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Local Existence and Uniqueness of Navier–Stokes–Schrödinger System

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Abstract
In this article, we prove that there exists a unique local smooth solution for the Cauchy problem of the Navier–Stokes–Schrödinger system. Our methods rely upon approximating the system with a sequence of perturbed system and parallel transport and are closer to the one in Ding and Wang (Sci China 44(11):1446–1464, 2001) and McGahagan (Commun Partial Differ Equ 32(1–3):375–400, 2007).

Keywords Initial value problem · Local solution · Navier–Stokes–Schrödinger system · Schrödinger maps

Mathematics Subject Classification 35Q55 · 35B65 · 35B30 · 35Q35

1 Introduction

In this paper, we consider the Navier–Stokes–Schödinger initial-value problem:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u - \operatorname{div}(\nabla\phi \odot \nabla\phi), \\ \operatorname{div} u = 0, \\ \partial_t \phi + u \cdot \nabla\phi = \phi \times \Delta\phi, \\ (u, \phi)\big|_{t=0} = (u_0, \phi_0). \end{cases} \tag{1.1}$$

Here, $d = 2, 3$, $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ represents the velocity field of the flow, P is the pressure function, and $\phi : \mathbb{R}^d \times [0, T] \to \mathbb{S}^2 \subset \mathbb{R}^3$ denotes the magnetization field.

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The notation $\times$ is the cross product for vectors in $\mathbb{R}^3$, and the term $\nabla\phi \odot \nabla\phi$ denotes the $d \times d$ matrix whose (i, j)th entry is given by $\partial_i \phi \cdot \partial_j \phi$ $(1 \le i, j \le d)$. This model is a coupled system of the incompressible Navier–Stokes equations and Schrödinger map flow which can be used to describe the dispersive theory of magnetization of ferromagnets with quantum effects.

The system (1.1) can be seen as a special case of Navier–Stokes–Landau–Lifshitz (NSLL) equation. For incompressible NSLL system with positive Gilbert constant in $\mathbb{R}^3$, the global existence of a unique solution in Besov spaces without any small conditions imposed on the third component of the initial velocity field was established by Zhai et al. [26]. Later, under the assumption of small initial data in Sobolev spaces, Wei et al. [25] proved the global solution by energy method and obtained the time decay rates of the higher-order spatial derivatives of the solutions by applying the Fourier splitting method introduced by Schonbek [20]. Fan et al. [9] studied the regularity criteria for the smooth solution to the inhomogeneous compressible NSLL equation in Besov spaces and the multiplier spaces. Wang and Guo [23] investigated the existence and uniqueness of the weak solution to the inhomogeneous compressible NSLL equation in two dimensions. Recently, they further investigate the global existence of the weak solutions to the compressible NSLL equations with density-dependent viscosity in two dimensions in [24].

If $u \equiv 0$, the model (1.1) is reduced to the Schrödinger flow of maps from $\mathbb{R}^d$ into $\mathbb{S}^2$, which is an interesting equation known as the ferromagnetic chain system, and has been intensely studied in the last decades. The local well-posedness of Schrödinger flow was established by Sulem, Sulem and Bardos [22] for $\mathbb{S}^2$ target, Ding and Wang [7,8] and McGahagan [17] for general Kähler manifolds. The first global well-posedness result for Schrödinger flow of maps into $\mathbb{S}^2$ with small data in the critical Besov spaces in dimensions $d \ge 3$ was proved by Ionescu and Kenig [13] and independently by Bejenaru [1]. This was further improved to global regularity for small data in the critical Sobolev spaces in dimensions $d \ge 2$ in [2] and [3]. Recently, Li [15,16] considered the Schrödinger flow of maps into compact Kähler manifolds and proved that the flow with small initial data in critical Sobolev space is global. However, the Schrödinger map equation with large data is a much more difficult problem. When the target is $\mathbb{S}^2$, there exists a collection of families $\mathcal{Q}^m$ ([5]) of finite energy stationary solutions for integer $m \ge 1$. Hence, the global well-posedness and scattering for equivariant Schrödinger flow with energy below the ground state were proved by Bejenaru, Ionescu, Kenig and Tataru in [4]. When the energy of maps is larger than that of ground state, the dynamic behaviors are complicated. The asymptotic stability and blowup for Schrödinger flow have been considered by many authors for instance [5,10–12,18,19]. We refer to [14] for more open problems in this field.

In this paper, we establish the local existence and uniqueness of (1.1) for large data by parabolic approximation, which has been shown to be successful in the study of the Schrödinger flow [8].

We start with some notations. Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $[q]$ be the integer part of a positive number q. For $k \in \mathbb{Z}_+$, $p \in [1, \infty]$, let $H^k(\mathbb{R}^d)$, $W^{k,p}(\mathbb{R}^d)$ denote the usual Sobolev spaces of functions on $\mathbb{R}^d$. It will be convenient to consider $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ as a submanifold of $\mathbb{R}^3$; then, the map ϕ can be represented as $\phi = (\phi_1, \phi_2, \phi_3)$ with ϕ_i being globally defined functions on $\mathbb{R}^d$. Denote ∇ as the usual derivative for functions on $\mathbb{R}^d$. Then for $Q \in \mathbb{S}^2$, we define the metric space

$$W_Q^{k,p} = \{f : \mathbb{R}^d \to \mathbb{R}^3 : |f(x)| \equiv 1 \text{ a.e. and } f - Q \in W^{k,p}\}, \tag{1.2}$$

with the induced distance $d_Q^{k,p}(f, g) = \|f - g\|_{W_Q^{k,p}}$. For simplicity of notation, let $\|f\|_{W_Q^{k,p}} = d_Q^{k,p}(f, Q)$, and further denote $H_Q^k := W_Q^{k,2}$.

