{P}_n) \qquad \begin{aligned} &\mathrm{Min}\ F_n(x) \\ &\text{s. t. } x \in S_n, \end{aligned}$$

where S_n is a nonempty subset of X and $F_n : X \rightrightarrows Y$ is a nonempty bounded set-valued map. Let the image set of F_n on S_n be $\mathcal{F}_n$, that is, $\mathcal{F}_n = \{F_n(x_n) : x_n \in S_n\}$.

We denote the family of all m-minimal (resp. m-weak minimal) solution sets of (P_n) by m-Min$(\mathcal{F}_n)$ (resp. m-WMin$(\mathcal{F}_n)$). The set m-Eff$(\mathcal{F}_n)$ (resp. m-WEff$(\mathcal{F}_n)$) refers to the set of m-efficient (resp. m-weak efficient) solutions of (P_n). Throughout, we assume that the set of m-efficient (resp. m-weak efficient) solutions of (P) and (P_n) are nonempty for every n.

In the following result, we derive the upper set order convergence of families of m-weak minimal solution sets.

Theorem 1 *If $\mathcal{F}_n \overset{ls\tau}{\to} \mathcal{F}$, then m-WMin$(\mathcal{F}_n) \overset{us\tau}{\to} m$-WMin$(\mathcal{F})$.*

Proof Let $F_n(x_n) \in m$-WMin$(\mathcal{F}_n)$ be such that there exists a subsequence $(F_{n_k}(x_{n_k}))_{k\in\mathbb{N}}$ of $(F_n(x_n))_{n\in\mathbb{N}}$ with $F_{n_k}(x_{n_k}) \overset{\tau}{\to} F(x)$. We need to show that $F(x) \in m$-WMin$(\mathcal{F})$. On the contrary, assume that there exists $u \in S$ such that $F(u) \prec^m F(x)$. Since $\mathcal{F}_{n_k} \overset{ls\tau}{\to} \mathcal{F}$ it follows that there exists $F_{n_k}(u_{n_k}) \in \mathcal{F}_{n_k}$ for all k such that $F_{n_k}(u_{n_k}) \overset{\tau}{\to} F(u)$. Thus, using Lemma 3(i) we have $F_{n_k}(u_{n_k}) \prec^m F_{n_k}(x_{n_k})$ for sufficiently large k, which contradicts the fact that $F_{n_k}(x_{n_k}) \in m$-WMin$(\mathcal{F}_{n_k})$.

Remark 2 Anh et al. [27, Corollary 4.1] considered a set optimization problem by perturbing the feasible set and established the upper convergence for a notion of weak minimal solution sets in terms of external stability via Hausdorff convergence using set less order relation. However, we consider a problem with perturbations in both the feasible set and the objective function, to establish the upper convergence for m-weak minimal solution sets in terms of topological convergence. It may be observed that topological convergence is compatible with the considered set order relation and the assumptions in Theorem 1 are different from those considered in [27, Corollary 4.1].

We next provide sufficient conditions for the lower set order convergence of a sequence of families of minimal solution sets of the perturbed problems to the corresponding family of minimal solution sets of the original problem.

Theorem 2 *If the following conditions hold*:

(a) $\mathcal{F}_n \to \mathcal{F}$;
(b) *there is a* τ*-compact set* $\mathcal{E} \in \mathcal{P}(\mathcal{B}^0(Y))$ *such that* $\mathcal{F}_n \subseteq \mathcal{E}$ *for sufficiently large* n;
(c) $\mathcal{F}_n$ *satisfies* m*-domination property on* S_n *for all* n;
(d) F *is closed-valued on* S;

then $m\text{-Min}(\mathcal{F}_n) \overset{ls\tau}{\to} m\text{-Min}(\mathcal{F})$.

Proof Let $F(x) \in m\text{-Min}(\mathcal{F})$. Since $\mathcal{F}_n \overset{ls\tau}{\to} \mathcal{F}$, there exists $F_n(x_n) \in \mathcal{F}_n$ for all n such that $F_n(x_n) \overset{\tau}{\to} F(x)$. As $\mathcal{F}_n$ satisfies m-domination property on S_n for all n, there exists $F_n(u_n) \in m\text{-Min}(\mathcal{F}_n)$ such that $F_n(u_n) \preceq^m F_n(x_n)$. Using the compactness assumption, it follows that $(F_n(u_n))_{n\in\mathbb{N}}$ has a convergent subsequence $(F_{n_k}(u_{n_k}))_{k\in\mathbb{N}}$ such that $F_{n_k}(u_{n_k}) \overset{\tau}{\to} F(u)$. Clearly, $F(u) \in \mathcal{F}$ as $\mathcal{F}_{n_k} \overset{us\tau}{\to} \mathcal{F}$. From Lemma 4(iii) we obtain that $F(u) \preceq^m F(x)$, which further implies that $F(u) = F(x)$ as $F(x) \in m\text{-Min}(\mathcal{F})$. Thus, there exists $F_{n_k}(u_{n_k}) \in m\text{-Min}(\mathcal{F}_{n_k})$ such that $F_{n_k}(u_{n_k}) \overset{\tau}{\to} F(x)$. Since the later convergence holds for every convergent subsequence, therefore the entire sequence $(F_n(u_n))_{n\in\mathbb{N}}$ converges to $F(x)$. Hence, $F(x) \in \text{Li}(m\text{-Min}(\mathcal{F}_n))$.

The following two examples illustrate that condition (a) cannot be relaxed in Theorem 2.

***Example* 2** Let $X = \mathbb{R}$, $S = [0, 1]$, $S_n = [0, 1 - \frac{1}{n}]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Consider the maps $F, F_n : X \rightrightarrows Y$ defined as

$$F(x) = \{(t, 0) : \min\{x, 1-x\} \leqslant t \leqslant 1\},$$
$$F_n(x) = \begin{cases} \{(t, -1) : 0 \leqslant t \leqslant 1\}, & \text{if } x = 0, \\ \{(t, 0) : \min\{x, 1-x\} \leqslant t \leqslant 1\}, & \text{if } x \neq 0. \end{cases}$$

It can be easily seen that $\mathcal{F}_n \overset{us\tau}{\nrightarrow} \mathcal{F}$ as $F_n(0) \overset{\tau}{\to} A$, where $A = \{(t, -1) : 0 \leqslant t \leqslant 1\}$ and $A \notin \mathcal{F}$, but all other assumptions of Theorem 2 are satisfied. Here $m\text{-Min}(\mathcal{F}) = \{F(0), F(1)\}$ and $m\text{-Min}(\mathcal{F}_n) = \{F_n(0)\}$. We observe that for any $F(x) \in m\text{-Min}(\mathcal{F})$ there does not exist any $F_n(x_n) \in m\text{-Min}(\mathcal{F}_n)$ such that $F_n(x_n) \overset{\tau}{\to} F(x)$.

