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On Vanishing Theorems for Locally Conformally Flat Riemannian Manifolds with an Integral Pinching Condition

Duc Thoan Pham[1] · Van Khien Tran[1] · Thi Hong Nguyen[2]

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Abstract
In this paper, we show some vanishing theorems for harmonic p-forms on a locally conformally flat Riemannian manifold. In the concrete, provided that the integral of the traceless Ricci tensor has a suitable bound, we obtain a vanishing theorem for them without any scalar curvature conditions. Another theorem is also given under the condition on nonpositive scalar curvature, which improves and extends the ones previous.

Keywords Harmonic p-form · Vanishing theorem · Locally conformally flat Riemannian manifold

Mathematics Subject Classification 58J05 · 58J35

1 Introduction

In the Riemannian geometry, the theory of L^2-harmonic forms has played an important role in the study of the structure of complete manifolds such as the topology at infinity of a complete Riemannian manifold or a complete orientable δ-stable minimal hypersurface in $\mathbb{R}^{n+1}$. Therefore, it have been studied very vigorously by many authors. One

✉ Duc Thoan Pham
thoanpd@huce.edu.vn

Van Khien Tran
khientv@huce.edu.vn

Thi Hong Nguyen
nguyenthihongmm@gmail.com

1 Department of Mathematics, Hanoi University of Civil Engineering, 55 Giai Phong Road, Dong Tam, Hai Ba Trung, Hanoi, Vietnam

2 Department of Foundation, Academy of Cryptography Techniques, 141 Chien Thang Road, Tan Trieu, Thanh Tri, Hanoi, Vietnam

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of the interesting problems of this theory is to find sufficient conditions on a locally conformally flat manifold M for the vanishing of harmonic forms. Noting that a locally conformally flat manifold may be regarded as a higher dimensional generalization of a Riemannian surface. When M is compact, the Hodge theory states that the space of harmonic p-forms on M is isomorphic to its p-th de Rham cohomology group. By this property, there have been a lot of remarkable results on vanishing theorems related to the Betti number. For example, Bourguignon [1] proved that a compact, locally conformally flat manifold of dimension $2m$ with positive scalar curvature has no nonzero harmonic m-forms and hence its m-th Betti number β_m must be zero. When M is noncompact, the Hodge theory is no longer true in general. But it remains valid for complete noncompact manifolds. So it is necessary to investigate the harmonic forms on the such manifolds.

For vanishing theorems, there are also many results of those for complete locally conformally flat manifolds endowed with special analysis structure (see [3, 4, 6, 7, 11, 13, 14, 17, 19] and others). By assuming that the Ricci curvature is bounded from below in terms of the dimension and the first eigenvalue, Li-Wang [16] obtained a vanishing-type theorem of L^2 harmonic 1-forms. Later, this result is generalized and extended by many authors (see [5, 8, 12, 18, 20] for details). Since the Riemannian curvature of a locally conformally flat manifold can be expressed by its Ricci curvature and scalar curvature, we can compute explicitly the Bochner-Weitzenböck formula for harmonic p-forms. Based on this formula and L^2-Sobolev inequality, Dong, Lin and Wei [5] established vanishing results for L^2 harmonic p-forms on the complete Riemannian manifolds with scalar curvature $R \geq 0$ under various $L^{n/2}$-integral curvature or pointwise curvature pinching conditions as follows.

Theorem 1.1 [5] *Let (M^n, g) $(n \geq 3)$ be a complete non-compact, simply connected, locally conformally flat Riemannian manifold of dimension n with the scalar curvature $R \geq 0$. Assume that the traceless Ricci tensor satisfies*

$$\left(\int_M |E|^{n/2}\right)^{2/n} \leq C(p),$$

where $C(p) = \frac{(n-2)\sqrt{n}}{p|n-2p|\sqrt{n-1}} \min\{1+k_p, \frac{4p(n-p)}{n(n-2)}\} Q(\mathbb{S}^n)$ *for* $1 \leq p \leq n-1$ *but* $p \neq \frac{n}{2}$ *and with* $k_p = \frac{1}{\max\{p, n-p\}}$. *Then* $\mathcal{H}^p(L^2(M)) = \{0\}$.

