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Some Remarks on Projective Representations of Compact Groups and Frames

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Abstract
In this paper, we study continuous frames with symmetries from projective representations of compact groups. In particular, we study maximal spanning vectors in detail and we prove the existence of maximal spanning vectors for irreducible projective representations of compact abelian groups by a dimension counting method.

Keywords Continuous frame · Fourier transform · Maximal spanning vector · Projective representation · The Peter–Weyl theorem

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1 Introduction

Let G be a locally compact group and $\alpha \in Z^2(G, \mathbb{S})$ be a multiplier. Let $\pi : G \to \mathbf{U}(V)$ be an irreducible α-representation of G on a complex Hilbert space V (cf. Definition 1.2). For $x \in V$, if the map $G \to V$ $(g \mapsto \pi(g)x)$ is a continuous frame (cf. Definition 1.5), we call x a frame vector for (π, G, V) and the associated frame a (G, α)-frame. Let $\mathrm{HS}(V)$ be the space of Hilbert–Schmidt operators on V, and let

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$x \otimes x \in \mathrm{HS}(V)$ be the one-dimensional projection defined by

$$\begin{aligned} x \otimes x : V &\to V \\ v &\mapsto \langle v, x \rangle x. \end{aligned}$$

From the perspective of frames and cyclic representations, a natural question arises: which triples (π, G, V) admit an element $x \in V$, such that

$$\overline{\mathrm{Span}}\{\pi(g)x \otimes \pi(g)x \mid g \in G\} = \mathrm{HS}(V)?$$

We call such an $x \in V$ a maximal spanning vector for (π, G, V). If $x \in V$ is a maximal spanning frame vector, then the associated frame is phase retrievable, i.e., the map

$$\begin{aligned} T_x : V &\to L^2(G) \\ v &\mapsto (g \mapsto |\langle v, \pi(g)x \rangle|) \end{aligned}$$

is injective modulo $\mathbb{S}$, where $\mathbb{S}$ is the set of complex numbers with absolute value 1.

In [18], Li et al. showed that if G is a finite abelian group and (π, V) is any irreducible projective representation of G, then the set of maximal spanning vectors for (π, G, V) is nontrivial and Zariski open in V. Moreover [18] conjectured that the same holds for all finite groups (cf. [18, Conjecture]). In [8], Cheng et al. conjectured that there exist infinitely many maximal spanning vectors for $(\pi, \widehat{G} \times G, L^2(G))$, where G is a Hausdorff and second countable locally compact abelian group and $\pi : \widehat{G} \times G \to \mathbf{U}(L^2(G))$ is the Weyl–Heisenberg representation of $\widehat{G} \times G$ (cf. [8, Conjecture 1.4]). Moreover, [8, Theorem 1.5] verified the conjecture for a large class of locally compact abelian groups. A recent computation [9] showed that [8, Conjecture 1.4] holds for G a local field with residue characteristic 2, hence removed the characteristic assumption of [8, Theorem 1.5]; the paper [11] studies the phase retrieval property of representations of nilpotent Lie groups and certain nilpotent p-groups. In this paper, we study the case where G is a compact group and π is an irreducible projective representation of G. The main result of this paper is the following.

Theorem 1.1 *Let G be a compact group and $\pi : G \to V$ be an irreducible projective representation of G.*

1. *Let V^* be the dual representation of V and suppose that each irreducible component of $V \otimes V^*$ has multiplicity one. Then, $x \in V$ is a maximal spanning vector for V if and only if the projection of $x \otimes x$ (as an element in $V \otimes V^*$ via the natural isomorphism $\mathrm{HS}(V) \cong V \otimes V^*$) in each irreducible component of $V \otimes V^*$ is nontrivial.*
2. *Suppose that G is abelian. Then, the set*

$$\{v \in V \mid v \text{ is maximal spanning for } (\pi, G, V)\}$$

is nontrivial and Zariski open in V.

By the twisted Peter–Weyl Theorem (cf. Sect. 2), irreducible projective representations of compact groups are finite dimensional and the following conditions are equivalent:

- $x \in V$ is a maximal spanning vector for (π, G, V).
- $g \mapsto \pi(g)x \otimes \pi(g)x$ is a continuous frame for $\mathrm{HS}(V)$.
- $\dim \mathrm{Span}\{\pi(g)x \otimes \pi(g)x \mid g \in G\} = (\dim V)^2$.
- $\dim \mathrm{Span}\{c^{\pi}_{\pi(g)x,\pi(g)x} \mid g \in G\} = (\dim V)^2$, where $c^{\pi}_{u,v} : G \to \mathbb{C}$ $(u, v \in V)$ is the matrix coefficient defined by

$$c^{\pi}_{u,v}(h) = \langle \pi(h)u, v\rangle.$$

Theorem 1.1(1) is then deduced from a structure result for (G, α)-frames (Proposition 3.8), and it provides in some cases a practical method to verify whether a vector is maximal spanning. As projective representations and linear representations are related by using the Mackey groups (cf. [16]), some results on (G, α)-frames we obtained in this paper could be deduced from the results in [14]. But we provide different proofs which are more explicit and more suitable for our application to the study of maximal spanning vectors. As an opportunity, for G compact, we deduce several properties of (G, α)-frames in Sect. 3, in particular we characterize the Gramian of (G, α)-frames (cf. Proposition 3.10 and Corollary 3.12). Theorem 1.1(2) is proved by dimension counting after a careful study of α-representations of compact abelian groups. For this part, considering projective representations of G directly is simpler than considering linear representations of the attached Mackey group $G(\alpha)$, since we could avoid the complications from the two-step filtration of $G(\alpha)$. Note that in case G is locally compact, the representation spaces are usually infinite dimensional and the counting method we used here does not apply.

Due to the reasons explained above, in this paper we consider projective representations directly (which include linear representations as a special case with the multiplier $\alpha = 1$). The contents of the paper are as follows. In Sect. 2, we review the twisted Peter–Weyl Theorem and explicitly compute the Fourier transform for α-representations. This is the main tool we use in Sect. 3, where we study (G, α)-frames in detail and obtain the necessary results for the study of maximal spanning vectors. In Sect. 4, we show that Theorem 1.1(1) is an easy consequence of Proposition 3.8 and prove Theorem 1.1(2). In Sect. 4.3, we study the spanning dimension in the reducible situation and justify the irreducibility condition in Conjecture 4.2. This is also related to [18, Problems C, D].

1.1 Notation and Convention

In the following, we recall the definitions of multipliers, α-representations, continuous frames, etc. The readers could find more details in [12, 16, 20, 23].

Let G be a compact group. A multiplier (or 2-cocyle) on G is a measurable function $\alpha : G \times G \to \mathbb{S}$, such that

1. $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for any $x, y, z \in G$;
2. $\alpha(x, 1) = \alpha(1, x) = 1$ for any $x \in G$.

Two multipliers α and α' are similar if there exists a measurable function $\beta : G \to \mathbb{S}$ such that $\alpha'(g, h) = \alpha(g, h)\frac{\beta(gh)}{\beta(g)\beta(h)}$ for all $g,\ h \in G$.

Definition 1.2 A projective representation of G with multiplier α (or an α-representation of G) is a map $\pi : G \to \mathbf{U}(V)$, where V is a complex Hilbert space, $\mathbf{U}(V)$ is the space of unitary operators on V, such that

1. for all $v \in V$, the map $G \to V$ $(g \mapsto \pi(g)v)$ is measurable;
2. $\pi(g)\pi(h) = \alpha(g, h)\pi(gh)$ for any $g, h \in G$.

Let $\pi : G \to \mathbf{U}(V)$ be a finite dimensional α-representation of G. The *character* of π is the map $\chi_\pi : G \to \mathbb{C}$ given by $\chi_\pi(g) = \mathrm{Tr}(\pi(g))$.

Remark 1.3 Let $\pi : G \to \mathbf{U}(V)$ be an α-representation and $\pi' : G \to \mathbf{U}(V)$ be an α'-representation. We say that π and π' are equivalent if there exist a measurable map $\beta : G \to \mathbb{S}$ and a unitary isomorphism $\ell : V \to V'$ such that $\ell(\pi(g)v) = \beta(g)\pi'(g)(\ell(v))$ for all $g \in G$ and $v \in V$. In particular, if π and π' are equivalent, then α and α' are similar multipliers.

Remark 1.4 Let G be a compact group and $\alpha \in Z^2(G, \mathbb{S})$ be a multiplier. Let $G(\alpha)$ be the associated Mackey group. It is the set $\mathbb{S} \times G$ provided with the group structure

$$(s, g)(s', g') = (ss'\sigma(g, g'), gg') \quad \text{for all } s, s' \in S,\ g, g' \in G.$$

If α is continuous, we equip $G(\alpha)$ with the product topology of $\mathbb{S}$ and G. If α is normalized in the sense that $\alpha(g, g^{-1}) = 1$ for all $g \in G$, we equip $G(\alpha)$ with a topology in which a basis for the neighborhoods of the identity is composed of the sets AA^{-1}, where A is a measurable set of finite positive measure for the product of right Haar measures on $\mathbb{S}$ and G. As explained in [16, Page 218], similar normalized multipliers give us isomorphic and homeomorphic Mackey groups. In both cases, $G(\alpha)$ is a compact topological group. Moreover, let $\pi : G \in \mathbf{U}(V)$ be an α-representation of G. Then, $\pi_\alpha : G(\alpha) \to \mathbf{U}(V)$ $((s, g) \mapsto s\pi(g))$ is a linear representation of $G(\alpha)$. The map $\pi \mapsto \pi_\alpha$ defines a bijection between the set of equivalent classes of α-representations of G and the set of equivalent classes of linear representations of $G(\alpha)$ with $\mathbb{S}$ acting as scalars (cf. [16, Page 223 Corollary]).

Definition 1.5 Let V be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to V$ is called a continuous frame with respect to (Ω, μ), if

1. F is weakly measurable, i.e., for all $v \in V$, the map $\omega \mapsto \langle v, F(\omega)\rangle$ is a measurable function on Ω;
2. there exist constants $A, B > 0$ such that

$$A\|v\|^2 \le \int_\Omega |\langle v, F(\omega)\rangle|^2 \,\mathrm{d}\mu(\omega) \le B\|v\|^2, \quad \text{for all } v \in V. \tag{1.1}$$

The constants A and B are called continuous frame bounds. The frame F is called tight if $A = B$. A tight frame F is called Parseval if $A = B = 1$.

For simplicity, we also say that the family of vectors $\{F(\omega) \mid \omega \in \Omega\}$ is a continuous frame for V.

Let $F : \Omega \to V$ be a continuous frame. The synthesis operator $T : L^2(\Omega, \mu) \to V$ of F is defined by

$$\langle T\phi, v\rangle = \int_\Omega \phi(\omega)\langle F(\omega), v\rangle \,\mathrm{d}\,\mu(\omega), \;\; v \in V.$$

The analysis operator $T^* : V \to L^2(\Omega, \mu)$ of F is the adjoint of T, i.e.,

$$(T^* v)(\omega) = \langle v, F(\omega)\rangle, \;\; \omega \in \Omega.$$

The frame operator $S : V \to V$ of F is given by $S = T^* \circ T$. By [20, Corollary 2.12], S is invertible and $S^{-1}F$ is a continuous frame. We call $S^{-1}F$ the standard dual frame of F.

2 The Twisted Peter–Weyl Theorem for Compact Groups

In this section, G is a compact group and $\alpha \in Z^2(G, \mathbb{S})$ is a multiplier. Let $\int_G \cdot \,\mathrm{d}\,g$ be the Haar measure on G with $\mathrm{vol}(G) = 1$. Denote by $\widehat{G}_\alpha$ the set of isomorphism classes of finite dimensional irreducible α-representations of G. Let (π, V_π, α) be a representative of an element in $\widehat{G}_\alpha$ and denote by $[\pi]$ the corresponding isomorphism class. Fix a G-invariant Hermitian inner product $\langle , \rangle$ on V_π, which exists by the averaging argument. Given $v, w \in V_\pi$, the function $f : g \mapsto \langle \pi(g)v, w\rangle$ is called a matrix coefficient of π. Let $\mathcal{A}_\alpha(G)$ be the space spanned by all matrix coefficients of finite dimensional irreducible α-representations of G.

For $1 \le p \le \infty$, let $L^p(G)$ be the L^p-Banach space with norm $\| \cdot \|_p$ for $(G, \int_G \cdot \,\mathrm{d}\,g)$. Given $f, f' \in L^2(G)$, define an inner product by

$$\langle f, f'\rangle_2 = \int_G f(g)\overline{f'(g)}\,\mathrm{d}\,g. \tag{2.1}$$

With this inner product, $L^2(G)$ is a Hilbert space. Furthermore, $ff' \in L^1(G)$ and we have the following inequalities.

$$\begin{aligned} &\|ff'\|_1 \le \|f\|_2\|f'\|_2, \\ &|\langle f, f'\rangle_2| \le \|f\|_2\|f'\|_2 \text{ (Schwarz inequality).} \end{aligned} \tag{2.2}$$

If $f : G \to \mathbb{C}$ and $g \in G$, define $R(g)f := R_\alpha(g)f : G \to \mathbb{C}$ by

$$(R(g)f)(g_0) = \alpha(g_0, g)f(g_0 g)$$

for all $g_0 \in G$. It is easy to check that $R(g)f \in L^2(G)$ if $f \in L^2(G)$ and $R(g)$ is an element in $\mathbf{U}(L^2(G))$. Then, $R : G \to \mathbf{U}(L^2(G))$ defines an α-representation of G

(cf. Lemma 2.3). We call it the right α-translation or right regular α-representation of G on $L^2(G)$.

As $\langle , \rangle_2$ is G-invariant, the α-representation $(R, L^2(G), \alpha)$ decomposes as a direct sum of irreducible α-representations. We have the following twisted version of the classical Peter–Weyl Theorem.

