Communications in Mathematics and Statistics (2022) 10:653–667
https://doi.org/10.1007/s40304-021-00242-x

# On Finite Non-Solvable Groups Whose Gruenberg–Kegel Graphs are Isomorphic to the Paw

**A. S. Kondrat'ev[1] · N. A. Minigulov[1]**

Received: 14 August 2020 / Revised: 8 March 2021 / Accepted: 29 March 2021 /
Published online: 13 September 2021


**Abstract**

The Gruenberg–Kegel graph (or the prime graph) $\Gamma(G)$ of a finite group $G$ is a graph, in which the vertex set is the set of all prime divisors of the order of $G$ and two different vertices $p$ and $q$ are adjacent if and only if there exists an element of order $pq$ in $G$. The paw is a graph on four vertices whose degrees are 1, 2, 2, 3. We consider the problem of describing finite groups whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw. For example, the Gruenberg–Kegel graph of the alternating group $A_{10}$ of degree 10 is isomorphic as abstract graph to the paw. In this paper, we describe finite non-solvable groups $G$ whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw in the case when $G$ has no elements of order 6 or the vertex of degree 1 of $\Gamma(G)$ divides the order of the solvable radical of $G$.

**Keywords** Finite group · Non-solvable group · Gruenberg–Kegel graph · The paw

**Mathematics Subject Classification** 20D06 · 20D60 · 05C25

## 1 Introduction

Let $G$ be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of $G$. If $|\pi(G)| = n$, then $G$ is called $n$-primary. The Gruenberg–Kegel graph (or the prime graph) $\Gamma(G)$ of $G$ is a graph with the vertex set $\pi(G)$, in which two different vertices $p$ and $q$ are adjacent if and only if there exists an element of order $pq$ in $G$. The

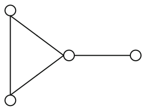
✉ N. A. Minigulov
nikola-minigulov@mail.ru

A. S. Kondrat'ev
a.s.kondratiev@imm.uran.ru

1 N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, 16 S.Kovalevskaya Str, Yekaterinburg 620108, Russia

graph $\Gamma(G)$ is one of significant arithmetical invariants of a group $G$. Studying finite groups by the properties of their Gruenberg–Kegel graphs is an important direction in finite group theory. In some papers, characterizations of finite groups in terms of graph-theoretical properties of their Gruenberg–Kegel graphs were obtained (see, for example, [1,2,9,10,19–21,26,33–35]). This paper is of such type.

The first author described finite groups with the same Gruenberg–Kegel graph as groups $Aut(J_2)$ [17] and $A_{10}$ [18], respectively. The Gruenberg–Kegel graphs of all these groups are isomorphic as abstract graphs to the paw, i. e., a graph on four vertices whose degrees are 1, 2, 2, 3. Thus, the paw has the following form:

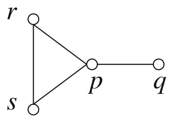


.

We establish a more general problem: to describe finite groups whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw.

As a part of the solution of this problem, we have proved in [23] that if $G$ is a finite non-solvable group and graph $\Gamma(G)$ as an abstract graph is isomorphic to the paw, then the quotient group $G/S(G)$ (where $S(G)$ is the solvable radical of $G$) is almost simple, and we have classified all finite almost simple groups whose the Gruenberg–Kegel graphs as abstract graphs are isomorphic to subgraphs of the paw. In this paper, we continue the study of the problem.

Let $G$ be a finite non-solvable group, and the graph $\Gamma(G)$ as an abstract graph is isomorphic to the paw. Then, the graph $\Gamma(G)$ has the following form:




,

where $p$, $q$, $r$, and $s$ are some pairwise distinct primes.

Let $S = S(G) \neq 1$, and $\overline{G} = G/S$. By [23, Theorem 1], $\overline{G}$ is almost simple. In this paper, we consider the case when $G$ has no elements of order 6 or $q$ divides $|S|$. We prove the following three theorems.

**Theorem 1.1** *If* 3 *does not divide* $|G|$, *then up to permutation of the numbers* $r$ *and* $s$ *one of the following statements holds*:

(1) $\overline{G} \cong Aut(Sz(32))$, $\{r, s\} = \{2, 5\}$, $\{p, q\} = \{31, 41\}$, $p \in \pi(S) \subseteq \{2, p\}$, $O_{2',2}(S)/O(S)$ *is an elementary abelian* 2-*group*, $F^*(G/O_{2',2}(S)) = P \times E$, *where* $P$ *is a* $p$-*group and* $E \cong Sz(32)$, *and either* $S = O(G)$ *or the group* $E$ *induces on* $O_{2',2}(S)/O_{2'}(S)$ *a direct sum of modules*, *each of which is isomorphic to the natural* 4-*dimensional* $GF(32)Sz(32)$-*module*;

(2) $\overline{G} \cong Sz(8)$, $r = 2$, $\{p, s\} = \{5, 7\}$, $q = 13$, $\pi(S) = \{2, p\}$, *every* 2-*chief factor of* $G$ *as a* $\overline{G}$-*module is isomorphic to the* 4-*dimensional or the* 16-*dimensional irreducible* $GF(8)Sz(8)$-*module*; *moreover*, *the second possibility always appears*;

(3) $\overline{G} \cong Sz(32)$ *or* $Aut(Sz(32))$, $r = 2$, $\{p, s\} \subseteq \{5, 31, 41\}$, $q \in \{31, 41\}$, $\pi(S) = \{2, p\}$, *every* 2*-chief factor of* $G'$ *as a* $\overline{G}'$*-module is isomorphic to the* 4*-dimensional, one of two* 16*-dimensional, or one of two* 64*-dimensional irreducible* $GF(32)Sz(32)$*-modules*;

(4) $\overline{G} \cong Sz(8)$, $\{r, s\} = \{5, 7\}$, $p = 2$, $q = 13$, $5 \in \pi(S) \subseteq \{5, p\}$, $G/O^2(S) = P \circ E$, *where* $P$ *is a* 2*-group and* $E \cong 2^{\cdot}Sz(8)$ *or* $(2 \times 2)^{\cdot}Sz(8)$, *and the group* $E$ *induces on every* 5*-chief factor of* $O^2(G)$ *the faithful irreducible 8-dimensional* $GF(5)2^{\cdot}Sz(8)$*-module.*

**Remark 1.2** We can prove that all cases from the conclusion of Theorem 1.1 are realizable. Statement (1) holds for a group $\mathbb{Z}_p \times (2^{20} \rtimes Aut(Sz(32)))$, where $p \in \{31, 41\}$. Statement (2) holds for groups $\mathbb{Z}_p \times (2^{12} \rtimes Sz(8))$ and $\mathbb{Z}_p \times (2^{48} \rtimes Sz(8))$, where $p \in \{5, 7\}$. Statement (3) holds for groups $\mathbb{Z}_p \times (2^m \rtimes Sz(32))$ and $\mathbb{Z}_p \times (2^m \rtimes Aut(Sz(32)))$, where $p \in \{5, 31, 41\}$ and $m \in \{20, 80, 320\}$. Statement (4) holds for a group $5^8 \rtimes 2^{\cdot}Sz(8)$.

**Theorem 1.3** *If* 3 *divides* $|G|$ *and* $G$ *has no elements of order* 6, *then one of the following statements holds*:

(1) $q = 2$, $\overline{G} \cong L_2(2^n)$, $S = O_{2',2}(G)$, $O(G) = O_p(G)$, $S/O(G)$ *is an elementary abelian* 2*-group which is either trivial or isomorphic as a* $\overline{G}$*-module to a direct sum of the natural* $GF(2^n)\overline{G}$*-modules, and one of the following statements holds*:

(1*a*) $n = 4$, $p = 17$ *and* $\{r, s\} = \{3, 5\}$;

(1*b*) $n$ *is a prime*, $n \geq 5$, $p = 2^n - 1$, $\{r, s\} = \{3, (2^n + 1)/3\}$;

(2) $q = 2$, $S = O_p(G)$, $\overline{G} \cong L_2(p)$, $p \geq 31$, $p \equiv \varepsilon 5 (mod\ 12)$, $\varepsilon \in \{+, -\}$, $p - \varepsilon 1$ *is a power of* 2, *and* $3 \in \{r, s\} = \pi((p + \varepsilon 1)/2)$;

(3) $q = 3$, $S = O_p(G)$, *and one of the following statements holds*:

(3*a*) $\overline{G} \cong PGL_2(9)$, $p > 5$, *and* $\{r, s\} = \{2, 5\}$;

(3*b*) $\overline{G}$ *is isomorphic to* $L_2(81)$, $PGL_2(81)$ *or* $L_2(81).2_3$, $p = 41$, *and* $\{r, s\} = \{2, 5\}$;

(3*c*) $\overline{G} \cong L_2(3^n)$ *or* $PGL_2(3^n)$, $n$ *is an odd prime*, $p = (3^n - 1)/2$, *and* $\{r, s\} = \pi(3^n + 1)$.