Here, we denote by $\mathcal{H}^p(L^2(M))$ the space of L^2 harmonic p-forms on M and denote $Q(\mathbb{S}^n) = \frac{n(n-2)\omega_n^{2/n}}{4}$ by the Yamabe constant of $\mathbb{S}^n$ with the volume ω_n of the unit sphere in $\mathbb{R}^n$.

Under the conditions similar to those of Theorem 1.1, but the scalar curvature $R \leq 0$, Han, Zhang and Liang [12] also obtained a vanishing theorem for l-harmonic 1-forms. When $l = 2$, their theorem is stated as follows.

Theorem 1.2 [12] *Let $(M^n, g)(n \geq 3)$ be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with the scalar curvature $R \leq 0$.*

If the traceless Ricci tensor satisfies

$$\left(\int_M |E|^{n/2}\right)^{2/n} < \left[\frac{n}{n-1} - \frac{4(n-1)}{\sqrt{n}(n-2)}\right] Q(\mathbb{S}^n),$$

then we have $\mathcal{H}^1(L^2(M)) = \{0\}$.

Similarly, Lin [14] also obtained the following result.

Theorem 1.3 [14] *Let* $(M^n, g)(n \geq 17)$ *be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with the scalar curvature* $R \leq 0$. *Assume that*

$$\left(\int_M |E|^{n/2}\right)^{2/n} < \left[\frac{n}{2(n-1)} - \frac{2(n-1)}{\sqrt{n}(n-2)}\right] Q(\mathbb{S}^n).$$

Then $\mathcal{H}^1(L^2(M)) = \{0\}$ *and M must have only one end.*

To obtain theorems above, they based on a precise estimate of the curvature operators which appear in the Bochner-Weitzenböck formula on harmonic p-forms and together with the Sobolev inequality induced by the positivity of the Yamabe constant as well as Kato's inequality. However, under the condition $\left(\int_M |E|^{n/2}\right)^{2/n}$ less than some certain value, Theorems 1.1, 1.2 and 1.3 require that the scalar curvature has only one sign. That is, either $R \geq 0$ or $R \leq 0$ on the manifold M. We would like to emphasize that under the condition on the traceless Ricci tensor similar to that in these theorems, some vanishing theorems for harmonic p-forms on the same manifold are also given, but they require an other condition on scalar curvature (see [10, Theorem 1.1], [12, Theorem 3.1], [14, Theorem 1.3] for example).

The following question arises naturally at this moment: do we obtain results on vanishing of harmonic p-forms on the complete, locally conformally flat Riemannian manifolds without any scalar curvature conditions?

The first aim of this paper is to give a positive answer for this question. Namely, we have the following result.

Theorem 1.4 *Let* (M^n, g) $(n \geq 3)$ *be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold and let p be a positive integer. Assume that the traceless Ricci tensor satisfies*

$$\left(\int_M |E|^{n/2}\right)^{2/n} < \frac{4(n-p)}{|n-2p|\sqrt{n(n-1)}} Q(\mathbb{S}^n).$$

If $\frac{4p(n-p)}{n(n-2)} < 1 + k_p$ *then* $\mathcal{H}^p(L^2(M)) = \{0\}$ *for all* $1 \leq p \leq n-1$ *but* $p \neq \frac{n}{2}$.

By further investigating Theorems 1.2 and 1.3, we found that the theorems may be still weak. The second aim of this paper is to give a vanishing theorem which is an improvement and an extension of those theorems in the case of n less than some certain value.

Theorem 1.5 *Let* (M^n, g) $(n \geq 3)$ *be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with the scalar curvature* $R \leq 0$ *and let p be a positive integer. Assume that the traceless Ricci tensor satisfies*

$$\left(\int_M |E|^{n/2}\right)^{2/n} \leq \frac{4(n-1)}{\sqrt{n}(n-2)} Q(\mathbb{S}^n).$$

If $\frac{4(n-1)}{\sqrt{n}(n-2)} < 1 + k_p$ *then* $\mathcal{H}^p(L^2(M)) = \{0\}$ *for all* $1 \leq p \leq n-1$.