***Example* 3** Let $X = \mathbb{R}$, $S = [0, 1]$, $S_n = [\frac{1}{n}, 1]$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Consider the maps $F, F_n : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} \{(t, 0) : -1 \leqslant t \leqslant 0\}, & \text{if } x = 0, \\ \{(t, 0) : 0 \leqslant t \leqslant \max\{x, 1-x\}\}, & \text{if } x \neq 0, \end{cases}$$
$$F_n(x) = \{(t, 0) : 0 \leqslant t \leqslant \max\{x, 1-x\}\}.$$

It can be seen that $\mathcal{F}_n \overset{ls\tau}{\nrightarrow} \mathcal{F}$ as for $F(0)$, there does not exists any $F_n(x_n) \in \mathcal{F}_n$ such that $F_n(x_n) \overset{\tau}{\to} F(0)$, but all other assumptions of Theorem 2 are satisfied.

Here m-Min$(\mathcal{F}) = \{F(0)\}$ and m-Min$(\mathcal{F}_n) = \{F_n(1)\}$. We observe that for $F(0) \in m$-Min$(\mathcal{F})$, the result fails to hold.

***Remark* 3** As stated in Remark 2 earlier, we also observe here that the lower convergence established in this paper is different from the one established by Anh et al. [27, Theorem 4.4].

The proof of the following theorem, which presents the upper stability of m-weak efficient solution sets in the given space, follows on the lines of Theorem 3.12 in [15].

Theorem 3 *If the following conditions hold*:

(a) $S_n \overset{K}{\rightharpoonup} S$;
(b) $\mathcal{F}_n \overset{ls\tau}{\to} \mathcal{F}$;
(c) *for every* $x_n \in S_n$, $x \in S$ *with* $x_n \to x$ *there exists a subsequence* $(F_{n_k}(x_{n_k}))_{k\in\mathbb{N}}$ *of* $(F_n(x_n))_{n\in\mathbb{N}}$ *such that* $F_{n_k}(x_{n_k}) \overset{\tau}{\to} F(x)$;

then m-WEff$(\mathcal{F}_n) \overset{K}{\rightharpoonup} m$-WEff$(\mathcal{F})$.

Next, we show that the upper stability of m-weak efficient solution sets can also be obtained by replacing condition (b) in Theorem 3 by the lower Painlevé–Kuratowski convergence of the sets S_n.

Theorem 4 *If the following conditions hold*:

(a) $S_n \overset{K}{\to} S$;
(b) *for every* $x_n \in S_n$, $x \in S$ *with* $x_n \to x$ *there exists a subsequence* $(F_{n_k}(x_{n_k}))_{k\in\mathbb{N}}$ *of* $(F_n(x_n))_{n\in\mathbb{N}}$ *such that* $F_{n_k}(x_{n_k}) \overset{\tau}{\to} F(x)$;

then m-WEff$(\mathcal{F}_n) \overset{K}{\rightharpoonup} m$-WEff$(\mathcal{F})$.

Proof Let $x_n \in m$-WEff$(\mathcal{F}_n)$ be such that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $x_{n_k} \to x$. As $S_n \overset{K}{\rightharpoonup} S$ it follows that $x \in S$. Using assumption (b), without loss of generality we assume that $F_{n_k}(x_{n_k}) \overset{\tau}{\to} F(x)$. We show that $x \in m$-WEff$(\mathcal{F})$. On the contrary, assume that there exists $u \in S$ such that $F(u) \prec^m F(x)$. Since $S_{n_k} \overset{K}{\rightharpoondown} S$, there exists $u_{n_k} \in S_{n_k}$ for all k such that $u_{n_k} \to u$. Again using assumption (b), without loss of generality we assume that $F_{n_k}(u_{n_k}) \overset{\tau}{\to} F(u)$. From Lemma 3(i), we have $F_{n_k}(u_{n_k}) \prec^m F_{n_k}(x_{n_k})$, which is a contradiction as $x_{n_k} \in m$-WEff$(\mathcal{F}_{n_k})$.

***Remark* 4** From [28], we recall that for $A, B \in \mathcal{P}^0(Y)$, $A \preceq^l B$ if and only if $B \subseteq A + K$ and $A \prec^l B$ if and only if $B \subseteq A + \text{int}K$. It may be observed from [20, Proposition 9] that if $A \preceq^m B$ (resp. $A \prec^m B$) then $A \preceq^l B$ (resp. $A \prec^l B$). Considering the solutions based on the strict relation $\prec^l$, Karuna and Lalitha [10] studied the upper stability of stronger notion of weak efficient solution sets. The problem considered in [10] involved just the perturbation of the feasible set and the stability results were obtained under continuity and closedness assumptions on the objective map.

Using the quasi-order relation $\preceq^l$, Geoffroy [15, Corollary 3.14] established the lower stability of minimal solution set by proving that every minimal solution is a limit of a sequence of approximate minimal solutions of (P_n). However, by considering the solutions based on partial order relation $\preceq^m$, we establish the lower Painlevé–Kuratowski convergence of a sequence of efficient solution sets of the perturbed problems to strict efficient solution set of the original problem.

Theorem 5 *If the conditions* (a) − (c) *of Theorem* 3 *are satisfied and the following conditions hold*:

(d) $S_n \overset{K}{\rightharpoonup} S$;
(e) *there exists a compact set E in X such that $S_n \subseteq E$ for sufficiently large n*;
(f) *$\mathcal{F}_n$ satisfies m-domination property on S_n for sufficiently large n*;
(g) *F is closed-valued on S*;

then m-Eff$(\mathcal{F}_n) \overset{K}{\rightharpoonup} m$-SEff$(\mathcal{F})$.

Proof Let $x \in m$-SEff$(\mathcal{F})$. Since $\mathcal{F}_n \overset{ls\tau}{\to} \mathcal{F}$, there exists $F_n(x_n) \in \mathcal{F}_n$ for all n such that $F_n(x_n) \overset{\tau}{\to} F(x)$. As $\mathcal{F}_n$ satisfies m-domination property on S_n for all n, there exists $u_n \in m$-Eff$(\mathcal{F}_n)$ such that

$$F_n(u_n) \preceq^m F_n(x_n).$$

By assumption (e), there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $u_{n_k} \to u$. Clearly $u \in S$ as $S_n \overset{K}{\rightharpoonup} S$, which further by assumption (c) implies that there exists a subsequence $(F_{n_{k_l}}(u_{n_{k_l}}))_{l\in\mathbb{N}}$ of $(F_{n_k}(u_{n_k}))_{k\in\mathbb{N}}$ such that $F_{n_{k_l}}(u_{n_{k_l}}) \overset{\tau}{\to} F(u)$. By virtue of Lemma 4(iii), we have $F(u) \preceq^m F(x)$. Since $x \in m$-SEff$(\mathcal{F})$, we obtain that $u = x$. Thus, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ with $u_{n_k} \in m$-Eff$(\mathcal{F}_{n_k})$ such that $u_{n_k} \to x$. Since this convergence holds for every convergent subsequence, therefore the entire sequence $(u_n)_{n\in\mathbb{N}}$ converges to x. Thus, $x \in$ Li$(m$-Eff$(\mathcal{F}_n))$.