**Remark 1.4** We can prove that all cases from the conclusion of Theorem 1.3 are realizable. Statement (1*a*) holds for the group $\mathbb{Z}_{17} \times L_2(16)$. Statement (1*b*) holds for groups $\mathbb{Z}_{2^n-1} \times L_2(2^n)$, where $n \geq 5$, $2^n - 1$ and $(2^n + 1)/3$ are primes. Statement (2) holds for the group $\mathbb{Z}_{31} \times L_2(31)$. Statement (3*a*) holds for groups $\mathbb{Z}_p \times PGL_2(9)$, where $p > 5$ is a prime. Statement (3*b*) holds for groups $\mathbb{Z}_{41} \times L_2(81)$, $\mathbb{Z}_{41} \times PGL_2(81)$ and $\mathbb{Z}_{41} \times L_2(81).2_3$. Statement (3*c*) holds for groups $\mathbb{Z}_p \times L_2(3^n)$ and $\mathbb{Z}_p \times PGL_2(3^n)$, where $n$ and $(3^n - 1)/2$ are odd primes, and $|\pi(3^n + 1)| = 2$.

In the proof of Theorem 1.3, we use the classification of the finite non-solvable groups without elements of order 6 obtained by the authors in [22, Theorem 2].

**Theorem 1.5** *If* $G$ *contains an element of order* 6 *and* $q$ *divides* $|S|$, *then one of the following statements holds*:

(1) $q$ *does not divide* $|\overline{G}|$, $G/O_p(G) = A \rtimes B$, *where* $A$ *is a non-cyclic abelian* $q$*-group*, $B = O_p(B) \rtimes B_1$, $F^*(B) = O_p(B) \times F^*(B_1)$, *and one of the following statements holds*:

(1a) $F^*(B_1) \cong SL_2(5)$, $p = 3$, *and* $\{r, s\} = \{2, 5\}$;
(1b) $F^*(B_1) \cong SL_2(7)$, $p = 3$, *and* $\{r, s\} = \{2, 7\}$;
(1c) $F^*(B_1) \cong SL_2(9)$, $p = 3$, *and* $\{r, s\} = \{2, 5\}$;
(1d) $F^*(B_1) \cong SL_2(17)$, $p = 3$, *and* $\{r, s\} = \{2, 17\}$;
(1e) $F^*(B_1) \cong SL_2(5)$, $p = 5$, $\{r, s\} = \{2, 3\}$, *and* $AB_1$ *is a Frobenius group with kernel* $A$ *and complement* $B_1$;
(2) $q$ *is a Mersenne or Fermat prime*, $q \geq 31$, $p = 2$, $\pi(q^2 - 1) = \{2, r, s\}$, $S = O_{2,2',2}(S)$, $O_{2,2'}(S)/O_2(S)$ *is a non-cyclic abelian* $q$*-group*, $G/O_{2,2'}(S) = P \circ E$, *where* $P$ *is a* 2*-group*, $E \cong SL_2(q)$, *and the group* $E$ *induces on every* $q$*-chief factor of* $O^2(G)$ *the* 2*-dimensional natural* $GF(q)SL_n(q)$*-module.*

**Remark 1.6** We can prove that all cases from the conclusion of Theorem 1.5 are realizable. Statement (1$a$) holds for a group $\mathbb{Z}_3 \times (q^2 \rtimes SL_2(5))$, where $q > 5$ is a prime and $q \equiv \pm 1 \pmod{10}$, and for a group $\mathbb{Z}_3 \times (q^4 \rtimes SL_2(5))$, where $q > 5$ is a prime. Statement (1$b$) holds for a group $\mathbb{Z}_3 \times (q^6 \rtimes SL_2(7))$, where $3 < q \neq 7$ is a prime and $q \equiv \pm 7 \pmod{16}$, and for a group $\mathbb{Z}_3 \times (q^{12} \rtimes SL_2(7))$, where $3 < q \neq 7$ is a prime and $q \equiv \pm 3, \pm 5 \pmod{16}$. Statement (1$c$) holds for a group $\mathbb{Z}_3 \times (q^4 \rtimes SL_2(9))$, where $q > 5$ is a prime. Statement (1$d$) holds for a group $\mathbb{Z}_3 \times (q^8 \rtimes SL_2(17))$, where $3 < q \neq 17$ is a prime and $q \equiv \pm 1, \pm 2 \pm 4, \pm 8 \pmod{16}$, and for a group $\mathbb{Z}_3 \times (q^{16} \rtimes SL_2(17))$, where $3 < q \neq 17$ is a prime. Statement (1$e$) holds for a group $\mathbb{Z}_5 \times (q^2 \rtimes SL_2(5))$, where $q > 5$ is a prime and $q \equiv \pm 1 \pmod{10}$, and for a group $\mathbb{Z}_5 \times (q^4 \rtimes SL_2(5))$, where $q > 5$ is a prime and $q \equiv \pm 3 \pmod{10}$. Statement (2) holds for a group $31^2 \rtimes SL_2(31)$.

In view of the obtained results, in the further study of our problem for a finite non-solvable group $G$, we can assume that $G$ has an element of order 6 and $q$ does not divide $|S|$.

## 2 Preliminaries

Our notation and terminology are mostly standard and can be found in [3,5,8,16]. For a finite group $G$, $G^{(\infty)}$, $Soc(G)$ and $E(G)$ denote the last member of the derived series, the socle and the layer (the subgroup generated by all subnormal quasi-simple subgroups) of $G$, respectively. If $A$ and $B$ are groups, then $A.B$, $A : B$ (or $A \rtimes B$), and $A^{\cdot}B$ denote an extension, a split extension (or a semidirect product), and a non-split extension of the group $A$ by the group $B$, respectively. By $A \circ B$ denote the central product of groups $A$ and $B$ over their largest common central subgroup. If $n$ is a positive integer and $p$ is a prime, then $p^n$ denote also the elementary abelian $p$-group of order $p^n$. A finite group $G$ is called a Frobenius group with kernel $A$ and complement $B$ if $G = A \rtimes B$, where groups $A$ and $B$ are non-trivial and $C_A(b) = 1$ for any non-trivial element $b$ of $B$. A finite group $G$ is called a 2-Frobenius group if there exist subgroups $A$, $B$, and $C$ in $G$ such that $G = ABC$, $A$ and $AB$ are normal subgroups in $G$, and $AB$ and $BC$ are Frobenius groups with kernels $A$ and $B$ and complements $B$ and $C$, respectively.

If $K$ and $L$ are two neighboring terms in a chief series of a finite group $G$ such that $K < L \leq S(G)$, then the (chief) factor $V = L/K$ is an elementary abelian $p$-group for some prime $p$; it is called a *$p$-chief factor* of $G$.

Consider some results, which are used in the proofs of the theorems.