Theorem 2.1 (*The Peter–Weyl theorem*) *Let G be a compact group, $\alpha \in Z^2(G, \mathbb{S})$ a multiplier of G. Then, the following claims hold.*

1. *$\mathcal{A}_\alpha(G)$ is dense in $L^2(G)$.*
2. *Every irreducible unitary α-representation of G is finite dimensional.*
3. *Fix an element ρ in each class in $\widehat{G}_\alpha$ and denote by d_ρ the dimension of ρ. Then, as α-representations of G,*

$$(R, L^2(G), \alpha) \cong \widehat{\bigoplus}_{[\rho]\in\widehat{G}_\alpha} \rho^{\oplus d_\rho}.$$

4. *If $\psi \in L^2(G)$, then*

$$\|\psi\|_2^2 = \sum_{[\rho]\in\widehat{G}_\alpha} d_\rho \cdot \mathrm{Tr}(\rho_\psi \rho_\psi^*) = \sum_{[\rho]\in\widehat{G}_\alpha} d_\rho \cdot \|\rho_\psi\|_{\mathrm{HS}}.$$

Here, $\rho_\psi = \int_G \psi(g)\rho(g)^{-1}\,\mathrm{d}\,g$; $\|M\|_{\mathrm{HS}} = \sum_{i,j}|m_{ij}|^2$ for a matrix $M = (m_{ij})$ of finite rank.

5. *The characters $(\chi_\rho)_{[\rho]\in\widehat{G}_\alpha}$ form an orthonormal basis of $\mathbb{H}_\alpha$ (Definition 2.7).*

From Sect. 2.2, for $\pi : G \to \mathbf{U}(V_\pi)$ a finite dimensional projective representation, we denote by V_π the associated Hilbert space and by d_π the dimension of V_π. We fix an orthonormal basis of the representation space of π for $[\pi] \in \widehat{G}_\alpha$ and denote it by $\{e_i^\pi \mid 1 \le i \le d_\pi\}$. Denote by π_{ij} $(1 \le i, j \le d_\pi)$ the matrix coefficients of π with respect to the fixed basis, i.e.,

$$\pi_{ij}(g) = \langle \pi(g)e_j^\pi, e_i^\pi \rangle \quad \text{for all } g \in G.$$

Then, $\{\sqrt{d_\pi}\pi_{ij} \mid [\pi] \in \widehat{G}_\alpha,\ 1 \le i, j \le d_\pi\}$ is an orthonormal basis of $L^2(G)$.

2.1 Proof of the Peter–Weyl Theorem

Let α and α' be two similar multipliers. There exists a measurable function $\beta : G \to \mathbb{S}$ with $\alpha'(g, h) = \alpha(g, h)\frac{\beta(gh)}{\beta(g)\beta(h)}$ for all $g,\ h \in G$. If $\pi : G \to \mathbf{U}(V)$ is an α-representation of G, then $\pi' : G \to \mathbf{U}(V)$ given by $\pi'(g) = \beta^{-1}(g)\pi(g)$ is an α'-representation. Hence, multiplication by β^{-1} induces an isomorphism from the space of matrix coefficients of π to that of π'. Therefore, $\mathcal{A}_\alpha(G) = \beta \cdot \mathcal{A}_{\alpha'}(G)$. Moreover, the right regular α-representation and the right regular α'-representation are equivalent, as for $g \in G$, we have the following commutative diagram

$$\begin{array}{ccc} L^2(G) & \xrightarrow{\beta^{-1}\cdot} & L^2(G) \\ {\scriptstyle R_\alpha(g)}\downarrow & & \downarrow{\scriptstyle \beta(g)\cdot R_{\alpha'}(g)} \\ L^2(G) & \xrightarrow[\beta^{-1}\cdot]{} & L^2(G). \end{array}$$

In other words, proving Theorem 2.1 for α is equivalent to proving it for α'. By [16, Lemma 1 and Corollary], in the following we may and do assume that either α is continuous or it is normalized and continuous in a neighborhood of $(1, 1)$.

Under the above assumption, parts of Theorem 2.1 are easy consequences of the classical Peter–Weyl Theorem using the Mackey group of G (cf. Theorem [16, Theorem 1]). For example, irreducible α-representations of G correspond to irreducible linear representations of $G(\alpha)$; hence, they are finite dimensional. One could also deduce Theorem 2.1 from the general Plancherel formula for projective representations of locally compact groups (cf. [17, Theorem 7.1]). Since we could not locate a reference where Theorem 2.1 is stated as in the above form, we provide a detail proof of part (1) of the Theorem. This section is independent of the other parts of this paper.

Proposition 2.2 *With the notation as above, if either α is continuous or α is normalized and continuous in a neighborhood of $(1, 1)$, then $\mathcal{A}_\alpha(G)$ is dense in $L^2(G)$.*

The following proof is adapted from the proof of the classical Peter–Weyl Theorem (see for example [19]), with an extra attention on the multiplier α. We start with some lemmas.

Lemma 2.3 *Let $f \in L^2(G)$. Then, the map $G \to L^2(G)$ $(g \mapsto R(g)f)$ is measurable and it is continuous in a neighborhood of $1 \in G$.*

Proof Note that for any $f \in L^2(G)$, the usual translation map $G \to L^2(G)$ $(g \mapsto (h \mapsto f(hg)))$ is continuous (e.g., [10, 2.42 Proposition]). The measurability follows as α is a measurable multiplier.

If $\alpha : G \times G \to \mathbb{S}$ is a continuous multiplier, then $g \mapsto R(g)f$ is a continuous map from G to $L^2(G)$. Otherwise, the multiplier α is normalized and continuous in a neighborhood of $(1, 1) \in G \times G$, the section from G to the associated Mackey group $G(\alpha)$ $(g \mapsto (1, g))$ is continuous in a neighborhood of $1 \in G$. The continuity claim follows from this and [16, Theorem 1]. ◻

Lemma 2.4 *Let $f : g \mapsto \langle \pi(g)v, w\rangle$ be a matrix coefficient of π. Then, the functions $g \mapsto \alpha(g, g^{-1})\overline{f(g^{-1})}$, $g \mapsto \alpha(g, h)f(gh)$, $g \mapsto \alpha(h, g)\alpha(h^{-1}, h)^{-1}f(hg)$ are matrix coefficients of π. We call them the adjoint of f, the right translation of f, the left translation of f, respectively.*

Proof Note that

$$\begin{aligned} \overline{f(g^{-1})} = \overline{\langle \pi(g^{-1})v, w\rangle} &= \langle w, \pi(g^{-1})v\rangle \\ &= \langle \pi(g)w, \pi(g)\pi(g^{-1})v\rangle = \alpha(g, g^{-1})^{-1}\langle \pi(g)w, v\rangle. \end{aligned} \tag{2.3}$$

This shows that $g \mapsto \alpha(g, g^{-1})\overline{f(g^{-1})}$ is a matrix coefficient. Similarly, it is easy to see that

$$\begin{aligned} f(gh) &= \alpha(g,h)^{-1}\langle \pi(g)(\pi(h)v), w\rangle, \\ f(hg) &= \alpha(h,g)^{-1}\alpha(h^{-1},h)\langle \pi(g)v, \pi(h^{-1})w\rangle. \end{aligned} \tag{2.4}$$

The other claims follow easily. □

Lemma 2.5 *Let $f \in L^2(G)$. For every $\epsilon > 0$, there exist finitely many $g_i \in G$ and Borel sets $B_i \subset G$ such that G is the disjoint union of the B_i's and $\|R(g)f - R(g_i)f\|_2 < \epsilon$ for all i and $g \in B_i$.*

Proof By Lemma 2.3, there exists an open neighborhood U of 1 such that $\|R(g)f - f\|_2 < \epsilon$ for all $g \in U$. Note that $\{hU \mid h \in G\}$ is an open cover of G and G is compact, there exist finitely many $g_1, \ldots, g_n$ such that $G = \cup_{i=1}^n g_i U$. Let $B_i = g_i U - \cup_{j=1}^{i-1} g_j U$. It is easy to check that these objects satisfy the property in the statement. □

Lemma 2.6 *Let $f \in L^2(G)$ and $f_1 \in L^1(G)$. Define $F : G \to \mathbb{C}$ by*

$$F(g') = \int_G \alpha(g', g) f(g'g) f_1(g)\, \mathrm{d}g.$$

Then, F is an element in $L^2(G)$ and it is a limit of a sequence of functions, each of which is a finite linear combination of right translations of f.

Proof Let $\epsilon > 0$. Choose g_i and B_i as in Lemma 2.5. Set $e_i = \int_{B_i} f_1(g)\, \mathrm{d}g$. Then,

$$\begin{aligned} \|F - \sum_{i=1}^n e_i R(g_i) f\|_2 &\le \sum_{i=1}^n \int_{B_i} |f_1(g)| \cdot \|R(g)f - R(g_i)f\|_2\, \mathrm{d}g \\ &\le \sum_{i=1}^n \int_{B_i} |f_1(g)| \epsilon\, \mathrm{d}g = \epsilon \|f_1\|_1. \end{aligned} \tag{2.5}$$

The lemma follows. □

Definition 2.7 A function $f : G \to \mathbb{C}$ is called an α-class function if for all $g, h \in G$,

$$f(hgh^{-1}) = \frac{\alpha(h,h^{-1})}{\alpha(h,gh^{-1})\alpha(g,h^{-1})} f(g) = \frac{\alpha(h,h^{-1})}{\alpha(h,g)\alpha(hg,h^{-1})} f(g).$$

Let $\mathbb{H}_\alpha$ denote the closed subspace of $L^2(G)$ spanned by square-integrable α-class functions on G. The characters of finite dimensional α-representations of G belong to $\mathbb{H}_\alpha$.

Lemma 2.8 *Let $f \in L^1(G)$. Set*

$$f'(g) = \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f(hgh^{-1})\, \mathrm{d}h.$$

Then, f' is an α-class function on G.

Proof Note that

$$\begin{aligned} f'(i^{-1}gi) &= \int_G \frac{\alpha(h, i^{-1}gih^{-1})\alpha(i^{-1}gi, h^{-1})}{\alpha(h, h^{-1})} f(hi^{-1}gih^{-1})\, \mathrm{d}h \\ &= \int_G \frac{\alpha(h'i, i^{-1}gii^{-1}(h')^{-1})\alpha(i^{-1}gi, i^{-1}(h')^{-1})}{\alpha(h'i, i^{-1}(h')^{-1})} f(h'g(h')^{-1})\, \mathrm{d}h', \end{aligned}$$

where $h' = hi^{-1}$. Then, to show that f' is an α-class function, it suffices to show that

$$\begin{aligned} &\alpha(i^{-1}, i)\alpha(h, gh^{-1})\alpha(g, h^{-1})\alpha(hi, i^{-1}h^{-1}) \\ &\quad = \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, h^{-1})\alpha(hi, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}). \end{aligned} \tag{2.6}$$

Since $\alpha(h, i)\alpha(hi, i^{-1}h^{-1}) = \alpha(h, h^{-1})\alpha(i, i^{-1}h^{-1})$, it suffices to show that

$$\begin{aligned} &\alpha(i^{-1}, i)\alpha(h, gh^{-1})\alpha(g, h^{-1})\alpha(i, i^{-1}h^{-1}) \\ &\quad = \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, i)\alpha(hi, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}). \end{aligned} \tag{2.7}$$

This follows from the following computation.

$$\begin{aligned} \text{RHS} &= \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, gh^{-1})\alpha(i, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}) \\ &= \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, gh^{-1})\alpha(i, i^{-1}gi)\alpha(gi, i^{-1}h^{-1}) \\ &= \alpha(h, gh^{-1})[\alpha(i^{-1}, gi)\alpha(i, i^{-1}gi)][\alpha(g, i)\alpha(gi, i^{-1}h^{-1})] \\ &= \alpha(h, gh^{-1})\alpha(i, i^{-1})\alpha(g, h^{-1})\alpha(i, i^{-1}h^{-1}) = \text{LHS}. \end{aligned} \tag{2.8}$$

The lemma follows. □

Lemma 2.9 *Let $f : G \to \mathbb{C}$ be an α-class function. Then, $f'(g) = \alpha(g, g^{-1})\overline{f(g^{-1})}$ is also an α-class function.*

Proof One needs to show that

$$f'(hgh^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})} f'(g).$$

This is equivalent to

$$\frac{\alpha(hgh^{-1}, hg^{-1}h^{-1})\alpha(h, g^{-1}h^{-1})\alpha(g^{-1}, h^{-1})}{\alpha(h, h^{-1})} = \frac{\alpha(h, h^{-1})\alpha(g, g^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})}.$$

Note that

$$\begin{aligned}
&\alpha(hgh^{-1}, hg^{-1}h^{-1})\alpha(h, g^{-1}h^{-1})\alpha(g^{-1}, h^{-1})\alpha(h, gh^{-1})\alpha(g, h^{-1}) \\
&\quad = \alpha(hgh^{-1}, h)\alpha(hg, g^{-1}h^{-1})\alpha(g^{-1}, h^{-1})\alpha(h, gh^{-1})\alpha(g, h^{-1}) \\
&\quad = \alpha(hgh^{-1}, h)\alpha(hg, g^{-1})\alpha(h, h^{-1})\alpha(h, g)\alpha(hg, h^{-1}) \\
&\quad = \alpha(h, h^{-1})[\alpha(h, g)\alpha(hg, g^{-1})][\alpha(hg, h^{-1})\alpha(hgh^{-1}, h)] \\
&\quad = \alpha(h, h^{-1})\alpha(g, g^{-1})\alpha(h, h^{-1}).
\end{aligned} \tag{2.9}$$

The lemma follows. □

Lemma 2.10 *Let $f : G \to \mathbb{C}$ be an α-class function. Then,*

$$\frac{f(h^{-1}g)}{\alpha(h, h^{-1}g)} = \frac{f(gh^{-1})}{\alpha(gh^{-1}, h)}.$$

Proof Since $gh^{-1} = h(h^{-1}g)h^{-1}$, it suffices to prove

$$\frac{\alpha(h, h^{-1})}{\alpha(h, h^{-1}gh^{-1})\alpha(h^{-1}g, h^{-1})} = \frac{\alpha(gh^{-1}, h)}{\alpha(h, h^{-1}g)}.$$

Note that

$$\begin{aligned}
&\alpha(gh^{-1}, h)\alpha(h, h^{-1}gh^{-1})\alpha(h^{-1}g, h^{-1}) \\
&\quad = \alpha(gh^{-1}, h)\alpha(h, h^{-1}g)\alpha(g, h^{-1}) = \alpha(h, h^{-1}g)\alpha(h^{-1}, h).
\end{aligned} \tag{2.10}$$

The lemma follows. □

With the above preparation, we can now prove Proposition 2.2.

