Journal of the Operations Research Society of China (2019) 7:501–509
https://doi.org/10.1007/s40305-019-00259-8

Conditional Edge Connectivity of the Locally Twisted Cubes

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Received: 21 November 2018 / Revised: 11 May 2019 / Accepted: 11 June 2019 / Published online: 13 July 2019

Abstract
The k-component edge connectivity $c\lambda_k(G)$ of a non-complete graph G is the minimum number of edges whose deletion results in a graph with at least k components. In this paper, we extend some results by Guo et al. (Appl Math Comput 334:401–406, 2018) by determining the component edge connectivity of the locally twisted cubes LTQ_n, i.e., $c\lambda_{k+1}(\mathrm{LTQ}_n) = kn - \frac{ex_k}{2}$ for $1 \leqslant k \leqslant 2^{[\frac{n}{2}]}$, $n \geqslant 7$, where $ex_k = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$, and k is a positive integer with decomposition $k = \sum_{i=0}^{s} 2^{t_i}$ such that $t_0 = \lfloor \log_2 k \rfloor$ and $t_i = \lfloor \log_2 (k - \sum_{r=0}^{i-1} 2^{t_r}) \rfloor$ for $i \geqslant 1$. As a by-product, we characterize the corresponding optimal solutions.

Keywords Fault tolerance · Locally twisted cubes · Component edge connectivity

Mathematics Subject Classification 05C40

1 Introduction

Fault tolerance concerns the capability of an interconnection network to transmit messages; it is a very important property to study. In general, the network structure is modeled as graphs, and the properties of the network can be evaluated by the parameters of the graphs. There are many parameters that have been introduced to

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This paper is dedicated to Professor Ding-Zhu Du in celebration of his 70th birthday.

The research was supported by the National Natural Science Foundation of China (No. 11531011).

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measure the reliability of a network structure. Perhaps, edge connectivity $\lambda(G)$ of a graph G is the most important one. The larger the edge connectivity is, the more reliable the interconnection network is.

However, this criterion has its shortcoming: The further properties of disconnected components are not depicted. Under this consideration, Harary [2] introduced the concept of *conditional connectivity* by attaching some conditions on connected components and Latifi et al. [3] generalized the concept conditional connectivity by introducing *restricted h-connectivity*. The concept considered here is slightly different from theirs.

As a natural extension of traditional edge connectivity $\lambda(G)$, Sampathkumar [4] proposed the concept of *component edge connectivity*. Let G be a non-complete graph. A *k-component edge cut* of G is a set of edges whose deletion results in a graph with at least k components. The *k-component edge connectivity* $c\lambda_k(G)$ of a graph G is the size of the smallest k-component edge cut of G. Obviously, $\lambda(G) = c\lambda_2(G) \leqslant c\lambda_3(G) \leqslant \cdots \leqslant c\lambda_k(G)$.

In recent years, as we know, the k-component edge connectivity has been studied for several famous networks (hypercubes Q_n, folded hypercubes FQ_n, twisted cubes TN_n) in [1,5–7]. Very recently, Guo et al. [1] determined $c\lambda_k(\mathrm{LTQ}_n)$ of the locally twisted cubes LTQ_n for $k \leqslant 4$. In this paper, we extend their results by determining $c\lambda_k(\mathrm{LTQ}_n)$ for $k \leqslant 2^{\lceil \frac{n}{2} \rceil}$.

The rest of the paper is organized as follows: In Sect. 2, we introduce some preliminary knowledge. In Sect. 3, we give the main result of this paper.

2 Preliminaries

For graph-theoretical terminology and notation not mentioned here, we follow [8]. Let $G = (V, E)$ be a graph. For each node $u \in V$, the *neighborhood* of u in a subgraph $H \subseteq G$, denoted by $N_H(u)$, is defined as the set of all nodes adjacent to u in H, and $d_H(u) = |N_H(u)|$ is the *degree* of u in H. We simply denote $N_H(u) = N(u)$ if $H = G$. For a node subset $S \subseteq V$, $G[S]$ (resp. $G - S$) denotes the subgraph of G induced by the node set S (resp. $V - S$), and E_S denotes the set of edges in which each edge contains exactly one end node in S. Similarly, for an edge subset $F \subseteq E$, $G - F$ denotes the subgraph of G induced by the edge set $E - F$. Let "$\oplus$" represent the modulo 2 addition.