The main result is the following.

Theorem 1.1 *The Cauchy problem* (1.1) *with* $(u_0, \phi_0) \in H^k \times H_Q^{k+1}$, *for any integer* $k \geq [\frac{d}{2}] + 1$, *admits a unique local solution* (u, ϕ) *satisfying*

$$\|u\|_{H^k} + \left(\int_0^t \|\nabla u\|_{H^k}^2 \mathrm{d}s\right)^{1/2} + \|\nabla\phi\|_{H_Q^k} \leq C\left(k, \|u_0\|_{H^k}, \|\nabla\phi_0\|_{H_Q^k}\right),$$

for any $t \in [0, T]$, *where* $T = T(\|u_0\|_{H^2}, \|\nabla\phi_0\|_{H_Q^2})$.

The proof of Theorem 1.1 follows closely that of [8,17,21]. We prove the local existence for system (1.1) with finite data by approximation of perturbed parabolic system. Precisely, we consider the perturbed system for $\epsilon > 0$ small

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u - \sum_{j=1}^3 \partial_j(\nabla\phi \partial_j \phi), \\ \mathrm{div} u = 0, \\ \partial_t \phi = \epsilon D_k \partial_k \phi + J(\phi) D_k \partial_k \phi - u \cdot \nabla\phi, \\ (u, \phi)(0) = (u_0, \phi_0) \in C^\infty(M \times M, \mathbb{R}^d \times \mathbb{S}^2), \end{cases} \tag{1.3}$$

where J is complex structure on $\mathbb{S}^2$ and D is the covariant differential on $\phi^\star T\mathbb{S}^2$. The perturbed system (1.3) is weakly parabolic and behaves similar as the Navier–Stokes–Landau–Lifshitz system with positive Gilbert coefficients. By standard parabolic argument, it is easy to find that the system (1.3) admits a local solution $(u_\epsilon, \phi_\epsilon)$ on some time interval $[0, T_\epsilon)$ for every $\epsilon > 0$. Then, we derive the uniform estimates of $(u_\epsilon, \phi_\epsilon)$ and a lower bound for the life span T_ϵ, and obtain the solution of (1.1) on M as $\epsilon \to 0$.

The rest of the paper is organized as follows: In Sect. 2, we recall the basic properties of Sobolev spaces. In Sect. 3, we apply the approximating scheme and obtain the uniform bound for energy and then give the proof of local existence. In Sect. 4, we use parallel transport to prove the uniqueness and hence complete the proof of Theorem 1.1.

2 Preliminaries

In this section, we introduce the definition of intrinsic Sobolev spaces and state some basic inequalities.

For geometric PDEs, it is convenient to work in both intrinsic Sobolev spaces and extrinsic Sobolev spaces. The extrinsic Sobolev spaces were defined in (1.2), and we introduce the intrinsic Sobolev spaces as follows. For smooth maps ϕ from (M, g) to $\mathbb{S}^2$, the pullback bundle $\phi^\star T\mathbb{S}^2$ is the vector bundle over (M, g) whose fiber at $x \in M$ is the tangent space $T_{\phi(x)}\mathbb{S}^2$. Let D denote the induced covariant derivative in $\phi^\star T\mathbb{S}^2$. Then, the intrinsic norm of vector bundle $\nabla\phi$ is defined by

$$\|\nabla\phi\|^p_{\mathbf{W}^{k,p}(M)} = \sum_{i=0}^{k} \int_M |D^i \nabla\phi|^p dvol_g,$$

where $p \in [1, \infty)$. For $p = \infty$, we also define

$$\|\nabla\phi\|_{\mathbf{W}^{k,\infty}(M)} = \max\{\|D^i \nabla\phi\|_{L^\infty} : 0 \le i \le k\}.$$

For simplicity of notation, we denote $\mathbf{H}^k := \mathbf{W}^{k,2}$.

Then, we have the interpolation inequality for sections on vector bundles and equivalent relation between $\|\nabla\phi\|_{H_Q^{k-1}}$ and $\|\nabla\phi\|_{\mathbf{H}^{k-1}}$.

Proposition 2.1 ([8], Theorem 2.1, Propostion 2.1) *Suppose $s \in C^\infty(E)$ is a section where E is a vector bundle over a closed m-dimensional Riemannian manifold M. Then, we have*

$$\|D^j s\|_{L^p(M)} \le C\|s\|^a_{\mathbf{W}^{k,q}(M)}\|s\|^{1-a}_{L^r(M)}, \tag{2.1}$$

where $1 \le p, q, r \le \infty$, and $j/k \le a \le 1$ ($j/k \le a < 1$ if $q = m/(k - j) \ne 1$) are numbers such that

$$\frac{1}{p} = \frac{j}{m} + \frac{1}{r} + a\left(\frac{1}{q} - \frac{1}{r} - \frac{k}{m}\right).$$

The constant C only depends on M and the numbers j, k, q, r, a. Moreover, if $M = \mathbb{T}^d = \mathbb{R}^d/(R \cdot \mathbb{Z})^d$, then the constant C does not depend on the diameter $R \ge 1$.

