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Even Character Degrees and Ito–Michler Theorem

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Abstract
Let $\mathrm{Irr}_2(G)$ be the set of linear and even-degree irreducible characters of a finite group G. In this paper, we prove that G has a normal Sylow 2-subgroup if $\sum_{\chi\in\mathrm{Irr}_2(G)}\chi(1)^m/\sum_{\chi\in\mathrm{Irr}_2(G)}\chi(1)^{m-1} < (1+2^{m-1})/(1+2^{m-2})$ for a positive integer m, which is the generalization of several recent results concerning the well-known Ito–Michler theorem.

Keywords Character degrees · Sylow subgroups

Mathematics Subject Classification 20C15

1 Introduction

For a finite group G, let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G. We write

$$\mathrm{Irr}_2(G) := \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1 \text{ or } 2 \mid \chi(1)\}$$

and

$$\mathrm{S}_2^m(G) := \sum_{\chi\in\mathrm{Irr}_2(G)}\chi(1)^m/\sum_{\chi\in\mathrm{Irr}_2(G)}\chi(1)^{m-1},$$

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where m is a positive integer. Ito–Michler theorem [3, 5] states that if a prime p does not divide the degree of every irreducible character of a finite group G, then G has a normal abelian Sylow p-subgroup. For the prime $p = 2$, the $\mathrm{S}_2^m(G)$-version of the Ito–Michler theorem asserts that if $\mathrm{S}_2^m(G) = 1$, then G has a normal abelian Sylow 2-subgroup. In this paper, we improve this for $p = 2$.

Main theorem. Let G inite group with $\mathrm{S}_2^m(G) < (1 + 2^{m-1})/(1 + 2^{m-2})$. Then, G has a normal Sylow 2-subgroup.

We note that $(1 + 2^{m-1})/(1 + 2^{m-2})$ is the exactly value of $\mathrm{S}_2^m(G)$ with $G = S_3$, the symmetric group of degree 3, and S_3 does not have a normal Sylow 2-subgroup. For G, the nonabelian group of order 8 or the alternating group A_4 of degree 4, the value of $\mathrm{S}_2^m(G)$ respectively is $(1+2^{m-2})/(1+2^{m-3})$ and 1, which are both less than $\mathrm{S}_2^m(S_3)$, indeed G has a normal Sylow 2-subgroup.

For $m = 1, 2$, we obtain the following corollary.

Corollary 1.1 *Let G be a finite group. If* $\mathrm{S}_2^1(G) < 4/3$ *or* $\mathrm{S}_2^2(G) < 3/2$, *then G has a normal Sylow* 2*-subgroup.*

Proof See [1, Theorem 1.1] and [6, Theorem A]. □

In the following of this paper, we prove the Main theorem in Sect. 2, and discuss other variations of the Main theorem in Sect. 3.

2 Proof of Main Theorem

We denote by $n_k(G)$ the number of irreducible complex characters of degree k of G. If N is a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ denotes the set of irreducible characters of G that lie over θ. We write $I_G(\theta)$ to denote the inertia subgroup of θ in G. For an M-invariant subgroup N, we write $M \ltimes N$ to denote a semidirect of M and N.

The first lemma is the following observation.

Lemma 2.1 *Let N be a normal subgroup of a group G contained in the derived subgroup G′ of G. If* $\mathrm{S}_2^m(G) \leqslant 2$, *then* $\mathrm{S}_2^m(G/N) \leqslant \mathrm{S}_2^m(G)$.

Proof We write $R = \mathrm{S}_2^m(G)$. Then, $1 \leqslant R \leqslant 2$ and

$$R = \frac{n_1(G) + \sum_{2|k} k^m n_k(G)}{n_1(G) + \sum_{2|k} k^{m-1} n_k(G)}.$$

It follows that

$$\sum_{2|k} (k - R) k^{m-1} n_k(G) = (R - 1) n_1(G).$$

Since $\mathrm{Irr}(G/N) \subseteq \mathrm{Irr}(G)$ and $N \subseteq G'$, we have

$$\sum_{2|k}(k-R)k^{m-1}n_k(G/N) \leqslant \sum_{2|k}(k-R)k^{m-1}n_k(G)$$

and $n_1(G) = n_1(G/N)$. Hence $\sum_{2|k}(k-R)k^{m-1}n_k(G/N) \leqslant (R-1)n_1(G/N)$, which is equivalent to $\mathrm{S}_2^m(G/N) \leqslant R = \mathrm{S}_2^m(G)$. □

