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A Kind of Unified Strict Efficiency via Improvement Sets in Vector Optimization

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Abstract In this paper, we propose a kind of unified strict efficiency named E-strict efficiency via improvement sets for vector optimization. This kind of efficiency is shown to be an extension of the classical strict efficiency and ε-strict efficiency and has many desirable properties. We also discuss some relationships with other properly efficiency based on improvement sets and establish the corresponding scalarization theorems by a base-functional and a nonlinear functional. Moreover, some examples are given to illustrate the main conclusions.

Keywords E-strict efficiency · Improvement sets · Linear scalarization · Nonlinear scalarization · Vector optimization

Mathematics Subject Classification 90C26 · 90C29 · 90C30

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1 Introduction

Both theory and practice of vector optimization are always closely related to scalarization procedures. However, a subset of an efficient decision set may not be satisfactorily characterized by a scalar minimization problem. Then, various concepts of proper efficiency were introduced. Borwein and Zhuang [1,2] introduced the concept of super efficient solutions in normed spaces. Super efficiency refines the notions of efficiency and almost all the classical properly efficiencies, while the condition of the existence results for the super efficiency is too strong. So, Fu et al. [3] firstly introduced the concept of strict efficiency whose existence is much weaker than the super efficiency in real normed spaces. Furthermore, they extended it to the locally convex vector spaces and presented a scalar characterization by base-functional in [4]. This efficiency is shown to have nice properties of super efficiency and is equivalent to Henig proper efficiency in [5], super efficiency and strong efficiency under some suitable assumptions in [6]. Hence the strict efficiency is more advantageous than super efficiency and Henig efficiency. Zhao and Rong [7] proposed the ε-strict efficiency and presented the corresponding scalarization theorems. Besides, many researchers [8,9] studied the properties of strict efficiency such as connectedness and stability.

Another hot topic related to vector optimization is the unified concept. In 2011, based on comprehensive sets, Chicco et al. [10] introduced E-optimal points concept and investigated some properties of improvement sets in Euclidean space. Gutiérre et al. [11] extended the definition of improvement sets to a general real locally convex topological linear space and obtained the scalar characterization for E-efficient solution. Based on improvement sets, Zhao et al. [12] proposed E-Benson proper efficient solution which unified some proper efficiency and approximate proper efficiency and obtained some linear scalarization characterizations under the nearly E-subconvexlikeness proposed in [13]. Recently, Zhou et al. [14] introduced the concept of E-super efficiency and presented the corresponding linear scalarization theorems and Lagrange multiplier theorems.

Motivated by the works of [4,12,14], the paper is organized as follows. In Sect. 2, we present some basic concepts and results that are required in the sequel. In Sect. 3, we propose a kind of unified strict efficiency of vector optimization which named E-strict efficiency based on improvement sets. This efficiency is shown to be an extension of the classical strict efficiency and ε-strict efficiency and have many nice properties. Moreover, we compare the E-strict efficiency with E-super efficiency and E-Benson efficiency. Sections 4 and 5 are devoted to establish the scalarization theorems of E-strict efficiency by base-functional and a nonlinear scalarization function proposed by Göpfert. Moreover, some examples are shown to illustrate the main conclusions.

2 Preliminaries

In this paper, let X be a linear space and Y and Z be real locally convex Hausedorff topological spaces with topological dual spaces Y^* and Z^*, respectively. For a subset A of Y, we denote the interior, closure, the generated cone and the convex hull of A by int$A$, cl$A$, cone$A$ and co$A$. The cone generated by A is defined as cone$A = \{\alpha a | \alpha \geqslant 0,\ a \in A\}$.

The family of the neighborhoods of zero in Y is denoted by $N(0)$. Let $K \subseteq Y$ and $P \subseteq Z$ be nontrivial pointed closed convex cones with nonempty interior. If K is a convex cone, the convex subset $B \subseteq K$ is said to be a base of K if $K = $ cone$B$ and $0 \notin$ cl$B$. The positive dual cone and strict positive dual cone of K are defined as

$$K^* = \left\{ y^* \in Y^* | \langle y^*, y \rangle \geqslant 0,\ \forall y \in K \right\},$$
$$K^{\#} = \left\{ y^* \in Y^* | \langle y^*, y \rangle > 0,\ \forall y \in K \setminus \{0\} \right\},$$

and write

$$B^{\mathrm{st}} = \left\{ y^* \in Y^* |\ \text{there exists } t > 0, \quad \text{s.t. } \langle b, y^* \rangle \geqslant t, \quad \forall b \in B \right\}.$$

Every functional in B^{st} is said to be a base-functional, and through the paper, the family of the base for K is denoted by $\mathcal{B}$.