***Remark* 5** Using the relation $\preceq^l$ and perturbing the feasible set only, a similar result has been proved for a different notion of efficient solution set in [10, Theorem 4.3] by assuming a strict quasiconvexity assumption (see Definition 2.6 in [17]) on the objective map.

From Theorems 3 and 5, we now conclude the following result.

Theorem 6 *If the following conditions hold*:

(a) $S_n \overset{K}{\to} S$;
(b) $\mathcal{F}_n \overset{ls\tau}{\to} \mathcal{F}$;
(c) *there exists a compact set E in X such that $S_n \subseteq E$ for sufficiently large n*;
(d) *for every $x_n \in S_n$, $x \in S$ with $x_n \to x$ there exists a subsequence $(F_{n_k}(x_{n_k}))_{k\in\mathbb{N}}$ of $(F_n(x_n))_{n\in\mathbb{N}}$ such that $F_{n_k}(x_{n_k}) \overset{\tau}{\to} F(x)$*;
(e) *$\mathcal{F}_n$ satisfies m-domination property on S_n for sufficiently large n*;
(f) *F is closed-valued on S*;

then m-SEff$(\mathcal{F}) \subseteq$ Li$(m$-Eff$(\mathcal{F}_n)) \subseteq$ Ls$(m$-WEff$(\mathcal{F}_n)) \subseteq m$-WEff$(\mathcal{F})$.

4 Stability Using Scalarization Approach

In this section, based on the scalar problem proposed by Karaman et al. [20], we study the stability of solution sets of set optimization problem by means of scalarization techniques. For this purpose, we first obtain the characterizations of m-strict efficient and m-weak efficient solutions of (P) in terms of strict optimal and optimal solutions of scalarized problem, respectively. Then we establish the Painlevé–Kuratowski convergence of optimal solutions of perturbed scalar problems to solution sets of (P).

We consider a scalar function $I_e^m(.,.) : \mathcal{F}_X \times \mathcal{F}_X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$I_e^m(F(x), F(\bar{x})) := \inf\{t \in \mathbb{R} : F(x) \preceq^m te + F(\bar{x})\},$$

where $e \in \text{int}K$, $F : X \rightrightarrows Y$ is a nonempty bounded set-valued map and $\mathcal{F}_X := \{F(x) : x \in X\}$.

We first recall the following properties of I_e^m from [20] required in the sequel.

Theorem 7 [20, Proposition 19] *Let $x, \bar{x} \in X$. Then the following assertions hold*:

(i) $F(x) \prec^m F(\bar{x})$ *if and only if* $I_e^m(F(x), F(\bar{x})) < 0$.

(ii) *Let* $I_e^m(F(x), F(\bar{x}))$ *be finite and* $F(\bar{x}) \stackrel{\cdot}{-} F(x)$ *be a compact set. Then* $F(x) \preceq^m F(\bar{x})$ *if and only if* $I_e^m(F(x), F(\bar{x})) \leqslant 0$.

The following notions of m-increasing and strictly m-increasing function are from [20].

Definition 8 [20, Definition 9] Let $\mathcal{A} \in \mathcal{P}(\mathcal{P}_0(Y))$. A function $T : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ is called

(a) *m-increasing* on $\mathcal{A}$ if for $A, B \in \mathcal{A}$, $A \preceq^m B$ implies that $T(A) \leqslant T(B)$.

(b) *strictly m-increasing* on $\mathcal{A}$ if for $A, B \in \mathcal{A}$, $A \prec^m B$ implies that $T(A) < T(B)$.

Lemma 5 (i) [20, Proposition 15] *Let $F(x) \in \mathcal{F}_X$. Then $I_e^m(., F(x))$ is an m-increasing function on $\mathcal{F}_X$.*

(ii) [20, Proposition 17] *Let $F(x) \in \mathcal{F}_X$ be a compact set. Then $I_e^m(., F(x))$ is a strictly m-increasing function on $\mathcal{F}_X$.*

For $\bar{x} \in S$, we now consider the scalar optimization problem

$$(\text{P}(F(\bar{x}))) \qquad \text{Min } I_e^m(F(x), F(\bar{x})) \quad \text{s.t. } x \in S.$$

We denote the set of optimal (resp. strict optimal) solutions of $(\text{P}(F(\bar{x})))$ by $\text{argmin}_S(\text{P}(F(\bar{x})))$ (resp. $\text{argmin}_S^{<}(\text{P}(F(\bar{x})))$) defined as

$$\begin{aligned} \text{argmin}_S(P(F(\bar{x}))) &:= \{\hat{x} \in S : I_e^m(F(\hat{x}), F(\bar{x})) \leqslant I_e^m(F(x), F(\bar{x})) \text{ for all } x \in S\} \\ &(\text{resp. } \text{argmin}_S^{<}(P(F(\bar{x}))) \\ &:= \{\hat{x} \in S : I_e^m(F(\hat{x}), F(\bar{x})) < I_e^m(F(x), F(\bar{x})) \text{ for all } x \in S \setminus \{\hat{x}\}\}). \end{aligned}$$

The following result relates m-weak efficient and m-strict efficient solutions of (P) with the optimal and strict optimal solutions of scalar optimization problem, respectively.

Theorem 8 *The following assertions hold*:

(i) $\bigcup_{x\in S} \operatorname{argmin}_S^{<}(\mathrm{P}(F(x))) \subseteq m\text{-SEff}(\mathcal{F})$.

(ii) *If F is a compact-valued map on S, then* $\bigcup_{x\in S} \operatorname{argmin}_S(\mathrm{P}(F(x))) \subseteq m\text{-WEff}(\mathcal{F})$.

(iii) *If F is a compact-valued map on S, then* $m\text{-SEff}(\mathcal{F}) \subseteq \bigcup_{x\in S} \operatorname{argmin}_S^{<}(\mathrm{P}(F(x)))$.