Lemma 2.2 *For a normal subgroup N of G, let $\chi \in \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$ with degree 2. If $G/\ker(\chi)$ is nonsolvable, then χ is a primitive character of $G/\ker(\chi)$.*

Proof We write $\overline{G} = G/\ker(\chi)$, and suppose that χ is not a primitive character of $\overline{G}$. Then, there is a proper subgroup $\overline{H}$ of $\overline{G}$ and some character ϕ of $\overline{H}$ such that $\chi = \phi^{\overline{G}}$. It follows that $2 = \chi(1) = |\overline{G} : \overline{H}|\phi(1)$, $|\overline{G} : \overline{H}| = 2$ and $\phi(1) = 1$. Hence, $\overline{H}$ is normal in $\overline{G}$ and all the irreducible constituents of $\chi_{\overline{H}}$ are linear. Then, $[\overline{H}, \overline{H}] \leqslant \ker(\chi) = 1$, and $\overline{H}$ is abelian. So $\overline{G}$ is solvable, a contradiction. □

The following lemma implies that G in our main theorem is solvable.

Lemma 2.3 *Let G be a finite group with $\mathrm{S}_2^m(G) < (1+4^m)/(1+4^{m-1})$. Then, G is solvable.*

Proof Suppose that G is nonsolvable and let G be a counterexample of minimal order. Let A be a minimal nonsolvable normal subgroup of G, and let N be a minimal normal subgroup of G contained in A. Then, $N \subseteq A = A' \subseteq G'$. If in addition $[A, \mathrm{Rad}(A)] > 1$, then we can choose $N \subseteq [A, \mathrm{Rad}(A)]$, where $\mathrm{Rad}(A)$ is the solvable radical (i.e., the unique largest solvable normal subgroup) of A. We now discuss the following two cases: N is abelian or not.

(1). N is abelian.

Then, G/N is nonsolvable. Since $|G/N| < |G|$, we have $\mathrm{S}_2^m(G) < (1+4^m)/(1+4^{m-1}) \leqslant \mathrm{S}_2^m(G/N)$. Hence, we obtain

$$\sum_{2|k,k\geqslant 4}(k-1+4^{m-1}k-4^m)k^{m-1}n_k(G) < 3\cdot 4^{m-1}n_1(G) + 2^{m-1}(2\cdot 4^{m-1}-1)n_2(G)$$

and

$$\begin{aligned}&3\cdot 4^{m-1}n_1(G/N) + 2^{m-1}(2\cdot 4^{m-1}-1)n_2(G/N)\\ &\leqslant \sum_{2|k,k\geqslant 4}(k-1+4^{m-1}k-4^m)k^{m-1}n_k(G/N).\end{aligned}$$

Since $n_1(G/N) = n_1(G)$ and $n_k(G/N) \leqslant n_k(G)$ for $k \geqslant 2$, we obtain $n_2(G/N) < n_2(G)$. Now there is $\chi \in \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$ such that $\chi(1) = 2$. Since $N \nsubseteq \ker(\chi)$, we know that $A\ker(\chi)/\ker(\chi)$ is a nontrivial subgroup of $G/\ker(\chi)$, and thus $G/\ker(\chi)$

is nonsolvable. Now by Lemma 2.2, we know that $G/\ker(\chi)$ is a nonsolvable primitive linear group of degree 2. Write $C/\ker(\chi) = \mathrm{Z}(G/\ker(\chi))$. According to the proof in [6, Theorem 3.1], we know that $G = AC$ is a central product, $G/C \cong A_5$, $N = A \cap C \cong \mathbb{Z}_2$, $A \cong SL(2,5)$, $n_1(G) = n_1(C/N)$, $n_2(G) = n_2(C/N) + 2n_1(G)$, $n_4(G) \geqslant 2n_1(G)$, and $n_6(G) \geqslant n_1(G) + 2n_2(C/N)$. Now