**Lemma 2.1** (*Gruenberg–Kegel Theorem* [31, Theorem A]). *If $G$ is a finite group with disconnected Gruenberg–Kegel graph, then one of the following statements holds*:

(1) *$G$ is a Frobenius group*:
(2) *$G$ is a 2-Frobenius group*:
(3) *$G$ is an extension of a nilpotent group by a group $A$, where $Inn(P) \leq A \leq Aut(P)$ for a simple non-abelian group $P$.*

**Lemma 2.2** ([8, Remark on p. 377]). *Suppose that $G$ is a finite group whose Sylow 2-subgroup is isomorphic to a (generalized) quaternion group and $\overline{G} = G/O(G)$. Then, one of the following statements holds*:

(a) *$\overline{G}$ is isomorphic to a Sylow 2-subgroup of $G$*;
(b) *$\overline{G}$ is isomorphic to the group $2{\cdot}A_7$*;
(c) *$\overline{G}$ is an extension of the group $SL_2(q)$, where $q$ is odd, by a cyclic group whose order is not divisible by 4.*

**Lemma 2.3** ([29, Proposition 3.2]). *Suppose that $G$ is a finite group, $H \trianglelefteq G$, $G/H \cong L_2(q)$, where $q > 5$ is odd, and $C_H(t) = 1$ for an element $t$ of order 3 from $G$. Then, $H = 1$.*

**Lemma 2.4** ([13, Theorem 8.2], [29, Proposition 4.2]). *Suppose that $G$ is a finite group, $1 \neq H \trianglelefteq G$, and $G/H \cong L_2(2^n)$ for $n \geq 2$. Assume that $C_H(t) = 1$ for some element $t$ of order 3 of $G$. Then, $H = O_2(G)$ and $H$ is the direct product of minimal normal subgroups of order $2^{2n}$ in $G$ such that each of them as a $G/H$-module is isomorphic to the natural $GF(2^n)SL_2(2^n)$-module.*

**Lemma 2.5** ([25, Theorem, Remark 1]). *Suppose that $G$ is a finite group, $1 \neq H \trianglelefteq G$, $G/H \cong Sz(q)$ for $q \geq 8$, and $C_H(t) = 1$ for some element $t$ of order 5 from $G$. Then, $H = O_2(G)$ and $H$ is the direct product of minimal normal subgroups of order $q^4$ of $G$ such that each of them as a $G/H$-module is isomorphic to the natural 4-dimensional $GF(q)Sz(q)$-module.*

**Lemma 2.6** ([12]). *If $G$ is a finite simple 3-primary group, then $G$ is isomorphic to $L_2(q)$ for $q \in \{5, 7, 8, 9, 17\}$, $L_3(3)$, $U_3(3)$, $U_4(2)$.*

Suppose that $G$ is a finite group and $V$ is a $kG$-module for a finite field $k$ of characteristic $t$. The action of $G$ on $V$ and the pair $(G, V)$ are called $p'$-semiregular for a fixed prime $p$ if any non-trivial $p'$-element of $G$ acts fixed point free on $V \setminus \{0\}$. This action and the pair $(G, V)$ are called separable if $t$ does not divide $|G|$ and inseparable otherwise (when $t = p$).

Let $\mathcal{R}$ be the set of all primes $r$ such that $r - 1 = 2^a \cdot 3^b$ for $a \geq 2$ and $b \geq 0$ and $(r + 1)/2$ is a prime. It is known that $5, 13, 37, 73, 193, 1153 \in \mathcal{R}$, but it is unknown whether $\mathcal{R}$ is infinite or not.

**Lemma 2.7** ([7, Theorem 5.6]). *Suppose that $G$ is a non-trivial finite group and $G' = G$. If $(G, V)$ is a separable $p'$-semiregular pair, then one of the following statements holds*:

(a) $p = 2$ *and there exists a family* $K_1, \dots, K_m$ *of normal* 2*-subgroups of* $G$ *with the following properties*:

(a1) $\bigcap_{i=1}^m K_i = 1$;

(a2) *any quotient group* $G/K_i$ *either is isomorphic to* $SL_2(5)$ *or has the form* $2_-^{1+4}.A_5$;

(a3) *if* $G/K_i \cong G/K_j \cong SL_2(5)$, *then* $K_i = K_j$;

(b) $p = 3$ *and* $G \cong SL_2(r)$, *where* $r \in \mathcal{R} \cup \{7, 9, 17\}$;

(c) $p \geq 5$ *and* $G \cong SL_2(5)$.

*Conversely, if* $(G, p)$ *satisfies any of conditions* (a)–(c), *then there exists a faithful irreducible* $G$*-module* $V$ *over a field of characteristic not dividing* $|G|$ *such that the pair* $(G, V)$ *is* $p'$*-semiregular.*

**Lemma 2.8** ([24, Theorem 1], [15, Theorem VII.1.16]). *Suppose that $q$ is a power of a prime $p$, $G$ is a finite group, $H := O_p(G) \neq 1$, and $G/H \cong SL_n(q)$ for $n \geq 2$. Assume that $C_H(t) = 1$ for some element $t$ of order 3 from $G$. Then, any $p$-chief factor of $G$ as a $H$-module is isomorphic to the $n$-dimensional natural $GF(q)SL_n(q)$-module or to the contragredient to it.*

**Lemma 2.9** (*Thompson Theorem* [8, Theorem 5.3.11]). *Let $p$ be a prime and $P$ be a finite $p$-group. Then, $P$ possesses a characteristic subgroup $C$, named a critical subgroup of $P$, with the following properties*:

(a) $C/Z(C)$ *is elementary abelian*;

(b) $[P, C] \leq Z(C)$;

(c) $C_P(C) = Z(C)$;

(d) *every non-trivial $p'$-automorphism of $P$ induces a non-trivial automorphism of $C$.*

**Lemma 2.10** ([15, Theorem VII.1.16]).*Suppose that $G$ is a finite group, $F = GF(p^m)$ is the field of definition of characteristic $p > 0$ for an absolutely irreducible $FG$-module $V$, $\langle\sigma\rangle = Aut(F)$, $V_0$ denotes the module $V$ considered as a $GF(p)G$-module, and $W = V_0 \otimes_{GF(p)} F$. Then,*

(1) $W = \bigoplus_{i=1}^m V^{\sigma^i}$, *where $V^{\sigma^i}$ is the module algebraically conjugate to $V$ by means of $\sigma^i$*;

(2) *$V_0$ is an irreducible $GF(p)G$-module and, in particular, $W$ is realized as the irreducible $GF(p)G$-module $V_0$*;

(3) *Up to isomorphism of modules, irreducible $GF(p)G$-modules are in one-to-one correspondence with algebraically conjugacy classes of irreducible $\overline{GF(p)}G$-modules, where $\overline{GF(p)}$ is an algebraic closure of the field $GF(p)$.*

**Lemma 2.11** *Let $p$, $q$, $r$ be pairwise distinct primes and $G$ be a finite group of the form $G = P \rtimes (T \rtimes \langle x\rangle)$, where $P$ is a non-trivial $p$-group, $T$ is a $q$-group, $|x| = r$ and $C_G(P) = Z(P)$. Let $C$ be a critical subgroup of $T$ and $[T, \langle x\rangle] \neq 1$. Then, either $C_P(x) \neq 1$, or $Z(T) \leq Z(C) \leq C_T(x)$, $q = 2$, $r = 1 + 2^n$ is a Fermat prime, and $[C, \langle x\rangle]$ is an extraspecial group of order $2^{2n+1}$.*

***Proof*** Suppose that $C_P(x) = 1$. Show first that $Z(T) \leq Z(C) \leq C_T(x)$. The inclusion $Z(T) \leq Z(C)$ follows from Lemma 9($c$). Suppose that $[Z(C), \langle x\rangle] \neq 1$. By [8, Theorem 5.2.3], we have $Z(C) = [Z(C), \langle x\rangle] \times C_{Z(C)}(x)$, hence $[Z(C), \langle x\rangle]\langle x\rangle$ is a Frobenius group. Now by [27, Lemma 1], we obtain that $C_P(x) \neq 1$, a contradiction. Therefore, $Z(T) \leq Z(C) \leq C_T(x)$.