When $p = 1$ then $1 + k_p = \frac{n}{n-1}$. Obviously, the conditions of Theorems 1.2 and 1.3 imply those of Theorem 1.5. Moreover, the upper bounds of the integral in these theorems are quite small compared to that in this theorem if $17 \leq n \leq 64$. Indeed, by simple calculation, $\frac{4(n-1)}{\sqrt{n}(n-2)} < 1 + k_1$ holds if and only if $n \geq 17$ and $\frac{4(n-1)}{\sqrt{n}(n-2)} \geq \frac{n}{n-1} - \frac{4(n-1)}{\sqrt{n}(n-2)}$ holds only when $n \leq 64$. Therefore, a direct corollary of Theorem 1.5 is stated as follows which improves both of Theorems 1.2 and 1.3 in the case of the such n.

Corollary 1.6 *Let* (M^n, g) $(n \geq 17)$ *be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with the scalar curvature* $R \leq 0$. *If the traceless Ricci tensor satisfies*

$$\left(\int_M |E|^{n/2}\right)^{2/n} \leq \frac{4(n-1)}{\sqrt{n}(n-2)} Q(\mathbb{S}^n)$$

then $\mathcal{H}^1(L^2(M)) = \{0\}$.

By investigating above, clearly Theorem 1.5 only becomes stronger when n is small. Thus, the final aim of this paper is to give a version of vanishing theorem for harmonic p-forms which is considered as a generalization of Theorem 1.2. Also, it may be stronger than Theorem 1.5 when n is big.

Theorem 1.7 *Let* (M^n, g) $(n \geq 3)$ *be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with the scalar curvature* $R \leq 0$ *and let p be a positive integer. If the traceless Ricci tensor satisfies*

$$\left(\int_M |E|^{n/2}\right)^{2/n} < \left(1 + k_p - \frac{4(n-1)}{\sqrt{n}(n-2)}\right) Q(\mathbb{S}^n)$$

then $\mathcal{H}^p(L^2(M)) = \{0\}$ *for all* $1 \leq p \leq n-1$.

2 Preliminaries

Let M be an n-dimensional Riemannian manifold. Let d be the exterior differential operator, so its dual operator δ is defined by

$$\delta = (-1)^{n(p+1)+1} * d*,$$

where $*$ is the Hodge star operator acting on the space of smooth p-forms $\Lambda^p(M)$. Then the Hogde-Laplace-Beltrami operator Δ acting on the space of smooth p-forms $\Lambda^p(M)$ is given by

$$\Delta = -(\delta d + d\delta).$$

Recall that a p-form ω on a Riemannian manifold M is said to be harmonic if it satisfies $d\omega = 0$ and $\delta\omega = 0$.

For each a harmonic p-form ω, the Bochner-Weitzenböck formula gives

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \mathrm{Ric}(\omega). \tag{2.1}$$

Then, Lin [15] had an estimate for $\mathrm{Ric}(\omega)$ as follows

$$\begin{aligned}\mathrm{Ric}(\omega) \geq & -\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}|W||\omega|^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E||\omega|^2 \\ & + \frac{p(n-p)}{n(n-1)}R|\omega|^2,\end{aligned} \tag{2.2}$$

where W, R and $E = \mathrm{Ric} - \frac{R}{n}$ denote the Weyl curvature tensor, the scalar curvature and the traceless Ricci tensor, respectively. Noting that when M is a locally conformally flat manifold, the Weyl conformal curvature tensor vanishes. Then combining (2.1) and (2.2), we get

$$\frac{1}{2}\Delta|\omega|^2 \geq |\nabla\omega|^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E||\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2. \tag{2.3}$$

When M is complete, Lin [14, Lemma 2.2] gave a relation between these curvature operators as follows

$$\mathrm{Ric} \geq -|E|g - \frac{|R|}{\sqrt{n}}g \tag{2.4}$$

in the sense of quadratic forms.