Proof (i) Let $\hat{x} \in \bigcup_{x\in S} \operatorname{argmin}_S^{<}(\mathrm{P}(F(x)))$. Let $\hat{x} \in \operatorname{argmin}_S^{<}(\mathrm{P}(F(\bar{x})))$ for some $\bar{x} \in S$. Thus, $I_e^m(F(\hat{x}), F(\bar{x})) < I_e^m(F(x), F(\bar{x}))$ for all $x \in S \setminus \{\hat{x}\}$. We show that $\hat{x} \in m\text{-SEff}(\mathcal{F})$. On the contrary, assume that there exists $u \in S \setminus \{\hat{x}\}$ such that $F(u) \preceq^m F(\hat{x})$. Since $I_e^m(., F(\bar{x}))$ is an m-increasing function, we have $I_e^m(F(u), F(\bar{x})) \leqslant I_e^m(F(\hat{x}), F(\bar{x}))$, which is a contradiction.

(ii) Let $\hat{x} \in \bigcup_{x\in S} \operatorname{argmin}_S(\mathrm{P}(F(x)))$. Thus, $\hat{x} \in \operatorname{argmin}_S(\mathrm{P}(F(\bar{x})))$ for some $\bar{x} \in S$ and so $I_e^m(F(\hat{x}), F(\bar{x})) \leqslant I_e^m(F(x), F(\bar{x}))$ for all $x \in S$. We show that $\hat{x} \in m\text{-WEff}(\mathcal{F})$. On the contrary, assume that there exists $u \in S$ such that $F(u) \prec^m F(\hat{x})$. As $I_e^m(., F(\bar{x}))$ is a strictly m-increasing function, we have $I_e^m(F(u), F(\bar{x})) < I_e^m(F(\hat{x}), F(\bar{x}))$, which is a contradiction.

(iii) Let $\hat{x} \in m\text{-SEff}(\mathcal{F})$, hence it follows that $F(x) \not\preceq^m F(\hat{x})$ for any $x \in S \setminus \{\hat{x}\}$. Thus, by Theorem 7(ii), we have $I_e^m(F(x), F(\hat{x})) > 0$ for all $x \in S \setminus \{\hat{x}\}$. As $I_e^m(F(\hat{x}), F(\hat{x})) = 0$ it follows that $I_e^m(F(\hat{x}), F(\hat{x})) < I_e^m(F(x), F(\hat{x}))$ for all $x \in S \setminus \{\hat{x}\}$. Hence, $\hat{x} \in \operatorname{argmin}_S^{<}(\mathrm{P}(F(\hat{x})))$, which implies that $\hat{x} \in \bigcup_{x\in S} \operatorname{argmin}_S^{<}(\mathrm{P}(F(x)))$.

***Remark* 6** As F is a bounded set-valued map, thus from [20, Corollary 10] we observe that $m\text{-WEff}(\mathcal{F}) \subseteq \bigcup_{x\in S} \operatorname{argmin}_S(\mathrm{P}(F(x)))$.

Thus, from Theorem 8 and Remark 6, we conclude the following result.

Theorem 9 *If F is compact-valued map on S, then the following assertions hold*:

(i) $m\text{-WEff}(\mathcal{F}) = \bigcup_{x\in S} \operatorname{argmin}_S(\mathrm{P}(F(x)))$.

(ii) $m\text{-SEff}(\mathcal{F}) = \bigcup_{x\in S} \operatorname{argmin}_S^{<}(\mathrm{P}(F(x)))$.

In the following result, we investigate the stability of optimal solutions of perturbed scalarized problems in the upper Painlevé–Kuratowski sense.

Theorem 10 *If the following conditions hold*:

(a) $S_n \xrightarrow{K} S$;

(b) $\mathcal{F}_n \xrightarrow{ls\tau} \mathcal{F}$;

(c) *for every $x \in S$, $x_n \in S_n$ with $x_n \to x$ there exists a subsequence $(F_{n_k}(x_{n_k}))_{k\in\mathbb{N}}$ of $(F_n(x_n))_{n\in\mathbb{N}}$ such that $F_{n_k}(x_{n_k}) \xrightarrow{\tau} F(x)$;*

(d) *F_n is a compact-valued map on S_n for all n;*

then $\bigcup_{x_n \in S_n} \operatorname{argmin}_{S_n}(\mathrm{P}(F_n(x_n))) \overset{K}{\rightharpoonup} \bigcup_{x \in S} \operatorname{argmin}_S(\mathrm{P}(F(x)))$.

Proof Let $\hat{x}_n \in \bigcup_{x_n \in S_n} \operatorname{argmin}_{S_n}(\mathrm{P}(F_n(x_n)))$ be such that there exists a subsequence $(\hat{x}_{n_k})_{k \in \mathbb{N}}$ of $(\hat{x}_n)_{n \in \mathbb{N}}$ with $\hat{x}_{n_k} \to \hat{x}$. Let $\hat{x}_n \in \operatorname{argmin}_{S_n}(\mathrm{P}(F_n(\bar{x}_n)))$ for some $\bar{x}_n \in S_n$. Since $S_n \overset{K}{\rightharpoonup} S$ it follows that $\hat{x} \in S$. As $\hat{x}_{n_k} \in \operatorname{argmin}_{S_{n_k}}(\mathrm{P}(F_{n_k}(\bar{x}_{n_k})))$, we have

$$I_e^m(F_{n_k}(\hat{x}_{n_k}), F_{n_k}(\bar{x}_{n_k})) \leqslant I_e^m(F_{n_k}(x_{n_k}), F_{n_k}(\bar{x}_{n_k})), \text{ for all } x_{n_k} \in S_{n_k}. \tag{1}$$

Let $x \in S$. Since $\mathcal{F}_{n_k} \overset{ls\tau}{\to} \mathcal{F}$, it follows that there exists $F_{n_k}(x_{n_k}) \in \mathcal{F}_{n_k}$ for all $k \in \mathbb{N}$ such that $F_{n_k}(x_{n_k}) \overset{\tau}{\to} F(x)$. As F_{n_k} is compact-valued on S_{n_k}, from (1) and Lemma 5(ii) it follows that $F_{n_k}(x_{n_k}) \not\prec^m F_{n_k}(\hat{x}_{n_k})$ for all $k \in \mathbb{N}$. Also, using assumption (c), without loss of generality we assume that $F_{n_k}(\hat{x}_{n_k}) \overset{\tau}{\to} F(\hat{x})$. By Lemma 3(i), we have $F(x) \not\prec^m F(\hat{x})$. Thus, by Theorem 7(i), we have $I_e^m(F(x), F(\hat{x})) \geqslant 0$. As $I_e^m(F(\hat{x}), F(\hat{x})) = 0$ we obtain that $I_e^m(F(\hat{x}), F(\hat{x})) \leqslant I_e^m(F(x), F(\hat{x}))$. Thus $\hat{x} \in \operatorname{argmin}_S(\mathrm{P}(F(\hat{x}))) \subseteq \bigcup_{x \in S} \operatorname{argmin}_S(\mathrm{P}(F(x)))$.