$$\begin{aligned}
&\sum_{2|k,k\geqslant 4} (k-1+4^{m-1}k-4^m)k^{m-1}n_k(G)\\
&\quad\geqslant 3\cdot 4^{m-1}n_4(G) + (5+2\cdot 4^{m-1})\cdot 6^{m-1}n_6(G)\\
&\quad\geqslant 6\cdot 4^{m-1}n_1(G) + (5+2\cdot 4^{m-1})6^{m-1}\cdot (n_1(G)+2n_2(C/N))\\
&\quad= (6\cdot 4^{m-1}+5\cdot 6^{m-1}+2\cdot 24^{m-1})n_1(G) + (10+4^m)6^{m-1}n_2(C/N)\\
&\quad= (6\cdot 4^{m-1}+5\cdot 6^{m-1}+2\cdot 24^{m-1})n_1(G) + (10+4^m)6^{m-1}(n_2(G)-2n_1(G))\\
&\quad= (6\cdot 4^{m-1}-15\cdot 6^{m-1}-6\cdot 24^{m-1})n_1(G) + (10+4^m)6^{m-1}n_2(G)\\
&\quad= (6\cdot 4^{m-1}-15\cdot 6^{m-1}-6\cdot 24^{m-1})n_1(G) + [(10+4^m)6^{m-1}\\
&\qquad -2^{m-1}(2\cdot 4^{m-1}-1)]n_2(G) + 2^{m-1}(2\cdot 4^{m-1}-1)n_2(G)\\
&\quad\geqslant (6\cdot 4^{m-1}-15\cdot 6^{m-1}-6\cdot 24^{m-1})n_1(G) + 2[(10+4^m)6^{m-1}\\
&\qquad -2^{m-1}(2\cdot 4^{m-1}-1)]n_1(G) + 2^{m-1}(2\cdot 4^{m-1}-1)n_2(G)\\
&\quad= (5\cdot 6^{m-1}+2\cdot 24^{m-1}+6\cdot 4^{m-1}-4\cdot 8^{m-1}+2^m)n_1(G)\\
&\qquad +2^{m-1}(2\cdot 4^{m-1}-1)n_2(G)\\
&\quad\geqslant 3\cdot 4^{m-1}n_1(G) + 2^{m-1}(2\cdot 4^{m-1}-1)n_2(G).
\end{aligned}$$

Hence, $\mathrm{S}_2^m(G) \geqslant (1+4^m)/(1+4^{m-1})$, a contradiction.

(2). N is not abelian.

Assume that $N \ncong A_5$. By [1, Theorem 2.2], there exists $\phi \in \mathrm{Irr}(N)$ of even degree such that $\phi(1) \geqslant 8$ and ϕ is extendible to $I := I_G(\phi)$. By [1, Proposition 2.3], we have $n_1(G) \leqslant n_d(G)|G:I|$ and $n_2(G) \leqslant n_{2d}(G)|G:I| + \frac{1}{2}n_d(G)|G:I|$, where $d = \phi(1)|G:I| \geqslant 8|G:I|$. Now

$$\begin{aligned}
&3\cdot 4^{m-1}n_1(G) + 2^{m-1}(2\cdot 4^{m-1}-1)n_2(G)\\
&\quad\leqslant 3\cdot 4^{m-1}n_d(G)|G:I| + 2^{m-1}(2\cdot 4^{m-1}-1)n_{2d}(G)|G:I|\\
&\qquad + 2^{m-2}(2\cdot 4^{m-1}-1)n_d(G)|G:I|\\
&\quad= (3\cdot 4^{m-1}+8^{m-1}-2^{m-2})n_d(G)|G:I| + 2^{m-1}(2\cdot 4^{m-1}-1)n_{2d}(G)|G:I|\\
&\quad\leqslant (3\cdot 4^{m-1}+8^{m-1}-2^{m-2})n_d(G)\frac{d}{8} + 2^{m-1}(2\cdot 4^{m-1}-1)n_{2d}(G)\frac{d}{8}\\
&\quad= \left(\frac{3}{2}\cdot 4^{m-2}+8^{m-2}-2^{m-5}\right)dn_d(G) + 2^{m-4}(2\cdot 4^{m-1}-1)dn_{2d}(G)\\
&\quad\leqslant (d-1+4^{m-1}\cdot d-4^m)d^{m-1}n_d(G) + (2d-1+4^{m-1}\cdot 2d-4^m)(2d)^{m-1}n_{2d}(G)\\
&\quad\leqslant \sum_{2|k,k\geqslant 4} (k-1+4^{m-1}k-4^m)k^{m-1}n_k(G),
\end{aligned}$$

so we have $S_2^m(G) \geqslant (1+4^m)/(1+4^{m-1})$, a contradiction.