We recall the refined Kato's inequality as follows.

Lemma 2.1 [2] *For $p \geq 1$, let ω be a harmonic p-form on a complete Riemannian manifold M of dimension n. The following inequality holds*

$$|\nabla\omega|^2 - |\nabla|\omega||^2 \geq k_p|\nabla|\omega||^2,$$

where $k_p = \frac{1}{\max\{p, n-p\}}$.

When M is simply connected locally conformally flat, then it has a conformal immersion into the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^n$ and according to [9], the Yamabe constant of M satisfies

$$Q(M) = Q(\mathbb{S}^n) = \frac{n(n-2)\omega_n^{2/n}}{4},$$

where ω_n is the volume of $\mathbb{S}^n$. Therefore, the following inequality

$$Q(\mathbb{S}^n)\left(\int_M f^{2n/(n-2)}\right)^{(n-2)/n} \le \int_M |\nabla f|^2 + \frac{n-2}{4(n-1)}\int_M Rf^2 \tag{2.5}$$

holds for all $f \in C_0^\infty(M)$. Here, $C_0^\infty(M)$ denotes the set of all smooth functions with a compact support in M.

Since (2.5), Lin [14] proved the following.

Lemma 2.2 [14, *Lemma* 2.1] *Let* (M, g) *be a complete, simply connected, locally conformally flat Riemannian manifold. If* $R \le 0$ *or* $\int_M |R|^{n/2} < \infty$ *then the following Sobolev inequality*

$$\left(\int_M f^{2n/(n-2)}\right)^{(n-2)/n} \le S\int_M |\nabla f|^2 \tag{2.6}$$

holds for all $f \in C_0^\infty(M)$ *with some constant* $S > 0$, *which is equal to* $Q(\mathbb{S}^n)^{-1}$ *in the case* $R \le 0$. *In particular,* M *has infinite volume.*

3 Proof of theorem 1.4

Let ω be arbitrary harmonic p-form on M^n with finite L^2 norm. Since the fact that

$$\frac{1}{2}\Delta|\omega|^2 = |\omega|\Delta|\omega| + |\nabla|\omega||^2, \tag{3.1}$$

inequality (2.3) implies

$$|\omega|\Delta|\omega| + ||\nabla|\omega||^2 \ge |\nabla\omega|^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E||\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2.$$

By applying Lemma 2.1, we have

$$|\omega|\Delta|\omega| \ge k_p|\nabla|\omega||^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E||\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2. \tag{3.2}$$

Choose a smooth nonnegative function φ with a compact support in M. Multiplying both sides of inequality (3.2) by φ^2 and integrating by parts over M gives

$$-\int_M \left\langle \nabla(|\omega|\varphi^2), \nabla|\omega|\right\rangle \geq \int_M k_p\varphi^2|\nabla|\omega||^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}\int_M |E||\omega|^2\varphi^2 + \frac{p(n-p)}{n(n-1)}\int_M R|\omega|^2\varphi^2. \tag{3.3}$$

On the other hand, by applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_M \left\langle \nabla(|\omega|\varphi^2), \nabla|\omega|\right\rangle &= \int_M |\nabla|\omega||^2\varphi^2 + 2\int_M \varphi|\omega|\left\langle\nabla\varphi, \nabla|\omega|\right\rangle \\ &\geq \int_M |\nabla|\omega||^2\varphi^2 - 2\int_M |\varphi||\omega||\nabla\varphi||\nabla|\omega|| \\ &\geq (1-\epsilon)\int_M |\nabla|\omega||^2\varphi^2 - \int_M \frac{1}{\epsilon}|\omega|^2|\nabla\varphi|^2 \end{aligned} \tag{3.4}$$

for any positive ϵ.