Definition 2.1 [9] For an integer $n \geqslant 2$, the locally twisted cubes LTQ_n with node set $\{0, 1\}^n$ were introduced by Yang et al. [9]. It can be defined recursively as follows: LTQ_2 is a 4-cycle with node set $\{00, 01, 10, 11\}$ and edge set $\{(00, 01), (01, 11), (11, 10), (10, 00)\}$. For $n \geqslant 3$, LTQ_n can be built from $0\mathrm{LTQ}_{n-1}$ and $1\mathrm{LTQ}_{n-1}$, where $0\mathrm{LTQ}_{n-1}$ (resp. $1\mathrm{LTQ}_{n-1}$) denotes the graph obtained from LTQ_{n-1} by prefixing the label of each node with 0 (resp. 1), according to the following rule. Connect each node $0x_2x_3 \cdots x_n$ of $0\mathrm{LTQ}_{n-1}$ to the node $1(x_2 \oplus x_n)x_3 \cdots x_n$ of $1\mathrm{LTQ}_{n-1}$ with an edge.

Locally twisted cubes LTQ_2, LTQ_3, LTQ_4 are depicted in Fig. 1. Yang et al. [9] introduced the n-dimensional ($n \geqslant 2$) two-twisted cubes $Q_{n,2}$ and showed that $Q_{n,2}$

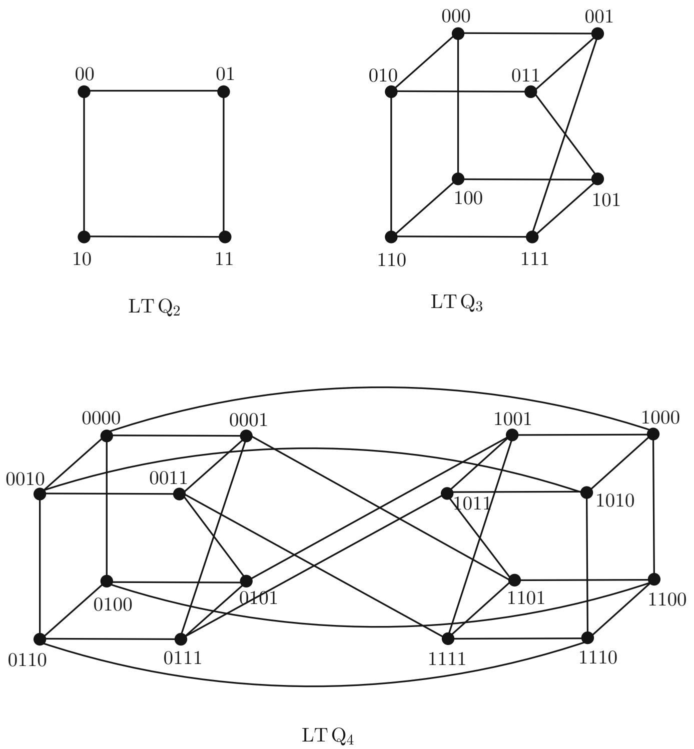

Fig. 1 Locally twisted cubes LTQ_2, LTQ_3 and LTQ_4

is isomorphic to Q_n. On the basis of the concepts of $Q_{n,2}$ and Q_n, the locally twisted cubes LTQ_n can also be defined as follows:

Definition 2.2 [9] The locally twisted cubes LTQ_n can be built from Q_{n-1} and $Q_{n-1,2}$ by the following steps:

(1) Let $Q_{n-1}0$ be the graph obtained from Q_{n-1} by suffixing the labels of all nodes with 0.
(2) Let $Q_{n-1,2}1$ be the graph obtained from $Q_{n-1,2}$ by suffixing the labels of all nodes with 1.
(3) Connect each node $x_1x_2\cdots x_{n-1}0$ of $Q_{n-1}0$ to the node $x_1x_2\cdots x_{n-1}1$ of $Q_{n-1,2}1$ by an edge.