Proof of Proposition 2.2 Let $\overline{\mathcal{A}_\alpha(G)}$ be the closure of $\mathcal{A}_\alpha(G)$ in $L^2(G)$. Since $\mathcal{A}_\alpha(G)$ is stable under the operations in Lemma 2.4, $\overline{\mathcal{A}_\alpha(G)}$ is also stable under those operations. Suppose that $\overline{\mathcal{A}_\alpha(G)} \neq L^2(G)$. Then, $\overline{\mathcal{A}_\alpha(G)}^\perp \neq \{0\}$ and it is stable under the operations in Lemma 2.4. Let $f_0 \in \overline{\mathcal{A}_\alpha(G)}^\perp$ and $f_0 \neq 0$. Let $\mathcal{U}$ be a compact and symmetric neighborhood base at 1 of G. For each $U \in \mathcal{U}$, let $\mathbb{I}_U$ be the characteristic function on U, $|U|$ the Haar measure of U, and

$$f_U(g) = |U|^{-1} \int_G \alpha(g, g_0)\mathbb{I}_U(g_0) f_0(gg_0)\, \mathrm{d}\, g_0.$$

Since $\mathbb{I}_U, f_0 \in L^2(G)$, by the Schwarz inequality, we see that f_U is a continuous function on G. Furthermore, $f_0 = \lim_{U \to \{1\}} f_U$ in $L^2(G)$ (cf. [10, Proposition 2.44]). Because $f_0 \neq 0$, there exist U such that $f_U \neq 0$. Since $\overline{\mathcal{A}_\alpha(G)}$ is G-stable by right translation and the right translation of G on $L^2(G)$ is unitary, $\overline{\mathcal{A}_\alpha(G)}^\perp$ is also G-stable. Hence, linear combinations of right translations of f_0 belong to $\overline{\mathcal{A}_\alpha(G)}^\perp$. By Lemma

2.6, $f_U \in \overline{\mathcal{A}_\alpha(G)}^\perp$. In particular, $\overline{\mathcal{A}_\alpha(G)}^\perp$ contains a nonzero continuous function. Let f_1 be such a function with $f_1(1) \in \mathbb{R} - \{0\}$. Define

$$f_2(g) = \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f_1(hgh^{-1})\,\mathrm{d}\,h.$$

By Lemma 2.8, f_2 is an α-class function. It is easy to see that $f_2(1) \in \mathbb{R} - \{0\}$. Moreover, for any $f' \in \overline{\mathcal{A}_\alpha(G)}$, $f''(g) = \alpha(h^{-1}, g)\alpha(h, h^{-1})^{-1}\alpha(h^{-1}g, h)f'(h^{-1}gh)$ is also an element in $\overline{\mathcal{A}_\alpha(G)}$ by Lemma 2.4. Since

$$\begin{aligned}
\langle f_2, f'\rangle_2 &= \int_G f_2(g)\overline{f'(g)}\,\mathrm{d}\,g \\
&= \int_G\int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f_1(hgh^{-1})\overline{f'(g)}\,\mathrm{d}\,h\,\mathrm{d}\,g \\
&= \int_G\int_G \frac{\alpha(h, h^{-1}g)\alpha(h^{-1}gh, h^{-1})}{\alpha(h, h^{-1})} f_1(g)\overline{f'(h^{-1}gh)}\,\mathrm{d}\,h\,\mathrm{d}\,g \\
&= \int_G\int_G f_1(g)\overline{f''(g)}\,\mathrm{d}\,g\,\mathrm{d}\,h = 0,
\end{aligned} \tag{2.11}$$

we have $f_2 \in \overline{\mathcal{A}_\alpha(G)}^\perp$. Define $f_3(g) = f_2(g) + \alpha(g, g^{-1})\overline{f_2(g^{-1})}$. Then, f_3 is in $\overline{\mathcal{A}_\alpha(G)}^\perp$ and is an α-class function by Lemma 2.9. Moreover, it is easy to check that $f_3(g) = \alpha(g, g^{-1})\overline{f_3(g^{-1})}$. Define

$$K(g, h) = f_3(gh^{-1})\alpha(gh^{-1}, h)^{-1}.$$

Since

$$\alpha(hg^{-1}, g)\alpha(gh^{-1}, h) = \alpha(hg^{-1}, gh^{-1})\alpha(1, h) = \alpha(hg^{-1}, gh^{-1}),$$

one gets $K(g, h) = \overline{K(h, g)}$. Define

$$(Tf)(g) = \int_G K(g, h)f(h)\,\mathrm{d}\,h.$$

Then, T is a nonzero self-adjoint Hilbert–Schmidt operator on $L^2(G)$. Hence, T has a nonzero real eigenvalue γ and the eigenspace $V_\gamma \subset L^2(G)$ is finite dimensional (see for example [3, I.8.4.1 and I.8.5.5]). Let $f \in V_\gamma$. Then,

$$\begin{aligned}
(T(R(g_0)f))(g) &= \int_G K(g, g_1)\alpha(g_1, g_0) f(g_1 g_0)\, \mathrm{d}\, g_1 \\
&= \int_G K(g, g_1 g_0^{-1})\alpha(g_1 g_0^{-1}, g_0) f(g_1)\, \mathrm{d}\, g_1 \\
&= \int_G f_3(g g_0 g_1^{-1}) \frac{\alpha(g_1 g_0^{-1}, g_0)}{\alpha(g g_0 g_1^{-1}, g_1 g_0^{-1})} f(g_1)\, \mathrm{d}\, g_1 \\
&= \int_G f_3(g g_0 g_1^{-1}) \frac{\alpha(g, g_0)}{\alpha(g g_0 g_1^{-1}, g_1)} f(g_1)\, \mathrm{d}\, g_1 \\
&= \int_G K(g g_0, g_1)\alpha(g, g_0) f(g_1)\, \mathrm{d}\, g_1 \\
&= \alpha(g, g_0)(Tf)(g g_0) = \gamma(R(g_0)f)(g).
\end{aligned} \tag{2.12}$$

The eigenspace V_γ is stable under right translation. Now, $R : G \to \mathbf{U}(V_\gamma)$ is a finite dimensional α-representation of G. Let $W \subset V_\gamma$ be an irreducible sub-α-representation and $\{e_1, \ldots, e_n\}$ an orthonormal basis of W. Then,

$$g \mapsto \langle R(g)e_i, e_j\rangle_2 = \int_G \alpha(g_0, g) e_i(g_0 g)\overline{e_j(g_0)}\, \mathrm{d}\, g_0$$

is a matrix coefficient in $\mathcal{A}_\alpha(G)$. Since $f_3 \in \overline{\mathcal{A}_\alpha(G)}^{\perp}$, we have

$$\begin{aligned}
0 &= \int_G f_3(g)\left(\int_G \overline{\alpha(g_0, g)e_j(g_0 g)} e_j(g_0)\, \mathrm{d}\, g_0\right) \mathrm{d}\, g \\
&= \int_G \left(\int_G f_3(g)\overline{\alpha(g_0, g)e_j(g_0 g)}\, \mathrm{d}\, g\right) e_j(g_0)\, \mathrm{d}\, g_0 \\
&= \int_G \left(\int_G f_3(g_0^{-1} g)\overline{\alpha(g_0, g_0^{-1} g)}\,\overline{e_j(g)}\, \mathrm{d}\, g\right) e_j(g_0)\, \mathrm{d}\, g_0 \\
&= \int_G \left(\int_G f_3(g_0^{-1} g)\overline{\alpha(g_0, g_0^{-1} g)} e_j(g_0)\, \mathrm{d}\, g_0\right) \overline{e_j(g)}\, \mathrm{d}\, g \\
&= \int_G \left(\int_G f_3(g g_0^{-1})\overline{\alpha(g g_0^{-1}, g_0)} e_j(g_0)\, \mathrm{d}\, g_0\right) \overline{e_j(g)}\, \mathrm{d}\, g \text{ (Lemma 2.10)} \\
&= \int_G (Te_j)(g)\overline{e_j(g)}\, \mathrm{d}\, g = \gamma\langle e_j, e_j\rangle_2.
\end{aligned} \tag{2.13}$$

Hence, $\gamma = 0$, which is a contradiction. Therefore, we must have $\overline{\mathcal{A}_\alpha(G)} = L^2(G)$. $\square$

2.2 The Fourier Transform and α-Convolution Operators

Fix a multiplier $\alpha \in Z^2(G, \mathbb{S})$, we equip $L^2(G)$ with a structure of $*$-algebra with respect to α and study it in detail. First, we define a convolution and an involution on $L^2(G)$ with respect to the multiplier α. Let $\mu, \nu \in L^2(G)$, the convolution of μ and ν is defined by

$$\begin{aligned}(\nu * \mu)(x) &= \int_G \frac{\alpha(x,s)}{\alpha(s,s^{-1})}\mu(xs)\nu(s^{-1})\,\mathrm{d}\,s \\ &= \int_G \frac{\alpha(t^{-1},x)}{\alpha(t^{-1},t)}\mu(t)\nu(t^{-1}x)\,\mathrm{d}\,t,\end{aligned} \tag{2.14}$$

and the involution is defined by

$$(\mu^*)(x) = \overline{\mu(x^{-1})}\alpha(x,x^{-1}). \tag{2.15}$$

Note that since G is compact, by the Minkowski's integral inequality (cf. [13, Section 6.13]), $\nu * \mu \in L^2(G)$ if $\mu, \nu \in L^2(G)$. We denote this $*$-algebra by $L^2(G,*)$. This is well-defined by the following lemma.

Lemma 2.11 *The following two identities hold.*

1. $(\nu * \mu)^* = \mu^* * \nu^*$.
2. $(\nu * \mu) * \phi = \nu * (\mu * \phi)$.

Proof Since α is a cocyle, it is easy to see that

$$\alpha(x,s)\alpha(xs,s^{-1}x^{-1})\alpha(s^{-1},x^{-1}) = \alpha(x,x^{-1})\alpha(s,s^{-1}). \tag{2.16}$$

The first identity then follows from

$$\begin{aligned}\overline{(\mu^* * \nu^*)(x)} &= \int_G \frac{\alpha(s,s^{-1})}{\alpha(x,s)}\overline{\nu^*(xs)}\,\overline{\mu^*(s^{-1})}\,\mathrm{d}\,s \\ &= \int_G \frac{\alpha(s,s^{-1})}{\alpha(x,s)}\frac{1}{\alpha(xs,s^{-1}x^{-1})}\nu(s^{-1}x^{-1})\frac{1}{\alpha(s,s^{-1})}\mu(s)\,\mathrm{d}\,s \\ &= \alpha(x,x^{-1})^{-1}\int_G \frac{\alpha(s^{-1},x^{-1})}{\alpha(s,s^{-1})}\mu(s)\nu(s^{-1}x^{-1})\,\mathrm{d}\,s \quad (\text{by } 2.16) \\ &= \alpha(x,x^{-1})^{-1}(\nu * \mu)(x^{-1}) \\ &= \overline{((\nu * \mu)^*(x))}.\end{aligned} \tag{2.17}$$

The second identity follows from

$$\begin{aligned}
(\nu*(\mu*\phi))(x) &= \int_G (\mu*\phi)(h)\frac{\alpha(h^{-1},x)}{\alpha(h,h^{-1})}\nu(h^{-1}x)\,\mathrm{d}\,h\\
&= \int_G\int_G \phi(g)\frac{\alpha(g^{-1},h)}{\alpha(g,g^{-1})}\frac{\alpha(h^{-1},x)}{\alpha(h,h^{-1})}\mu(g^{-1}h)\nu(h^{-1}x)\,\mathrm{d}\,g\,\mathrm{d}\,h\\
(h=xs)\quad &= \int_G\int_G \phi(g)\frac{\alpha(g^{-1},xs)}{\alpha(g,g^{-1})}\frac{\alpha((xs)^{-1},x)}{\alpha(xs,(xs)^{-1})}\mu(g^{-1}xs)\nu(s^{-1})\,\mathrm{d}\,g\,\mathrm{d}\,s\\
&= \int_G\int_G \phi(g)\frac{\alpha(g^{-1},x)}{\alpha(g,g^{-1})}\frac{\alpha(g^{-1}x,s)}{\alpha(s,s^{-1})}\mu(g^{-1}xs)\nu(s^{-1})\,\mathrm{d}\,g\,\mathrm{d}\,s \qquad (2.18)\\
&= \int_G \frac{\alpha(g^{-1},x)}{\alpha(g,g^{-1})}\phi(g)\left(\int_G \frac{\alpha(g^{-1}x,s)}{\alpha(s,s^{-1})}\mu(g^{-1}xs)\nu(s^{-1})\,\mathrm{d}\,s\right)\mathrm{d}\,g\\
&= \int_G \frac{\alpha(g^{-1},x)}{\alpha(g,g^{-1})}\phi(g)(\nu*\mu)(g^{-1}x)\,\mathrm{d}\,g\\
&= ((\nu*\mu)*\phi)(x).
\end{aligned}$$

□

Definition 2.12 Let $\eta \in L^2(G)$. We call the operator $O_\eta : L^2(G) \to L^2(G)$ ($\phi \mapsto \eta * \phi$) an α-convolution operator. Hence,

$$O_\eta(\phi)(h) = \int_G \phi(g)\frac{\alpha(g^{-1},h)}{\alpha(g,g^{-1})}\eta(g^{-1}h)\,\mathrm{d}\,g.$$

By the associativity of $*$, we have the following identity.

Lemma 2.13 $O_{\nu*\mu} = O_\nu \circ O_\mu$.

Denote by M^* the conjugate transpose of a matrix M. For the vector space $M_n(\mathbb{C})$ of $n \times n$ matrices, we fix an inner product by $\langle A, B\rangle = n\,\mathrm{Tr}(A^*B)$. Fix $[\pi] \in \widehat{G}_\alpha$ and an orthonormal basis $\{e_i^\pi \mid 1 \le i \le d_\pi\}$ of V_π. The Fourier transform with respect to π is the linear map

$$\begin{aligned}
F_\pi : L^2(G) &\to \mathrm{End}(V_\pi) \cong M_{d_\pi}(\mathbb{C})\\
f &\mapsto \widehat{f_\pi} := \int_G f(g)\pi(g)^*\,\mathrm{d}\,g = \int_G f(g)\pi(g)^{-1}\,\mathrm{d}\,g.
\end{aligned} \qquad (2.19)$$

Define $F : L^2(G) \to \widehat{\bigoplus}_{[\pi]\in\widehat{G}_\alpha} M_{d_\pi}(\mathbb{C})$ by $F := \widehat{\bigoplus}_{[\pi]\in\widehat{G}_\alpha} F_\pi$. This is an isometry by the Peter–Weyl theorem. Moreover, it is an isomorphism of $*$-algebras by the following lemma.