Proposition 2.2 ([8], Proposition 2.2) *Assume that $k > d/2$; (M, g) is a closed Riemannian manifold. Then, there exists a constant $C = C(\mathbb{S}^2, k)$ such that for all maps $\phi \in C^\infty(M, \mathbb{S}^2)$,*

$$\|\nabla\phi\|_{H_Q^{k-1}(M)} \le C\sum_{l=1}^{k}\|D\phi\|^l_{\mathbf{H}^{k-1}(M)}$$

and

$$\|D\phi\|_{\mathbf{H}^{k-1}(M)} \le C\sum_{l=1}^{k}\|\nabla\phi\|^l_{H_Q^{k-1}(M)}.$$

Finally, we state the density property of Sobolev spaces $H_Q^k(\mathbb{R}^d, \mathbb{S}^2)$.

Lemma 2.3 ([8], Lemma 3.4) *Let* $k > d/2$ *and* $\phi \in H^k_Q(\mathbb{R}^d, \mathbb{S}^2)$. *Then, there exists a sequence of maps* $\phi_i - Q \in H^k(\mathbb{R}^d, \mathbb{S}^2) \cap C_0^\infty(\mathbb{R}^d, \mathbb{R}^3)$ *such that* $\phi_i \to \phi$ *in* $H^k_Q(\mathbb{R}^d, \mathbb{S}^2)$.

3 Local Existence of Navier–Stokes–Schrödinger System

In this section, we first prove the local existence of smooth solutions for the initial-value problem of the Navier–Stokes–Schrödinger system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u - \sum_{j=1}^3 \partial_j (\nabla\phi \partial_j \phi), \\ \operatorname{div} u = 0, \\ \partial_t \phi + u \cdot \nabla \phi = \phi \times \Delta \phi, \\ (u, \phi)(0) = (u_0, \phi_0) \in C^\infty(M \times M, \mathbb{R}^d \times \mathbb{S}^2), \end{cases} \tag{3.1}$$

where M is a flat closed d-dimensional Riemannian manifold. Then, we use the smooth solutions (u_i, ϕ_i) on $\mathbb{T}_i^{2d} = \mathbb{R}^{2d}/(2R_i \cdot \mathbb{Z})^{2d}$ to give the smooth solution of system (1.1) and finish the proof of Theorem 1.1.

Since $(\mathbb{S}^2, J, h)$ is a compact Kähler manifold with complex structure J and Kähler metric h, the term $\phi \times \Delta\phi$ can be rewritten as

$$J(\phi) D_k \partial_k \phi,$$

where we implicitly sum over repeated indices. Then, we may employ an approximate procedure and solve first the following perturbed problem:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u - \sum_{j=1}^3 \partial_j (\nabla\phi \partial_j \phi), \\ \operatorname{div} u = 0, \\ \partial_t \phi = \epsilon D_k \partial_k \phi + J(\phi) D_k \partial_k \phi - u \cdot \nabla \phi, \\ (u, \phi)(0) = (u_0, \phi_0) \in C^\infty(M \times M, \mathbb{R}^d \times \mathbb{S}^2), \end{cases} \tag{3.2}$$

where $\epsilon > 0$ small.

For the initial-value problem (3.2), we have

Lemma 3.1 *Let* $m_0 = [d/2] + 1 = 2$, *and let* $u_0 \in C^\infty(M, \mathbb{R}^d)$, $\phi_0 \in C^\infty(M, \mathbb{S}^2)$. *There exists a constant* $T = T(\|u_0\|_{H^2(M)}, \|\nabla\phi_0\|_{\mathbf{H}^2(M)}) > 0$, *independent of* $\epsilon \in (0, 1]$, *such that if* $(u, \phi) \in C^\infty(M \times [0, T_\epsilon])$ *is a solution of* (3.2) *with* $\epsilon \in (0, 1]$, *then*

$$T_\epsilon \geq T(\|u_0\|_{H^2(M)}, \|\nabla\phi_0\|_{\mathbf{H}^2(M)})$$

and

$$\begin{aligned} &\|u(t)\|_{H^k(M)} + \|\nabla u\|_{L^2([0,t]; H^k(M))} + \|\nabla\phi\|_{\mathbf{H}^k(M)} \\ &\leq C(k, \|u_0\|_{H^k(M)}, \|\nabla\phi_0\|_{\mathbf{H}^k(M)}), \quad t \in [0, T], \end{aligned}$$

for all $k \geq 2$.