Let the support functional of Q be defined as

$$\sigma_Q(\varphi) = \sup_{y \in Q} \langle y,\ \varphi \rangle, \quad \varphi \in Y^*.$$

In this paper, we consider the following vector optimization problem:

$$\begin{aligned} &\text{(VP)} \quad \min \ \ F(x) \\ &\quad \text{s.t. } x \in D = \left\{ x \in I | G(x) \bigcap (-P) \neq \varnothing \right\}, \end{aligned}$$

where $I \subseteq X$, $F : I \rightrightarrows Y$ and $G : I \rightrightarrows Z$ are set-valued maps with nonempty value.

Definition 2.1 [11] Let E be a nonempty subset in Y. E is called an improvement set with respect to K iff $0 \notin E$ and $E + K = E$.

The family of improvement sets is denoted by $\mathfrak{T}_Y$.

Definition 2.2 [4] Let A be a nonempty subset in Y. We say that $\overline{y} \in A$ is a strictly efficient point of A, if there exists $U \in N(0)$ such that

$$\mathrm{cl}(\mathrm{cone}(A - \overline{y})) \cap (U - B) = \varnothing,$$

which is denoted by $\overline{y} \in O_{\mathrm{FE}}(A, B)$. $\overline{y} \in A$ is called the proper strictly efficient point of A if

$$\overline{y} \in \cap \{O_{\mathrm{FE}}(A, B) : B \in \mathcal{B}\},$$

which is denoted by $\overline{y} \in O_{\mathrm{FE}}(A, K)$.

Definition 2.3 [7] Let A be a nonempty subset in Y. We say $\overline{y} \in A$ is an ε-strictly efficient point of A, if there exists $U \in N(0)$ such that

$$\mathrm{cl}(\mathrm{cone}(A + \varepsilon - \overline{y})) \cap (U - B) = \varnothing,$$

which is denoted by $\overline{y} \in \varepsilon - O_{\mathrm{FE}}(A, B)$. $\overline{y} \in A$ is called the ε-proper strictly efficient point of A if

$$\overline{y} \in \cap\{\varepsilon - O_{\mathrm{FE}}(A, B) : B \in \mathcal{B}\},$$

which is denoted by $\varepsilon - O_{\mathrm{FE}}(A, K)$.

Definition 2.4 [12] Let $E \in \mathfrak{T}_Y$ and $A \subseteq Y$ be a nonempty subset. We say that $\overline{y} \in A$ is an E-efficient point of A, if

$$(\overline{y} - E - K \setminus \{0\}) \cap A = \varnothing.$$

We denote the set of E-efficient point by $O^E(A, K)$.

Definition 2.5 [12] Let $E \in \mathfrak{T}_Y$ and a set $A \subseteq Y$ be given. We say $\overline{y} \in A$ is an E-Benson proper efficient point of A if

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (-K) = \{0\}.$$

We denote the set of E-Benson proper efficient point by $O^E_{\mathrm{Be}}(A, K)$.

Definition 2.6 [14] Let $A \subseteq Y$ and $E \in \mathfrak{T}_Y$ be given. We say that $\overline{y} \in A$ is an E-super efficient point of A if for any $V \in N(0)$, there exists $U \in N(0)$, such that

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - K) \subseteq V.$$

We denote the set of E-super proper efficient point by $O^E_{\mathrm{Se}}(A, K)$.

3 E-Strict Efficiency and Its Properties

In this section, we propose a kind of unified strict efficiency which named E-strict efficiency and study the relations with the classical strict efficiency, ε-strict efficiency and other proper efficiency proposed via improvement sets. Moreover, we deduce some properties of E-strict efficiency for vector optimization.

Definition 3.1 Let a set $A \subseteq Y$ be given. We say that $\overline{y} \in A$ is an E-strictly efficient point of A with respect to B (E-strictly efficient point for short), if there exists $U \in N(0)$, such that

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B) = \varnothing, \tag{3.1}$$

and we denote this by $\overline{y} \in O^E_{\mathrm{FE}}(A, B)$.

$\overline{y} \in A$ is called the proper E-strictly efficient point of A if $\overline{y} \in \cap\{O^E_{\mathrm{FE}}(A, B) : B \in \mathcal{B}\}$ and we denote this by $\overline{y} \in O^E_{\mathrm{FE}}(A, K)$.