Let $C_1 := [C, \langle x\rangle]$. By Lemma 2.9 and [8, Theorem 5.3.6], we have $C_1 = [C_1, \langle x\rangle] \neq 1$, and $C_1/Z(C_1) \cong C_1Z(C)/Z(C)$ has exponent $q$. By [8, Theorems 5.1.4, 5.3.2], we can assume that $P$ is elementary abelian and $P$ is a faithful irreducible $GF(p)C_1\langle x\rangle$-module. Let $K$ be an algebraic closure of the field $GF(p)$. By Lemma 2.10, there exists an algebraically conjugacy class $\{W_1, \ldots, W_m\}$ of (absolutely) irreducible $KC_1\langle x\rangle$-modules such that $P \otimes_{GF(p)} K = \bigoplus_{i=1}^m W_i$ (here $GF(p^m)$ is the field of definition for the $KC_1\langle x\rangle$-modules $W_1, \ldots, W_s$). It is clear that $W_1$ is a faithful $KC_1\langle x\rangle$-module and $C_{W_1}(x) = 1$. Now, arguing as in the proof of lemma from [11], we obtain all remaining statements of lemma.

Lemma is proved. □

**Lemma 2.12** ([6, Lemma 4]). *Suppose that $G$ is a finite quasi-simple group, $F$ is a field of characteristic $p > 0$, $V$ is a faithful absolutely irreducible $FG$-module, and $\beta$ is a Brauer character of the module $V$. If $g$ is an element of $G$ of prime order coprime to $p|Z(G)|$, then*

$$\dim C_V(g) = (\beta|_{\langle g\rangle}, 1|_{\langle g\rangle}) = \frac{1}{|g|} \sum_{x \in \langle g\rangle} \beta(x).$$

**Lemma 2.13** ([32, Lemma 6.(iii)]). *Let $a$, $s$, $t$ be positive integers. Then,*

(a) $(a^s - 1, a^t - 1) = a^{(s,t)} - 1$,

(b)

$$\left(a^s + 1, a^t + 1\right) = \begin{cases} a^{(s,t)} + 1, & \text{if } s/(s,t) \text{ and } t/(s,t) \text{ are odd},\\ (2, a + 1), & \text{otherwise,}\end{cases}$$

(c)

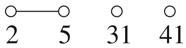
$$\left(a^s - 1, a^t + 1\right) = \begin{cases} a^{(s,t)} + 1, & \text{if } s/(s,t) \text{ is even and } t/(s,t) \text{ is odd},\\ (2, a + 1), & \text{otherwise.}\end{cases}$$

**Lemma 2.14** (*Zsigmondy's Theorem* [36]). *Let $q$ and $n$ be integers both greater than 1. Then, there exists a prime $r$ dividing $q^n - 1$ and not dividing $q^i - 1$ for each $1 \leq i < n$ such that $r \equiv 1 \pmod n$, except for the following cases: $q = 2$ and $n = 6$; $q = 2^k - 1$ for some prime $k$ and $n = 2$.*

## 3 Proof of Theorem 1.1

Let $G$ be a group satisfying the conditions of Theorem 1.1 and $T$ be a Sylow 2-subgroup of $G$. By [23, Theorem 1], $\overline{G} = G/S \cong Sz(8)$, $Sz(32)$ or $Aut(Sz(32))$. By

[5], $\Gamma(\overline{G})$ is a completely disconnected graph (coclique) if $\overline{G} = Soc(\overline{G})$ and has the form




if $\overline{G} \cong Aut(Sz(32))$.

Suppose that $r$ and $s$ both do not divide $|S|$. Then, $r$ and $s$ are adjacent vertices of the graph $\Gamma(\overline{G})$. Hence, $\overline{G} \cong Aut(Suz(32))$, $\{r, s\} = \{2, 5\}$ and $\pi(S) \subseteq \{p, q\} = \{31, 41\}$, therefore $S = O(G)$.

Suppose that $q \in \pi(S)$ and $Q \in Syl_q(S)$. By the Frattini argument, $G = SN_G(Q)$, and we can assume that $T < N_G(Q)$. Then, $T$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$; therefore, some involution from $T$ centralizes some elements from $Q$, and hence, 2 and $q$ are adjacent in $\Gamma(G)$, a contradiction.

Thus, $q \notin \pi(S)$, hence $S = O_p(G)$ and statement (1) of Theorem 1.1 holds.

Suppose that $r$ or $s$ divides $|S|$. Without loss of generality, let $r \in \pi(S)$. Let $Q \in Syl_q(G)$. The solvable group $SQ$ contains a $\{r, q\}$-Hall subgroup $U$. Since the graph $\Gamma(U)$ is disconnected, by Lemma 2.1, $U$ is either a Frobenius group or a 2-Frobenius group, and the subgroup $F(U)$ is either $O_r(U)$ or $O_q(U)$.

Suppose that $F(U) = O_q(U)$. Then, a Sylow $r$-subgroup $R$ of $S$ is either a cyclic group or a (generalized) quaternion group. Then, $\overline{C_G(\Omega_1(R))} \geq Soc(\overline{G})$, hence $r = p$, a contradiction.

Thus, $F(U) = O_r(U)$; hence, $Q$ is either a cyclic group or a (generalized) quaternion group. Arguing as in a previous paragraph, we obtain that $q$ does not divide $|S|$, hence $q \neq 2$. Furthermore, $U$ is a Frobenius group with kernel $U \cap S$ and cyclic complement $Q$.

Suppose that $p \notin \pi(S)$. If $s \notin \pi(S)$, then $S = O_r(G)$ and therefore $p$ and $s$ are adjacent vertices of the graph $\Gamma(\overline{G})$, where $\overline{G} \cong Aut(Suz(32))$, $\{p, s\} = \{2, 5\}$ and $\{r, q\} = \{31, 41\}$. From the table of the $r$-modular Brauer characters of $Sz(32)$ (see [16]) and Lemma 2.12, we obtain that $C_S(x) \neq 1$ for an element $x$ of order $q$ of $G$, a contradiction. Thus, $\pi(S) = \{r, s\}$. An element of order $q$ of $G$ acts on $S \setminus \{1\}$ fixed point free, hence, by Lemma 2.1, $S = F(G)$. By Lemma 2.5, $q \neq 5$, and hence $q > 5$. Therefore, $p$ and $q$ are adjacent in $\Gamma(\overline{G})$, hence $\{p, q\} = \{2, 5\}$, a contradiction.

Thus, $p \in \pi(S)$. Arguing as above, we obtain that a $\{r, s, q\}$-Hall subgroup $V$ of the solvable group $SQ$ is a Frobenius group with kernel $W := F(V) = V \cap S$ and complement $Q$. We have $G = SN_G(W)$, therefore, $N_G(W)/N_S(W) \cong \overline{G}$. Let $N = N_G(W)$. Then, $S(N) = N_S(W) = W \rtimes P$, where $P \in Syl_p(S(N))$. It is clear that $F(N) = W \times C_P(W) = WC_N(W)$ and $C_P(W) = O_p(N)$. Put $\widetilde{N} = N/O_p(N)$.

Suppose that $S(\widetilde{N}) = \widetilde{W}$. Then, $\widetilde{N}/\widetilde{W} \cong \overline{G}$ and $\widetilde{W} = F(\widetilde{N})$.

Suppose that the graph $\Gamma(\widetilde{N})$ is connected. Then, $p$ and $q$ are adjacent vertices of the graph $\Gamma(\overline{G})$. Therefore, $\overline{G} \cong Aut(Sz(32))$, $\{p, q\} = \{2, 5\}$, and $\{r, s\} = \{31, 41\}$. Then, $\widetilde{W} = O_{\{2,5\}'}(\widetilde{N})$ and a Sylow 2-subgroup and a Sylow 5-subgroup of $\overline{G}$ contain subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$, respectively. Therefore, $q$ and $r$ are adjacent in $\Gamma(\widetilde{N})$, a contradiction.

Therefore, the graph $\Gamma(\widetilde{N})$ is disconnected and by [21, Theorems 3 and 4], one of statements $(1) - (3)$ of Theorem 1.1 holds.