***Remark* 7** We observe that as in Theorem 4, we can replace the condition $\mathcal{F}_n \overset{ls\tau}{\to} \mathcal{F}$ by $S_n \overset{K}{\rightharpoonup} S$ in Theorem 10 also.

The next theorem is an immediate consequence of Theorem 9(i) and Theorem 10 that establishes the upper Painlevé–Kuratowski convergence of the optimal solution sets of perturbed scalarized problems to m-weak efficient solution set of (P).

Theorem 11 *If the conditions* (a)-(d) *of Theorem* 10 *are satisfied and*

(e) *F is compact-valued on S, then* $\bigcup_{x_n \in S_n} \operatorname{argmin}_{S_n}(\mathrm{P}(F_n(x_n))) \overset{K}{\rightharpoonup} m\text{-WEff}(\mathcal{F})$.

Next, we prove the lower Painlevé–Kuratowski convergence of the optimal solution sets of perturbed scalarized problems to the strict optimal solution set of a scalar problem.

Theorem 12 *If the following conditions hold*:

(a) $S_n \overset{K}{\to} S$;
(b) *for every $x \in S$, $x_n \in S_n$ with $x_n \to x$ there exists a subsequence $(F_{n_k}(x_{n_k}))_{k \in \mathbb{N}}$ of $(F_n(x_n))_{n \in \mathbb{N}}$ such that $F_{n_k}(x_{n_k}) \overset{\tau}{\to} F(x)$;*
(c) *there is a compact set E in X such that $S_n \subseteq E$ for sufficiently large n;*
(d) *$\mathcal{F}_n$ satisfies m-domination property on S_n for sufficiently large n;*

then

$$\bigcup_{x_n \in S_n} \operatorname{argmin}_{S_n}(P(F_n(x_n))) \overset{K}{\rightharpoondown} \bigcup_{x \in S} \operatorname{argmin}^{<}_S(P(F(x))).$$

Proof Let $\hat{x} \in \bigcup_{x \in S} \text{argmin}_S^{<}(\text{P}(F(x)))$. Let $\hat{x} \in \text{argmin}_S^{<}(P(F(\bar{x})))$ for some $\bar{x} \in S$. Thus, $I_e^m(F(\hat{x}), F(\bar{x})) < I_e^m(F(x), F(\bar{x}))$ for all $x \in S \setminus \{\hat{x}\}$. Since $S_n \xrightarrow{K} S$, it follows that there exists $\hat{x}_n \in S_n$ for all $n \in \mathbb{N}$ such that $\hat{x}_n \to \hat{x}$. By assumption (d), there exists $u_n \in m\text{-Eff}(\mathcal{F}_n)$ such that $F_n(u_n) \preceq^m F_n(\hat{x}_n)$. From [20, Corollary 8], we have $m\text{-Eff}(\mathcal{F}_n) \subseteq \bigcup_{x_n \in S_n} \text{argmin}_{S_n}(\text{P}(F_n(x_n)))$. By virtue of assumption (c), there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ such that $u_{n_k} \to u$. Clearly, $u \in S$ as $S_n \xrightarrow{K} S$. Using assumption (b), without loss of generality we assume that $F_{n_k}(\hat{x}_{n_k}) \xrightarrow{\tau} F(\hat{x})$ and $F_{n_k}(u_{n_k}) \xrightarrow{\tau} F(u)$. From Lemma 4(iii), it follows that $F(u) \preceq^m F(\hat{x})$, which by Lemma 5(i) implies that $I_e^m(F(u), F(\bar{x})) \leqslant I_e^m(F(\hat{x}), F(\bar{x}))$. Since $\hat{x} \in \text{argmin}_S^{<}(\text{P}(F(\bar{x})))$, it follows that $u = \hat{x}$. Thus, by using the same argument given in Theorem 5, we have $\hat{x} \in \text{Li}(\bigcup_{x_n \in S_n} \text{argmin}_{S_n}(\text{P}(F_n(x_n))))$.

As an outcome of Theorem 9(ii) and Theorem 12, we establish the lower Painlevé–Kuratowski convergence of the optimal solution sets of perturbed scalarized problems to m-strict efficient solution set of (P).

Theorem 13 *If the conditions* (a)-(d) *of Theorem* 12 *are satisfied and*

(e) *F is compact-valued on S, then* $\bigcup_{x_n \in S_n} \text{argmin}_{S_n}(\text{P}(F_n(x_n))) \xrightarrow{K} m\text{-SEff}(\mathcal{F})$.

We now conclude this section with the following result, which improves Theorem 6 in Sect. 3.

Theorem 14 *If the conditions* (a)-(e) *of Theorem* 13 *are satisfied and the following conditions hold*:

(f) $\mathcal{F}_n \xrightarrow{ls\tau} \mathcal{F}$;
(g) *F_n is a compact-valued map on S_n for all n*;

then $m\text{-SEff}(\mathcal{F}) \subseteq \text{Li}(\bigcup_{x_n \in S_n} \text{argmin}_{S_n}(\text{P}(F_n(x_n)))) \subseteq \text{Ls}(\bigcup_{x_n \in S_n} \text{argmin}_{S_n}(\text{P}(F_n(x_n)))) \subseteq m\text{-WEff}(\mathcal{F})$.

***Remark* 8** We observe from Theorem 14 that for compact-valued maps, we obtain stability results for the set optimization problem (P) in terms of solutions of perturbed scalarized problems which are much easier to achieve in comparison with the stability results obtained via solution sets of perturbed set optimization problems.

5 Continuity of Solution Set Map

This section deals with another stability aspect pertaining to continuity of solution set maps in parametric set optimization problems both in the image space and given space. For this, we first consider a family of parametric set optimization problems by perturbing both the objective map F and the feasible set S of problem (P) over a nonempty set $U \subseteq Z$, where Z is a real normed linear space. Let the set-valued map

$S : U \rightrightarrows X$ be the perturbation map for the feasible set and $F : X \times Z \rightrightarrows Y$ be a nonempty bounded set-valued map. For parameter u, we now consider the following parametric set optimization problem

$$(\mathrm{P}(u)) \qquad m\text{-Min } F(x,u)$$
$$\text{s.t. } x \in S(u),$$

where $S(u)$ is a nonempty subset of X. For each $u \in U$, let $\mathcal{F}_u$ denote the family of image sets of $F(.,u)$ on $S(u)$, that is, $\mathcal{F}_u := \{F(x,u) : x \in S(u)\}$. We denote the set of m-minimal (resp. m-weak minimal) solution sets of (P(u)) by m-Min(u) (resp. m-WMin(u)) and the set m-Eff(u) (resp. m-WEff(u) and m-SEff(u)) denote the set of m-efficient (resp. m-weak efficient and m-strict efficient) solutions of (P(u)). We refer to the map m-Min : $U \rightrightarrows \mathcal{B}^0(Y)$ (resp. m-WMin : $U \rightrightarrows \mathcal{B}^0(Y)$) as m-minimal (resp. m-weak minimal) solution set map and m-Eff : $U \rightrightarrows X$ (resp. m-WEff : $U \rightrightarrows X$) as m-efficient (resp. m-weak efficient) solution set map. Throughout, we assume that m-Eff(u), m-WEff(u) and m-SEff(u) are nonempty for every $u \in U$.