As an attractive alternative to hypercubes Q_n, LTQ_n is a member of hypercube-like networks HL_n, and has been studied for many years and found many good properties, see, for example, [10–18] and the references therein.

Each node $u \in V(LTQ_n)$ can be denoted by an n-bit binary string, i.e., $u = u_n u_{n-1}\cdots u_1$, and also can be represented by decimal number, i.e., $u = \sum_{i=1}^{n} u_i 2^{i-1}$.

Let m be an integer and $\sum_{i=0}^{s} 2^{t_i}$ be the decomposition of m such that $t_0 = \lfloor \log_2 m \rfloor$, and $t_i = \lfloor \log_2(m - \sum_{r=0}^{i-1} 2^{t_r}) \rfloor$ for $i \geqslant 1$. We denote by $\frac{ex_m}{2}$ the maximum size of the subgraph of LTQ_n induced by m nodes, i.e., ex_m =max$\{2|E(LTQ_n[S])| : S \subseteq$

$V(\mathrm{LTQ}_n)$ and $|S| = m\}$ is the maximum possible sum of degrees of the nodes in the subgraph of LTQ_n induced by m nodes.

Zhang et al. [19] and Yang et al. [20] have proved the following important results.

Lemma 2.3 [19] *Let S be a node subset of LTQ_n, where $|S| = m$ and $m = \sum_{i=0}^{s} 2^{t_i}$. Then, $ex_m(\mathrm{LTQ}_n) = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$.*

Lemma 2.4 [20] *Let $1 \leqslant i, j \leqslant 2^n$ and $i + j \leqslant 2^n$. Then, $ex_i + ex_j + 2min\{i, j\} \leqslant ex_{i+j}$.*

In the same paper, they introduced a method to pick a connected subgraph G_0 in LTQ_n such that $|V(G_0)| = m = \sum_{i=0}^{s} 2^{t_i}$ and $|E(G_0)| = \frac{ex_m}{2}$. Take $(s + 1)$ t_i-dimensional sub-LTQ_n (we use the notation sub-LTQ_n for the lower dimensional LTQ_n) for $i = 0, 1, \cdots, s$ as follows:

$$\begin{aligned} &\mathrm{LTQ}^0 : \underbrace{X_1 \cdots X_{t_o}}_{t_0} 0 \cdots 0, \\ &\mathrm{LTQ}^1 : \underbrace{\underbrace{X_1 \cdots X_{t_1}}_{t_1} 0 \cdots 0}_{t_0} 1 0 \cdots 0, \\ &\mathrm{LTQ}^2 : \underbrace{\underbrace{X_1 \cdots X_{t_2}}_{t_2} 0 \cdots 0}_{t_1} 1 0 \cdots 0 1 0 \cdots 0, \\ &\cdots . \end{aligned} \tag{2.1}$$

Note that LTQ^0 is given and LTQ^i is taken from a t_{i-1}-dimensional sub-LTQ_n which is obtained from LTQ^{i-1} by changing the 0 of $(t_{i-1} + 1)th$-coordinate of LTQ^{i-1} to 1. Denote $G_0 = \mathrm{LTQ}_n[V(\mathrm{LTQ}^0) \cup \cdots \cup V(\mathrm{LTQ}^s)]$. It is not difficult to count the number of edges of G_0 by considering the edges within LTQ^i's $(\frac{1}{2} \cdot \sum_{i=0}^{s} t_i 2^{t_i})$ and the edges between LTQ^i's $(\frac{1}{2} \cdot \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i})$.

Remark 1 Note that each LTQ^i is connected for $0 \leqslant i \leqslant s$. Then, G_0 is connected.

3 Main Result

In this section, we extend Theorem 3.1 of Guo et al. [1] by determining $c\lambda_{k+1}(\mathrm{LTQ}_n)$ for $k = 1, 2, \cdots, 2^{[\frac{n}{2}]}, n \geqslant 7$.