Generally speaking, the set of $O^E_{\mathrm{FE}}(A, B)$ is changing, while the base B is changing. If necessary, we can assume the neighborhood $U \in N(0)$ to be open, or closed, or balanced, or convex.

Remark 3.1 Let K be a cone and B be a base of K. Obviously, we have that $O^E_{\mathrm{FE}}(A, K) \subseteq O^E_{\mathrm{FE}}(A, B)$.

Remark 3.2 From Definition 3.1, $\overline{y} \in O^E_{\mathrm{FE}}(A, B)$ iff there exists a neighborhood U of zero such that

$$\mathrm{cone}(A + E - \overline{y}) \cap (U - B) = \varnothing. \tag{3.2}$$

In fact, we only need to show that, Assuming the Eq. (3.2) is correct, then

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B) = \varnothing.$$

If not, there exists $y \in \mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B)$. Therefore, there exists $\{y_n\} \subseteq \mathrm{cone}(A + E - \overline{y})$ such that $y_n \to y$. Because $U - B$ is open and $y \in (U - B)$, $y_n \in U - B$ for sufficiently large n. Therefore, $y_n \in \mathrm{cone}(A + E - \overline{y}) \cap (U - B)$ for sufficiently large n, which contradicts to (3.2).

Next, we discuss the relationship between E-strict efficiency and some other proper efficiency.

Remark 3.3 Let $E = \varepsilon + K$. Then, E is an improvement set with respect to K. Then, E-strict efficiency reduces to ε-strict efficiency.

In fact, if $\overline{y} \in O^E_{\mathrm{FE}}(A, B)$, then we have

$$\begin{aligned} \mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B) &= \mathrm{cl}(\mathrm{cone}(A + \varepsilon + K - \overline{y})) \cap (U - B) = \varnothing \\ &\Leftrightarrow \mathrm{cone}(A + \varepsilon + K - \overline{y}) \cap (U - B) = \varnothing \\ &\Leftrightarrow \mathrm{cl}(\mathrm{cone}(A + \varepsilon - \overline{y})) \cap (U - B) = \varnothing. \end{aligned}$$

Then, $\overline{y}$ is an ε-strictly efficient point of A.

Remark 3.4 Let $E = K \setminus \{0\}$. Then, E is an improvement set with respect to K. Then, E-strict efficiency reduces to the strict efficiency. In fact, if $\overline{y} \in O^E_{\mathrm{FE}}(A, B)$, then we have

$$\begin{aligned} \mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B) &= \mathrm{cl}(\mathrm{cone}(A + K \setminus \{0\} - \overline{y})) \cap (U - B) = \varnothing \\ &\Leftrightarrow \mathrm{cone}(A + K - \overline{y}) \cap (U - B) = \varnothing \\ &\Leftrightarrow \mathrm{cl}(\mathrm{cone}(A - \overline{y})) \cap (U - B) = \varnothing. \end{aligned}$$

Then, $\overline{y}$ is a strictly efficient point of A.

In the following, we study the properties of E-strictly efficient solution for vector optimization.

Proposition 3.1 *Let B be a base of K. Then, $O^E_{FE}(A, B) \subseteq O^E(A, K)$.*

Proof Let $\overline{y} \in O^E_{FE}(A, B)$. If $\overline{y} \notin O^E(A, K)$, from Definition 2.4, then there exists $y \in A,\ y \neq 0$, s.t. $y \in (A + E - \overline{y}) \cap (-K)$. Since B is the base of K, there exists $\lambda > 0,\ b \in B$ such that $y = -\lambda b$. Therefore, $\frac{1}{\lambda} y = -b \in -B \subseteq U - B$. From $y \in (A + E - \overline{y})$, we have $\frac{1}{\lambda} y \in \text{cone}(A + E - \overline{y}) \cap (U - B)$, which contradicts to (3.2). This implies that $O^E_{FE}(A, B) \subseteq O^E(A, K)$.