Assume that $N \cong A_5$. Since the irreducible character of N of degree 4 is extendible to G, by [1, Proposition 2.3] again, we have $n_1(G) \leqslant n_4(G)$ and $n_2(G) \leqslant n_8(G)$. Now

$$\begin{aligned}
&3 \cdot 4^{m-1} n_1(G) + 2^{m-1}(2 \cdot 4^{m-1} - 1) n_2(G) \\
&\quad \leqslant 3 \cdot 4^{m-1} n_4(G) + 2^{m-1}(2 \cdot 4^{m-1} - 1) n_8(G) \\
&\quad \leqslant 3 \cdot 4^{m-1} n_4(G) + (7 + 4^m) 8^{m-1} n_8(G) \\
&\leqslant \sum_{2|k, k \geqslant 4} (k - 1 + 4^{m-1}k - 4^m) k^{m-1} n_k(G),
\end{aligned}$$

and thus $S_2^m(G) \geqslant (1+4^m)/(1+4^{m-1})$. This contradiction completes the proof. □

Lemma 2.4 *Let $G = M \ltimes N$. If λ is a G-invariant linear character of N, then λ is extendible to G.*

Proof Clearly, we may assume that $\ker(\lambda) = 1$. Then, N is cyclic. Let $\chi \in \mathrm{Irr}(\lambda^G)$, then $\chi_N = \chi(1)\lambda$. Hence, $N \subseteq Z(\chi) = \{g \in G \big| |\chi(g)| = \chi(1)\}$ and $N\ker(\chi)/\ker(\chi) \subseteq Z(\chi)/\ker(\chi) = Z(G/\ker(\chi))$. Then, $[G, N] \subseteq \ker(\chi)$. It follows that $[G, N] \subseteq N \cap \ker(\chi) = \ker(\lambda) = 1$. Hence, $N \subseteq Z(G)$. Now $G = M \times N$. Let $\psi = 1_m \times \lambda$. Then, $\psi \in \mathrm{Irr}(G)$ and $\psi_N = \lambda$. So λ is extendible to G. □

The following lemma is important in the proof of our Main theorem.

Lemma 2.5 *Let $G = M \ltimes N$, where $N \leqslant G'$ is an abelian group. Assume that no nontrivial irreducible character of N is invariant under M. If $S_2^m(G) < (1 + 2^{m-1})/(1 + 2^{m-2})$, then there is no orbit of even size in the action of M on the set of irreducible characters of N.*

Proof Let $\{\theta_0 = 1_N, \theta_1, \cdots, \theta_t\}$ be a set of representatives of M-orbits on $\mathrm{Irr}(N)$. Let $I_i = I_G(\theta_i)$ for $i \in [1, t]$. By hypothesis, we have $I_i < G$ for $i \geqslant 1$. Suppose that there is some orbit of even size in the action of M on $\mathrm{Irr}(N)$. Then, we can find an integer k such that $2 \mid |G : I_k|$. For $0 \leqslant i \leqslant t$, we set $n_{i,1} = n_1(I_i/N)$ and $T_{i,m} = \sum_{\lambda \in \mathrm{Irr}(I_i/N), 2|\lambda(1)} \lambda(1)^m$.

By Lemma 2.4, every θ_i has an extension ψ to I_i. By Gallagher theorem [2, Corollary 6.17], we have bijections $\lambda \mapsto \lambda\psi_i$ from $\mathrm{Irr}(I_i/N)$ to $\mathrm{Irr}(I_i|\theta_i)$. By Clifford correspondence, we have a bijections $\lambda\psi_i \mapsto (\lambda\psi_i)^G$ from $\mathrm{Irr}(I_i|\theta_i)$ to $\mathrm{Irr}(G|\theta_i)$. Observe that $(\lambda\theta_i)^G(1) = |G : I_i|\lambda(1)$ is even if and only if $|G : I_i|$ is even or $|G : I_i|$ is odd and $\lambda(1)$ is even. Then,

$$\sum_{\chi \in \mathrm{Irr}_2(G)} \chi(1)^{m-1} = n_1(G/N) + \sum_{2||G:I_i|} |G : I_i|^{m-1} \sum_{\lambda \in \mathrm{Irr}(I_i/N)} \lambda(1)^{m-1}$$
$$+ \sum_{2\nmid|G:I_i|} |G : I_i|^{m-1} T_{i,m-1}$$

and

$$\sum_{\chi \in \mathrm{Irr}_2(G)} \chi(1)^m = n_1(G/N) + \sum_{2 \mid |G:I_i|} |G:I_i|^m \sum_{\lambda \in \mathrm{Irr}(I_i/N)} \lambda(1)^m + \sum_{2 \nmid |G:I_i|} |G:I_i|^m \mathrm{T}_{i,m}$$
$$\geqslant n_1(G/N) + \sum_{2 \mid |G:I_i|} |G:I_i|^m \sum_{\lambda \in \mathrm{Irr}(I_i/N)} \lambda(1)^{m-1} + \sum_{2 \nmid |G:I_i|} |G:I_i|^m \mathrm{T}_{i,m-1}.$$