Proof By standard argument, the initial-value problem (3.2) has a unique smooth solution $(u_\epsilon, \phi_\epsilon)$ for some $T_\epsilon > 0$. For simplicity, denote $(u, \phi) := (u_\epsilon, \phi_\epsilon)$ be a solution of (3.2), and denote $H^l := H^l(M)$, $\mathbf{H}^l := \mathbf{H}^l(M)$ for any integer $l \geq 0$. It is easy to obtain that the energy

$$E(u, \phi) := \frac{1}{2}\|u\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 ds + \frac{1}{2}\|\nabla \phi\|_{L^2}^2$$

is uniformly bounded for $t \in [0, T_\epsilon)$. Precisely, by (3.2) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E &= \int_M u(\Delta u - \nabla P - u \cdot \nabla u - \sum_{j=1}^3 \partial_j(\nabla\phi\partial_j\phi))\mathrm{d}x + \|\nabla u\|_{L^2}^2 \\ &\quad + \int_M \sum_{i=1}^3 \langle \nabla_i \phi, D_i(-u \cdot \nabla\phi + \epsilon D_k\partial_k\phi + J D_k\partial_k\phi)\rangle \mathrm{d}x.\end{aligned}$$

Then, integration by parts gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E &= \int_M \sum_{j=1}^3 \partial_j u \cdot \langle \nabla\phi, \partial_j\phi\rangle \mathrm{d}x - \epsilon \int_M |D_k\partial_k\phi|^2 \mathrm{d}x \\ &\quad + \int_M \langle D_k\partial_k\phi, J D_k\partial_k\phi\rangle \mathrm{d}x \int_M \\ &\quad - \sum_{j=1}^3 \partial_j u \cdot \langle \nabla\phi, \partial_j\phi\rangle - u \cdot \langle D_j\nabla\phi, \partial_j\phi\rangle \mathrm{d}x \\ &= -\epsilon\|D_k\partial_k\phi\|_{L^2}^2 - \int_M u \cdot \langle D\partial_j\phi, \partial_j\phi\rangle \mathrm{d}x \\ &= -\epsilon\|D_k\partial_k\phi\|_{L^2}^2 + \int_M \frac{1}{2}\nabla \cdot u|\nabla\phi|^2 \mathrm{d}x \\ &= -\epsilon\|D_k\partial_k\phi\|_{L^2}^2.\end{aligned}$$

Fix an $N \geq m_0$, and let n be any integer with $1 \leq n \leq N$. Suppose that $\mathbf{a}$ is a multi-index of length n, i.e., $\mathbf{a} = (a_1, \ldots, a_n)$. For $t \leq T_\epsilon$, we define the energy functional by

$$E_n(u, \phi) := \sum_{|a|=n} \left(\frac{1}{2}\|\nabla_{\mathbf{a}} u\|_{L^2}^2 + \int_0^t |\nabla_{\mathbf{a}}\nabla u|_{L^2}^2 \mathrm{d}s + \frac{1}{2}\|D_{\mathbf{a}}\nabla\phi\|_{L^2}^2 \right).$$

Then by (3.2) and integration by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_n = \sum_{|\mathbf{a}|=n} \int_M -\nabla_{\mathbf{a}} u \cdot \nabla_{\mathbf{a}}(u \cdot \nabla u + \partial_j(D\phi \cdot \partial_j\phi))\mathrm{d}x$$

$$+\sum_{|\mathbf{a}|=n}\int_M \langle D_{\mathbf{a}}\nabla\phi, D_t D_{\mathbf{a}}\nabla\phi\rangle \mathrm{d}x =: I + II. \tag{3.3}$$

By incompressible condition $\nabla\cdot u = 0$, Hölder and integration by parts, we get

$$I \lesssim \|\nabla u\|_{H^n}\|u\|_{H^n}^2 + \|\nabla u\|_{H^n}\|\nabla\phi\|_{\mathbf{H}^n}^2. \tag{3.4}$$

Next, we estimate the term II. By ϕ-equation in (3.2), we obtain

$$\begin{aligned} D_t D_{\mathbf{a}}\partial_i\phi &= D_{\mathbf{a}} D_i \partial_t\phi + [D_t, D_{\mathbf{a}} D_i]\phi \\ &= D_{\mathbf{a}} D_i\partial_t\phi + \sum D_{\mathbf{b}} R(\phi)(D_{\mathbf{c}}\phi, D_{\mathbf{d}}\partial_t\phi) D_{\mathbf{e}}\partial_i\phi, \end{aligned} \tag{3.5}$$

where the sum is over all multi-indices $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ with possible zero lengths, except that $|\mathbf{c}| > 0$ always holds, such that $(\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = \sigma(\mathbf{a})$ is a permutation of $\mathbf{a}$. Replacing $\partial_t\phi$ in the second term by the right-hand side of ϕ-equation in (3.2), the second term can be rewritten as

$$\begin{aligned} &\sum D_{\mathbf{b}} R(\phi)(D_{\mathbf{c}}\phi, D_{\mathbf{d}}\partial_t\phi) D_{\mathbf{e}}\partial_i\phi \\ &\quad = \sum D_{\mathbf{b}} R(\phi)(D_{\mathbf{c}}\phi, D_{\mathbf{d}}(\epsilon D_k\partial_k\phi + J(\phi) D_k\partial_k\phi)) D_{\mathbf{e}}\partial_i\phi \\ &\qquad - \sum D_{\mathbf{b}} R(\phi)(D_{\mathbf{c}}\phi, D_{\mathbf{d}}(u\cdot\nabla\phi)) D_{\mathbf{e}}\partial_i\phi \\ &\quad =: Q_1 + Q_2. \end{aligned} \tag{3.6}$$