Let $S(\widetilde{N}) \neq \widetilde{W}$. Then, $S(\widetilde{N}) = \widetilde{W} \rtimes \widetilde{P}$, where $\widetilde{P}$ is a non-trivial $p$-group and $C_{\widetilde{N}}(\widetilde{W}) \leq \widetilde{W}$. We can assume that $S(\widetilde{N})\widetilde{Q} = \widetilde{W} \rtimes (\widetilde{P} \rtimes \widetilde{Q})$. Set $\langle x \rangle = \Omega_1(\widetilde{Q})$.

Show that $[\widetilde{P}, \langle x\rangle] = 1$. Suppose to the contrary that $[\widetilde{P}, \langle x\rangle] \neq 1$. Let $C$ be a critical subgroup of $\widetilde{P}$. Apply Lemmas 2.9 and 2.11 to the group $\widetilde{W} \rtimes (\widetilde{P} \rtimes \langle x\rangle)$. Then, we obtain the following: $\overline{G} \cong Sz(8)$ or $Sz(32)$, $p = 2$, $q = 5$, $\langle [\widetilde{P}, C], Z(\widetilde{P}), \Phi(C)\rangle \leq Z(C) \leq C_{\widetilde{P}}(x)$, $[C, \langle x\rangle]$ is an extraspecial group of order 32. The subgroups $\widetilde{W}C$ and $\widetilde{W}Z(C)$ are normal in $\widetilde{N}$. Set $H = \widetilde{N}/\widetilde{W}Z(C)$ and $V = \widetilde{W}C/Z(C)$. Then, $V$ is a normal elementary abelian 2-subgroup of $H$, $C_H(V) = O_2(H)$, $H/O_2(H) \cong \overline{G}$ and $|[V, \langle t\rangle]| = 16$ for some element $t$ of order 5 of $H$. In particular, $V$ is a faithful $GF(2)\overline{G}$-module. It is clear that the module $V$ has a composition factor $V_0$ of dimension at least 2. Let $K$ be an algebraic closure of the field $GF(2)$. By Lemma 2.10, for the faithful irreducible $GF(2)\overline{G}$-module $V_0$, there exists an algebraically conjugacy class $\{W_1, \ldots, W_m\}$ of faithful (absolutely) irreducible $K\overline{G}$-modules with the field of definition $GF(2^m)$ such that $V_0 \otimes_{GF(2)} K = \bigoplus_{i=1}^{m} W_i$. Denote by $W_0$ the module $W_1$ considered as a $GF(2^m)$-module. Then, the module $V_0$ can be identified with the module $W_0$ considered as a $GF(2)\overline{G}$-module. Therefore, for an element $g$ of order 5 of $\overline{G}$, we have $\dim V_0 = m \dim W_0$ and $\dim C_{V_0}(g) = m \dim C_{W_0}(g)$, hence

$$\dim[V_0, \langle g\rangle] = \dim V_0 - \dim C_{V_0}(g) = m(\dim W_0 - \dim C_{W_0}(g)) = 4.$$

By the tables of 2-modular Brauer characters of $Sz(8)$ and $Sz(32)$ (see [16]) and Lemma 2.12, we obtain the following: if $\dim W_0 = 4$, then $\dim C_{W_0}(g) = 0$ and $m$ equals 3 or 5 for $Sz(8)$ or $Sz(32)$, respectively; if $\dim W_0 \neq 4$, then $\dim W_0 - \dim C_{W_0}(g) > 4$. In any case, $\dim[V_0, \langle g\rangle] > 4$, a contradiction.

So, $[\widetilde{P}, \langle x\rangle] = 1$, hence $Soc(\widetilde{N}/S(\widetilde{N})) \leq C_{\widetilde{N}}(\widetilde{P})S(\widetilde{N})/S(\widetilde{N})$. Denote by $L$ the last member of the derived series of $C_{\widetilde{N}}(\widetilde{P})\widetilde{W}/\widetilde{W}$. By [5], $L \cong Sz(8), Sz(32), 2^{\cdot}Sz(8)$ or $2^{2\cdot}Sz(8)$. Let $K$ be the complete pre-image of $L$ in $\widetilde{N}$.

If $Z(L) = 1$, then $K/\widetilde{W}$ is a simple group. Arguing as above, we get that $\widetilde{W} = O_2(\widetilde{N})$, $p > 2$ and one of statements (1)–(3) of Theorem 1.1 holds.

Let $Z(L) \neq 1$. Then, $p = 2$ and $\overline{G} \cong Sz(8)$. We can assume that $L$ acts irreducibly on $O_r(\widetilde{W})/\Phi(O_r(\widetilde{W}))$. Therefore, $L \cong 2^{\cdot}Sz(8)$. By the Brauer character tables of the group $2^{\cdot}Sz(8)$ (see [16]) and Lemma 2.12, we obtain that $\widetilde{W} = O_5(S(\widetilde{N}))$, $\{r, s\} = \{5, 7\}$, $q = 13$ and 5-chief factors of $G$ are isomorphic to the faithful irreducible 8-dimensional $GF(5)2^{\cdot}Sz(8)$-module, since $2^{\cdot}Sz(8) < \Omega_8^+(5)$ by [4, Table 8.50]. Therefore, statement (4) of Theorem 1.1 holds.

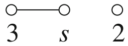
Theorem 1.1 is proved.

## 4 Proof of Theorem 1.3

Let $G$ be a group satisfying the conditions of Theorem 1.3. Then, $p > 3$ and we can assume that $\{r, q\} = \{2, 3\}$. By [22, Theorem 2] and [23, Theorems 2 and 3], $\overline{G} = G/S$ is an almost simple group whose the graph $\Gamma(\overline{G})$ is disconnected, $\varnothing \neq \pi(O(S)) \subseteq \{p, s\}$, and $S = O_{2',2}(G)$.

Suppose that $q = 2$. Then, $r = 3$. If $2 \in \pi(S)$, then [22, Theorem 2], [17] and [18] imply that $O(S) = O_p(G)$ and statement (1) of Theorem 1.3 holds. Let $2 \notin \pi(S)$. Then, by [22, Theorem 2], $S = O(G)$ and $\pi(S) \subseteq \{p, s\}$. Since a Sylow 2-subgroup of the almost simple group $\overline{G}$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the vertex

2 in the graph $\Gamma(G)$ is adjacent to each vertex from $\pi(S)$. Therefore, $S = O_p(P)$, hence




is an induced subgraph of the disconnected graph $\Gamma(\overline{G})$. Thus, by [20] and [21], statement (2) of Theorem 1.3 holds.

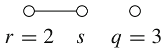
Suppose that $q = 3$. Then, $r = 2$ and, by [22, Theorem 2], 3 does not divide $|S|$. Therefore, $3 \in \pi(\overline{G})$.

Suppose that $s \in \pi(S)$. Let $U$ be a $\{2, s\}$-Hall subgroup of $S$. Then, $G = SN_G(U)$, hence $N_G(U)/N_S(U) \cong \overline{G}$. Thus, $N_G(U)$ contains an element $x$ of order 3. Since $C_U(x) = 1$, the subgroup $U$ is nilpotent.

Suppose that $O^s(O(S)) < O(S)$. Since the group $S/O^s(O(S))$ is nilpotent, we have $O^s(S) < S$. We can assume that $S$ is a non-trivial elementary abelian $s$-group. Every element of order 3 of $\overline{G}$ acts on $S\backslash\{1\}$ fixed point free, hence a Sylow 3-subgroup of $\overline{G}$ is cyclic. By [22, Theorem 2], the socle of $\overline{G}$ is isomorphic to $L_2(2^n)$, $L_3(2^n)$, $U_3(2^n)$ or $L_2(q)$, where $q \equiv \pm 5 (mod\ 12)$. By Lemma 2.3, the last case is impossible. By [4], the groups $L_3(2^n)$ and $U_3(2^n)$ contain subgroups isomorphic to $L_2(2^n)$; hence, we can assume that $\overline{G} \cong L_2(2^n)$. By Lemma 2.4, $S = O_2(G)$, a contradiction with $S = O_s(G)$ for $s > 2$.