Lemma 2.14 *Denote by R_ξ the right translation with respect to α. The Fourier transform satisfies the following properties.*

1. $\widehat{(R_\xi f)}_\pi = \pi(\xi)\widehat{f_\pi}$.

2. $\widehat{(f^*)}_\pi = (\widehat{f}_\pi)^*$.
3. $\widehat{(\nu * \mu)}_\pi = \widehat{\nu}_\pi \widehat{\mu}_\pi$.

Proof From $\pi(g)\pi(\xi) = \alpha(g,\xi)\pi(g\xi)$, we have $\alpha(g,\xi)\pi(g)^{-1} = \pi(\xi)\pi(g\xi)^{-1}$. Then,

$$\begin{aligned}\widehat{(R_\xi f)} &= \int_G (R_\xi f)(g)\pi(g)^{-1}\,\mathrm{d}\,g \\ &= \int_G \alpha(g,\xi) f(g\xi)\pi(g)^{-1}\,\mathrm{d}\,g \\ &= \pi(\xi)\int_G f(g\xi)\pi(g\xi)^{-1}\,\mathrm{d}\,g = \pi(\xi)\widehat{f}_\pi.\end{aligned} \tag{2.20}$$

Note that

$$\begin{aligned}\widehat{(f^*)}_\pi = \int_G f^*(g)\pi(g)^{-1}\,\mathrm{d}\,g &= \int_G \overline{f(g^{-1})}\pi(g^{-1})\,\mathrm{d}\,g \\ &= \left(\int_G f(g^{-1})\pi(g^{-1})^{-1}\,\mathrm{d}\,g\right)^* = (\widehat{f}_\pi)^*.\end{aligned} \tag{2.21}$$

This is the second identity. The third identity follows from

$$\begin{aligned}\widehat{\nu}_\pi\widehat{\mu}_\pi &= \int_G \nu(g)\pi(g)^{-1}\,\mathrm{d}\,g \int_G \mu(g)\pi(g)^{-1}\,\mathrm{d}\,g \\ &= \int_G\int_G \nu(g)\pi(g)^{-1}\mu(t)\pi(t)^{-1}\,\mathrm{d}\,g\,\mathrm{d}\,t \\ &= \int_G\int_G \nu(t^{-1}g)\mu(t)\pi(t^{-1}g)^{-1}\pi(t)^{-1}\,\mathrm{d}\,g\,\mathrm{d}\,t \\ &= \int_G\int_G \nu(t^{-1}g)\mu(t)\frac{1}{\alpha(t,t^{-1}g)}\pi(g)^{-1}\,\mathrm{d}\,g\,\mathrm{d}\,t \\ &= \int_G (\nu*\mu)(g)\pi(g)^{-1}\,\mathrm{d}\,g.\end{aligned} \tag{2.22}$$

□

As a consequence of Schur's Lemma (cf. [6, Lemma 2.1] and [21, Chapter 4]), we have

$$F_{\pi'}(\pi_{ij}) = \begin{cases} \frac{1}{d_\pi} E_{d_\pi, ji} & \text{if } [\pi'] = [\pi], \\ O & \text{otherwise.} \end{cases}$$

Here, π_{ij} is the matrix coefficient defined right before Sect. 2.1, $E_{d,ij}$ is the $d \times d$ matrix with (i, j)-entry 1 and other entries 0. An easy computation shows that the inverse of F is given by $E = \widehat{\bigoplus}_{[\pi]\in\widehat{G}_\alpha} E_\pi$, where

$$\begin{aligned} E_\pi : M_{d_\pi}(\mathbb{C}) &\to L^2(G,\alpha) \\ A &\mapsto (g \mapsto \mathrm{Tr}(A\pi(g))). \end{aligned}$$

The isomorphism of $*$-algebras $L^2(G,*) \to \widehat{\bigoplus}_{[\pi]\in\widehat{G}_\alpha} M_{d_\pi}(\mathbb{C})$ allows us to study $L^2(G,*)$ via the matrix algebra. An immediate consequence, which is useful in the study of central frames, is the following result.

Lemma 2.15 *The center of $L^2(G,*)$ is spanned by χ_π, where χ_π is the character of $[\pi] \in \widehat{G}_\alpha$.*

Proof This follows from the fact that the center of $M_n(\mathbb{C})$ is the set of scalar matrices. □

Remark 2.16 Fix a matrix $A \in M_{d_\pi}(\mathbb{C})$. Let $f : G \to \mathbb{C}$ be the function defined by

$$f(g) = \mathrm{Tr}(A\pi(g)) = \mathrm{Tr}(\pi(g)A).$$

Then,

$$F_{\pi'}(f(g)) = \begin{cases} A, & \text{if } [\pi'] = [\pi], \\ O, & \text{otherwise.} \end{cases}$$

In particular, we may choose $A = vv^*$, where $v \in V_\pi$. Then,

$$f(g) = \mathrm{Tr}(\pi(g)vv^*) = \mathrm{Tr}(v^*\pi(g)v) = \langle \pi(g)v, v\rangle.$$

This is closely related to maximal spanning vectors (cf. Lemma 4.1).

Remark 2.17 [A remark on the left translation] There is a corresponding left α-translation as well given by

$$L_g(f)(g') = \frac{\alpha(g,g^{-1})}{\alpha(g^{-1},g')} f(g^{-1}g').$$

By Eq. (2.16), $L : G \to \mathbf{U}(L^2(G))$ is an α-representation.

It is easy to check that left α-translation does not behave nicely under the Fourier transform F_π. But it behaves well under the Fourier transform with multiplier $1/\alpha$ and this has important applications (e.g., Lemma 2.18(4), Sect. 3.3).

For our application in frame theory, it is convenient to use another $*$-algebra structure on $L^2(G)$, which is constructed as above with α being replaced by $1/\alpha$. More precisely, the convolution is defined by

$$\begin{aligned}(\nu \diamond \mu)(x) &= \int_G \frac{\alpha(s, s^{-1})}{\alpha(x, s)} \mu(xs)\nu(s^{-1})\,\mathrm{d}s \\ &= \int_G \frac{\alpha(t^{-1}, t)}{\alpha(t^{-1}, x)} \mu(t)\nu(t^{-1}x)\,\mathrm{d}t,\end{aligned} \tag{2.23}$$

and the involution is defined by

$$(\mu^{\diamond})(x) = \overline{\mu(x^{-1})}\alpha(x, x^{-1})^{-1}. \tag{2.24}$$

Denote this $*$-algebra by $L^2(G, \diamond)$.

Note that there is a correspondence between $\widehat{G}_\alpha$ and $\widehat{G}_{1/\alpha}$ given by $\pi \mapsto \pi^* \cong \overline{\pi}$, where π^* is the dual representation of π, $\overline{\pi}$ is the complex conjugation of π, and they are isomorphic as they have the same character. The Fourier transform with respect to $1/\alpha$ and $\overline{\pi}$ is defined by

$$\begin{aligned}\mathcal{F}_{\overline{\pi}} : L^2(G) &\to M_{d_{\overline{\pi}}}(\mathbb{C}) \\ f &\mapsto \widehat{f}_{\overline{\pi}} := \int_G f(g)\overline{\pi}(g)^{-1}\,\mathrm{d}g.\end{aligned} \tag{2.25}$$

We collect the properties of $L^2(G, \diamond)$ and $\mathcal{F}_{\overline{\pi}}$. Denote by R' the right regular $1/\alpha$-representation of G on $L^2(G)$.

Lemma 2.18 *The following claims hold.*

1. $(\nu \diamond \mu)^{\diamond} = \mu^{\diamond} \diamond \nu^{\diamond}$.
2. $(\nu \diamond \mu) \diamond \phi = \nu \diamond (\mu \diamond \phi)$.
3. $\widehat{(R'_\xi f)}_{\overline{\pi}} = \overline{\pi}(\xi)\widehat{f}_{\overline{\pi}}$.
4. $\widehat{(L_\xi f)}_{\overline{\pi}} = \widehat{f}_{\overline{\pi}}\overline{\pi}(\xi)^{-1}$.
5. $\widehat{(f^{\diamond})}_{\overline{\pi}} = (\widehat{f}_{\overline{\pi}})^*$.
6. $\widehat{(\nu \diamond \mu)}_{\overline{\pi}} = \widehat{\nu}_{\overline{\pi}}\widehat{\mu}_{\overline{\pi}}$.
7. $\mathcal{F} = \widehat{\bigoplus}_{[\pi]}\mathcal{F}_{\overline{\pi}} : L^2(G, \diamond) \to \widehat{\bigoplus}_{[\pi]} M_{d_{\overline{\pi}}}(\mathbb{C})$ *is an isomorphism. The inverse of* $\mathcal{F}$ *is given by* $\mathcal{E} = \widehat{\bigoplus}\mathcal{E}_{\overline{\pi}}$, *where*

$$\begin{aligned}\mathcal{E}_{\overline{\pi}} : M_{d_{\overline{\pi}}}(\mathbb{C}) &\to L^2(G, \diamond) \\ A &\mapsto (g \mapsto \mathrm{Tr}(A\overline{\pi}(g))).\end{aligned} \tag{2.26}$$

8. *The center of* $L^2(G, \diamond)$ *is spanned by* $\overline{\chi}_\pi = \chi_{\overline{\pi}}$.

3 The (G, α)-Frames

In this section, for G compact, we study (G, α)-frames. In particular, we give a characterization of the Gramians of (G, α)-frames and classify tight (G, α)-frames.

3.1 The Gramian

Let V be a finite dimensional Hilbert space (cf. Remark 3.9). Let $\Phi = \{\phi_g \mid g \in G\}$ be a continuous frame for V indexed by elements of G. The Gramian of Φ is given by

$$\begin{aligned} L^2(G) &\to L^2(G) \\ \varphi &\mapsto \left(h \mapsto \int_G \varphi(g)\langle \phi_g, \phi_h\rangle \, \mathrm{d}\, g\right). \end{aligned} \tag{3.1}$$

In particular, if Φ is a (G, α)-frame, i.e., $\Phi = \{\phi_g := g\phi_1 \mid g \in G\}$, then the Gramian $\mathrm{Gram}(\Phi)$ is the operator given by

$$\begin{aligned} L^2(G) &\to L^2(G) \\ \varphi &\mapsto \left(h \mapsto \int_G \varphi(g)\frac{\alpha(g, g^{-1})}{\alpha(g^{-1}, h)}\langle \phi_1, \phi_{g^{-1}h}\rangle \, \mathrm{d}\, g\right) = \eta \diamond \varphi. \end{aligned} \tag{3.2}$$

Here, $\eta : G \to \mathbb{C}$ is defined by $\eta(g) = \langle \phi_1, \phi_g\rangle$. One sees that the Gramian of a (G, α)-frame is an $1/\alpha$-convolution operator. The following result shows that the converse also holds. It is a projective version of [22, Theorem 4.1]. See also [7, Theorem 2.2], [24, Theorem 5.2] and [15, Theorem 3.2]. This result gives a crude description of the Gramians of (G, α)-frames. A more explicit description is given by Proposition 3.10 and Corollary 3.12.

Theorem 3.1 *Let G be a compact group and V be a finite dimensional Hilbert space. Assume that $\Phi = \{\phi_g \mid g \in G\}$ is a continuous frame for V indexed by elements of G. If* $\mathrm{Gram}(\Phi)$ *is an $1/\alpha$-convolution operator, then there exists an α-representation $\pi : G \to \mathbf{U}(V)$ such that $\phi_g = \pi(g)\phi_1$.*

Proof The proof is similar to that of [7, Theorem 2.2]. Let $\eta : G \to \mathbb{C}$ be the function such that $\mathrm{Gram}(\Phi)(\varphi) = \eta \diamond \varphi$. It suffices to construct a projective representation $U : G \to \mathbf{U}(V)$ with multiplier α such that $U_g\phi_h = \alpha(g, h)\phi_{gh}$.

Let $\tilde{\Phi} = \{\tilde{\phi}_g := S^{-1}\phi_g \mid g \in G\}$ be the canonical dual frame of Φ (cf. [5, Definition 1.19, Proposition 1.13] and [20, Section 3]). Here, S is the frame operator of Φ. For any $v \in V$, we have

$$v = \int_{h_1 \in G} \langle v, \tilde{\phi}_{h_1}\rangle \phi_{h_1} \, \mathrm{d}\, h_1 = \int_{h_1 \in G} \langle v, \phi_{h_1}\rangle \tilde{\phi}_{h_1} \, \mathrm{d}\, h_1. \tag{3.3}$$

Define $U : G \to \mathrm{GL}(V)$ by

$$U_g(v) = \int_{h_1 \in G} \langle v, \phi_{h_1}\rangle \alpha(g, h_1)\tilde{\phi}_{gh_1} \, \mathrm{d}\, h_1.$$

This U satisfies the following properties.