Moreover, we have

$$|Q_1| \lesssim \sum_{(j_1,\ldots,j_s)\in\mathcal{J}} |D^{j_1}\phi|\cdots|D^{j_s}\phi|, \tag{3.7}$$

where

$$\begin{aligned} \mathcal{J} := \big\{ j_1, \ldots, j_s \in \mathbb{N} : j_1 \geq j_2 \geq \cdots \geq j_s,\ n+1 \\ \geq j_i \geq 1,\ j_1 + \cdots + j_s = n+3,\ s\geq 3\big\}. \end{aligned} \tag{3.8}$$

Similarly, we also have

$$|Q_2| \lesssim \sum_{(\tilde{j}_0,\ldots,\tilde{j}_s)\in\tilde{\mathcal{J}}} |\partial^{\tilde{j}_0} u||D^{\tilde{j}_1}\phi|\cdots|D^{\tilde{j}_s}\phi|, \tag{3.9}$$

where

$$\begin{aligned} &\tilde{\mathcal{J}} := \big\{ \tilde{j}_0, \ldots, \tilde{j}_s \in \mathbb{N} : \tilde{j}_1 \geq \tilde{j}_2 \geq \cdots \geq \tilde{j}_s,\ \tilde{j}_0 + \cdots + \tilde{j}_s = n+2,\ s\geq 3, \\ &n-1 \geq \tilde{j}_0 \geq 0,\ n \geq \tilde{j}_i \geq 1,\ \text{for } s \geq i \geq 1\big\}. \end{aligned} \tag{3.10}$$

For the first term in the right-hand side of (3.5), it follows from (3.2) that

$$\begin{aligned} D_{\mathbf{a}} D_i \partial_t \phi &= D_{\mathbf{a}} D_i (\epsilon D_k \partial_k \phi + J D_k \partial_k \phi - u \cdot \nabla \phi) \\ &= \epsilon D_k D_k D_{\mathbf{a}} \partial_i \phi + J D_k D_k D_{\mathbf{a}} \partial_i \phi + u \cdot D D_{\mathbf{a}} \partial_i \phi \\ &\quad + \sum_{(\mathbf{b},\mathbf{c})=\sigma(\mathbf{a})} \nabla_{\mathbf{b}} \partial_i u \cdot D_{\mathbf{c}} \nabla \phi + \sum_{(\mathbf{b},\mathbf{c})=\sigma(\mathbf{a}),|\mathbf{b}|\geq 1} \nabla_{\mathbf{b}} u \cdot D_{\mathbf{c}} D_i \nabla \phi + Q_3 + Q_4, \end{aligned} \tag{3.11}$$

where Q_3, Q_4 satisfy (3.7), (3.9), respectively.

Thus, we obtain from (3.5), (3.6) and (3.11):

$$\begin{aligned} D_t D_{\mathbf{a}} \partial_i \phi = \epsilon D_k D_k D_{\mathbf{a}} \partial_i \phi &+ J D_k D_k D_{\mathbf{a}} \partial_i \phi + u \cdot D D_{\mathbf{a}} \partial_i \phi \\ &+ \sum_{(\mathbf{b},\mathbf{c})=\sigma(\mathbf{a})} \nabla_{\mathbf{b}} \partial_i u \cdot D_{\mathbf{c}} \nabla \phi \\ &+ \sum_{(\mathbf{b},\mathbf{c})=\sigma(\mathbf{a}),|\mathbf{b}|\geq 1} \nabla_{\mathbf{b}} u \cdot D_{\mathbf{c}} D_i \nabla \phi + Q_1 + Q_2 + Q_3 + Q_4. \end{aligned}$$

Substituting this into II in (3.3) and integrating by parts, we have

$$\begin{aligned} II &= \sum_{|\mathbf{a}|=n} \int_M -\epsilon |D_k D_{\mathbf{a}} \partial_i \phi|^2 + \langle D_k D_{\mathbf{a}} \partial_i \phi, J D_k D_{\mathbf{a}} \partial_i \phi \rangle + \langle D_{\mathbf{a}} \partial_i \phi, u \cdot D D_{\mathbf{a}} \partial_i \phi \rangle \mathrm{d}x \\ &\quad + \sum_{|\mathbf{a}|=n} \int_M \langle D_{\mathbf{a}} \partial_i \phi, \sum_{(\mathbf{b},\mathbf{c})=\sigma(\mathbf{a})} \nabla_{\mathbf{b}} \partial_i u \cdot D_{\mathbf{c}} \nabla \phi + \sum_{(\mathbf{b},\mathbf{c})=\sigma(\mathbf{a}),|\mathbf{b}|\geq 1} \nabla_{\mathbf{b}} u \cdot D_{\mathbf{c}} D_i \nabla \phi \rangle \mathrm{d}x \\ &\quad + \sum_{|\mathbf{a}|=n} \int_M \langle D_{\mathbf{a}} \partial_i \phi, Q_1 + Q_3 \rangle \mathrm{d}x + \sum_{|\mathbf{a}|=n} \int_M \langle D_{\mathbf{a}} \partial_i \phi, Q_2 + Q_4 \rangle \mathrm{d}x. \end{aligned}$$