Thus, $O^s(O(S)) = O(S)$, whence $p \in \pi(S)$, $O^p(O(S)) < O(S)$ and $O^{p,p'}(O(S)) < O^p(O(S))$. We can assume that $O^p(O(S)) = O_s(O(S)) = F(O(S))$. Let $\widetilde{G} = G/O^p(O(S))$. Then, by Lemma 2.11, every element of order 3 from $\widetilde{G}$ acts trivially on $O_p(\widetilde{G})$, therefore $Soc(\overline{G})$ acts trivially on $O_p(\widetilde{G})$. By [22, Theorem 2] and [5], the order of the Schur multiplier of the simple group $Soc(\overline{G})$ divides 6. Therefore, $\widetilde{G}^{(\infty)} \cong Soc(\overline{G})$. Let $K$ be the complete pre-image of $\widetilde{G}^{(\infty)}$ in $G$. Then, $S(K) = O_s(K \neq 1$. Arguing as in the previous paragraph, we obtain a contradiction.

Thus, $s \notin \pi(S)$, and hence $S = O_p(G)$. Therefore,




is an induced subgraph of the disconnected graph $\Gamma(\overline{G})$. By [20] and [21], statement (3) of Theorem 1.3 holds.

Theorem 1.3 is proved.

## 5 Proof of Theorem 1.5

Let $G$ be a group satisfying the conditions of Theorem 1.5. Let $q \in \pi(S)$, $Q \in Syl_q(S)$, and $N = N_G(Q)$. By the Frattini argument, $G = SN$. Therefore, $\overline{G} = G/S \cong N/N \cap S$ is an almost simple group, and hence $S(N) = S \cap N$.

The subgroup $Q$ contains a subgroup isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_q$. Otherwise, $Soc(\overline{N}) \leq \overline{C_N(\Omega_1(Q))}$, and hence the degree of the vertex $q$ in $\Gamma(G)$ is at least 2, that is not so. Therefore, $O_{q'}(S) = O_p(S)$, and $Q_0 := Q \cap O_{q',q}(S)$ is a non-trivial Sylow $q$-subgroup of $O_{q',q}(S)$, which is a normal subgroup of $N$. By [8, Theorem 6.3.3], $C_S(Q_0) \leq O_{q',q}(S)$. If $C_G(Q_0) \nleq S$, then $Soc(\overline{N}) \leq \overline{C_N(Q_0)}$, which is impossible. Therefore, $C_G(Q_0) = C_S(Q_0) = Q_0 \times C_{O_p(G)}(Q_0)$.

Let $G_r \in Syl_r(G)$ and $G_s \in Syl_s(G)$. Since $G = O_p(G)N_G(Q_0)$, we can assume that $G_r$ and $G_s$ are contained in $N_G(Q_0)$. Since $Q_0G_r$ and $Q_0G_s$ are Frobenius groups with the kernel $Q_0$ and complements $G_r$ and $G_s$, respectively, each of the groups $G_r$ and $G_s$ is a cyclic group or a (generalized) quaternion group.

Suppose that $2 \in \{r, s\}$. Without loss of generality we can assume that $r = 2$. By [8, Theorem 10.3.1], $G_r$ is a (generalized) quaternion group, and $G_s$ is a cyclic group. By Lemma 2.2, $S = Z^*(G)$, and the group $Soc(G/O(G))$ is isomorphic to either $2{\cdot}A_7$ or $SL_2(t)$, where $t$ is odd and $t \geq 5$. Since the degree of the vertex $r$ in $\Gamma(G)$ is 2, we have $|\pi(Soc(\overline{G}))| = 3$, and hence by Lemma 2.6, $Soc(G/O(G)) \cong SL_2(t)$, where $t \in \{5, 7, 9, 17\}$. From here, $\{2, 3\} \subset \pi(\overline{G}) = \pi(Soc(G)) = \{r, s, p\}$.

The group $SG_2$ is solvable, hence by [8, Theorem 6.4.1], we can assume that $QG_2$ is a $\{2, q\}$-Hall subgroup of $SG_2$. By Lemma 2.1, $QG_2$ is a Frobenius group with kernel $Q$ and complement $G_2$. Since the (unique) involution of $G_2$ acts on $Q \setminus \{1\}$ fixed point free, this involution inverts $Q$, and hence the group $Q$ is abelian. Thus, $Q = Q_0$, and $G = O_p(G)N$.

Show that statement (1) of Theorem 1.5 holds for $G$. We can assume that $O_p(G) = 1$, and hence $G = N_G(Q)$, $C_G(Q) = Q$, and $Q \in Syl_q(G)$. By the Schur–Zassenhaus theorem (see [8, Theorem 6.2.1]), $G = Q \rtimes G_0$ for a group $G_0$, and hence $S = QS(G_0)$, where $S(G_0) = Z^*(G_0)$ and $\pi(O(G_1)) \subseteq \{p, s\}$.

Suppose that $s \in \pi(O(G_0))$, $U \in Syl_s(O(G_0))$, and $K = C_{G_0}(U)$. Then, $O(K) = O_s(K) \times O_p(K)$, and $F^*(K/O(K)) \cong SL_2(t)$, where $t \in \{5, 7, 9, 17\}$. If $t = 9$, then a Sylow 3-subgroup of $G$ is a non-cyclic, and hence $s \neq 3$. Since the order of the Schur multiplier of $L_2(t)$ divides 6 (see [5]), we have $F^*(K/O_p(K)) \cong O_s(K) \times SL_2(t)$, it implies that Sylow $s$-subgroups of $K$ are non-cyclic, this contradicts the cyclicity of $G_s$.

Thus, $O(G_0) = O_p(G_0)$. Let $x$ be an element of order $s$ from $G_0$. Applying Lemma 2.11 to the group $Q \rtimes (O_p(G_1) \rtimes \langle x\rangle)$, we obtain that $F^*(G_0) \cong O_p(G_0) \times SL_2(t)$. Let $E = E(G_0)$. Then, $E = (G_0)^{(\infty)}$, and $(E, \Omega_1(Q))$ is a separable $p'$-regular pair; therefore, by Lemma 2.7, either $p = 3$ and $\{r, s\} = \{2, 5\}$ or $p = t = 5$, $\{r, s\} = \{2, 3\}$ and $E \cong SL_2(5)$. If $p = 5$, then the ordinary character table of $SL_2(5)$ (see [5]), and Lemma 2.12 imply that $QE$ is a Frobenius group with kernel $Q$ and complement $E$. Thus, by Lemma 2.2, statement (1) of Theorem 1.5 holds.

Further, we will assume that $r$ and $s$ are odd; hence, subgroups $G_r$ and $G_s$ are cyclic and $2 \in \{p, q\}$.

Show that $r$ and $s$ both do not divide $|S|$. Suppose the contrary. Without loss of generality, we can assume that $r$ divides $|S|$. Then, the vertex $r$ is adjacent to each vertex from $\pi(Soc(\overline{G})) \setminus \{r\}$ in $\Gamma(G)$. Since the vertices $r$ and $q$ are no-adjacent in the graph $\Gamma(G)$, $\pi(Soc(\overline{G})) = \{r, s, p\}$, and hence $p = 2$. Since subgroups $G_r$ and $G_s$ are cyclic, Lemma 2.6 implies that $Soc(\overline{G}) \cong L_2(t)$, where $t \in \{5, 7, 8, 17\}$. Thus, $\{r, s\}$ is equal to $\{3, 5\}$, $\{3, 7\}$ or $\{3, 17\}$, and hence $q > 3$.

Let $W$ be a $\{r, s, q\}$-Hall subgroup of $S$. Since all such subgroups are conjugate in $S$, we can assume that $Q \in Syl_q(W)$, $G = SN_G(W)$, and hence $\overline{G} = G/S \cong N_G(W)/N_S(W)$. It follows that $S(N_G(W)) = N_S(W)$. The graph $\Gamma(W)$ is disconnected; hence, by Lemma 2.1, $W$ is either a Frobenius group or a 2-Frobenius group.