Proposition 3.2 *If B is a bounded base of cone K, then $O^E_{FE}(A, K) = O^E_{FE}(A, B)$.*

Proof Let $\overline{y} \in O^E_{FE}(A, B)$. There exists a balanced neighborhood U of zero such that

$$\text{cone}(A + E - \overline{y}) \cap (U - B) = \varnothing. \tag{3.3}$$

Next, we will show that $\overline{y} \in O^E_{FE}(A, \hat{B})$ for any $\hat{B} \in \mathcal{B}$. To the contrary, if there exist $B_1 \in \mathcal{B}, \overline{y} \notin O^E_{FE}(A, B_1)$, then for any $n \in \mathbb{N}$, there exists $y_n \in \text{cone}(A + E - \overline{y}) \cap (\frac{1}{n} U - B_1)$. Then, there exists $\lambda_n \geqslant 0,\ a_n \in A,\ e_n \in E,\ u_n \in U,\ b^1_n \in B_1$ such that $y_n = \lambda_n(a_n + e_n - \overline{y}) = \frac{1}{n} u_n - b^1_n$. Since $b^1_n \in B_1, b^1_n \neq 0$ and B is the base of K, we get $b^1_n = \mu_n b_n$ where $\mu_n > 0, b_n \in B$. Then,

$$\frac{1}{\mu_n} y_n = \frac{\lambda_n}{\mu_n}(a_n + e_n - \overline{y}) = \frac{1}{n}\frac{1}{\mu_n} u_n - b_n. \tag{3.4}$$

From Lemma 1 in [4] and B_1 is a base for K, so $B^{st}_1 \neq \varnothing$. Take $f \in B^{st}_1$, then there exists $t > 0$ such that $f(b_1) \geqslant t > 0$ for any $b_1 \in B_1$. On the other hand, from Proposition 2 in [4], $f \in B^{st}_1 \subset K^*$ and B is bounded, so $m_2 = \sup\{f(b) \big| b \in B\} < +\infty$. Thus, the equation $b^1_n = \mu_n b_n$ implies that $\frac{1}{\mu_n} = \frac{f(b_n)}{f(b^1_n)} \leqslant \frac{m_2}{t}$. So when $n \to +\infty$, it follows that $\frac{1}{n}(\frac{1}{\mu_n}) \to 0$. Hence, for sufficiently large n, we have $\frac{1}{n}(\frac{1}{\mu_n}) < 1$. By (3.4), we have

$$\frac{1}{\mu_n} y_n \in \text{cone}(A + E - \overline{y}) \cap (U - B),\ \text{for sufficiently large } n,$$

which contradicts to (3.3).

Proposition 3.3 *Let $E \in \mathfrak{T}_Y$ be an improvement set with respect to K. For any $B_1, B_2 \in \mathcal{B}$, then the statements below hold:*

(i) $O^E_{FE}(A, B_1) \cup O^E_{FE}(A, B_2) \subseteq O^E_{FE}(A, B_1 + B_2)$;
(ii) $O^E_{FE}(A, co(B_1 \cup B_2)) \subseteq O^E_{FE}(A, B_1) \cap O^E_{FE}(A, B_2)$.

Proof From Proposition 3.1 in [15], if B_1, B_2 are the bases of K, then $B_1 + B_2$ and $\text{co}(B_1 \cup B_2)$ must be the bases for K.

(i) Let $\overline{y} \in O^E_{FE}(A, B_1)$, but $\overline{y} \notin O^E_{FE}(A, B_1 + B_2)$. Then from Remark 3.2, for any neighborhood $U \in N(0)$, we have $\text{cone}(A + E - \overline{y}) \cap (U - (B_1 + B_2)) \neq \varnothing$.

Thus, there exist $\lambda_1 \geqslant 0,\ a_1 \in A,\ e_1 \in E,\ u_1 \in U,\ b_1 \in B_1,\ b_2 \in B_2$ such that $\lambda_1(a_1 + e_1 - \overline{y}) = u_1 - b_1 - b_2$. Since $b_2 \in B_2 \subseteq K = \text{cone}B_1$, there exists $\lambda \geqslant 0,\ b \in B_1$ such that $b_2 = \lambda b$. Therefore, $\lambda_1(a_1 + e_1 - \overline{y}) = u_1 - b_1 - \lambda b$. Since B_1 is a convex set, hence we have

$$\frac{\lambda_1}{1+\lambda}(a_1 + e_1 - \overline{y}) = \frac{1}{1+\lambda}u_1 - \left(\frac{1}{1+\lambda}b_1 + \frac{\lambda}{1+\lambda}b\right) \in \text{cone}(A + E - \overline{y}) \cap (U - B_1)$$

So, we can obtain that $\text{cone}(A + E - \overline{y}) \cap (U - B_1) \neq \varnothing$, which contradicts to $\overline{y} \in O^E_{\text{FE}}(A, B_1)$.