Theorem 3.1 [1] *$c\lambda_3(\mathrm{LTQ}_n) = 2n - 1$, $c\lambda_4(\mathrm{LTQ}_n) = 3n - 2$ for $n \geqslant 2$.*

Lemma 3.2 [21] *Let $S = \{0, 1, \cdots, m - 1\}$ be a node subset of Q_n. Then, $|E(Q_n[S])| = \frac{ex_m(Q_n)}{2}$, where $ex_m(Q_n) = \max\{2|E(Q_n[S])| : S \subseteq V(Q_n)$ and $|S| = m\}$.*

Lemma 3.3 *Let S be a node subset of* LTQ_n, *where* $|S| = m$ *and* $m = \sum_{i=0}^{s} 2^{t_i}$. *Then,* $|E_S| \geqslant nm - ex_m$. *Moreover, the function* $\xi(m) = nm - \frac{ex_m}{2}$ *is strictly increasing (with respect to* m*) if* $m \leqslant 2^{n-1} - 1$.

Proof By Lemma 2.3, we can immediately obtain $|E_S| \geqslant nm - ex_m$. Note that the inequality $\xi(m+1) - \xi(m) = n - (s+1) > 0$ is equivalent to $s < n - 1$. Since $m = \sum_{i=0}^{s} 2^{t_i} \leqslant 2^{n-1} - 1 = 2^{n-2} + 2^{n-3} + \cdots + 2^1 + 2^0$, $s < n-1$, which implies that $\xi(m)$ is strictly increasing for $m \leqslant 2^{n-1} - 1$.

Note that LTQ_n is n-regular. Let G_0 be a subgraph of LTQ_n induced by m nodes. By Lemma 2.3, $|E(G_0)| \leqslant \frac{ex_m}{2}$. Moreover, if $m \leqslant 2^{n-2}$, then one can pick the subgraph G_0 in an $(n-2)$-dimensional sub-LTQ_n such that $|E(G_0)| = \frac{ex_m}{2}$. Thus, we have the following observation.

Observation *If* $m \leqslant 2^{n-2}$, *then* $(n-2)m - ex_m \geqslant 0$.

Zhao et al. showed the following results in [6]:

Lemma 3.4 [6] *Let q and q_i be positive integers. If* $q = \sum_{i=1}^{k} q_i$, *then* $\sum_{i=1}^{k} ex_{q_i} \leqslant ex_{q-k+1}$.

Lemma 3.5 $c\lambda_{k+1}(\mathrm{LTQ}_n) \leqslant nk - \frac{ex_k}{2}$ *for* $k \leqslant 2^n$ *and* $n \geqslant 2$.

Proof To show that $c\lambda_{k+1}(\mathrm{LTQ}_n) \leqslant nk - \frac{ex_k}{2}$, it suffices to find an edge subset F of LTQ_n with $|F| = nk - \frac{ex_k}{2}$ such that $\mathrm{LTQ}_n - F$ is disconnected and has at least $k+1$ components.

Case 1. $k \leqslant 2^{n-1}$.

Let $S = \{0, 2, \cdots, 2k-2\}$ (or $S = \{1, 3, \cdots, 2k-1\}$) be a node subset of LTQ_n and $G_0 = \mathrm{LTQ}_n[S]$. By Definition 2.2 and Lemma 3.2, it is clear that $G_0 \subseteq Q_{n-1}0$ (or $G_0 \subseteq Q_{n-1,2}1$) and $|E(G_0)| = \frac{ex_k}{2}$. Take $F = E(G_0) \cup E_{V(G_0)}$ which is required.

Case 2. $2^{n-1} < k \leqslant 2^n$.

Let $S = \{0, 2, \cdots, 2^n - 2, 1, 3, \cdots, 2(k - 2^{n-1}) - 1\}$ (or $S = \{1, 3, \cdots, 2^n - 1, 0, 2, \cdots, 2(k - 2^{n-1}) - 2\}$) be a node subset of LTQ_n and $G_0 = \mathrm{LTQ}_n[S]$. Then, G_0 is the subgraph of LTQ_n in (2.1), and $|E(G_0)| = \frac{ex_k}{2}$. Take $F = E(G_0) \cup E_{V(G_0)}$ which is required.