In literature (for instance see [23, 29, 30]), various continuity notions have been given for set-valued maps. Thanks to the topology τ on $\mathcal{P}(\mathcal{B}^0(Y))$, we now propose the following notions of upper and lower semicontinuity for m-minimal (resp. m-weak minimal) solution set map with image set as family of sets.

Definition 9 The map m-Min : $U \rightrightarrows \mathcal{B}^0(Y)$ is said to be *upper semicontinuous* (*resp. lower semicontinuous*) at $\bar{u} \in U$ if for any τ-open set $\mathcal{V} \in \mathcal{P}(\mathcal{B}^0(Y))$ with m-Min($\bar{u}$) $\subseteq \mathcal{V}$ (resp. m-Min($\bar{u}$) $\cap \mathcal{V} \neq \emptyset$) there is a neighborhood W of $\bar{u}$ such that m-Min(u) $\subseteq \mathcal{V}$ (resp. m-Min(u) $\cap \mathcal{V} \neq \emptyset$) for all $u \in W \cap U$.

Similarly, we can define the upper and lower semicontinuity of m-weak minimal solution set map by replacing m-Min by m-WMin in the above definition.

We now provide sufficient conditions for the upper semicontinuity of m-weak minimal solution set map.

Theorem 15 *Let $\bar{u} \in U$. If the following conditions hold*:

(a) *for every sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ with $u_n \to \bar{u}$, we have $\mathcal{F}_{u_n} \to \mathcal{F}_{\bar{u}}$*;
(b) *there exists a compact set E in Y and $\delta > 0$ such that* $\bigcup_{u\in B_\delta(\bar{u})} \bigcup_{x\in S(u)} F(x,u) \subseteq E$;

*then m-*WMin *is upper semicontinuous at $\bar{u}$.*

Proof On the contrary, assume that m-WMin is not upper semicontinuous at $\bar{u}$. Then there exist a τ-open set $\mathcal{V}$ in $\mathcal{P}(\mathcal{B}^0(Y))$ with m-WMin($\bar{u}$) $\subseteq \mathcal{V}$, a sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ with $u_n \to \bar{u}$ such that m-WMin(u_n) $\nsubseteq \mathcal{V}$ for sufficiently large n. Let $F(x_n,u_n) \in m$-WMin(u_n) be such that $F(x_n,u_n) \notin \mathcal{V}$ for sufficiently large n. By compactness assumption, it follows that $(F(x_n,u_n))_{n\in\mathbb{N}}$ has a convergent subsequence $(F(x_{n_k},u_{n_k}))_{k\in\mathbb{N}}$ such that $F(x_{n_k},u_{n_k}) \xrightarrow{\tau} F(\bar{x},\bar{u})$. Clearly, $F(\bar{x},\bar{u}) \in \mathcal{F}_{\bar{u}}$ as $\mathcal{F}_{u_n} \xrightarrow{us\tau} \mathcal{F}_{\bar{u}}$. We next show that $F(\bar{x},\bar{u}) \in m$-WMin($\bar{u}$). On the contrary, assume that there exists $\bar{y} \in S(\bar{u})$ such that $F(\bar{y},\bar{u}) \prec^m F(\bar{x},\bar{u})$. Now, proceeding as in Theorem 1, we obtain that $F(\bar{x},\bar{u}) \in m$-WMin($\bar{u}$) which further implies that $F(x_{n_k},u_{n_k}) \in \mathcal{V}$ for sufficiently large k, which is a contradiction.

Under suitable assumptions, we next investigate the lower semicontinuity of m-minimal solution set map.

Theorem 16 *Let $\bar{u} \in U$. If the conditions* (a) *and* (b) *of Theorem 15 are satisfied and the following conditions hold*:

(c) *there exists $\delta > 0$ such that $\mathcal{F}_u$ satisfies m-domination property on $S(u)$ for every $u \in B_\delta(\bar{u})$;*
(d) *$F(., \bar{u})$ is closed-valued on $S(\bar{u})$;*

*then m-*Min *is lower semicontinuous at $\bar{u}$.*

Proof On the contrary, assume that m-Min is not lower semicontinuous at $\bar{u}$. Then there exist a τ-open set $\mathcal{V}$ in $\mathcal{P}(\mathcal{B}^0(Y))$ with $m\text{-Min}(\bar{u}) \cap \mathcal{V} \neq \varnothing$ and a sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ with $u_n \to \bar{u}$ such that

$$m\text{-Min}(u_n) \cap \mathcal{V} = \varnothing \text{ for sufficiently large } n. \tag{2}$$

Let $F(\bar{x}, \bar{u}) \in m\text{-Min}(\bar{u}) \cap \mathcal{V}$. Further, proceeding as in Theorem 2, it follows that there exists $F(\bar{y}, \bar{u}) \in \mathcal{F}_{\bar{u}}$ such that $F(\bar{y}, \bar{u}) \preceq^m F(\bar{x}, \bar{u})$, which further implies that $F(\bar{y}, \bar{u}) = F(\bar{x}, \bar{u})$ as $F(\bar{x}, \bar{u}) \in m\text{-Min}(\bar{u})$. Since $F(\bar{x}, \bar{u}) \in \mathcal{V}$ we have $F(y_{n_k}, u_{n_k}) \in \mathcal{V}$ for sufficiently large k, which contradicts (2).

We next discuss the continuity of solution set maps in the given space. Before that we provide the notion of continuity of a set-valued map with respect to topology τ. Similar notion of continuity has been considered by Zhang and Huang [18, Definition 2.4] by using the topology introduced by Geoffroy [15].

Definition 10 The map $F : X \rightrightarrows Y$ is said to be *τ-continuous* at $\bar{x} \in X$ if for every sequence $x_n \to \bar{x}$ we have $F(x_n) \xrightarrow{\tau} F(\bar{x})$. The map F is τ-continuous on X if it is τ-continuous at every $x \in X$.

The following theorem depicts the upper semicontinuity of m-weak efficient solution set map.