The proof of $O^E_{\text{FE}}(A, B_2) \subseteq O^E_{\text{FE}}(A, B_1 + B_2)$ is similar.

(ii) Let $\overline{y} \in O^E_{\text{FE}}(A, \text{co}(B_1 \cup B_2)$, thus there exists $U_1 \in N(0)$ such that

$$\text{cone}(A + E - \overline{y}) \cap (U_1 - \text{co}(B_1 \cup B_2)) = \varnothing.$$

since $B_1 \subseteq \text{co}(B_1 \cup B_2)$, then $U_1 - B_1 \subseteq U_1 - \text{co}(B_1 \cup B_2)$. Thus, $\text{cone}(A + E - \overline{y}) \cap (U_1 - B_1) = \varnothing$. That is $\overline{y} \in O^E_{\text{FE}}(A, B_1)$. It follows that $O^E_{\text{FE}}(A, \text{co}(B_1 \cup B_2)) \subseteq O^E_{\text{FE}}(A, B_1)$.

The proof of $O^E_{\text{FE}}(A, \text{co}(B_1 \cup B_2)) \subseteq O^E_{\text{FE}}(A, B_2)$ is similar. The proof is completed.

Remark 3.5 Proposition 3.3 is a generalization of Theorem 3.2 in [16], and from Lemma 3.1 in [16], this proposition is also a generalization of Proposition 3.1 in [15].

Proposition 3.4 *Let $A \subseteq Y$ and $E \in \mathfrak{T}_Y$, K have a bounded base B. Then,*

$$O^E_{\text{Se}}(A, K) \subseteq O^E_{\text{FE}}(A, K) \subseteq O^E_{\text{Be}}(A, K).$$

Proof Firstly, we improve $O^E_{\text{Se}}(A, K) \subseteq O^E_{\text{FE}}(A, K)$. Since B is a base of K, $0 \notin \text{cl}B$. Therefore, there exists a convex circled neighborhood $V \in N(0)$ such that $0 \notin B + V$. Let $V_1 = \frac{1}{2}V$, we can easily have

$$(-B) \cap (V_1 + V_1) = \varnothing.$$

From the fact that V_1 is circled, then

$$(V_1 - B) \cap V_1 = \varnothing.$$

Let $\overline{y} \in O^E_{\text{Se}}(A, K)$, $V_1 \in N(0)$, there exists a convex neighborhood $U_1 \in N(0)$ such that

$$\text{cl}(\text{cone}(A + E - \overline{y})) \cap (U_1 - K) \subseteq V_1.$$

Let $U := U_1 \cap V_1$. Clearly, U is a convex neighborhood in $N(0)$. Since $U \subseteq U_1$, we have

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - K) \subseteq V_1.$$

Furthermore, $U \subset V_1$, then $U - B \subseteq V_1 - B$. So, $(U - B) \cap V_1 = \varnothing$. Hence, we obtain that

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B) \subseteq (U - B).$$

Then, it follows that

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B) \subseteq \mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - K) \subseteq V_1.$$

Then,

$$\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (U - B) = \varnothing.$$

Hence, $\overline{y} \in O^E_{\mathrm{FE}}(A, B)$, which implies $\overline{y} \in O^E_{\mathrm{FE}}(A, K)$. Therefore, $O^E_{\mathrm{Se}}(A, K) \subseteq O^E_{\mathrm{FE}}(A, K)$.

Next, we prove $O^E_{\mathrm{FE}}(A, K) \subseteq O^E_{\mathrm{Be}}(A, K)$. From $\overline{y} \in O^E_{\mathrm{FE}}(A, B)$, we have $\mathrm{cl}(\mathrm{cone}(A+E-\overline{y})) \cap \mathrm{cone}(U-B) = \{0\}$. Since $-K = \mathrm{cone}(-B) \subseteq \mathrm{cone}(U-B)$, we get $\mathrm{cl}(\mathrm{cone}(A + E - \overline{y})) \cap (-K) = \{0\}$. Therefore $O^E_{\mathrm{FE}}(A, K) \subseteq O^E_{\mathrm{Be}}(A, K)$.

It is clear that E-strict efficiency of (VP) implies E-Benson efficiency of (VP), but the converse is not necessarily true. The following example illustrates this point.