A k-component edge cut F of LTQ_n is called $c\lambda_k$*-cut* if $|F| = c\lambda_k(\mathrm{LTQ}_n)$.

Lemma 3.6 $c\lambda_{k+1}(\mathrm{LTQ}_n) \geqslant nk - \frac{ex_k}{2}$ *for* $k \leqslant 2^{[\frac{n}{2}]}$ *and* $n \geqslant 7$.

Proof To show that $c\lambda_{k+1}(\mathrm{LTQ}_n) \geqslant nk - \frac{ex_k}{2}$, it suffices to prove $|F| \geqslant nk - \frac{ex_k}{2}$, where F is a $c\lambda_{k+1}$-cut of LTQ_n. For convenience sake, we assume that n is even. Let F be a $c\lambda_{k+1}$-cut of LTQ_n, then $\mathrm{LTQ}_n - F$ has exactly $k+1$ components. We use $C_1, C_2, \cdots, C_{k+1}$ to denote the above $k+1$ components, and suppose C_{k+1} be the largest one.

Case 1. $|V(C_{k+1})| < 2^{n-2}$.

This implies that there exist r components $\{C_1^{'}, C_2^{'}, \cdots, C_r^{'}\} \subseteq \{C_1, C_2, \cdots, C_{k+1}\}$ such that $2^{n-2} \leqslant \sum_{i=1}^{r} |V(C_i^{'})| < 2^{n-1}$. Let $G^{'} = \text{LTQ}_n[\cup_{i=1}^{r} V(C_i^{'})]$ and $|V(G^{'})| = m = \sum_{i=0}^{s} 2^{t_i}$. Obviously, $E_{V(G^{'})} \subseteq F$. From Lemma 3.3, we know that $|E_{V(G^{'})}| \geqslant nm - ex_m$ and $nk - \frac{ex_k}{2} = \xi(k) \leqslant \xi(2^{\frac{n}{2}}) = n2^{\frac{n}{2}} - \frac{ex_{2^{\frac{n}{2}}}}{2} = \frac{3n}{4} \cdot 2^{\frac{n}{2}}$. Next, we show that $nm - ex_m \geqslant \frac{3n}{4} \cdot 2^{\frac{n}{2}}$. Let $m^{'} = m - 2^{t_0}$, then $m^{'} < 2^{n-2}$. By the observation, it is easy to get the following:

$$
\begin{aligned}
nm - ex_m &= n2^{t_0} + n(2^{t_1} + \cdots + 2^{t_s}) - t_0 2^{t_0} - \left(\sum_{i=1}^{s} t^i 2^{t_i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{t_i}\right) \\
&= (n - t_0)2^{t_0} + \left(n \sum_{i=1}^{s} 2^{t_i} - \left(\sum_{i=1}^{s} t_i 2^{t_i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{t_i}\right)\right) \\
&= (n - t_0)2^{t_0} + nm^{'} - ex_{m^{'}} - 2m^{'} \\
&\geqslant (n - t_0)2^{t_0} \\
&= (n - t_0) \cdot 2^{t_0 - \frac{n}{2}} \cdot 2^{\frac{n}{2}}.
\end{aligned}
$$

Since $t_0 = n - 2$, we have $(n - t_0) \cdot 2^{t_0 - \frac{n}{2}} \geqslant \frac{3n}{4}$ for $n \geqslant 7$. Thus, $|F| \geqslant |E_{V(G^{'})}| \geqslant \frac{3n}{4} \cdot 2^{\frac{n}{2}} \geqslant nk - \frac{ex_k}{2}$ when $|V(C_{k+1})| < 2^{n-2}$.

Case 2. $|V(C_{k+1})| \geqslant 2^{n-2}$.

Denote $|V(C_i)| = q_i$ and $\sum_{i=1}^{k} q_i = q$. The case $q \geqslant 2^{n-2}$ can be proved by a similar discussion as Case 1. So we assume $q < 2^{n-2}$.

Case 2.1. $q = k$.