1. For any $g, h, h_1 \in G$, we have

$$\langle \phi_h, \phi_{h_1} \rangle = \langle \phi_{gh}, \phi_{gh_1} \rangle \alpha(g,h)\alpha(g,h_1)^{-1}. \tag{3.4}$$

Indeed, consider both sides as functions on h, it suffices to check that

$$\int_G \varphi(h)\langle \phi_h, \phi_{h_1} \rangle \,\mathrm{d}\, h = \int_G \varphi(h)\langle \phi_{gh}, \phi_{gh_1} \rangle \alpha(g,h)\alpha(g,h_1)^{-1} \,\mathrm{d}\, h \tag{3.5}$$

for any $\varphi \in L^2(G)$. Let $\varphi' \in L^2(G)$ given by

$$\varphi'(h) = \varphi(g^{-1}h)\alpha(g, g^{-1}h)\alpha(g,h_1)^{-1}.$$

Then, the claim follows from

$$\begin{aligned}
&\int_G \varphi(h)\langle \phi_{gh}, \phi_{gh_1} \rangle \alpha(g,h)\alpha(g,h_1)^{-1} \,\mathrm{d}\, h \\
&= \int_G \varphi(g^{-1}h)\langle \phi_h, \phi_{gh_1} \rangle \alpha(g, g^{-1}h)\alpha(g,h_1)^{-1} \,\mathrm{d}\, h \\
&= (\eta \diamond \varphi')(gh_1) = \int_G \frac{\alpha(i,i^{-1})}{\alpha(gh_1,i)} \varphi'(gh_1 i)\eta(i^{-1}) \,\mathrm{d}\, i \\
&= \int_G \frac{\alpha(i,i^{-1})}{\alpha(gh_1,i)} \frac{\alpha(g,h_1 i)}{\alpha(g,h_1)} \varphi(h_1 i)\eta(i^{-1}) \,\mathrm{d}\, i \\
&= \int_G \frac{\alpha(i,i^{-1})}{\alpha(h_1,i)} \varphi(h_1 i)\eta(i^{-1}) \,\mathrm{d}\, i \\
&= (\eta \diamond \varphi)(h_1) = \int_G \varphi(h)\langle \phi_h, \phi_{h_1} \rangle \,\mathrm{d}\, h.
\end{aligned}$$

2. We have

$$\begin{aligned}
U_g(\phi_h) &= \int_G \langle \phi_h, \phi_{h_1} \rangle \alpha(g,h_1)\tilde{\phi}_{gh_1} \,\mathrm{d}\, h_1 \\
&= \int_G \langle \phi_{gh}, \phi_{gh_1} \rangle \alpha(g,h)\tilde{\phi}_{gh_1} \,\mathrm{d}\, h_1 = \alpha(g,h)\phi_{gh}.
\end{aligned}$$

3. For any $\phi_{h_1}, \phi_{h_2} \in \Phi$,

$$\langle U_g \phi_{h_1}, U_g \phi_{h_2} \rangle = \frac{\alpha(g,h_1)}{\alpha(g,h_2)} \frac{\alpha(gh_1,(gh_1)^{-1})}{\alpha((gh_1)^{-1}, gh_2)} \eta(h_1^{-1}h_2).$$

We claim that $U_g \in \mathbf{U}(V)$, i.e., $\langle U_g \phi_{h_1}, U_g \phi_{h_2} \rangle = \langle \phi_{h_1}, \phi_{h_2} \rangle$. To prove this, it suffices to check that

$$\frac{\alpha(g,h_1)}{\alpha(g,h_2)} \frac{\alpha(gh_1,(gh_1)^{-1})}{\alpha((gh_1)^{-1}, gh_2)} = \frac{\alpha(h_1,h_1^{-1})}{\alpha(h_1^{-1},h_2)}.$$

Note that

$$\alpha(g, h_1)\alpha(gh_1, (gh_1)^{-1}) = \alpha(g, g^{-1})\alpha(h_1, h_1^{-1}g^{-1})$$

and

$$\alpha((gh_1)^{-1}, gh_2)\alpha(g, h_2) = \alpha(h_1^{-1}g^{-1}, g)\alpha(h_1^{-1}, h_2),$$

it suffices to check that

$$\alpha(g, g^{-1})\alpha(h_1, h_1^{-1}g^{-1}) = \alpha(h_1^{-1}g^{-1}, g)\alpha(h_1, h_1^{-1}). \tag{3.6}$$

Multiplying both sides with $\alpha(h_1^{-1}, g^{-1})$, it is easy to see that Eq. (3.6) holds, hence, the claim follows.

4. For any $\phi_h \in \Phi$,

$$\begin{aligned} U_{g_1}(U_{g_2}\phi_h) &= U_{g_1}(\alpha(g_2, h)\phi_{g_2h}) \\ &= \alpha(g_2, h)\alpha(g_1, g_2h)\phi_{g_1g_2h} \\ &= \alpha(g_1, g_2)(\alpha(g_1g_2, h)\phi_{g_1g_2h}) = \alpha(g_1, g_2)U_{g_1g_2}\phi_h. \end{aligned}$$

5. For any $v \in V$, the map $g \mapsto U_g(v)$ is a measurable function on G as Φ is a continuous frame.

Hence, $U : G \to \mathbf{U}(V)$ defines a projective representation with multiplier α. It is the desired projective representation and the theorem follows. □

3.2 The Tight (G, α)-Frames

Let $\Phi = \{\phi_g \mid g \in G\}$ be a (G, α)-frame. Let $\eta : G \to \mathbb{C}$ be the function $g \mapsto \langle\phi_1, \phi_g\rangle$. Let $O_\eta := \mathrm{Gram}(\Phi) : L^2(G) \to L^2(G)$ be the associated Gramian and $F_\pi(\eta) \in M_{d_\pi}(\mathbb{C})$ be the associated matrix. We have the following result.

Lemma 3.2 *With the notation as above, the following conditions are equivalent.*

1. Φ *is a tight* (G, α)*-frame.*
2. $\eta \diamond \eta = \lambda\eta$. *Here,* $\lambda \in \mathbb{R}_{>0}$.
3. $O_\eta \circ O_\eta = \lambda O_\eta$. *Here,* $\lambda \in \mathbb{R}_{>0}$.
4. $(F_\pi(\eta))^2 = \lambda F_\pi(\eta)$ *for any* $[\pi] \in \widehat{G}_\alpha$. *Here,* $\lambda \in \mathbb{R}_{>0}$.

Proof Let $T : L^2(G) \to V$ and $T^* : V \to L^2(G)$ be the synthesis operator and analysis operator, respectively. Then, the frame operator S is $T \circ T^*$ and the Gramian is $T^* \circ T$. Note that Φ is tight if and only if $S = T \circ T^* = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{R}_{>0}$, the equivalence between (1) and (3) is obvious. The other equivalences follow from Lemmas 2.13, 2.14, 2.18. □

As the projective representations of compact groups behave as the projective representations of finite groups, the argument in [7, Sections 2.2, 2.3], which uses the equivalence of the first condition and the third condition in Lemma 3.2, works well for (G, α)-frames with G compact. Let (π, V, α) be a finite dimensional α-representation of G. Write $V = \oplus_{i \in I} V_i$ as an orthogonal direct sum of irreducible α-representations. Let $v \in V$ be a nonzero vector. Write $v = \sum_{i \in I} v_i$ with $v_i \in V_i$ $(i \in I)$. We then have the following result (cf. [7, Section 2], [14, Corollary 5.8]). To ease notation, in the rest of this section, we write gv for $\pi(g)v$.

Theorem 3.3 *With the notation as above, $Gv = \{gv \mid g \in G\}$ is a tight (G, α)-frame for V if and only if the following two conditions hold*:

1. $\frac{\|v_i\|^2}{\|v_j\|^2} = \frac{\dim V_i}{\dim V_j}$ *for all* $i, j \in I$;
2. $\langle \sigma v_i, v_j \rangle = 0$ *for any* $\sigma \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_i, V_j)$ *and* $i \neq j$.

We adapt the idea in [7, Section 2] and divide the proof of Theorem 3.3 into several lemmas. We provide a complete proof for completeness and for the proof of Proposition 3.8. Note that by using the equivalence of the first condition and the fourth condition in Lemma 3.2, one may obtain another proof by arguing as [24, Theorem 8.1].

Lemma 3.4 *Let (π_i, V_i) $(i = 1, 2)$ be α-representations of G. Let $v_i \in V_i$ be a nonzero vector $(i = 1, 2)$. Define $S : V_1 \to V_2$ by*

$$\begin{aligned} S := S_{v_1, v_2} : V_1 &\to V_2 \\ u &\mapsto \int_G \langle u, gv_1 \rangle g v_2 \, \mathrm{d}g. \end{aligned}$$

Then, $S(hu) = hS(u)$ for all $h \in G$ and $u \in V_1$, i.e., $S \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_1, V_2)$.

Proof The lemma follows from the following computation.

$$\begin{aligned} S(hu) &= \int_G \langle hu, gv_1 \rangle g v_2 \, \mathrm{d}g \\ &= \int_G \langle h^{-1}(hu), h^{-1}(gv_1) \rangle g v_2 \, \mathrm{d}g \\ &= \int_G \frac{\alpha(h^{-1}, h)}{\alpha(h^{-1}, g)\alpha(h, h^{-1}g)} \langle u, (h^{-1}g)v_1 \rangle h((h^{-1}g)v_2) \, \mathrm{d}g \\ &= h\left(\int_G \langle u, (h^{-1}g)v_1 \rangle (h^{-1}g) v_2 \, \mathrm{d}g \right) = h(S(u)). \end{aligned}$$

□

Lemma 3.5 *Let (π, V) be an irreducible α-representation of G and $v \in V$ be a nonzero vector. Then, the family $\{gv \mid g \in G\}$ (in other words, the map $g \mapsto gv$) is a tight (G, α)-frame for V with frame bounds $A = B = \frac{\|v\|^2}{\dim V}$.*

Proof The map $g \mapsto gv$ is measurable by the definition of α-representations. Define

$$\begin{aligned} S_v : V &\to V \\ u &\mapsto \int_G \langle u, gv \rangle g v \, \mathrm{d}g. \end{aligned}$$

By Lemma 3.4, $S_v \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V, V)$. By Schur's Lemma, $S_v = \lambda_v \,\mathrm{id}_V$. Moreover, for any $u \in V$,

$$\begin{aligned}\int_G |\langle u, gv\rangle|^2 \,\mathrm{d}\, g &= \int_G \langle u, gv\rangle\langle gv, u\rangle \,\mathrm{d}\, g \\ &= \int_G \langle\langle u, gv\rangle gv, u\rangle \,\mathrm{d}\, g \\ &= \left\langle \int_G \langle u, gv\rangle gv \,\mathrm{d}\, g, u \right\rangle = \lambda_v \|u\|^2.\end{aligned} \tag{3.7}$$

To finish the proof, it suffices to show that $\lambda_v = \frac{\|v\|^2}{\dim V}$. Fix an orthonormal basis $\{w_j\}_{j\in J}$ of V. By Eq. (3.7), $\lambda_v = \int_G |\langle w_j, gv\rangle|^2 \,\mathrm{d}\, g$ for all $j \in J$. Therefore,

$$\begin{aligned}\lambda_v \dim V &= \sum_{j\in J} \int_G |\langle w_j, gv\rangle|^2 \,\mathrm{d}\, g \\ &= \int_G \sum_{j\in J} |\langle w_j, gv\rangle|^2 \,\mathrm{d}\, g = \int_G \|gv\|^2 \,\mathrm{d}\, g = \mathrm{vol}(G)\|v\|^2 = \|v\|^2.\end{aligned}$$

The lemma then follows and S_v is the frame operator of the attached frame. □

Lemma 3.6 *In the situation as in Theorem 3.3, $Gv = \{gv \mid g \in G\}$ is a tight (G, α)-frame for V if and only if the following two conditions hold*:

1. $\frac{\|v_i\|^2}{\|v_j\|^2} = \frac{\dim V_i}{\dim V_j}$ *for all* $i,\ j \in I$;
2. $\int_G \langle v_i, gv_i\rangle gv_j \,\mathrm{d}\, g = 0$ *for all* $i \neq j$.

Proof Note that Gv is tight if and only if there exists $\lambda \in \mathbb{R}_{>0}$ with $\int_{g\in G} \langle f, gv\rangle gv = \lambda f$ for all $f \in V$. Assume that Gv is tight. Take $f_i \in V_i$, then

$$\begin{aligned}\int_G \langle f_i, gv\rangle gv \,\mathrm{d}\, g &= \int_G \langle f_i, gv_i\rangle gv \,\mathrm{d}\, g \\ &= \int_G \langle f_i, gv_i\rangle gv_i \,\mathrm{d}\, g + \int_G \sum_{j\neq i} \langle f_i, gv_i\rangle gv_j \,\mathrm{d}\, g.\end{aligned} \tag{3.8}$$

We must have

$$\begin{cases} \int_G \langle f_i, gv_i\rangle gv_i \,\mathrm{d}\, g = \lambda_i f_i, \\ \int_G \langle f_i, gv_i\rangle gv_j \,\mathrm{d}\, g = 0 \text{ for any } j \neq i. \end{cases} \tag{3.9}$$

The second condition holds by taking $f_i = v_i$. Since V_i is irreducible and $v_i \neq 0$, by Lemma 3.5, we have $\lambda_i = \frac{\|v_i\|^2}{\dim V_i}$. Moreover, $\lambda_i = \lambda = \lambda_j$. Hence, $\frac{\|v_i\|^2}{\|v_j\|^2} = \frac{\dim V_i}{\dim V_j}$ for all $i,\ j \in I$. We obtain the first condition as well.