Example 3.1 Let $Y = l^1 = \{y = (y_1, y_2, \cdots) | \sum_{i=1}^{\infty} |y_i| < +\infty\}$, $K = \{y \in Y | y_i \geqslant 1, y_i \geqslant 0, i = 1, 2, \cdots\}$, $B = \{y \in Y | \sum_{i=1}^{\infty} y_i = 1, y_i \geqslant 0, i = 1, 2, \cdots\}$ and $E = \{y \in Y | \sum_{i=1}^{\infty} y_i \geqslant 3^{-1}, y_i \geqslant 0, i = 1, 2, \cdots\}$. The real linear space l^1 is separable from Banach space with respect to the norm $\|\cdot\|$ given by $\|y\| := \sum_{i=1}^{\infty} |y_i|$ for all $y \in Y$. Let $A = \{-3e_i + 3^{-(i-1)}e_1 | i = 2, 3, \cdots\} \cup \{0\}$, where $e_i = (0, 0, \cdots, 0, 1, 0 \cdots) \in Y$. Since there exists $\varphi = (1, 3^{-3}, 3^{-4}, \cdots) \in K^\#$ such that $\langle a, \varphi\rangle \geqslant \sigma_{-E}(\varphi)$, $\forall a \in A$. It follows from Theorem 3.2 in [17] that $0 \in O^E_{\mathrm{Be}}(A, K)$. However,

$$- e_i + 3^{-(i-1)}e_1 \in (A + E) \cap (3^{-(i-2)}U - B), \quad i = 2, 3, \cdots,$$

where U is a unit ball in Y. Hence, it follows from Definition 3.1 that $0 \notin O^E_{\mathrm{FE}}(A, B)$.

4 Scalarization by Base-Functional

In this section, we establish a scalarization theorem of E-strictly efficiency by base-functional under the nearly E-subconvexlikeness.

Let the feasible set D of (VP) be nonempty and $\varphi \in Y^* \setminus \{0\}$. The scalar minimization problem of (VP) is defined as follows:

$$(\mathrm{VP})_\varphi \quad \min\langle F(x), \varphi\rangle \quad \text{s.t.} \quad x \in D.$$

Definition 4.1 Let B be the base of K and $E \in \mathfrak{T}_Y$. $\overline{x} \in D$ is called an E-strictly efficient solution of (VP) with respect to B (E-strictly efficient solution for short), if there exists a neighborhood U of zero such that

$$\mathrm{cl}(\mathrm{cone}(F(D) + E - F(\overline{x}))) \cap (U - B) = \varnothing.$$

The set of E-strictly efficient solutions of (VP) is denoted by $O^E_{\mathrm{FE}}(F(D), B)$.

The point pair $(\overline{x}, \overline{y})$ is called an E-strictly efficient element of (VP) with respect to B.

$\overline{x} \in D$ is called an E-proper strictly efficient solution of (VP) with respect to K, if $\overline{x}$ is the E-strictly efficient solution of (VP) with respect to every base of K, which is denoted by $O^E_{\mathrm{FE}}(F(D), K)$.

Obviously, $O^E_{\mathrm{FE}}(F(D), K) = \cap\{O^E_{\mathrm{FE}}(F(D), B) : B \in \mathcal{B}\}$.

Definition 4.2 [17] $\overline{x} \in D$ is called an optimal solution of $(\mathrm{VP})_\varphi$ with respect to E, if there exists $\overline{y} \in F(\overline{x})$ such that

$$\langle y - \overline{y}, \varphi\rangle \geqslant \sigma_{-E}(\varphi), \quad \forall x \in D, \quad \forall y \in F(x).$$

The point pair $(\overline{x}, \overline{y})$ is called an optimal element of $(\mathrm{VP})_\varphi$ with respect to E.

Definition 4.3 [17] Let $D \subseteq X$ and $E \in \mathfrak{T}_Y$. The set-valued map $F : D \rightrightarrows Y$ is called nearly E-subconvexlike on D iff $\mathrm{cl}(\mathrm{cone}(F(D) + E))$ is a convex set in Y.

The following theorems provide scalar characterizations of E-strictly efficient points.