Now since $n \geq 3$ and by applying Hölder inequality, Sobolev inequality (2.5) and Cauchy-Schwarz inequality again, we obtain that

$$\begin{aligned}\int_M |E|\varphi^2|\omega|^2 &\leq \left(\int_{\mathrm{supp}(\varphi)} |E|^{n/2}\right)^{2/n}\left(\int_M (\varphi|\omega|)^{2n/(n-2)}\right)^{(n-2)/n} \\ &\leq \phi(E)\left(\int_M |\nabla(\varphi|\omega|)|^2 + \frac{n-2}{4(n-1)}\int_M R\varphi^2|\omega|^2\right) \\ &\leq \phi(E)\left((1+\epsilon)\int_M \varphi^2|\nabla|\omega||^2 + \left(1+\frac{1}{\epsilon}\right)\int_M |\omega|^2|\nabla\varphi|^2\right) \\ &\quad + \phi(E)\frac{n-2}{4(n-1)}\int_M R\varphi^2|\omega|^2 \end{aligned} \tag{3.5}$$

for any positive ϵ, where $\phi(E) := \frac{1}{Q(\mathbb{S}^n)}\left(\int_{\mathrm{supp}(\varphi)} |E|^{n/2}\right)^{2/n}$. Together (3.3) with (3.4) and (3.5), we obtain

$$\begin{aligned}&\left(1+k_p-\epsilon-(1+\epsilon)\Phi(E)\right)\int_M |\nabla|\omega||^2\varphi^2 \\ &\quad + \left(\frac{p(n-p)}{n(n-1)} - \Phi(E)\frac{n-2}{4(n-1)}\right)\int_M R\varphi^2|\omega|^2 \\ &\quad \leq \left(\frac{1}{\epsilon} + \left(1+\frac{1}{\epsilon}\right)\Phi(E)\right)\int_M |\omega|^2|\nabla\varphi|^2, \end{aligned} \tag{3.6}$$

where $\Phi(E) := \frac{p|n-2p|}{n-2} \cdot \sqrt{\frac{n-1}{n}}\phi(E) = \frac{p|n-2p|}{n-2} \cdot \sqrt{\frac{n-1}{n}} \cdot \frac{1}{Q(\mathbb{S}^n)} \left(\int_{\mathrm{supp}(\varphi)} |E|^{n/2}\right)^{2/n}$. By the assumption, we have

$$\Phi(E) < \frac{4p(n-p)}{n(n-2)}.$$

This implies that

$$\frac{p(n-p)}{n(n-1)} - \Phi(E)\frac{n-2}{4(n-1)} > 0. \tag{3.7}$$

Now by inequality (2.5), we get

$$-\frac{n-2}{4(n-1)}\int_M Rf^2 \leq \int_M |\nabla f|^2 \tag{3.8}$$

for all $f \in C_0^\infty(M)$. Replacing f by $\varphi|\omega|$ in (3.8) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} -\frac{n-2}{4(n-1)}\int_M R\varphi^2|\omega|^2 &\leq \int_M |\nabla(\varphi|\omega|)|^2 \\ &\leq (1+\epsilon)\int_M \varphi^2|\nabla|\omega||^2 + \left(1+\frac{1}{\epsilon}\right)\int_M |\omega|^2|\nabla\varphi|^2 \end{aligned} \tag{3.9}$$

with any $\epsilon > 0$. Then by (3.7), inequality (3.9) implies

$$\begin{aligned} &\left(\frac{p(n-p)}{n(n-1)} - \Phi(E)\frac{n-2}{4(n-1)}\right)\int_M R\varphi^2|\omega|^2 \\ &\qquad \geq -\left(\frac{4p(n-p)}{n(n-2)} - \Phi(E)\right)\left((1+\epsilon)\int_M \varphi^2|\nabla|\omega||^2 \right. \\ &\qquad\quad \left. + \left(1+\frac{1}{\epsilon}\right)\int_M |\omega|^2|\nabla\varphi|^2\right). \end{aligned} \tag{3.10}$$