This implies that $|V(C_i)| = q_i = 1$ for $1 \leqslant i \leqslant k$. Then, $|F| \geqslant nk - \frac{ex_k}{2}$ by Lemmas 2.3 and 3.3.

Case 2.2. $q > k$.

Let $G^* = \text{LTQ}_n[\cup_{i=1}^{k} V(C_i)]$ and $F^{'} = F \cap E(G^*)$. Obviously, $E_{V(C_i)} \subseteq F$ $(1 \leqslant i \leqslant k)$. Then, $|F| \geqslant |\cup_{i=1}^{k} E_{V(C_i)}| = |E_{V(C_1)}| + |E_{V(C_2)}| + \cdots + |E_{V(C_k)}| - |F^{'}|$. Note that $|E_{V(C_i)}| = nq_i - 2|E(C_i)|$, $|F^{'}| \leqslant \frac{ex_q}{2} - |\cup_{i=1}^{k} E(C_i)|$ and $E(C_i) \cap E(C_j) = \varnothing$ for $1 \leqslant i \neq j \leqslant k$. By Lemma 3.4, we can obtain

$$
\begin{aligned}
|F| \geqslant |\cup_{i=1}^{k} E_{V(C_i)}| &= |E_{V(C_1)}| + |E_{V(C_2)}| + \cdots + |E_{V(C_k)}| - |F^{'}| \\
&\geqslant \sum_{i=1}^{k} (nq_i - 2|E(C_i)|) - \left(\frac{ex_q}{2} - |\cup_{i=1}^{k} E(C_i)|\right) \\
&= nq - 2\sum_{i=1}^{k} |E(C_i)| - \frac{ex_q}{2} + \sum_{i=1}^{k} |E(C_i)| \\
&= nq - \frac{ex_q}{2} - \sum_{i=1}^{k} |E(C_i)|
\end{aligned}
$$

$$\geqslant nq - \frac{ex_q}{2} - \sum_{i=1}^{k} \frac{ex_{q_i}}{2}$$
$$\geqslant nq - \frac{ex_q}{2} - \frac{ex_{q-k+1}}{2}.$$

Next, we show that $nq - \frac{ex_q}{2} - \frac{ex_{q-k+1}}{2} \geqslant nk - \frac{ex_k}{2}$.

Let $S = \{v_1, v_2, \cdots, v_q\} \subseteq V(\mathrm{LTQ}_n)$. By Definition 2.1 and Lemma 2.3, we may pick a subgraph $G_0 = \mathrm{LTQ}_n[S]$ in LTQ_{n-1}, where G_0 is the subgraph of LTQ_n in (2.1). Then, $|E(G_0)| = \frac{ex_q}{2}$. Since $q-k+1 < q$, we can pick a subgraph $G_1 \subseteq G_0$ such that $|V(G_1)| = q-k+1$ and $|E(G_1)| = \frac{ex_{q-k+1}}{2}$ (here we pick the subgraph G_1 that has the same structural property as G_0 in (2.1)).

Claim. There exists a node $u \in V(G_1)$ such that $d_{G_1}(u) = s+1$, where $q-k = \sum_{i=0}^{s} 2^{t_i}$.

If $q-k$ is even, then $|V(G_1)| = q-k+1 = 2^{t_0}+\cdots+2^{t_s}+2^{t_{s+1}}$ and $t_{s+1} = 0$. From (2.1), we know that $G_1 = \mathrm{LTQ}_n[V(\mathrm{LTQ}^0)\cup\cdots\cup V(\mathrm{LTQ}^s)\cup V(\mathrm{LTQ}^{s+1})]$ and LTQ^{s+1} is isomorphic to K_1. Let $V(\mathrm{LTQ}^{s+1}) = \{u_1\}$. Clearly, $|N_{G_1}(u_1) \cap V(\mathrm{LTQ}^j)| = 1$ for $0 \leqslant j \leqslant s$. Thus, $d_{G_1}(u_1) = s+1$.