Theorem 4.1 *Let B be a base of K and $E \in \mathfrak{T}_Y$. Suppose that $(\overline{x}, \overline{y})$ is an E-strictly efficient element of (VP) and $F - \overline{y}$ is nearly E-subconvexlike on D. Then, there exists $\varphi \in B^{st}$ such that $(\overline{x}, \overline{y})$ is an optimal element of $(\mathrm{VP})_\varphi$ with respect to E.*

Proof By Definition 4.1, if $\overline{y} \in O^E_{\mathrm{FE}}(F(D), B)$, there exists a convex neighborhood $U \in N(0)$ such that $\mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y})) \cap (U - B) = \varnothing$. Since $F - \overline{y}$ is nearly E-subconvexlike on D, $\mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y}))$ is a convex set in Y. Clearly, $U - B$ is a nonempty open convex set in Y. By the separation theorem for convex sets, there exists $\varphi \in Y^* \setminus \{0\}$ such that

$$\langle y_1, \quad \varphi\rangle \geqslant \langle y_2, \varphi\rangle, \quad \forall y_1 \in \mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y})), \quad \forall y_2 \in U - B. \quad (4.1)$$

Since $0 \in \mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y}))$, it follows that

$$\langle y_2, \varphi\rangle \leqslant 0, \quad \forall y_2 \in U - B.$$

From Proposition 2.1 in [6], $\varphi \in B^{\mathrm{st}}$. By Eq. (4.1), φ is bounded below on the closed convex cone $\mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y}))$ in Y; it follows from (4.1) that

$$\langle y_1, \quad \varphi\rangle \geqslant 0, \ \forall y_1 \in \mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y})).$$

Clearly

$$\langle y_1, \varphi \rangle \geqslant 0, \quad \forall y_1 \in F(D) + E - \overline{y}.$$

Hence, we have

$$\langle y - \overline{y}, \varphi \rangle \geqslant \sigma_{-E}(\varphi), \quad \forall x \in D, \quad \forall y \in F(x),$$

which implies that $(\overline{x}, \overline{y})$ is an optimal element of $(\mathrm{VP})_\varphi$ with respect to E.

Theorem 4.2 *Let B be a bounded base of K and $E \in \mathfrak{T}_Y$. If there exists $\varphi \in B^{\mathrm{st}}$ such that $(\overline{x}, \overline{y})$ is an optimal element of $(VP)_\varphi$ with respect to E. Then, $(\overline{x}, \overline{y})$ is an E-strictly efficient element of (VP) with respect to B.*

Proof Let $\varphi \in B^{\mathrm{st}}$. Then, there exists $t > 0$ such that $\langle b, \varphi \rangle \geqslant t > 0, \ \forall b \in B$. Let $U = \{x \in X, \langle x, \varphi \rangle < t\}$. Then, $U \in N(0)$ and $\langle u, \varphi \rangle < 0, \ \forall u \in U - B$. Since$(\overline{x}, \overline{y})$ is the E-optimal element of $(\mathrm{VP})_\varphi$, we have

$$\langle y - \overline{y}, \varphi \rangle \geqslant \sigma_{-E}(\varphi), \quad \forall y \in F(D).$$

Hence

$$\langle y_1, \varphi \rangle \geqslant 0, \quad \forall y_1 \in \mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y})).$$

So

$$\mathrm{cl}(\mathrm{cone}(F(D) + E - \overline{y})) \cap (U - B) = \varnothing.$$

Then, $(\overline{x}, \overline{y})$ is an E-strictly efficient element of (VP).

Remark 4.1 Theorem 4.1 and Theorem 4.2 are the generalizations of Lemma 6 in [3], Theorem 1 in [4], Theorem 1 in [8] and Theorem 3.3 in [15].

Theorem 4.3 *If B is a bounded base for K, then $O^E_{\mathrm{Se}}(A, K) = O^E_{\mathrm{FE}}(A, K)$.*

Proof By the scalar property of E-super efficiency in [14] and E-strict efficiency above, it is clear that $O^E_{\mathrm{Se}}(A, B) = O^E_{\mathrm{FE}}(A, B)$. From Lemma 3.2, we have $O^E_{\mathrm{Se}}(A, K) = O^E_{\mathrm{FE}}(A, K)$.

Remark 4.2 By Theorem 3.4 in [14], $O_{GE}(A, K) = O_{FE}(A, K) = O_{SE}(A, K)$ when A is a convex set and K has a bounded base. And from Proposition 3.5 in [15], E-strict efficiency also generalizes Henig proper efficiency, strong efficiency proposed in [6], and coincides with E-super efficiency in [14] under the suitable assumptions.

5 Scalarization by Nonlinear Function $\xi_{q,E}(y)$

In this section, we obtain a nonlinear scalarization for E-strictly efficient solution by the nonlinear function proposed by Göpfert et al. [18].