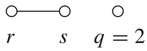
Since $O_{q'}(W) = 1$, $q > 3$, and the automorphism groups of Sylow $r$-subgroups and Sylow $s$-subgroups of $W$ are a $\{2, 3\}$-groups, the latter case is impossible. Therefore,

$W$ is a Frobenius group with kernel $Q$ and some complement $D$. By [14, Theorem V.8.18], $D$ is a metacyclic $\{r, s\}$-group with the non-trivial center. By the Schur–Zassenhaus Theorem (see [8, Theorem 6.2.1]), $N_G(W) = Q \rtimes X$ for some subgroup $X$ containing $D$. Since $S(N_G(W)) = Q \rtimes D$ and $N_G(W)/S(N_G(W)) \cong \overline{G}$, we have $S(X) = D$ and $X/D \cong \overline{G}$. We have $C_X(D)/Z(D) \cong Soc(X/D)$, since $D$ is a metacyclic group, and hence $C_X(D)D/D$ contains $Soc(X/D)$. Since $Z(D) \neq 1$, and the order of the Schur multiplier of $L_2(t)$ divides 2 (see [5]), the group $C_X(D)$ contains a subgroup isomorphic to $Z(D) \times L_2(t)$. This implies that at least one of $G_r$ or $G_s$ is non-cyclic, a contradiction.

Thus, $r$ and $s$ do not divide $|S|$.

If the group $G$ does not contain elements of order 6, then by Theorems 1.1 and 1.3, statement (2) of Theorem 1.5 holds. Therefore, in the sequel we will assume that $G$ contains an element of order 6.

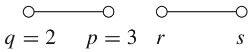
Assume that $q = 2$. Then,




is an induced subgraph of $\Gamma(\overline{G})$. If $3 \notin \pi(\overline{G})$, then $3 \in \pi(S)$, and hence $p = 3$ and $\pi(\overline{G}) = \{2, r, s\}$, this contradicts to Lemma 2.6. Therefore, $3 \in \pi(\overline{G})$. If $p \neq 3$, then $G$ does not contain elements of order 6, which contradicts to our assumption. Therefore, $\pi(G) = \pi(\overline{G})$.

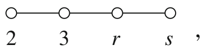
Suppose that $\overline{G}$ does not contain elements of order 6. Then, 2 is an isolated vertex in $\Gamma(\overline{G})$, and hence, by [30], $\overline{G}$ is isomorphic to one of the following groups: $L_2(2^n)$, where $n \geq 2$; $L_2(t)$, where $t$ is a Mersenne or Fermat prime; $L_3(4)$. By [21], $r$ and $s$ are non-adjacent vertices in $\Gamma(\overline{G})$, a contradiction.

Thus, $\overline{G}$ contains an element of order 6, and hence

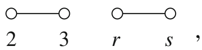



is a subgraph of $\Gamma(\overline{G})$. By [21], the graph $\Gamma(\overline{G})$ is connected.

Suppose that $\Gamma(\overline{G}) \neq \Gamma(G)$. Then, without loss of generality, we can assume that $\Gamma(\overline{G})$ has the form




hence, by [1], $\Gamma(Soc(\overline{G}))$ has the form:



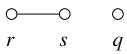

which is impossible by [21].

Thus, $\Gamma(\overline{G}) = \Gamma(G)$, and therefore by [22], $q > 2$; a contradiction.

So, $q \neq 2$, and therefore $p = 2$.

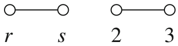
Suppose that $q \notin \pi(\overline{G})$. Then, $\pi(\overline{G}) = \{r, s, 2\}$. By [21], $3 \in \{r, s\}$, and hence, a Sylow 3-subgroup of $\overline{G}$ is cyclic. Since $r$ and $s$ are adjacent vertices of $\Gamma(\overline{G})$, this contradicts to [20].

Thus, $q \in \pi(\overline{G})$, and hence $\pi(\overline{G}) = \pi(G)$. In particular,

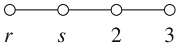


is an induced subgraph of the graph $\Gamma(\overline{G})$. It is clear that $3 \in \{r, s, q\}$.

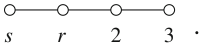
Suppose that $q = 3$. If 2 and 3 are non-adjacent in $\Gamma(\overline{G})$, then the graph $\Gamma(\overline{G})$ has a form, which contradicts to [21]. Therefore,




is a subgraph of $\Gamma(\overline{G})$. By [21], the graph $\Gamma(\overline{G})$ is connected. Therefore, either $\Gamma(\overline{G}) = \Gamma(G)$ or $\Gamma(\overline{G})$ has one of the following forms:



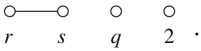

or




This contradicts to [28].

Thus, $q > 3$, and hence $3 \in \{r, s\}$.

Suppose that 2 and 3 are non-adjacent in $\Gamma(\overline{G})$. Then, by [28], $\Gamma(\overline{G})$ is disconnected. By [21], and taking into account that $G_r$ and $G_s$ are cyclic, we obtain that $\overline{G} \cong L_2(t)$, where either $t = 2^m$, where m=4 or $m \geq 5$ is prime, or $t \geq 31$ is a Mersenne or Fermat prime, and the graph $\Gamma(\overline{G})$ has the form:




By Lemmas 2.3 and 2.4, $O^q(S) = S$, and therefore $2 \in \pi(S)$, $O^2(S) < S$, and $O^{2,q}(S) < O^2(S)$. Since $Q \in Syl_q(O^2(S))$, by the Frattini argument, we have $G = O^2(S)N_G(Q)$. We can assume that $Q = O^2(S) \lhd G$. Put $\widetilde{G} = G/Q$.

Suppose that $C_{\widetilde{G}}(\widetilde{S}) \nleq \widetilde{S}$. Then, $\widetilde{G} = \widetilde{S} \circ C_{\widetilde{G}}(\widetilde{S})$, $\widetilde{S} = O_2(\widetilde{G}) = F(\widetilde{G}) \neq 1$, and $E(\widetilde{G}) = C_{\widetilde{G}}(\widetilde{S})^{(\infty)} \cong L_2(t)$ or $SL_2(t)$. Let $K$ be the complete pre-image of $E(\widetilde{G})$ in $G$. Then, $O(K) = O_q(K) = Q$, and $K/Q \cong E(\widetilde{G})$. It is clear, that $C_K(Q) \leq S(K)$. If $C_K(Q) \nleq Q$, then $O^q(S) < S$, that is not so. Therefore, $C_K(Q) \leq Q$. An element of order 3 from $K$ acts on $Q \setminus \{1\}$ fixed point free, hence, by Lemmas 2.3 and 2.4, $K/Q \cong SL_2(t)$, where $t = q \geq 31$ is a Mersenne or a Fermat prime. Let $\tau$ be an involution from $Z^*(K)$. Then, $K = QC_K(\tau)$.

Suppose that $C_Q(\tau) \neq 1$, and put $L = C_K(\tau)/\langle\tau\rangle$. Then, $O(L) \cong C_Q(\tau) \neq 1$, $L/O(L) \cong L_2(t)$, and an element of order 3 from $L$ acts on $O(L) \setminus \{1\}$ fixed point free, a contradiction to Lemma 2.3.

Thus, $C_Q(\tau) = 1$, and hence $Q$ is an abelian group and $(K/Q, \Omega_1(Q))$ is a inseparable $q'$-semiregular pair. Therefore, by Lemma 2.8, statement (3) of Theorem 1.5 holds.

Suppose that $C_{\widetilde{G}}(\widetilde{S}) \leq \widetilde{S}$. We have that $C_G(Q) = O_2(G) \times Z(Q)$. If $S = O_2(G)Q$, then $O^q(S) < S$, that is not so. Therefore, we can assume that $O_2(G) = 1$, and hence $C_G(Q) = Z(G)$. Let $x$ and $y$ be some elements of orders $r$ and $s$ from $G$, respectively. Applying Lemma 2.11 to the groups $S\langle x\rangle$ and $S\langle y\rangle$, we obtain that $r$ and $s$ are Fermat primes. Since $3 \in \{r, s\}$, we can assume that $r = 3 = 1 + 2$ and $s = 1 + 2^k$ for $k = 2^l > 1$.

Let $t = 2^m$ for some prime $m \geq 5$. Then, by [20], $s = (2^m + 1)/3$ and $q = 2^m - 1$. The number $s = 1 + 2^k$ divides $2^m + 1$, moreover, $1 < k = 2^l < m$. But $(k, m) = 1$, therefore, by Lemma 2.13, we obtain that $(2^m + 1, 2^k + 1) = 1$; a contradiction.