Together (3.6) with (3.10), we get

$$C_\varepsilon \int_M |\nabla|\omega||^2\varphi^2 \leq D_\varepsilon \int_M |\omega|^2|\nabla\varphi|^2 \tag{3.11}$$

for any $\varphi \in C_0^\infty(M)$, where

$$C_\varepsilon := 1 + k_p - \varepsilon - (1+\varepsilon)\frac{4p(n-p)}{n(n-2)}$$

and

$$D_\varepsilon := \frac{1}{\epsilon} + \left(1+\frac{1}{\varepsilon}\right)\frac{4p(n-p)}{n(n-2)} > 0.$$

Choose a sufficiently small $\epsilon > 0$ such that if $\frac{4p(n-p)}{n(n-2)} < 1 + k_p$ then $C_\varepsilon > 0$.

Fix a point $x_0 \in M$ and let $\zeta(x)$ be the geodesic distance on M from x_0 to x. Let us choose a nonnegative smooth function φ which is called the cut-off function such that

$$\varphi = \begin{cases} 1, & \text{if } \zeta(x) \leq r, \\ 0, & \text{if } 2r \leq \zeta(x), \end{cases}$$

and $|\nabla\varphi| \leq \frac{2}{r}$. Then inequality (3.11) implies

$$C_\varepsilon \int_M |\nabla|\omega||^2 \leq \frac{4D_\varepsilon}{r^2} \int_M |\omega|^2.$$

Letting $r \to \infty$, this inequality yields that $|\omega| \in L^2(M)$ is constant on M.

Assume that $|\omega|$ is a nonzero constant on M, then $\mathrm{Vol}(M) < \infty$. For each $r > 0$, choose a cut-off function φ_r as in above and substitute it into (2.6) to obtain

$$0 < Q(\mathbb{S}^n)(\mathrm{Vol}(B(r)))^{(n-2)/n} \leq \frac{4}{r^2}\mathrm{Vol}(M) + \frac{n-2}{4(n-1)} \lim_{r\to\infty} \int_M R\varphi_r^2.$$

Letting $r \to \infty$, we get

$$0 < Q(\mathbb{S}^n)(\mathrm{Vol}(M))^{(n-2)/n} \leq \frac{n-2}{4(n-1)} \lim_{r\to\infty} \int_M R\varphi_r^2,$$

which yields that $\lim\limits_{r\to\infty} \int_M R\varphi_r^2 > 0$.

On the other hand, after substituting the cut-off function φ_r into (3.6) and noting that $|\omega|$ is a nonzero constant, we deduce that

$$\left(\frac{p(n-p)}{n(n-1)} - \Phi(E)\frac{n-2}{4(n-1)}\right) \lim_{r\to\infty} \int_M R\varphi_r \leq \frac{4}{r^2}\left(\frac{1}{\epsilon} + (1+\frac{1}{\epsilon})\Phi(E)\right)\mathrm{Vol}(M).$$

Letting $r \to \infty$, we get $\lim\limits_{r\to\infty} \int_M R\varphi_r \leq 0$ which is a contradiction. Therefore, ω must be a zero constant on M. This helps us obtain the conclusion of Theorem 1.4. □

4 Proof of theorem 1.5

Let ω be any harmonic p-form on M with finite L^2 norm. It follows from (2.1) and (2.4) that

$$\frac{1}{2}\Delta|\omega|^2 \geq |\nabla\omega|^2 - |E||\omega|^2 - \frac{|R|}{\sqrt{n}}|\omega|^2.$$

Using (3.1) and Lemma 2.1, above inequality implies that

$$|\omega|\Delta|\omega| \geq k_p|\nabla|\omega||^2 - |E||\omega|^2 - \frac{|R|}{\sqrt{n}}|\omega|^2.$$

By multiplying both sides of this inequality by φ^2 where φ is a nonnegative compact support on M and integrating by parts over M, we get

$$-\int_M \left\langle \nabla(|\omega|\varphi^2), \nabla|\omega|\right\rangle \geq k_p \int_M \varphi^2|\nabla|\omega||^2 - \int_M \varphi^2|E||\omega|^2 - \int_M \frac{|R|}{\sqrt{n}}\varphi^2|\omega|^2. \tag{4.1}$$