Definition 5.1 Let K is a cone and $E \in \mathfrak{T}_Y$ with respect to $K, q \in \text{int}K$. The function $\xi_{q,E} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\xi_{q,E}(y) = \inf\{t \in \mathbb{R} | y \in tq - E\}, \quad y \in Y,$$

where $\inf \varnothing = +\infty$.

Lemma 5.1 [19] *Let $E \in \mathfrak{T}_Y$ with respect to K and $q \in \text{int}K$. Then, $\xi_{q,E}(y)$ is a continuous function and satisfies $\{y \in Y | \xi_{q,E}(y) < c\} = cq - \text{int}E,\ \forall c \in \mathbb{R}$.*

We consider the scalar optimization problem:

$$(P_{q,y}) \min_{x \in D} \xi_{q,E}(F(x) - y),$$

where $y \in Y$, $q \in \text{int}K$, F is a vector-valued map with nonempty value and denote $\xi_{q,E}(F(x) - y)$ by $(\xi_{q,E,y} \circ F)(x)$.

Let $\varepsilon \geqslant 0, \overline{x} \in D$. If $(\xi_{q,E,y} \circ F)(x) \geqslant (\xi_{q,E,y} \circ F)(\overline{x}) - \varepsilon,\ \forall x \in D$, then $\overline{x}$ is called an ε-optimal solution of $(P_{q,y})$. Denote the set of ε-optimal solutions for $(P_{q,y})$ by $\text{AMin}(\xi_{q,E,y} \circ F, \varepsilon)$.

Theorem 5.1 *Let B be a base of cone K and $q \in intK$, $E \in \mathfrak{T}_Y$, $\beta = \inf\{t \in \mathbb{R}_+ | tq \in E\}$. Then,*

$$\overline{x} \in O^E_{FE}(F(D), B) \Rightarrow \overline{x} \in AMin\ (\xi_{q,E,f(\overline{x})} \circ F, \beta).$$

Proof For each $\overline{x} \in O^E_{FE}(F(D), B)$, we have

$$\text{cl}(\text{cone}(F(D) + E - F(\overline{x}))) \cap (U - B) = \varnothing. \tag{5.1}$$

Then, we have

$$\text{cl}(\text{cone}(F(D) + E - F(\overline{x}))) \cap \text{cone}(U - B) = \{0\}. \tag{5.2}$$

Otherwise, if there exists $y \neq 0$ and $y \in \text{cl}(\text{cone}(F(D)+E-F(\overline{x}))) \cap \text{cone}(U-B)$, then there exists $u \in U,\ b \in B,\ \lambda > 0$ such that $y = \lambda(u - b)$ and

$$\frac{y}{\lambda} \in \text{cl}(\text{cone}(F(D) + E - F(\overline{x}))),$$

which contradicts to (5.1). It is clear that $-K = \text{cone}(-B) \subseteq \text{cone}(U - B)$. Hence by (5.2), we have

$$\text{cl}(\text{cone}(F(D) + E - F(\overline{x}))) \cap (-K) = \{0\}.$$

Hence

$$(F(D) - F(\overline{x})) \cap -(E + \mathrm{int}K) = \varnothing.$$

By Lemma 2.1 in [17], we have

$$(F(D) - F(\overline{x})) \cap (-\mathrm{int}E) = \varnothing.$$

From Lemma 5.1 and let $c = 0$, we have

$$(\xi_{q,E,F(\overline{x})\circ F})(x) = \xi_{q,E}(F(x) - F(\overline{x})) \geqslant 0, \quad \forall x \in D. \tag{5.3}$$

On the other hand,

$$\begin{aligned}\left(\xi_{q,E,F(\overline{x})\circ F}\right)(\overline{x}) &= \xi_{q,E}\left(F(\overline{x}) - F(\overline{x})\right)\\ &= \inf\{t \in \mathbb{R} | tq \in E\} \leqslant \inf\{t \in \mathbb{R}_+ | tq \in E\} = \beta.\end{aligned}$$

Hence by (5.3), we have

$$(\xi_{q,E,F(\overline{x})\circ F})(x) \geqslant (\xi_{q,E,F(\overline{x})\circ F})(\overline{x}) - \beta, \quad \forall x \in D.$$

6 Conclusions

In this paper, we have proposed a kind of new unified strict efficiency which is a generalization of strict efficiency, ε-strict efficiency, Henig proper efficiency and strongly proper efficiency under suitable condition. We also deduce some scalarization characterizations for the unified strict efficiency. It is also meaningful to study the section property, density property and connectedness of E-strictly efficiency.

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