From $\sum_{\chi \in \mathrm{Irr}_2(G)} \chi(1)^m = \mathrm{S}_2^m(G) \sum_{\chi \in \mathrm{Irr}_2(G)} \chi(1)^{m-1}$, we have

$$\mathrm{S}_2^m(G)(n_1(G/N) + \sum_{2 \mid |G:I_i|} |G:I_i|^{m-1} \sum_{\lambda \in \mathrm{Irr}(I_i/N)} \lambda(1)^{m-1} + \sum_{2 \nmid |G:I_i|} |G:I_i|^{m-1} \mathrm{T}_{i,m-1})$$
$$\geqslant n_1(G/N) + \sum_{2 \mid |G:I_i|} |G:I_i|^m \sum_{\lambda \in \mathrm{Irr}(I_i/N)} \lambda(1)^{m-1} + \sum_{2 \nmid |G:I_i|} |G:I_i|^m \mathrm{T}_{i,m-1}.$$

Then,

$$\begin{aligned} &\sum_{2 \mid |G:I_i|} (|G:I_i| - \mathrm{S}_2^m(G))|G:I_i|^{m-1} \sum_{\lambda \in \mathrm{Irr}(I_i/N)} \lambda(1)^{m-1} \\ &\quad + \sum_{2 \nmid |G:I_i|} (|G:I_i| - \mathrm{S}_2^m(G))|G:I_i|^{m-1} \mathrm{T}_{i,m-1} \\ &\leqslant (\mathrm{S}_2^m(G) - 1) n_1(G/N). \end{aligned}$$

In particular, for k, we obtain

$$(|G:I_k| - \mathrm{S}_2^m(G))|G:I_k|^{m-1} \sum_{\lambda \in \mathrm{Irr}(I_k/N)} \lambda(1)^{m-1} \leqslant (\mathrm{S}_2^m(G) - 1) n_1(G/N).$$

Since $n_1(G/N) = |G/N : G'/N|$ and $n_{k,1} = n_1(I_k/N) = |I_k/N, (I_k/N)'|$, so $n_1(G/N) \leqslant |G:I_k| n_{k,1}$. In addition, $\sum_{\lambda \in \mathrm{Irr}(I_k/N)} \lambda(1)^{m-1} \geqslant n_1(I_k/N)$. Hence, we get that

$$(|G:I_k| - \mathrm{S}_2^m(G))|G:I_k|^{m-1} n_1(I_k/N) \leqslant (\mathrm{S}_2^m(G) - 1)|G:I_k| n_1(I_k/N).$$

It then follows that

$$\mathrm{S}_2^m(G) \geqslant \frac{|G:I_k|^m + |G:I_k|}{|G:I_k|^{m-1} + |G:I_k|} \geqslant \frac{2^{m-1}+1}{2^{m-2}+1},$$

which is a contradiction. □

We now prove our main theorem, which is restated.

Theorem 2.6 *Let G be a finite group. Suppose that $\mathrm{S}_2^m(G) < (1+2^{m-1})/(1+2^{m-2})$. Then, G has a normal Sylow 2-subgroup.*