If $q-k$ is odd, then $q-k = 2^{t_0}+\cdots+2^{t_s}$ and $t_s = 0$. This implies that $|V(G_1)| = q-k+1 = 2^{t_0}+\cdots+2^{t_s}$ and $t_s = 1$. Similarly, we have $G_1 = \mathrm{LTQ}_n[V(\mathrm{LTQ}^0)\cup \cdots \cup V(\mathrm{LTQ}^s)]$ and LTQ^s is isomorphic to K_2 by (2.1). Let $V(\mathrm{LTQ}^s) = \{u_1, u_2\}$, then $|N_{G_1}(u_i) \cap V(\mathrm{LTQ}^j)| = 1$ for $i = 1, 2$ and $0 \leqslant j \leqslant s$. Thus, $d_{G_1}(u_i) = s+1$ for $i = 1, 2$.

Label the nodes of G_0 by $v_1, v_2, \cdots, v_q$ and the nodes of G_1 by $v_q, v_{q-1}, \cdots, v_{k+1}, v_k$ such that $d_{G_1}(v_k) = s+1$. Let $k < q$, $X = \{v_1, v_2, \cdots, v_k\}$, $X' = \{v_1, v_2, \cdots, v_{k-1}\}$ and $|E(\mathrm{LTQ}_n[X'])| = f_0$. Clearly, $|\cup_{i=1}^{k} E_{v_i}| = nk - |E(\mathrm{LTQ}_n[X])| \geqslant nk - \frac{ex_k}{2}$. Thus, by Fig. 2, we can get

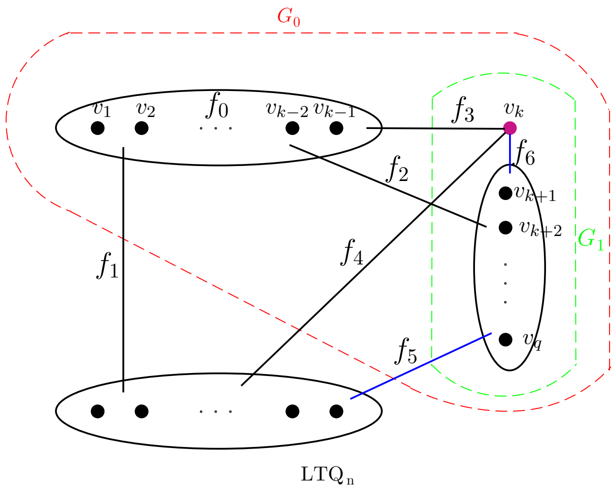

Fig. 2 The edges between the components

$$\begin{aligned} f_0 + f_1 + f_2 + f_3 + f_4 + f_5 &= nq - 2|E(G_0)| + (|E(G_0)| - |E(G_1)|) \\ &= nq - \frac{ex_q}{2} - \frac{ex_{q-k+1}}{2} \end{aligned} \tag{3.1}$$

and

$$f_0 + f_1 + f_2 + f_3 + f_4 + f_6 = |\cup_{i=1}^{k} E_{v_i}| \geqslant nk - \frac{ex_k}{2}. \tag{3.2}$$

From (3.1) and (3.2), we know that the inequality $nq - \frac{ex_q}{2} - \frac{ex_{q-k+1}}{2} \geqslant nk - \frac{ex_k}{2}$ is equivalent to $f_5 \geqslant f_6$. Note that $G_0 \subseteq \mathrm{LTQ}_{n-1}$ and $q - k = \sum_{i=0}^{s} 2^{t_i}$, then $f_5 \geqslant q - k \geqslant s + 1 = d_{G_1}(v_k) = f_6$.

When n is odd, the argument is similar. We omit it.

Combining Lemmas 3.5 and 3.6, we obtain the following main result immediately.

Theorem 3.7 *$c\lambda_{k+1}(\mathrm{LTQ}_n) = nk - \frac{ex_k}{2}$ for $k \leqslant 2^{[\frac{n}{2}]}$ and $n \geqslant 7$. Moreover, for any $c\lambda_{k+1}$-cut F of LTQ_n, $\mathrm{LTQ}_n - F$ has one large component plus k singletons.*

Setting $k = 2 = 2^1$ and $k = 3 = 2^1 + 2^0$ in Theorem 3.7, respectively, we obtain Theorem 3.1 for $n \geqslant 7$.

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