Conversely, assume that conditions (1) and (2) hold. From Eqs. (3.8) and (3.9), to show that $Gv = \{gv \mid g \in G\}$ is a tight (G, α)-frame for V, it suffices to show that

$\int_G \langle f_i, gv_i\rangle gv_j \, \mathrm{d}\, g = 0$ for any $f_i \in V_i$ and $j \neq i$. First, take $f_i = hv_i$ for $h \in G$, by Lemma 3.4,

$$\int_G \langle f_i, gv_i\rangle gv_j \, \mathrm{d}\, g = \int_G \langle hv_i, gv_i\rangle gv_j \, \mathrm{d}\, g = h(\int_G \langle v_i, (h^{-1}g)v_i)\rangle (h^{-1}g)v_j \, \mathrm{d}\, g) = 0.$$

The lemma then follows since $\int_G \langle f_i, gv_i\rangle gv_j \, \mathrm{d}\, g$ is linear as a map on f_i. □

Lemma 3.7 *Let* (π_i, V_i) $(i = 1, 2)$ *be two irreducible* α*-representations of* G*. Let* $v_i \in V_i$ *be a nonzero vector* $(i = 1, 2)$*. Define* $S : V_1 \to V_2$ *by*

$$\begin{aligned} S := S_{v_1,v_2} : V_1 &\to V_2 \\ f &\mapsto \int_G \langle f, gv_1\rangle gv_2 \, \mathrm{d}\, g. \end{aligned}$$

Then, $S = 0$ *if* V_1 *and* V_2 *are not isomorphic in* Rep_G^α*. If* $\sigma : V_1 \to V_2$ *is an isomorphism in* Rep_G^α*, then*

$$S(f) = \frac{\|v_1\|^2}{(\dim V_1)\|\sigma v_1\|^2} \langle v_2, \sigma v_1\rangle \sigma(f) \textit{ for all } f \in V_1.$$

Proof By Lemma 3.4, $S \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_1, V_2)$. By Schur's Lemma, $S = 0$ if V_1 and V_2 are not isomorphic. Now, suppose that $\sigma : V_1 \to V_2$ is an isomorphism of α-representations. Then, S must be $\lambda \cdot \sigma$ for some constant λ. Take $f = v_1$, we have

$$\begin{aligned} \lambda\|\sigma v_1\|^2 = \langle Sv_1, \sigma v_1\rangle &= \left\langle \int_G \langle v_1, gv_1\rangle gv_2 \, \mathrm{d}\, g, \sigma v_1 \right\rangle = \int_G \langle v_1, gv_1\rangle \langle gv_2, \sigma v_1\rangle \, \mathrm{d}\, g \\ &= \int_G \langle v_1, gv_1\rangle \alpha(g^{-1}, g)\langle v_2, g^{-1}(\sigma v_1)\rangle \, \mathrm{d}\, g \\ &= \int_G \langle g^{-1}v_1, v_1\rangle \langle v_2, \sigma(g^{-1}v_1)\rangle \, \mathrm{d}\, g \\ &= \left\langle v_2, \sigma\left(\int_G \langle v_1, gv_1\rangle gv_1 \, \mathrm{d}\, g\right)\right\rangle \\ &= \frac{\|v_1\|^2}{\dim V_1}\langle v_2, \sigma v_1\rangle \quad \text{(by Lemma 3.5)}. \end{aligned}$$

The lemma follows. □

Theorem 3.3 follows easily from Lemmas 3.6 and 3.7. This result is stronger than [14, Corollary 5.8], which treats the multiplicity free case. Here, an α-representation is multiplicity free if all of its irreducible components have multiplicity one.

Using the above idea, we prove a result for general (G, α)-frames (not necessarily tight), which has an application in the study of maximal spanning vectors in Sect. 4.

Proposition 3.8 *Let* V *be a finite dimensional* α*-representation of* G*. Write* $V = \oplus_{i\in I} V_i$ *as an orthogonal direct sum of irreducible* α*-representations. Let* $v \in V$ *be a* *nonzero vector. Write* $v = \sum_{i \in I} v_i$ *with* $v_i \in V_i$ $(i \in I)$*. Assume that* V *is multiplicity free, i.e.,* V_i *and* V_j *are not isomorphic for any* $i \neq j$*. Then,* v *is a frame vector for* V *if and only if* v_i *is a frame vector for* V_i*.*

Proof Assume that v is a frame vector for V. Let S be the frame operator of the frame $\{gv \mid g \in G\}$. Then, by Lemma 3.4, $S \in \mathrm{Hom}_{\mathrm{Rep}_G^{\alpha}}(V, V)$. In particular, for each $i \in I$, $S|_{V_i} = \lambda_i \,\mathrm{id}_{V_i}$ $(\lambda_i \neq 0)$. Then, for $f_i \in V_i$, we have

$$\begin{aligned}\lambda_i f_i = S(f_i) &= \int_G \langle f_i, gv\rangle gv \,\mathrm{d}\, g \\ &= \int_G \langle f_i, gv_i\rangle gv_i \,\mathrm{d}\, g + \sum_{j \neq i} \int_G \langle f_i, gv_i\rangle gv_j \,\mathrm{d}\, g.\end{aligned}$$

We then have $\lambda_i = \frac{\|v_i\|^2}{\dim V}$ and v_i is a frame vector for V_i.

Conversely, assume that v_i is a frame vector for V_i $(i \in I)$. By Lemma 3.7, we have $\int_G \langle v_i, gv_i\rangle gv_j \,\mathrm{d}\, g = 0$ for all $i \neq j$. The argument in Lemma 3.6 then shows that $\int_G \langle f_i, gv_i\rangle gv_j \,\mathrm{d}\, g = 0$ for all $f_i \in V_i$ and $i \neq j$. Then, for any $f \in V$, let f_i be the projection of f in V_i, we have

$$\begin{aligned}\int_G |\langle f, gv\rangle|^2 \,\mathrm{d}\, g &= \int_G |\sum_{i \in I} \langle f_i, gv_i\rangle|^2 \,\mathrm{d}\, g \\ &= \sum_{i \in I} \int_G |\langle f_i, gv_i\rangle|^2 \,\mathrm{d}\, g + \sum_{i \neq j} \int_G \langle f_i, gv_i\rangle \langle gv_j, f_j\rangle \,\mathrm{d}\, g \\ &= \sum_{i \in I} \int_G |\langle f_i, gv_i\rangle|^2 \,\mathrm{d}\, g + \sum_{i \neq j} \left\langle \int_G \langle f_i, gv_i\rangle gv_j \,\mathrm{d}\, g, f_j \right\rangle \\ &= \sum_{i \in I} \int_G |\langle f_i, gv_i\rangle|^2 \,\mathrm{d}\, g.\end{aligned}$$

Then, $g \mapsto gv$ is a frame for V with frame bounds $A = \min_{i \in I}\{\frac{\|v_i\|^2}{\dim V}\}$ and $B = \max_{i \in I}\{\frac{\|v_i\|^2}{\dim V}\}$. □

Remark 3.9 We give an explanation that justifies the assumption $\dim V < \infty$ (cf. [14, Theorem 5.2]). Let (π, V) be an α-representation and $v \in V$ be a (G, α)-frame vector with lower frame bound A. Assume that $V = \oplus_{i \in I} V_i^{\oplus r_i}$, where $I \subset \widehat{G}_\alpha$, $r_i \in \mathbb{Z}_{>0}$, V_i is an irreducible α-representation with dimension d_i, $V_i \ncong V_j$ if $i \neq j$. Let v_{ij} $(i \in I, 1 \leq j \leq r_i)$ be the projection of v in the j-th component of $V_i^{\oplus r_i}$. We claim that $r_i \leq d_i$ for all $i \in I$ and I is a finite set.

Indeed, as v is a (G, α)-frame vector, we know that $\overline{\mathrm{Span}}\{gv \mid g \in G\} = V$. Suppose that $r_i > d_i$ for some $i \in I$, then $v_{i1}, v_{i2}, \ldots, v_{ir_i}$ are linearly dependent. Therefore, the projection of $\overline{\mathrm{Span}}\{gv \mid g \in G\}$ to $V_i^{\oplus r_i}$ is not onto and this is a contradiction.

Moreover, let f be any element in the j-th component of $V_i^{\oplus r_i}$. Then,

$$A\|f\|^2 \leq \int_G |\langle f, gv\rangle|^2 \,\mathrm{d}\,g = \int_G |\langle f, gv_{ij}\rangle|^2 \,\mathrm{d}\,g = \frac{\|v_{ij}\|^2}{d_i}\|f\|^2.$$

Therefore, $\inf\{\|v_{ij}\|^2 \mid i \in I,\ 1 \leq j \leq r_i\} \geq A > 0$. Hence, I is a finite set.

3.3 Gramian and Left Regular Translation

Combining the results in Sects. 3.1 and 3.2, in this section we show that an α-representation (π, G, V) admits a frame vector if and only if it is a finite dimensional sub-representation of the left regular α-representation. Let (π, V, α) be a sub-α-representation of the left regular representation $(L, L^2(G), \alpha)$. Let $v \in V$ be a nonzero element. For any $u \in L^2(G)$, we have

$$\begin{aligned}
\langle u, L_\xi v\rangle &= \int_G u(g)\overline{(L_\xi v)(g)}\,\mathrm{d}\,g \\
&= \int_G u(g)\frac{\alpha(\xi^{-1}, g)}{\alpha(\xi, \xi^{-1})}\overline{v(\xi^{-1}g)}\,\mathrm{d}\,g \\
&= \int_G \frac{\alpha(\xi^{-1}, g)\alpha(g^{-1}\xi, \xi^{-1}g)}{\alpha(\xi, \xi^{-1})} u(g)v^\diamond(g^{-1}\xi)\,\mathrm{d}\,g \\
&= \int_G \frac{\alpha(g^{-1}, g)}{\alpha(g^{-1}, \xi)} u(g)v^\diamond(g^{-1}\xi)\,\mathrm{d}\,g \quad \text{(Eq. 2.16)} \\
&= (v^\diamond \diamond u)(\xi).
\end{aligned} \tag{3.10}$$

We then have the following result, which is more precise than the description in Sect. 3.1.

Proposition 3.10 *Let (π, V, α) be a sub-α-representation of the left regular α-representation $(L, L^2(G), \alpha)$. Let $v \in V$ be a nonzero element. If $\Phi_v = \{\pi(g)v \mid g \in G\}$ is a (G, α)-frame, then the Gramian of Φ_v is O_η with $\eta = v^\diamond \diamond v \in L^2(G)$.*

Assume that $\eta = v^\diamond \diamond v$ for some $v \in L^2(G)$ and $v \neq 0$. Consider the space $V = \overline{\mathrm{Span}}\{L_\xi v \mid \xi \in G\} \subset L^2(G)$. This is clearly a sub-$\alpha$-representation of the left regular representation $(L, L^2(G), \alpha)$. Consider the family $\{L_\xi v \in V \mid \xi \in G\}$, if this is a compact (G, α)-frame for V with frame vector v, then Eq. (3.10) tells us that the Gramian of this frame is O_η. We provide a characterization of such frame vectors in $L^2(G)$ via Fourier transform with respect to $1/\alpha$. Let e_i^d be the i-th standard basis in the vector space $\mathbb{C}^d$. Recall that the Fourier transform $\mathcal{F}$ gives us an isometry

$$\mathcal{F} := \widehat{\bigoplus}_{[\overline{\pi}]\in\widehat{G}_{1/\alpha}} \mathcal{F}_{\overline{\pi}} : L^2(G) \to \widehat{\bigoplus}_{[\overline{\pi}]\in\widehat{G}_{1/\alpha}} M_{d_{\overline{\pi}}}(\mathbb{C}).$$

For each $\overline{\pi}$, let $J(\overline{\pi})$ be the subspace of $\mathbb{C}^{d_{\overline{\pi}}}$ spanned by the columns of the matrices $\mathcal{F}_{\overline{\pi}}(u)$ with $u \in V$, i.e.,

$$J(\overline{\pi}) = \mathrm{Span}\{\mathcal{F}_{\overline{\pi}}(u)e_i^{d_{\overline{\pi}}} \mid 1 \leq i \leq d_{\overline{\pi}},\ u \in V\}.$$

By Lemma 2.18(4), we have $J(\overline{\pi}) = \mathrm{Span}\{\mathcal{F}_{\overline{\pi}}(v)e_i^{d_{\overline{\pi}}} \mid 1 \leq i \leq d_{\overline{\pi}}\}$. For the vector space $\mathbb{C}^{d_{\overline{\pi}}}$, we write an element as a column vector and fix an inner product by $\langle u, v\rangle = d_{\overline{\pi}}(v^* u)$. Note that V is finite dimensional if and only if $\dim J(\overline{\pi}) = 0$ for all but finitely many $\overline{\pi}$. We then have the following result.

Theorem 3.11 *Let $v \in L^2(G)$ be a nonzero element and $V = \overline{\mathrm{Span}}\{L_\xi v \mid \xi \in G\} \subset L^2(G)$. Assume that V is finite dimensional. The following two conditions are equivalent.*

1. *The family $\Phi_v = \{L_\xi v \mid \xi \in G\}$ is a (G, α)-frame for V with frame bounds (A, B).*
2. *For each $[\pi] \in \widehat{G}_{1/\alpha}$ with $\dim J(\overline{\pi}) > 0$, the finite family $\Phi_{v,\overline{\pi}} = \{\mathcal{F}_{\overline{\pi}}(v)e_i^{d_{\overline{\pi}}} \mid 1 \leq i \leq d_{\overline{\pi}}\}$ is a frame for $J(\overline{\pi})$ with frame bounds (A, B).*

Proof Assume that condition (1) holds. Fix $[\pi] \in \widehat{G}_{1/\alpha}$. For any $w \in J(\overline{\pi})$, define $u \in L^2(G)$ by

$$\mathcal{F}_\sigma(u) = \begin{cases} d_{\overline{\pi}}^{-1/2}(w\ w\ \cdots\ w) & \text{if } [\sigma] = [\overline{\pi}], \\ O & \text{otherwise.} \end{cases}$$

Then, $\|u\| = \|d_{\overline{\pi}}^{-1/2}(w\ w\ \cdots\ w)\| = \|w\|$. Note that

$$\begin{aligned} \int_G |\langle u, L_\xi v\rangle|^2 \,\mathrm{d}\, g &= \|v^\diamond \diamond u\|^2 \quad (\text{Eq. } 3.10) \\ &= \sum_{[\sigma]\in\widehat{G}_{1/\alpha}} \|\mathcal{F}_\sigma(v^\diamond)\mathcal{F}_\sigma(u)\|^2 \\ &= \|\mathcal{F}_{\overline{\pi}}(v^\diamond)\mathcal{F}_{\overline{\pi}}(u)\|^2 = \sum_{i=1}^{d_{\overline{\pi}}} |\langle w, \mathcal{F}_{\overline{\pi}}(v)e_i^{d_{\overline{\pi}}}\rangle|^2. \end{aligned} \tag{3.11}$$

Condition (2) then follows from

$$\begin{aligned} A\|w\|^2 = A\|u\|^2 \leq \int_G |\langle u, L_\xi v\rangle|^2 &= \sum_{i=1}^{d_{\overline{\pi}}} |\langle w, \mathcal{F}_{\overline{\pi}}(v)e_i^{d_{\overline{\pi}}}\rangle|^2 \\ &\leq B\|u\|^2 = B\|w\|^2. \end{aligned}$$

Conversely, suppose that condition (2) holds. For every $u \in V \subset L^2(G)$,

$$\|u\|^2 = \|\mathcal{F}(u)\|^2 = \sum_{[\sigma]\in\widehat{G}_{1/\alpha}} \|\mathcal{F}_\sigma(u)\|^2 = \sum_{[\sigma]\in\widehat{G}_{1/\alpha}} \sum_{i=1}^{d_\sigma} \|\mathcal{F}_\sigma(u)e_i^{d_\sigma}\|^2.$$

Then, condition (1) follows from

$$\begin{aligned} A\|u\|^2 &= \sum_{[\sigma]\in\widehat{G}_{1/\alpha}} \sum_{i=1}^{d_\sigma} A\|\mathcal{F}_\sigma(u)e_i^{d_\sigma}\|^2 \\ &\le \sum_{[\sigma]\in\widehat{G}_{1/\alpha}} \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} |\langle \mathcal{F}_\sigma(u)e_i^{d_\sigma}, \mathcal{F}_\sigma(v)e_j^{d_\sigma}\rangle|^2 \\ &= \int_G |\langle u, L_\xi v\rangle|^2 \,\mathrm{d}\,\xi \quad (\text{Eq. } 3.10) \\ &\le \sum_{[\sigma]\in\widehat{G}_{1/\alpha}} \sum_{i=1}^{d_\sigma} B\|\mathcal{F}_\sigma(u)e_i^{d_\sigma}\|^2 = B\|u\|^2. \end{aligned}$$

The theorem follows. □

Combining with Remark 3.9 and the Peter–Weyl theorem, we obtain the following result.