Theorem 17 *Let $\bar{u} \in U$. If the following conditions hold*:

(a) *S is continuous at $\bar{u}$ and $S(\bar{u})$ is compact;*
(b) *$F(., .)$ is τ-continuous on $S(\bar{u}) \times \{\bar{u}\}$;*

*then m-*WEff *is upper semicontinuous at $\bar{u}$ and m-*WEff$(\bar{u})$ *is a compact set.*

Proof On the contrary, assume that m-WEff is not upper semicontinuous at $\bar{u}$. Then there exist an open set V in X with $m\text{-WEff}(\bar{u}) \subseteq V$ and a sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ with $u_n \to \bar{u}$ such that $m\text{-WEff}(u_n) \nsubseteq V$ for sufficiently large n. Hence, there exists $x_n \in m\text{-WEff}(u_n)$ such that $x_n \notin V$ for sufficiently large n. Since S is upper semicontinuous at $\bar{u}$ and $S(\bar{u})$ is compact, there exist a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $\bar{x} \in S(\bar{u})$ such that $x_{n_k} \to \bar{x}$. We need to show that $\bar{x} \in m\text{-WEff}(\bar{u})$. On the contrary, let $\bar{y} \in S(\bar{u})$ be such that $F(\bar{y}, \bar{u}) \prec^m F(\bar{x}, \bar{u})$. Using the lower semicontinuity of S at $\bar{u}$, there exists a sequence $(y_{n_k})_{k\in\mathbb{N}}$, $y_{n_k} \in S(u_{n_k})$ such that $y_{n_k} \to \bar{y}$. By virtue of τ-continuity of

$F(.,.)$ on $S(\bar{u}) \times \{\bar{u}\}$ we have $F(y_{n_k}, u_{n_k}) \xrightarrow{\tau} F(\bar{y}, \bar{u})$ and $F(x_{n_k}, u_{n_k}) \xrightarrow{\tau} F(\bar{x}, \bar{u})$. From Lemma 3(i) we obtain that $F(y_{n_k}, u_{n_k}) \prec^m F(x_{n_k}, u_{n_k})$ for sufficiently large k, which is a contradiction to the fact that $x_{n_k} \in m$-WEff(u_{n_k}). Thus, $\bar{x} \in m$-WEff$(\bar{u})$ and hence $x_{n_k} \in V$ for sufficiently large k, which is again a contradiction.

Next, to prove that m-WEff$(\bar{u})$ is a compact set, it is enough to show that m-WEff$(\bar{u})$ is a closed set. Let $x_n \in m$-WEff$(\bar{u})$ be such that $x_n \to \bar{x}$. As S is upper semicontinuous at $\bar{u}$ and $S(\bar{u})$ is compact, it follows that $\bar{x} \in S(\bar{u})$. We now claim that $\bar{x} \in m$-WEff$(\bar{u})$. On the contrary, assume that there exists $\bar{y} \in S(\bar{u})$ such that $F(\bar{y}, \bar{u}) \prec^m F(\bar{x}, \bar{u})$. Using the τ-continuity of $F(.,.)$ at $(\bar{x}, \bar{u})$ we have $F(x_n, \bar{u}) \xrightarrow{\tau} F(\bar{x}, \bar{u})$. Thus, by Lemma 3(i) it follows that $F(\bar{y}, \bar{u}) \prec^m F(x_n, \bar{u})$ for sufficiently large n. This contradicts the fact that $x_n \in m$-WEff$(\bar{u})$ for all n.

***Remark* 9** Zhang and Huang [18, Theorem 3.1] proved a similar result for relaxed minimal solution set map under the assumptions of K-closedness and topological continuity of the map F with respect to topology given by Geoffroy [15].

We next provide sufficient conditions for the lower semicontinuity of m-efficient solution set map.

Theorem 18 *Let $\bar{u} \in U$. If the conditions* (a) *and* (b) *of Theorem* 17 *are satisfied and the following conditions hold*:

(c) *there exists $\delta > 0$ such that $\mathcal{F}_u$ satisfies m-domination property on $S(u)$ for every $u \in B_\delta(\bar{u})$;*
(d) *$F(., \bar{u})$ is closed-valued on $S(\bar{u})$;*
(e) m-Eff$(\bar{u}) = m$-SEff$(\bar{u})$;

then m-Eff is lower semicontinuous at $\bar{u}$.

Proof On the contrary, assume that m-Eff is not lower semicontinuous at $\bar{u}$. Thus, there exists an open set V in X with m-Eff$(\bar{u}) \cap V \neq \emptyset$ and a sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ with $u_n \to \bar{u}$ such that

$$m\text{-Eff}(u_n) \cap V = \emptyset \text{ for sufficiently large } n. \tag{3}$$

Let $\bar{x} \in m$-Eff$(\bar{u}) \cap V$. Using lower semicontinuity of S at $\bar{u}$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in S(u_n)$ such that $x_n \to \bar{x}$. By m-domination property, there exists $y_n \in m$-Eff(u_n) for sufficiently large n such that $F(y_n, u_n) \preceq^m F(x_n, u_n)$. Since S is upper semicontinuous at $\bar{u}$ and $S(\bar{u})$ is compact, there exist a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ and $\bar{y} \in S(\bar{u})$ such that $y_{n_k} \to \bar{y}$. Since $F(.,.)$ is τ-continuous on $S(\bar{u}) \times \{\bar{u}\}$ we have $F(y_{n_k}, u_{n_k}) \xrightarrow{\tau} F(\bar{y}, \bar{u})$ and $F(x_{n_k}, u_{n_k}) \xrightarrow{\tau} F(\bar{x}, \bar{u})$. From Lemma 4(iii), we obtain that $F(\bar{y}, \bar{u}) \preceq^m F(\bar{x}, \bar{u})$. As m-Eff$(\bar{u}) = m$-SEff$(\bar{u})$, we have $\bar{x} = \bar{y}$ and so $y_{n_k} \in m$-Eff$(u_{n_k}) \cap V$ for sufficiently large k, which contradicts (3).

***Remark* 10** We recall from [18, Definition 2.6] that F is strictly K-quasiconvex on a nonempty convex subset S of X if for any $x, \bar{x} \in S$ with $x \neq \bar{x}$ and $\lambda \in (0, 1)$, $F(x) \preceq^l F(\bar{x})$ implies that $F(\lambda x + (1-\lambda)\bar{x}) \prec^{sl} F(\bar{x})$. Zhang and Huang [18, Theorem 3.3] derived the lower semicontinuity of minimal solution set map (denoted by $\text{Min}(F, S(.), \preceq^l))$ by assuming F to be strictly K-quasiconvex. However, it can be seen from the following example that considering the solutions on the basis of relation $\preceq^m$, the lower semicontinuity of efficient solution set map can also be obtained even though the objective map is not strictly K-quasiconvex.