Proof Suppose the theorem is false and let G be a counterexample of minimal order. By Lemma 2.3, G is solvable. Let N be a minimal normal subgroup of G contained in the derived subgroup G' of G. Then, N is elementary abelian. By Lemma 2.1, $\mathrm{S}_2^m(G/N) \leqslant \mathrm{S}_2^m(G) < (1+2^{m-1})/(1+2^{m-2})$, and by the choice of G, G/N has a normal Sylow 2-subgroup R/N. If N is a 2-group, then R is a normal Sylow 2-subgroup of G, a contradiction. Hence, N is a $2'$-group. By the Schur–Zassenhaus theorem, $R = P \ltimes N$, where P is a Sylow 2-subgroup of R. By the Frattini argument, $G = RN_G(P) = NN_G(P)$. Clearly, $N \nsubseteq N_G(P)$. Since $N_G(P) \cap N < N$ and $N_G(P) \cap N \trianglelefteq G = NN_G(P)$, we obtain $N_G(P) \cap N = 1$, and thus $G = N_G(P) \ltimes N$. If $N \subseteq Z(G)$, then $R = P \times N$ and $P \trianglelefteq G$, a contradiction. Hence, $N \nsubseteq Z(G)$. Then, $[N, G] = N$. Let $\lambda \in \mathrm{Irr}(N) - 1_N$. Assume that λ is G-invariant. Then by Lemma 2.4, λ has an extension χ to G. Now we have $\chi_N = \lambda$, $N \subseteq \ker(\chi)$, and $\lambda = 1_N$, this contradiction implies that no nontrivial irreducible character of N is G-invariant. By Lemma 2.5, there is no orbit of even size in the action of P on N, that is, P acts trivially on N, hence $R = P \times N$ and $P \trianglelefteq G$, this final contradiction completes the proof. □

3 Variations of Main Theorem

Recall that a character $\chi \in \mathrm{Irr}(G)$ is real if $\chi(g) \in \mathbb{R}$ for every element $g \in G$, and $\chi \in \mathrm{Irr}(G)$ is strongly real if χ is afforded by a real representation, or equivalently, its Frobenius–Schur indicator $\upsilon_2(\chi)$ is 1. We denote by $n_{k,+}$ the number of irreducible strongly real characters of degree k of G. We write

$$\mathrm{Irr}_{2,\mathbb{R}}(G) := \{\chi \in \mathrm{Irr}_2(G) \mid \chi \text{ is real}\},$$

$$\mathrm{S}^m_{2,\mathbb{R}}(G) := \sum_{\chi \in \mathrm{Irr}_{2,\mathbb{R}}(G)} \chi(1)^m \Big/ \sum_{\chi \in \mathrm{Irr}_{2,\mathbb{R}}(G)} \chi(1)^{m-1},$$

$$\mathrm{Irr}_{2,+}(G) := \{\chi \in \mathrm{Irr}_2(G) \mid \upsilon_2(\chi) = 1\},$$

$$\mathrm{S}^m_{2,+}(G) := \sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^m \Big/ \sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^{m-1}.$$

Similar to Lemma 2.1, we have the following observation.

Lemma 3.1 *Let N be a normal subgroup of a group G such that $N \subseteq G'$. If $\mathrm{S}^m_{2,+}(G) \leqslant 2$, then $\mathrm{S}^m_{2,+}(G/N) \leqslant \mathrm{S}^m_{2,+}(G)$.*

Proof Similar to Lemma 2.1. □

For a normal subgroup with odd index, we have the next lemma.

Lemma 3.2 *Let N be a normal subgroup of a group G with G/N odd order. If $\mathrm{S}^m_{2,+}(G) \leqslant 2$, then $\mathrm{S}^m_{2,+}(N) \leqslant \mathrm{S}^m_{2,+}(G)$.*

Proof Firstly, every strongly real linear character of G restricts to a strongly real linear character of N. Secondly, by [4, Lemma 2.1], every strongly real linear character of

N lies under a unique strongly real linear character of G. Thus, we obtain $n_{1,+}(N) = n_{1,+}(G)$. In addition, from [4, Lemma 2.1], we also have

$$\sum_{2|k,k\geqslant 2} n_{k,+}(G) \geqslant \sum_{2|k,k\geqslant 2} n_{k,+}(N).$$

Denote by $V = \mathrm{S}^m_{2,+}(G)$. Then

$$V = \frac{n_{1,+}(G) + \sum_{2|k} k^m n_{k,+}(G)}{n_{1,+}(G) + \sum_{2|k} k^{m-1} n_{k,+}(G)},$$

and we obtain

$$(V-1)n_{1,+}(G) = \sum_{2|k}(k^m - Vk^{m-1})n_{k,+}(G).$$

It then follows that

$$(V-1)n_{1,+}(N) = \sum_{2|k}(k^m - Vk^{m-1})n_{k,+}(G) \geqslant \sum_{2|k}(k^m - Vk^{m-1})n_{k,+}(N),$$

which is equivalent to $\mathrm{S}^m_{2,+}(N) \leqslant V = \mathrm{S}^m_{2,+}(G)$, the proof is complete. □

Now we give a variation of Main theorem for strongly real character.