Corollary 3.12 1. *Let $v \in L^2(G)$ be a nonzero element and $V = \overline{\mathrm{Span}}\{L_\xi v \mid \xi \in G\}$. Then, $\Phi_v = \{L_\xi v \mid \xi \in G\}$ is a (G, α)-frame for V if and only if V is finite dimensional.*
2. *Let $\pi : G \to \mathbf{U}(V)$ be an α-representation. Then, π admits a frame vector if and only if π is isomorphic to a finite dimensional sub-representation of the left regular α-representation $(L, L^2(G), \alpha)$. Therefore, π admits a frame vector if and only if π admits a Parseval frame vector.*

4 Maximal Spanning Vectors

In this section, we study maximal spanning vectors for α-representations of compact groups.

4.1 Basic Properties

Let $\pi : G \to \mathbf{U}(V)$ be a finite dimensional α-representation of G over a complex Hilbert space V. Denote by d the dimension of V. Let (π^*, V^*) be the dual projective representation of (π, V) and $(\overline{\pi}, \overline{V} = V)$ be the complex conjugation of (π, V). Then, $\pi^* \cong \overline{\pi}$ as they have the same character.

For any $x \in V$, denote by $x^* \in V^*$ the linear functional defined by $u \mapsto \langle u, x\rangle$. For any $u,\ v \in V$, we have a matrix coefficient $c^\pi_{u,v} : G \to \mathbb{C}$ defined by

$$c^\pi_{u,v}(h) = v^*(\pi(h)u) = \langle \pi(h)u, v\rangle.$$

Denote by $C_\pi \subset L^2(G)$, the spanning space of matrix coefficients of V, i.e.,

$$C_\pi = \mathrm{Span}\{c^\pi_{u,v} \mid u \in V,\ v \in V\} \subset L^2(G).$$

Fix a basis $\{u_i \mid 1 \le i \le d\}$ of V, then $\{c^\pi_{u_i,u_j} \mid 1 \le i, j \le d\}$ is a spanning set of C_π. Define

$$C_{u,v} = \mathrm{Span}\{c^\pi_{\pi(g)u,\pi(g)v} \mid g \in G\} \subset L^2(G).$$

Certainly, every element $c^\pi_{\pi(g)u,\pi(g)v}$ is a matrix coefficient of V. Hence, $\dim C_{u,v} \le (\dim V)^2$. If the equality holds, we say that (u, v) is a maximal spanning pair for V. The following lemma is an easy consequence of the results in Sects. 2 and 3.

Lemma 4.1 *With the above notation, the following conditions are equivalent.*

1. $x \in V$ *is a maximal spanning vector for* (π, G, V).
2. $g \mapsto \pi(g)x \otimes \pi(g)x$ *is a continuous frame for* $\mathrm{HS}(V)$.
3. $dim\ Span\{\pi(g)x \otimes \pi(g)x \mid g \in G\} = (dim\ V)^2$.
4. *Fix an isomorphism* $V \cong \mathbb{C}^d$, *Span* $\{\pi(g)x(\pi(g)x)^* \mid g \in G\} = M_d(\mathbb{C})$.
5. $dim\ Span\{c^\pi_{\pi(g)x,\pi(g)x} \mid g \in G\} = (dim\ V)^2$.

A maximal spanning vector $x \in V$ is automatically phase retrievable in the sense that the function $g \mapsto |\langle v, \pi(g)x\rangle|$ in $L^2(G)$ uniquely (up to a unimodular scalar) determines v. We refer to [1, 2, 4, 18] for more information on the relation between maximal spanning vectors and the phase retrieval problems. In this section, we focus on a generalized version of [18, Conjecture].

Conjecture 4.2 *Let G be a compact group and $\alpha \in Z^2(G, \mathbb{S})$ be a multiplier. If $\pi : G \to \mathbf{U}(V)$ is an irreducible α-representation, then there exists a maximal spanning vector for V.*

Remark 4.3 If $\pi : G \to \mathbf{U}(V)$ is an irreducible α-representation, $V \otimes V^*$ is a subrepresentation of the linear left regular representation. From Corollary 3.12, $V \otimes V^*$ admits frame vectors. The conjecture claims that it admits a frame vector of the form $x \otimes x^*$. Moreover, if $x \in V$ is a maximal spanning vector for V, then for any $g \in G$, $\pi(g)x$ is also a maximal spanning vector for V.

In [18], Li et al. verified this conjecture for finite abelian groups and certain metacyclic groups. We remark that the irreducibility condition in Conjecture 4.2 is necessary (cf. Section 4.3).

Denote by $\widehat{G}(= \widehat{G}_1)$ the dual space of G. If $V \otimes V^*$ is multiplicity free, then for each $[\rho] \in \widehat{G}$, we have a canonical projection $V \otimes V^* \to V_\rho$ given by

$$P_\rho = \int_{t \in G} \chi_\rho(t^{-1})\pi \otimes \pi^*(t)\,\mathrm{d}\,t.$$

Here, χ_ρ is the character of ρ. The following result and Theorem 1.1(1) follow immediately from Proposition 3.8 and Lemma 4.1.

Proposition 4.4 *If $V \otimes V^*$ is multiplicity free, then $(u, v) \in V \oplus V$ is a maximal spanning pair for (π, V) if and only if $P_\rho(u \otimes v^*) \neq 0$ for any $\rho \in \widehat{G}$ with Hom $(\rho, \pi \otimes \pi^*) \neq 0$.*

4.2 The Abelian Case

We show that Conjecture 4.2 holds for compact abelian groups. In the following, G is a compact abelian group, (π, V, α) is an irreducible α-representation of G, $\widehat{G}$ is the dual group of G. Assume further that the cohomology class $[\alpha] \in H^2(G, \mathbb{S})$ is nontrivial. Otherwise, α-representations are equivalent to linear representations; hence, the irreducible ones are all one-dimensional and the conjecture is obviously true. We introduce two subgroups of G.

Group H_α Let (π', V', α) be another irreducible α-representation of G. The tensor product $V \otimes \overline{V}'$ is a projective representation of G with trivial multiplier; hence, it is a linear representation. In particular, $V \otimes \overline{V}'$ is a direct sum of one-dimensional linear representations. If χ is one of the direct summands, then we must have $V \cong V' \otimes \chi$ and χ has multiplicity one in $V \otimes \overline{V}'$. (Indeed, as explained in [21, Chapter 4], the properties of characters of finite groups hold for compact groups as well. Hence, for any linear character ψ of G, $\dim \mathrm{Hom}(\psi, V \otimes \overline{V}') = \dim \mathrm{Hom}(V' \otimes \psi, V) \leq 1$.) Denote the dimension of V (which depends only on α) by d_α. Let us take $V' = V$ and define

$$\mathcal{H}(V) = \{\chi \mid \dim \mathrm{Hom}(\chi, V \otimes \overline{V}) = 1\}.$$

Then, $\mathcal{H}(V)$ is a finite subgroup of $\widehat{G}$ and it is independent of V. Denote this group by $\mathcal{H}_\alpha$. Define

$$H_\alpha = \mathcal{H}_\alpha^\perp := \{g \in G \mid \chi(g) = 1 \text{ for all } \chi \in \mathcal{H}_\alpha\}.$$

Then, H_α is a closed subgroup of G with index d_α^2.

Group K_α Consider

$$\begin{aligned} G \times G &\to \mathbb{S} \\ (g, h) &\mapsto \frac{\alpha(g, h)}{\alpha(h, g)}. \end{aligned} \tag{4.1}$$

Since G is abelian, this map is a bi-homomorphism. It induces a morphism

$$\begin{aligned} \lambda : G &\to \widehat{G} \\ g &\mapsto \lambda_g = \frac{\alpha(g, \cdot)}{\alpha(\cdot, g)} \end{aligned} \tag{4.2}$$

Let K_α be the kernel of λ. Then, $K_\alpha \neq \{1\}$ as $[\alpha] \in H^2(G, \mathbb{S})$ is nontrivial and it is an open subgroup of G with finite index as λ is continuous and $\widehat{G}$ is discrete.

By adapting the argument in [7, Lemmas 3.9, 3.10], we prove the following result, which is the key ingredient of our proof of Theorem 1.1(2).

Proposition 4.5 *With the notation as above, $H_\alpha = K_\alpha$.*

Proof First, for any $k \in K_\alpha$ and $g \in G$, we have

$$\pi(k)\pi(g) = \alpha(k, g)\pi(kg) = \alpha(g, k)\pi(gk) = \pi(g)\pi(k).$$

Hence, $\pi(k) \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V, V)$. By Schur's lemma, $\pi(k)$ must be a scalar. Therefore, $(\pi \otimes \overline{\pi})|_{K_\alpha}$ is trivial. By construction, H_α is the maximal subgroup of G with this property, and we obtain one inclusion $K_\alpha \subset H_\alpha$.

Now, we prove the other inclusion. By the same argument of [7, Lemma 3.9], $\alpha|_{H_\alpha \times H_\alpha}$ is a coboundary. Then, the set $\{B \leq G \text{ closed} : \alpha|_{B\times B} \text{ is a coboundary and } H_\alpha \subset B\}$ is nonempty. Let K be a maximal element in this set. Therefore, $[G : K] < \infty$ and $V|_K \cong \oplus\sigma_i$ is a direct sum of one-dimensional α-representations. Let $\sigma \in \{\sigma_i\}$ be a fixed element and $\alpha\,\mathrm{Ind}_K^G\,\sigma$ be the α-induction of σ. We claim that $\alpha\,\mathrm{Ind}_K^G\,\sigma$ is irreducible, hence $V \cong \alpha\,\mathrm{Ind}_K^G\,\sigma$ by Frobenius reciprocity.

Indeed, fix a subset $S \subset G$ of representatives of G/K, then $(\alpha\,\mathrm{Ind}_K^G\,\sigma)|_K = \oplus_{s\in S}\sigma^s$, where σ^s is the α-twist of σ, i.e., $\sigma^s(k) = \frac{\alpha(s^{-1},k)}{\alpha(k,s^{-1})}\sigma(k)$ for all $k \in K$. By Mackey's criterion, $\alpha\,\mathrm{Ind}_K^G\,\sigma$ is irreducible if and only if $\sigma \ncong \sigma^s$ for $s \notin K$. Suppose that $\sigma \cong \sigma^s$ for some $s \in S - K$, then $\sigma \cong \sigma^{s^i}$ for $i \in \mathbb{Z}$. Therefore, $\alpha(s^{-i}, k) = \alpha(k, s^{-i})$ for all $k \in K$ and $i \in \mathbb{Z}$. By [6, Lemmas 2.12, 2.13], $\alpha|_{K'\times K'}$ is a coboundary, where $K' = \langle K, s\rangle \leq G$. This contradicts to the fact that K is maximal. The claim then follows.

As $(V \otimes \overline{V})|_K = \oplus_{s,t\in S}(\sigma^s \otimes \overline{\sigma^t})$ and $H_\alpha \subset K$, we have

$$(V \otimes \overline{V})|_{H_\alpha} = \oplus_{s,t\in S}(\sigma^s|_{H_\alpha} \otimes \overline{\sigma^t}|_{H_\alpha}).$$

Thus, $\sigma^s|_{H_\alpha} \otimes \overline{\sigma^t}|_{H_\alpha}$ is trivial. In particular, let t be the representative of the unity of G/K, we obtain $\sigma(s^{-1}, h) = \alpha(h, s^{-1})$ for all $h \in H_\alpha$ and $s \in S$. Then, $H_\alpha \subset \mathrm{Ker}(\lambda) = K_\alpha$. The proposition follows. □

Corollary 4.6 *The following claims hold.*

1. $[G : K_\alpha] = d_\alpha^2$.
2. *$\dim C_{u,v} = d_\alpha^2$ if and only if $c_{u,v}(g) \neq 0$ for all $g \in G$.*

Proof We have $[G : K_\alpha] = [G : H_\alpha] = |\mathcal{H}_\alpha| = d_\alpha^2$. The first claim holds.

For any $u, v \in V$, we have

$$\begin{aligned} c_{\pi(g)u,\pi(g)v}(h) &= \langle \pi(h)(\pi(g)u), \pi(g)v\rangle \\ &= \alpha(g, g^{-1})^{-1}\alpha(g^{-1}, h)\alpha(g^{-1}h, g)c_{u,v}(g^{-1}hg). \end{aligned}$$

Since G is abelian, it is easy to see that $C_{u,v} = \mathrm{Span}\{\lambda_g c_{u,v} \mid g \in G\}$. By the orthogonality of characters, if $c_{u,v}(g) \neq 0$ for all $g \in G$, then $\dim C_{u,v} = |\mathrm{Im}(\lambda)| = d_\alpha^2$.

Suppose that $\dim C_{u,v} = d_\alpha^2$ and $c_{u,v}(g) = 0$ for some $g \in G$. Then, $c_{\pi(h)u,\pi(h)v}(g) = 0$ for all $h \in G$. This implies $\pi(g) = O$, which is impossible. The second claim then follows. □

Remark 4.7 From the construction of H_α, we see that $V \otimes V^*$ is multiplicity free. Therefore, we may apply Proposition 4.4 to verify whether (u, v) is a maximal spanning pair. An interesting consequence is that $c_{u,v}(g) \neq 0$ for all $g \in G$ if and only if $\int_{t\in G} \chi(t^{-1})\pi \otimes \pi^*(t)(u \otimes v^*)\,\mathrm{d}\,t \neq 0$ for all $\chi \in \mathcal{H}_\alpha$.