Note that in the first integrand, the first term is non-positive by $\epsilon > 0$ and the second term vanishes by complex structure J. Then by integration by parts, (3.7) and (3.9), we get

$$\begin{aligned} II &\leq \sum_{|\mathbf{a}|=n} \int_M \frac{1}{2} u \cdot \nabla |D_{\mathbf{a}} \partial_i \phi|^2 \mathrm{d}x + \sum_{n_1+n_2=n+1, n_1\geq 1} \int_M |D^{n+1}\phi| |\nabla^{n_1} u| |D^{n_2} \nabla \phi| \mathrm{d}x \\ &\quad + \sum_{(j_1,\ldots,j_s)\in\mathcal{J}} \int_M |D^{n+1}\phi| |D^{j_1}\phi| \cdots |D^{j_s}\phi| \mathrm{d}x \\ &\quad + \sum_{(\tilde{j}_0,\tilde{j}_1,\ldots,\tilde{j}_s)\in\tilde{\mathcal{J}}} \int_M |D^{n+1}\phi| |\partial^{\tilde{j}_0} u| |D^{\tilde{j}_1}\phi| \cdots |D^{\tilde{j}_s}\phi| \mathrm{d}x \\ &= II_1 + II_2 + II_3 + II_4. \end{aligned}$$

Since $\nabla \cdot u = 0$, the first term II_1 vanishes. It suffices to estimate II_2, II_3 and II_4, respectively.

Step 1. We prove the bound

$$II_2 \leq \begin{cases} C\|\nabla\phi\|_{\mathbf{H}^2}^2\|\nabla u\|_{H^2}, & \text{if } n \leq 2, \\ C\|\nabla\phi\|_{\mathbf{H}^n}^2\|\nabla u\|_{H^n}, & \text{if } n \geq 3, \end{cases} \tag{3.12}$$

where the constant C depends on n.

By Hölder, it suffices to estimate

$$\||\nabla^{n_1}u||D^{n_2}\nabla\phi|\|_{L^2}, \tag{3.13}$$

for $n_1 + n_2 = n + 1, n_1 \geq 1$. When $n \leq 2$, by Hölder, Sobolev embedding and Proposition 2.1, we have

$$\begin{aligned}(3.13) &\leq \|\nabla^2 u\|_{L^2}\|\nabla\phi\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|D\nabla\phi\|_{L^2} + \|\nabla^3 u\|_{L^2}\|\nabla\phi\|_{L^\infty} \\ &\quad + \|\nabla^2 u\|_{L^4}\|D\nabla\phi\|_{L^4} + \|\nabla u\|_{L^\infty}\|D^2\nabla\phi\|_{L^2} \\ &\leq C\|\nabla u\|_{H^2}\|\nabla\phi\|_{\mathbf{H}^2},\end{aligned}$$

which is acceptable. When $n \geq 3$, by Hölder, Sobolev embedding and Proposition 2.1, we have

$$\begin{aligned}(3.13) &\leq \sum_{1\leq n_1\leq\frac{n+1}{2}} \|\nabla^{n_1}u\|_{L^\infty}\|D^{n+1-n_1}\nabla\phi\|_{L^2} \\ &\quad + \sum_{\frac{n+1}{2}<n_1\leq n+1} \|\nabla^{n_1}u\|_{L^2}\|D^{n+1-n_1}\nabla\phi\|_{L^\infty} \\ &\leq C\|\nabla u\|_{H^n}\|\nabla\phi\|_{\mathbf{H}^n},\end{aligned}$$

which is also acceptable.

Step 2. We prove the bound

$$II_3 \leq \begin{cases} C\displaystyle\sum_{s=3}^{n+3}\|\nabla\phi\|_{\mathbf{H}^2}^{s+1}, & \text{if } n \leq 2, \\ C(1+\|\nabla\phi\|_{\mathbf{H}^n}^2)(1+\|\nabla\phi\|_{\mathbf{H}^{n-1}})^{n+2}, & \text{if } \quad n \geq 3. \end{cases} \tag{3.14}$$

The integral II_3 is the same as (3.10) in [8]. Hence, this bound (3.14) is obtained immediately by the following lemma which was proved in [8].

Lemma 3.2 ([8], Lemmas 3.2 and 3.3) *If* $1 \leq n \leq 2$, *then there exists* $C(M, n)$ *such that*

$$II_3 \leq C\|\nabla\phi\|_{\mathbf{H}^2}^A\|\nabla\phi\|_{L^2}^B\|D^n\partial\phi\|_{L^2},$$

where $A(m, n) = [n + 3 + (m/2 - 1)s - m/2]/m_0$ *and* $B = s - A$.

If $n \geq 3$, *then there exists a constant* $C = C(M, n)$ *such that*

$$II_3 \leq \begin{cases} C\|D^n \partial\phi\|_{L^2} \|\nabla\phi\|_{\mathbf{H}^{m_0}}^{m/m_0} \|\nabla\phi\|_{L^2}^{2-m/m_0}, & \text{for } j_1 = n+1, \\ C(1 + \|\nabla\phi\|_{\mathbf{H}^n}^2)(1 + \|\nabla\phi\|_{\mathbf{H}^{n-1}}^A), & \text{for } j_1 \leq n, \end{cases}$$

where $A = A(m, n)$.