Thus, $t$ is a Mersenne or Fermat prime, i.e., $t-\varepsilon 1 = 2^m$, where $\varepsilon \in \{+,-\}$, $m \geq 5$, and $\{r, s\} = \pi((t+\varepsilon 1)/2)$.

Suppose that $\varepsilon = +$. Then $t = 2^m + 1$, where $m = 2^n \geq 8$, and $\{r, s\} = \pi((t+1)/2)$. We have $(t+1)/2 = 2^m + 1$. Therefore, by Lemma 2.11, $s$ divides $(2^{m-1}+1, 2^k+1) = 2^{(m-1,k)} + 1 = 3$; a contradiction.

Thus, $\varepsilon = -$, and hence $t = 2^m - 1$, where $m$ is a prime, $m \geq 5$, and $\{r, s\} = \pi((t-1)/2)$. We have $(t-1)/2 = 2^{m-1} - 1$. The number $s$ divides $(2^{m-1}-1, 2^k+1)$, hence by Lemma 2.13, $k/(m-1,k)$ is odd, and $(m-1)/(m-1,k)$ is even. This implies that $2k$ divides $m-1$, and therefore $2^{2k}-1$ divides $2^{m-1}-1$. But $2^{2k}-1 = (2^k-1)(2^k+1)$, and $(2^k-1, 2^k+1) = 1$, hence $2^k - 1 = 3^v$ for some $v \in \mathbb{N}$. Then, by the Lemma 2.14, $v = 1$, and hence $k = 2$ and $s = 2$. If $m > 5$, then $m - 1 > 4$, and hence, by Lemma 2.14, $2^{m-1} - 1$ has a prime divisor, which is not equal to 3 of 5; a contradiction.

Therefore, $m = 5$ and $t = 31$. Arguing as in the proof of Theorem 1.1 and using Lemmas 2.9-2.12 for Fermat primes 3 or 5 and the table of 2-modular Brauer characters of $L_2(31)$ (see [16]), we obtain a contradiction.

Theorem 1.5 is proved.

**Acknowledgements** The work is supported by the Russian Science Foundation (project 19-71-10067).

## References

1. Alekseeva, O.A., Kondrat'ev, A.S.: Finite groups whose prime graphs do not contain triangles. I. Proc. Steklov Inst. Math. **295**(Suppl. 1), 11–20 (2016)
2. Alekseeva, O.A., Kondrat'ev, A.S.: Finite groups whose prime graphs do not contain triangles. II. Proc. Steklov Inst. Math. **296**(Suppl. 1), 19–30 (2017)
3. Aschbacher, M.: Finite group theory. Cambridge University Press, Cambridge (1986)
4. Bray, J.N., Holt, D.F., Roney-Dougal, C.M.: The maximal subgroups of the low-dimensional finite classical groups. Cambridge University Press, Cambridge (2013)
5. Conway, J.H., et al.: Atlas of finite groups. Clarendon Press, Oxford (1985)
6. Dolfi, S., Jabara, E., Lucido, M.S.: C55-groups. Siberian Math. J. **45**(6), 1053–1062 (2004)
7. Fleischmann, P., Lempken, W., Tiep, P.H.: Finite $p'$-semiregular groups. J. Algebra **188**(2), 547–579 (1997)
8. Gorenstein, D.: Finite groups. Harper and Row, New York (1968)
9. Gorshkov, I.B., Maslova, N.V.: Finite almost simple groups with Gruenberg-Kegel graphs as for solvable groups. Algebra Log. **57**(2), 115–129 (2018)
10. Gruber, A., Keller, T.M., Lewis, M.L., Naughton, K., Strasser, B.: A characterization of the prime graphs of solvable groups. J. Algebra **442**, 397–422 (2015)
11. Hartley, B., Meixner, T.: Finite soluble groups containing an element of prime order whose centralizer is small. Arch. Math. **36**(3), 211–213 (1981)
12. Herzog, M.: On finite simple groups of order divisible by three primes only. J. Algebra **10**(3), 383–388 (1968)
13. Higman, G.: Odd characterizations of finite simple groups: Lecture Notes. Univ. Michigan, Ann Arbor (1968)
14. Huppert, B.: Endliche Gruppen I. Springer-Verlag, Berlin (1967)
15. Huppert, B., Blackburn, N.: Finite groups II. Springer-Verlag, Berlin (1982)
16. Jansen, C., et al.: An atlas of Brauer characters. Clarendon Press, Oxford (1995)
17. Kondrat'ev, A.S.: Finite groups with prime graph as in the group $Aut(J_2)$. Proc. Steklov Inst. Math. **283**(Suppl. 1), 1–8 (2013)
18. Kondrat'ev, A.S.: Finite groups that have the same prime graph as the group $A_{10}$. Proc. Steklov Inst. Math. **285**(Suppl. 1), 1–9 (2014)

19. Kondrat'ev, A.S.: Finite groups with given properties of their prime graphs. Algebra Log. **55**(1), 77–82 (2016)
20. Kondrat'ev, A.S., Khramtsov, I.V.: On finite triprimary groups. Trudy Inst. Mat. Mech. UrO RAN **16**(3), 150–158 (2010). (In Russian)
21. Kondrat'ev, A.S., Khramtsov, I.V.: On finite tetraprimary groups. Proc. Steklov Inst. Math. **279**(Suppl. 1), 43–61 (2012)
22. Kondrat'ev, A.S., Minigulov, N.A.: Finite groups without elements of order six. Math. Notes **104**(5), 696–701 (2018)
23. Kondrat'ev, A.S., Minigulov, N.A.: Finite almost simple groups whose Gruenberg-Kegel graphs as abstract graphs are isomorphic to subgraphs of the Gruenberg-Kegel graph of the alternating group $A_{10}$. Siberian Electr. Math. Rep. **15**, 1378–1382 (2018)
24. Kondrat'ev, A.S., Osinovskaya, A.A., Suprunenko, I.D.: On the behavior of elements of prime order from Singer cycles in representations of special linear groups. Proc. Steklov Inst. Math. **285**(Suppl. 1), 108–115 (2014)
25. Martineau, R.P.: On 2-modular representations of the Suzuki groups. Am. J. Math. **94**, 55–72 (1972)
26. Maslova, N.V., Pagon, D.: On the realizability of a graph as the Gruenberg-Kegel graph of a finite group. Siberian Electr. Math. Rep. **13**, 89–100 (2016)
27. Mazurov, V.D.: Characterizations of finite groups by sets of orders of their elements. Algebra Log. **36**(1), 23–32 (1997)
28. Minigulov, N.A.: Finite almost simple 4-primary groups with connected Gruenberg-Kegel graph. Proc. Steklov Inst. Math. **309**(Suppl. 1), 93–97 (2020)
29. Stewart, W.B.: Groups having strongly self-centralizing 3-centralizers. Proc. London Math. Soc. **426**(4), 653–680 (1973)
30. Suzuki, M.: On a class of doubly transitive groups. Annl. Math. **75**(1), 105–145 (1962)
31. Williams, J.S.: Prime graph components of finite groups. J. Algebra **69**(2), 487–513 (1981)
32. Zavarnitsine, A.V.: Recognition of the simple groups $L_3(q)$ by element orders. J. Group Theory **7**(1), 81–97 (2004)
33. Zavarnitsine, A.V.: Recognition of finite groups by the prime graph. Algebra Log. **45**(4), 220–231 (2006)
34. Zavarnitsine, A.V.: Finite groups with a five-component prime graph. Sib. Math. J. **54**(1), 40–46 (2013)
35. Zinov'eva, M.R., Kondrat'ev, A.S.: Finite almost simple groups with prime graphs all of whose connected components are cliques. Proc. Steklov Inst. Math. **295**(Suppl. 1), 178–188 (2016)
36. Zsigmondy, K.: Zur Theorie der Potenzreste. Monatsch. Math. Phys. **3**(Suppl. 1), 265–284 (1892)