Theorem 4.8 *Let G be a compact abelian group and $\alpha \in Z^2(G, \mathbb{S})$ be a multiplier. If $\pi : G \to \mathbf{U}(V)$ is an irreducible α-representation, then the set $\{x \in V \mid x \text{ is maximal}\}$ is open dense in V. In particular, Conjecture 4.2 holds for compact abelian groups.*

Proof We show that $\{x \in V \mid c_{x,x}(g) \neq 0 \text{ for all } g \in G\}$ is open dense in V. As explained in the proof of Proposition 4.5, $\pi(k)$ is a scalar for $k \in K_\alpha$. Let $S \subset G$ be a set of representatives of G/K_α. Let $V_s \subset V$ be the set $\{v \in V \mid c_{v,v}(s) \neq 0\}$. Since $c_{v,v}(s) = 0$ is given by a nontrivial quadratic equation, V_s is open dense in V. Moreover,

$$\{x \in V \mid c_{x,x}(g) \neq 0 \text{ for all } g \in G\} = \bigcap_{s\in S} V_s.$$

By Corollary 4.6, the theorem follows since S is a finite set. □

Remark 4.9 From the knowledge on K_α, we may reduce the compact group case to finite group case, hence obtain a slightly different proof of Theorem 4.8. Indeed, for any $t \in G/K_\alpha$, fix a lifting $\tilde{t} \in G$ of t. Define a map $\omega : G/K_\alpha \to \mathrm{GL}(V)$ by $t \mapsto \pi(\tilde{t})$. We obtain a map $P_\omega : G/K_\alpha \to \mathrm{PGL}(V)$, which is a homomorphism as $\pi(k)$ is scalar for $k \in K_\alpha$. Hence, P_ω induces a projective representation $P_\omega : G/K_\alpha \to \mathrm{GL}(V)$ with respect to some multiplier. Moreover, it is irreducible as π is irreducible.

4.3 Some Remarks on the Reducible Case

This part is related to [18, Problem D]. We show that, if $\pi : G \to \mathbf{U}(V)$ is a reducible and finite dimensional projective representation, there is no maximal spanning vector in the sense of [18]. More precisely, write $\pi = \oplus_{j\in J}\pi_j$ as a direct sum of irreducible sub-representations, for any $v \in V$, we show that

$$\dim \mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} < \sum_{j\in J} n_j^2, \tag{4.3}$$

where n_j is the degree of the irreducible projective representation π_j. Equation (4.3) is equivalent to

$$\dim \operatorname{Span}\{c^{\pi}_{\pi(g)v,\pi(g)v} \mid g \in G\} < \sum_{j\in J} n_j^2. \tag{4.4}$$

We call $\dim \operatorname{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\}$ the spanning dimension of (π, V, v).

Proposition 4.10 *Let G be a compact group and $(\pi = \pi_1 \oplus \pi_2, V = V_1 \oplus V_2)$ be a direct sum of two irreducible projective representations of G. Here, V_i is the representation space of π_i with dimension n_i $(i = 1, 2)$. Assume that $v = v_1 \oplus v_2 \in V_1 \oplus V_2$ is a frame vector for (π, V). Then,*

$$\dim \operatorname{Span}\{c^{\pi}_{\pi(g)v,\pi(g)v} \mid g \in G\} < n_1^2 + n_2^2.$$

Proof Choose a finite subset H of G such that $1 \in H$ and

$$\operatorname{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = \operatorname{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in H\}.$$

By enlarging H, we may assume that

$$\begin{aligned} \operatorname{Span}\{c^{\pi}_{u,v} \mid g \in G\} &\to \mathbb{C}^{|H|} \\ c^{\pi}_{u,v} &\mapsto (c^{\pi}_{u,v}(h))_{h\in H} \end{aligned}$$

is injective.

Fix an ordering of the elements of H with the identity at the first place and define Ω_i to be the $|H| \times |H|$ matrix, whose (g, h)-entry is given by $c^{\pi_i}_{\pi_i(g)v_i,\pi_i(g)v_i}(h)$. Denote by $C(M)$ the column space of a matrix M. Then, $C(\Omega_1)\cap C(\Omega_2)$ contains a nontrivial vector from the identity element. Moreover, $\dim C(\Omega_i) \le \dim \operatorname{Span}\{c^{\pi_i}_{\pi_i(g)v_i,\pi_i(g)v_i} \mid g \in G\} \le n_i^2$. As

$$\begin{aligned} c^{\pi}_{\pi(g)v,\pi(g)v}(h) &= \langle \pi(h)\pi(g)v, \pi(g)v\rangle \\ &= \langle \pi_1(h)\pi_1(g)v_1, \pi_1(g)v_1\rangle + \langle \pi_2(h)\pi_2(g)v_2, \pi_2(g)v_2\rangle \\ &= c^{\pi_1}_{\pi_1(g)v_1,\pi_1(g)v_1}(h) + c^{\pi_2}_{\pi_2(g)v_2,\pi_2(g)v_2}(h), \end{aligned}$$

we have

$$\begin{aligned} \dim \operatorname{Span}\{c^{\pi}_{\pi(g)v,\pi(g)v} \mid g \in G\} &= \operatorname{rank}(\Omega_1 + \Omega_2) \\ &= \dim C(\Omega_1 + \Omega_2) \\ &\le \dim(C(\Omega_1) + C(\Omega_2)) \\ &= \dim C(\Omega_1) + \dim C(\Omega_2) - \dim(C(\Omega_1) \cap C(\Omega_2)) \\ &< \dim C(\Omega_1) + \dim C(\Omega_2) \\ &\le n_1^2 + n_2^2. \end{aligned}$$

The proposition follows.

□

We thank the referee for the simple proof of the proposition. For (π, V) as in Proposition 4.10, we compute the spanning dimensions for two examples in the following and the computation shows that the invariant

$$\max_{v\in V}\{\dim \operatorname{Span}\{\pi(g)v\otimes\pi(g)v \mid g\in G\}\}$$

depends on not only the degrees n_1 and n_2, but also the structures of π_1 and π_2.

4.3.1 *G* is Abelian

In this part, G is a compact abelian group. Let (π_1, V_1) and (π_2, V_2) be two irreducible α-representations of G and $\pi = \pi_1 \oplus \pi_2$. Then, $\pi_2 = \pi_1 \otimes \chi$, where χ is a linear character. Let $v = v_1 + v_2$ be a frame vector for $(\pi, V = V_1 \oplus V_2)$, where $v_1 \in V_1$ and $v_2 \in V_2$. Then,

$$\begin{aligned}
c^{\pi}_{\pi(g)v,\pi(g)v}(h) &= \langle \pi(h)\pi(g)v, \pi(g)v\rangle \\
&= \langle \pi_1(h)\pi_1(g)v_1, \pi_1(g)v_1\rangle + \langle \pi_2(h)\pi_2(g)v_2, \pi_2(g)v_2\rangle \\
&= \alpha(g, g^{-1})^{-1}\alpha(g^{-1}, h)\alpha(g^{-1}h, g)(\langle \pi_1(h)v_1, v_1\rangle + \langle \pi_2(h)v_2, v_2\rangle) \\
&= \frac{\alpha(g^{-1}, h)}{\alpha(h, g^{-1})}(c^{\pi_1}_{v_1,v_1}(h) + \chi(h)c^{\pi_1}_{v_2,v_2}(h)).
\end{aligned}$$

As G is abelian, $h \mapsto \frac{\alpha(g^{-1},h)}{\alpha(h,g^{-1})}$ is a character of G. Denote this character by λ_g. By Sect. 4.2 or [7, Section 3.2],

$$\dim \operatorname{Span}\{\lambda_g \mid g \in G\} = (\dim V_1)^2.$$

Hence,

$$\begin{aligned}
\dim \operatorname{Span}\{c^{\pi}_{\pi(g)v,\pi(g)v} \mid g \in G\} &\leq \dim \operatorname{Span}\{\lambda_g \mid g \in G\} \\
&= (\dim V_1)^2 < (\dim V_1)^2 + (\dim V_2)^2.
\end{aligned}$$

Note that $\dim \operatorname{Span}\{c^{\pi}_{\pi(g)v,\pi(g)v} \mid g \in G\} = (\dim V_1)^2$ if $c^{\pi_1}_{v_1,v_1}(h) + \chi(h)c^{\pi_1}_{v_2,v_2}(h) \neq 0$ for all $h \in G$, and this is true for infinitely many $v \in V$.

4.3.2 $G = D_6$

In this part, $G = D_6$ is the dihedral group with 6 elements. Fix a presentation of G, say

$$G = \langle a,\ b \mid a^3 = 1,\ b^2 = 1,\ ab = ba^{-1}\rangle.$$

Let $\rho : G \to \mathbf{U}(\mathbb{C}^2)$ be the two-dimensional irreducible representation of G given by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad a^2 \mapsto \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix},$$
$$b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad ba \mapsto \begin{pmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{pmatrix}, \quad ba^2 \mapsto \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix}.$$

Here, $\zeta = e^{2\pi i/3}$. Let $v = (x\ y)' \in \mathbb{C}^2$ be a nontrivial vector. Let $\alpha = x\bar{x} + y\bar{y}$, $\beta = \zeta x\bar{x} + \zeta^{-1} y\bar{y}$, $\gamma = \zeta^{-1} x\bar{x} + \zeta y\bar{y}$, $\delta = \bar{x}y + x\bar{y}$, $\epsilon = \zeta^{-1}\bar{x}y + \zeta x\bar{y}$, $\eta = \zeta\bar{x}y + \zeta^{-1}x\bar{y}$. Let Ω_v be the 6×6 matrix, whose (g, h)-entry is given by $c^{\rho}_{\rho(g)v,\rho(g)v}(h)$. Direct computation shows that

$$\Omega_v = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \epsilon & \eta \\ \alpha & \beta & \gamma & \eta & \delta & \epsilon \\ \alpha & \beta & \gamma & \epsilon & \eta & \delta \\ \alpha & \gamma & \beta & \delta & \eta & \epsilon \\ \alpha & \gamma & \beta & \eta & \epsilon & \delta \\ \alpha & \gamma & \beta & \epsilon & \delta & \eta \end{pmatrix}.$$

Making the following operations $r_2 - r_1$, $r_3 - r_1$, $r_5 - r_4$, $r_6 - r_4$, $c_4 + (c_5 + c_6)$, $r_4 - r_1$, $r_4 \leftrightarrow r_3$, $r_3 \leftrightarrow r_2$, $r_6 + r_3$, $r_5 + r_4$, we obtain the matrix

$$\tilde{\Omega}_v = \begin{pmatrix} \alpha & \beta & \gamma & \delta + \epsilon + \eta & \epsilon & \eta \\ 0 & \gamma - \beta & \beta - \gamma & 0 & \eta - \epsilon & \epsilon - \eta \\ 0 & 0 & 0 & 0 & \delta - \epsilon & \epsilon - \eta \\ 0 & 0 & 0 & 0 & \eta - \epsilon & \delta - \eta \\ 0 & 0 & 0 & 0 & \epsilon - \eta & \delta - \epsilon \\ 0 & 0 & 0 & 0 & \delta - \eta & \eta - \epsilon \end{pmatrix}.$$

It is easy to see that $\operatorname{rank} \Omega_v \leq 4$. There are infinitely many v with $\operatorname{rank} \Omega_v = 4$; hence, each of those $v \in \mathbb{C}^2$ is maximal spanning and the frame $\{\rho(g)v \mid g \in G\}$ is phase retrievable (cf. [18, Conjecture]). For example, this happens if $\beta \neq \gamma$ and $2\delta - \epsilon - \eta \neq 0$.

For $\pi = \text{triv} \oplus \rho : G \to \mathbf{U}(\mathbb{C}^3)$, let $u = (w\ x\ y)' \in \mathbb{C}^3$. Then, $\dim \operatorname{Span}\{c^{\pi}_{\pi(g)u,\pi(g)u} \mid g \in G\} = \operatorname{rank} \Delta_v$, where Δ_v is obtained from Ω_v by replacing each entry z with $z + c$, and $c = w\bar{w}$ is the constant from the trivial character. Making the same operations as above on Δ_v, we obtain the matrix

$$\tilde{\Delta}_v = \begin{pmatrix} c + \alpha & c + \beta & c + \gamma & 3c + \delta + \epsilon + \eta & c + \epsilon & c + \eta \\ 0 & \gamma - \beta & \beta - \gamma & 0 & \eta - \epsilon & \epsilon - \eta \\ 0 & 0 & 0 & 0 & \delta - \epsilon & \epsilon - \eta \\ 0 & 0 & 0 & 0 & \eta - \epsilon & \delta - \eta \\ 0 & 0 & 0 & 0 & \epsilon - \eta & \delta - \epsilon \\ 0 & 0 & 0 & 0 & \delta - \eta & \eta - \epsilon \end{pmatrix},$$

whose rank is at most four. Hence,

$$\dim \operatorname{Span}\{c^{\pi}_{\pi(g)u,\pi(g)u} \mid g \in G\} \leq 4.$$

The equality holds if for example $\beta \neq \gamma$ and $2\delta - \epsilon - \eta \neq 0$.

Remark 4.11 There is no phase retrievable frame vector for the $(\pi, \mathbb{C}^3)$ above. Indeed, let $v = (w\ x\ y)' \in \mathbb{C}^3$ be a frame vector for $(\pi, \mathbb{C}^3)$, then $w(|x| + |y|) \neq 0$. Consider the two vectors $\phi_1 = (e\ 1\ 1)'$ and $\phi_2 = (\frac{1}{2}e\ 2\ 2)' \in \mathbb{C}^3$, where $|e|^2|w|^2 = 3(|x|^2 + |y|^2 + \bar{x}y + x\bar{y})$. It is straightforward to check that $|\langle \phi_1, \pi(g)v\rangle| = |\langle \phi_2, \pi(g)v\rangle|$ for all $g \in G$.

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Declarations

Conflict of interest The authors declare that they do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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