Step 3. We prove the bound

$$II_4 \leq \begin{cases} C\sum_{s=3}^{n+2} \|\nabla\phi\|_{\mathbf{H}^2}^{s+1} \|\nabla u\|_{H^2}, & \text{if } n \leq 2, \\ C\|\nabla\phi\|_{\mathbf{H}^n}^2 (1 + \|u\|_{H^{n-1}} + \|\nabla\phi\|_{\mathbf{H}^{n-1}})^{n+2}, & \text{if } \ n \geq 3. \end{cases} \tag{3.15}$$

Case 3.1. $n \leq 2$. By (3.10) and $n \leq 2$, we have

$$\tilde{j}_1 \leq 2, \quad \tilde{j}_2, \ldots, \tilde{j}_s \leq 1.$$

Then by Hölder and Proposition 2.1, we may estimate II_4 by

$$\begin{aligned} II_4 &\leq \sum_{(\tilde{j}_0, \tilde{j}_1, \ldots, \tilde{j}_s) \in \tilde{\mathcal{J}}} \|D^{n+1}\phi\|_{L^2} \|\partial^{\tilde{j}_0} u\|_{L^\infty} \|D^{\tilde{j}_1}\phi\|_{L^2} \cdots \|D^{\tilde{j}_s}\phi\|_{L^\infty} \\ &\leq C\|\nabla\phi\|_{\mathbf{H}^2} \|\nabla u\|_{H^2} \|\nabla\phi\|_{\mathbf{H}^2}^s. \end{aligned}$$

Case 3.2. $n > 2$.

First, if $\tilde{j}_1 = n$, (3.10) implies $s = 3$, $\tilde{j}_0 = 0$, and $\tilde{j}_2 = \tilde{j}_3 = 1$. Then, we may use Hölder and Proposition 2.1 to bound II_4 by

$$\begin{aligned} II_4 &\leq \sum_{(\tilde{j}_0, \tilde{j}_1, \ldots, \tilde{j}_s) \in \tilde{\mathcal{J}}, j_1 = n} \|D^{n+1}\phi\|_{L^2} \|u\|_{L^\infty} \|D^n\phi\|_{L^2} \|D\phi\|_{L^\infty}^2 \\ &\leq C\|\nabla\phi\|_{\mathbf{H}^n} \|u\|_{H^2} \|\nabla\phi\|_{\mathbf{H}^{n-1}}^3. \end{aligned}$$

Second, if $\tilde{j}_1 \leq n-1$, $\tilde{j}_0 \leq [n/2]$, from (3.10), we obtain

$$\tilde{j}_2 \leq n-1, \tilde{j}_3, \ldots, \tilde{j}_s \leq n-2.$$

Then by Hölder and Proposition 2.1, II_4 can be bounded by

$$\begin{aligned} II_4 &\leq \sum_{(\tilde{j}_0, \tilde{j}_1, \ldots, \tilde{j}_s) \in \tilde{\mathcal{J}}, \tilde{j}_1 \leq n-1, \tilde{j}_0 \leq [n/2]} \|D^{n+1}\phi\|_{L^2} \|\partial^{\tilde{j}_0} u\|_{L^4} \|D^{\tilde{j}_1}\phi\|_{L^4} \|D^{\tilde{j}_2}\phi\|_{L^\infty} \cdots \|D^{\tilde{j}_s}\phi\|_{L^\infty} \\ &\leq C\|\nabla\phi\|_{\mathbf{H}^n} \|u\|_{H^{[n/2]+1}} \|\nabla\phi\|_{\mathbf{H}^{n-1}} \|\nabla\phi\|_{\mathbf{H}^n} \|\nabla\phi\|_{\mathbf{H}^{n-1}}^{s-2} \\ &\leq C\|\nabla\phi\|_{\mathbf{H}^n}^2 \|u\|_{H^{n-1}} \|\nabla\phi\|_{\mathbf{H}^{n-1}}^{s-1}. \end{aligned}$$

Finally, we consider the remainder case $\tilde{j}_1 \leq n-1$, $\tilde{j}_0 > [n/2]$. By (3.10), we get

$$\tilde{j}_1, \ldots, \tilde{j}_s \leq n-2.$$

Then, it follows from Hölder and Proposition 2.1 that

$$
\begin{aligned}
II_4 &\le \sum_{(\tilde{j}_0,\tilde{j}_1,\ldots,\tilde{j}_s)\in\tilde{\mathcal{J}},\tilde{j}_1\le n-1,\tilde{j}_0>[n/2]} \|D^{n+1}\phi\|_{L^2}\|\partial^{\tilde{j}_0}u\|_{L^2}\|D^{\tilde{j}_1}\phi\|_{L^\infty}\cdots\|D^{\tilde{j}_s}\phi\|_{L^\infty}\\
&\le C\|\nabla\phi\|_{\mathbf{H}^n}\|u\|_{H^{n-1}}\|\nabla\phi\|^s_{\mathbf{H}^{n-1}},
\end{aligned}
$$

which concludes the bound (3.15).