Together this with (3.4) and (3.5) and noting that $R \leq 0$, we get

$$\begin{aligned}\left(1 + k_p - \epsilon - (1+\epsilon)\phi(E)\right) \int_M |\nabla|\omega||^2\varphi^2 + \left(\frac{1}{\sqrt{n}} - \phi(E)\frac{n-2}{4(n-1)}\right)\int_M R\varphi^2|\omega|^2 \\ \leq \left(\frac{1}{\epsilon} + (1+\frac{1}{\epsilon})\phi(E)\right)\int_M |\omega|^2|\nabla\varphi|^2.\end{aligned} \tag{4.2}$$

Since the assumption, it is easy to see that $\frac{1}{\sqrt{n}} - \phi(E)\frac{n-2}{4(n-1)} \geq 0$. Combining (4.2) and (3.9) gives

$$\begin{aligned}\left(1 + k_p - \epsilon - (1+\epsilon)\frac{4(n-1)}{\sqrt{n}(n-2)}\right)\int_M |\nabla|\omega||^2\varphi^2 \\ \leq \left(\frac{1}{\epsilon} + (1+\frac{1}{\epsilon})\frac{4(n-1)}{\sqrt{n}(n-2)}\right)\int_M |\omega|^2|\nabla\varphi|^2.\end{aligned}$$

By the assumption of the theorem, we can choose an enough small $\epsilon > 0$ such that

$$1 + k_p - \epsilon - (1+\epsilon)\frac{4(n-1)}{\sqrt{n}(n-2)} > 0.$$

Using the same arguments as in the proof of Theorem 1.4, we can show that $|\omega|$ is a constant on M. If $|\omega|$ is a nonzero constant then M has finite volume. But, Lemma 2.2 implies from $R \leq 0$ that M has infinity volume. This is a contradiction. Hence, ω must be a zero constant. The proof of Theorem 1.5 is complete. □

5 Proof of theorem 1.7

Let ω be arbitrary harmonic p-form on M with finite L^2 norm. By applying Hölder inequality, Sobolev inequality (2.6) in Lemma 2.2 and Cauchy-Schwarz inequality, we obtain that

$$\int_M |E|\varphi^2|\omega|^2 \leq \left(\int_{\text{supp}(\varphi)} |E|^{n/2}\right)^{2/n}\left(\int_M \left(\varphi|\omega|\right)^{2n/(n-2)}\right)^{(n-2)/n}$$

$$\begin{aligned} &\leq \phi(E)\int_M |\nabla(\varphi|\omega|)|^2 \\ &\leq \phi(E)\left((1+\epsilon)\int_M \varphi^2|\nabla|\omega||^2 + \left(1+\frac{1}{\epsilon}\right)\int_M |\omega|^2|\nabla\varphi|^2\right), \end{aligned} \tag{5.1}$$

for any positive ϵ. Together (5.1) with (3.4) and (3.9) and (4.1), we get

$$\begin{aligned} &\left(1+k_p-\epsilon-(1+\epsilon)(\phi(E)+\frac{4(n-1)}{\sqrt{n}(n-2)})\right)\int_M |\nabla|\omega||^2\varphi^2 \\ &\qquad \leq \left(\frac{1}{\epsilon}+(1+\frac{1}{\epsilon})(\phi(E)+\frac{4(n-1)}{\sqrt{n}(n-2)})\right)\int_M |\omega|^2|\nabla\varphi|^2. \end{aligned} \tag{5.2}$$

By the assumption, we can take an enough $\epsilon > 0$ such that

$$1+k_p-\epsilon-(1+\epsilon)(\phi(E)+\frac{4(n-1)}{\sqrt{n}(n-2)}) > 0.$$

Hence, inequality (5.2) and the proof of Theorem 1.5 help us obtain the conclusion of Theorem 1.7. □

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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