***Example* 4** Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $U = [0, 1]$ and $K = \mathbb{R}^2_+$. Define the set-valued map $S : U \rightrightarrows X$ as $S(u) = [0, 1]$ for all $u \in U$. Consider the map $F : X \times Z \rightrightarrows Y$ defined as

$$F(x, u) = \text{co}\{(0, 0), (ux, 2 - x), (x + 1, u(x - 1))\}.$$

It can be easily seen that for $\bar{u} = 0$, F is not strictly K-quasiconvex on $S(\bar{u})$. Here,

$$\text{Min}(F, S(.), \preceq^l) = \begin{cases} [0, 1], & \text{if } u = 0, \\ \{0\}, & \text{if } u \neq 0, \end{cases}$$

and
$$m\text{-Eff}(u) = [0, 1], \text{ for every } u \in U.$$

We observe that $\text{Min}(F, S(.), \preceq^l)$ is not lower semicontinuous at $\bar{u}$, but m-Eff is lower semicontinuous at $\bar{u}$.

6 Conclusion

In this paper, we investigated two aspects of stability namely, convergence of solution sets and continuity of solution set maps, both in the given space and image space. In an attempt to solve the open problem given by Geoffroy [15], we considered the preference relation $\prec^m$ to define a topology τ on the whole set $\mathcal{P}^0(Y)$ and studied the related concepts of τ-convergence and τ-continuity to establish the stability results. Further, taking the advantage of the topology τ, we introduced the concept of semicontinuity for solution set maps in the image space. Moreover, in the setting of τ-convergence and the preference relations $\prec^m$ and $\preceq^m$, we also studied the convergence aspect of stability in set optimization using scalarization techniques.

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References

[1] Xiang, S., Zhou, Y.: On essential sets and essential components of efficient solutions for vector optimization problems. J. Math. Anal. Appl. **315**, 317–326 (2006)
[2] Song, Q.Q., Tang, G.Q., Wang, L.S.: On essential stable sets of solutions in set optimization problems. J. Optim. Theory Appl. **156**, 591–599 (2013)
[3] Xu, Y.D., Li, S.J.: Continuity of the solution set mappings to a parametric set optimization problem. Optim. Lett. **8**, 2315–2327 (2014)

[4] Xu, Y.D., Li, S.J.: On the solution continuity of parametric set optimization problems. Math. Methods Oper. Res. **84**, 223–237 (2016)
[5] Han, Y., Huang, N.J.: Well-posedness and stability of solutions for set optimization problems. Optimization **66**, 17–33 (2017)
[6] Han, Y., Zhang, K.: Semicontinuity of the minimal solution mappings to parametric set optimization problems on Banach lattices. Optimization https://doi.org/10.1080/02331934.2022.2045985 (2022)
[7] Zhang, C.L., Huang, N.J.: Well-posedness and stability in set optimization with applications. Positivity **25**, 1153–1173 (2021)
[8] Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Convergence of solutions of a set optimization problem in the image space. J. Optim. Theory Appl. **170**, 358–371 (2016)
[9] Han, Y., Zhang, K., Huang, N.J.: The stability and extended well-posedness of the solution sets for set optimization problems via the Painlevé-Kuratowski convergence. Math. Methods Oper. Res. **91**, 175–196 (2020)
[10] Karuna, Lalitha, C.S.: External and internal stability in set optimization. Optimization **68**, 833–852 (2019)
[11] Anh, L.Q., Duy, T.Q., Hien, D.V.: Stability of efficient solutions to set optimization problems. J. Glob. Optim. **78**, 563–580 (2020)
[12] Gupta, M., Srivastava, M.: On Levtin-Polyak well posedness and stability in set optimization. Positivity **25**, 1903–1921 (2021)
[13] Dhingra, M., Lalitha, C.S.: Approximate solutions and scalarization in set-valued optimization. Optimization **66**, 1793–1805 (2017)
[14] Karuna, Lalitha, C.S.: External and internal stability in set optimization using gamma convergence. Carpathian J. Math. **35**, 393–406 (2019)
[15] Geoffroy, M.H.: A topological convergence on power sets well-suited for set optimization. J. Global Optim. **73**, 567–581 (2019)
[16] Chen, J., Wang, G., Ou, X., Zhang, W.: Continuity of solutions mappings of parametric set optimization problems. J. Ind. Manag. Optim. **16**, 25–36 (2020)
[17] Karuna, Lalitha, C.S.: Continuity of approximate weak efficient solution set map in parametric set optimization. J. Nonlinear Convex Anal. **19**, 1247–1262 (2018)
[18] Zhang, C.L., Huang, N.J.: On the stability of minimal solutions for parametric set optimization problems. Appl. Anal. **100**, 1533–1543 (2021)
[19] Liu, P.P., Wei, H.Z., Chen, C.R., Li, S.J.: Continuity of solutions for parametric set optimization problems via scalarization methods. J. Oper. Res. Soc. China **11**, 1–19 (2018)
[20] Karaman, E., Soyertem, M., Güvenç, İA., Tozkan, D., KüçüK, M., KüçüK, Y.: Partial order relations on family of sets and scalarizations for set optimization. Positivity **22**, 783–802 (2018)
[21] Lalitha, C.S., Chatterjee, P.: Stability and scalarization of weak efficient, efficient and Henig proper efficient sets using generalized quasiconvexities. J. Optim. Theory Appl. **155**, 941–961 (2012)
[22] Pallaschke, D., Urbański, R.: Pairs of Compact Convex Sets: Fractional arithmetic with Convex Sets. Volume 548 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht (2002)
[23] Khan, A.A., Tammer, C., Zălinescu, C.: Set-valued Optimization: An Introduction with Applications. Springer, Berlin (2015)
[24] Gutiérrez, C., Miglierina, E., Molho, E., Novo, V.: Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. **75**, 1822–1833 (2012)
[25] Khushboo, Lalitha, C.S.: Scalarizations for a set optimization problem using generalized oriented distance function. Positivity **23**, 1195–1213 (2019)
[26] Hernández, E., Rodríguez-Marín, L.: Existence theorems for set optimization problems. Nonlinear Anal. **67**, 1726–1736 (2007)
[27] Anh, L.Q., Duy, T.Q., Hien, D.V., Kuroiwa, D., Petrot, N.: Convergence of solutions to set optimization problems with the set less order relation. J. Optim. Theory Appl. **185**, 416–432 (2020)
[28] Kuroiwa, D.: On duality of set-valued optimization, Research on nonlinear analysis and convex analysis (Japanese) (Kyoto, 1998) Sūrikaisekikenkyūsho Kōkyūroku. 1071, 12–16 (1998)
[29] Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis: Pure and applied mathematics. Wiley, New York (1984)

[30] Kuroiwa, D.: The natural criteria in set-valued optimization, Research on nonlinear analysis and convex analysis (Japanese) (Kyoto, 1997) Sūrikaisekikenkyūsho Kōkyūroku. 1031, 85–90 (1998)