Theorem 3.3 *Let G be a finite group. Suppose that $\mathrm{S}^m_{2,+}(G) < (1+2^{m-1})/(1+2^{m-2})$. Then, G has a normal Sylow $2-$subgroup.*

Proof Suppose that the theorem is not true, and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G contained in G'. Observe that

$$\begin{aligned}
\frac{\sum_{\chi\in \mathrm{Irr}_{2,+}(G)} \chi(1)}{|\mathrm{Irr}_{2,+}(G)|} &\leqslant \frac{n_{1,+}(G) + \sum_{2|k} k^2 n_{k,+}(G)}{n_{1,+}(G) + \sum_{2|k} k n_{k,+}(G)} \\
&\leqslant \cdots \\
&\leqslant \frac{n_{1,+}(G) + \sum_{2|k} k^m n_{k,+}(G)}{n_{1,+}(G) + \sum_{2|k} k^{m-1} n_{k,+}(G)} \\
&= \mathrm{S}^m_{2,+}(G) \\
&< \frac{1+2^{m-1}}{1+2^{m-2}} < 2,
\end{aligned}$$

by [1, Theorem 5.1] and Lemma 3.1, we know that G is solvable, N is an elementary abelian group with odd order, G/N has a normal Sylow 2-subgroup R/N, and $G = R = P \ltimes N$, where $P \in \mathrm{Syl}_2(G)$.

From the proof of [6, Theorem 5.1], we have that $n_{1,+}(G) \leqslant |N| - 1 \leqslant \chi_1(1) + \cdots + \chi_s(1)$, where $\chi_1, \cdots, \chi_s$ are the strongly real irreducible characters of G. Now

$$\begin{aligned}\frac{n_{1,+}(G) + \sum_{i=1}^{s} \chi_i(1)^m n_{\chi_i(1),+}}{n_{1,+}(G) + \sum_{i=1}^{s} \chi_i(1)^{m-1} n_{\chi_i(1),+}} &\geqslant \frac{n_{1,+}(G) + \sum_{i=1}^{s} \chi_i(1)^m}{n_{1,+}(G) + \sum_{i=1}^{s} \chi_i(1)^{m-1}} \\ &\geqslant \frac{n_{1,+}(G) + 2^{m-1} \sum_{i=1}^{s} \chi_i(1)}{n_{1,+}(G) + 2^{m-2} \sum_{i=1}^{s} \chi_i(1)} \\ &\geqslant \frac{n_{1,+}(G) + 2^{m-1} n_{1,+}(G)}{n_{1,+}(G) + 2^{m-2} n_{1,+}(G)} \\ &= \frac{1 + 2^{m-1}}{1 + 2^{m-2}}.\end{aligned}$$

Then for any $\chi \in \mathrm{Irr}_{2,+}(G)$ with $\chi(1) \geqslant 2 \geqslant (1 + 2^{m-1})/(1 + 2^{m-2})$, we have

$$\mathrm{S}_{2,+}^{m}(G) = \frac{\sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^m}{\sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^{m-1}} \geqslant \frac{1 + 2^{m-1}}{1 + 2^{m-2}},$$

and this contradiction completes the proof. □

Note that for any $\chi \in \mathrm{Irr}_{2,\mathbb{R}}(G) - \mathrm{Irr}_{2,+}(G)$, we have $\chi(1) \geqslant 2$. If $\mathrm{S}_{2,+}^{m}(G) \leqslant 2$, then we have

$$\frac{\sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^m}{\sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^{m-1}} \leqslant \frac{\sum_{\chi \in \mathrm{Irr}_{2,\mathbb{R}}(G)} \chi(1)^m - \sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^m}{\sum_{\chi \in \mathrm{Irr}_{2,\mathbb{R}}(G)} \chi(1)^{m-1} - \sum_{\chi \in \mathrm{Irr}_{2,+}(G)} \chi(1)^{m-1}}.$$

Therefore, we obtain the following corollary, which is a variation of Main theorem for real characters.

Corollary 3.4 *Let G be a finite group with $\mathrm{S}_{2,\mathbb{R}}^{m}(G) < (1 + 2^{m-1})/(1 + 2^{m-2})$. Then, G has a normal Sylow $2-$subgroup.*

Declarations

Conflict of interest All authors disclosed no relevant relationships.

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