Thus, from (3.12), (3.14), (3.15) and Hölder, we obtain the bound when $1\le n\le 2$,

$$
\begin{aligned}
II &\le C\left(\|\nabla\phi\|^2_{\mathbf{H}^2}\|\nabla u\|_{H^2}+\sum_{s=3}^{n+3}\|\nabla\phi\|^{s+1}_{\mathbf{H}^2}+\sum_{s=3}^{n+2}\|\nabla\phi\|^{s+1}_{\mathbf{H}^2}\|\nabla u\|_{H^2}\right)\\
&\le \frac{1}{4}\|\nabla u\|^2_{H^2}+C(1+\|\nabla\phi\|^2_{\mathbf{H}^2})^5,
\end{aligned}
\tag{3.16}
$$

and when $n\ge 3$,

$$
\begin{aligned}
II &\le C[\|\nabla\phi\|^2_{\mathbf{H}^n}\|\nabla u\|_{H^n}+(1+\|\nabla\phi\|_{\mathbf{H}^n})^2(1+\|u\|_{H^{n-1}}+\|\nabla\phi\|_{\mathbf{H}^{n-1}})^{n+2}]\\
&\le \frac{1}{4}\|\nabla u\|^2_{H^n}+C(1+\|u\|^2_{H^n}+\|\nabla\phi\|^2_{\mathbf{H}^n})^2(1+\|u\|_{H^{n-1}}+\|\nabla\phi\|_{\mathbf{H}^{n-1}})^{n+2}.
\end{aligned}
\tag{3.17}
$$

Next, we continue to bound the energy of u and ϕ. We first consider the case $1\le n\le 2$. Then, (3.3), together with (3.4) and (3.16), leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2_{H^2}+\|\nabla\phi\|^2_{\mathbf{H}^2})+\|\nabla u\|^2_{H^2}\\
&\quad\le C\|\nabla u\|_{H^2}(\|u\|^2_{H^2}+\|\nabla\phi\|^2_{\mathbf{H}^2})+\frac{1}{4}\|\nabla u\|^2_{H^2}+C(1+\|\nabla\phi\|^2_{\mathbf{H}^2})^5\\
&\quad\le \frac{1}{2}\|\nabla u\|^2_{H^2}+C(1+\|u\|^2_{H^2}+\|\nabla\phi\|^2_{\mathbf{H}^2})^5.
\end{aligned}
\tag{3.18}
$$

If we set $f(t)=1+\|u\|^2_{H^2}+\|\nabla\phi\|^2_{\mathbf{H}^2}$, then we have

$$
f'\le Cf^5,\ \ f(0)=1+\|u_0\|^2_{H^2}+\|\nabla\phi_0\|^2_{\mathbf{H}^2},
\tag{3.19}
$$

where constant C depends only on M and $\mathbb{S}^2$. It follows from (3.19) that there exists $T=T(\mathbb{S}^2,\|u_0\|_{\mathbb{H}^2},\|\nabla\phi_0\|_{\mathbf{H}^2})>0$ and $\tilde{K}_2>0$ such that

$$
\|u\|_{H^2}+\|\nabla\phi\|_{\mathbf{H}^2}\le\tilde{K}_2,\ \ t\in[0,T].
$$

Hence, by this and (3.18) there exists $K_2>0$ such that

$$
\|u\|_{H^2}+\|\nabla\phi\|_{\mathbf{H}^2}+\left(\int_0^t\|\nabla u\|^2_{H^2}\mathrm{d}s\right)^{1/2}\le K_2,\ \ t\in[0,T].
\tag{3.20}
$$

For the higher-order energy of u and ϕ, (3.3), (3.4) and (3.17) imply

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^n}^2 + \|\nabla\phi\|_{\mathbf{H}^n}^2) + \|\nabla u\|_{H^n}^2 \\
&\quad \le C\|\nabla u\|_{H^n}(\|u\|_{H^n}^2 + \|\nabla\phi\|_{\mathbf{H}^n}^2) + \frac{1}{4}\|\nabla u\|_{H^n}^2 \\
&\quad\quad + C(1 + \|u\|_{H^n}^2 + \|\nabla\phi\|_{\mathbf{H}^n}^2)^2(1 + \|u\|_{H^{n-1}} + \|\nabla\phi\|_{\mathbf{H}^{n-1}})^{n+2} \\
&\quad \le \frac{1}{2}\|\nabla u\|_{H^n}^2 + C(1 + \|u\|_{H^n}^2 + \|\nabla\phi\|_{\mathbf{H}^n}^2)^2(1 + \|u\|_{H^{n-1}} + \|\nabla\phi\|_{\mathbf{H}^{n-1}})^{n+2}.
\end{aligned} \tag{3.21}$$

From (3.20), we may assume that for any $2 \le l \le n-1$, there exists $K_l > 0$ such that

$$\|u\|_{H^l}^2 + \|\nabla\phi\|_{\mathbf{H}^l}^2 + \int_0^t \|\nabla u\|_{H^l}^2 \mathrm{d}s \le K_l,\ t \in [0, T]. \tag{3.22}$$

Let $f_n = 1 + \|u\|_{H^n}^2 + \|\nabla\phi\|_{\mathbf{H}^n}^2$, then by (3.21) and (3.22), we have

$$f_n' \le C K_{n-1}^{n+2} f_n^2,$$

which further implies that there exists $\tilde{K}_n > 0$ such that

$$\|u\|_{H^n} + \|\nabla\phi\|_{\mathbf{H}^n} \le \tilde{K}_n,\ t \in [0, T].$$

Hence, this, together with (3.21), yields

$$\|u\|_{H^n} + \|\nabla\phi\|_{\mathbf{H}^n} + \left(\int_0^t \|\nabla u\|_{H^n}^2 \mathrm{d}s\right)^{1/2} \le K_n,\ t \in [0, T],$$

which completes the proof of lemma. □

Next, we use the above lemma to prove the local existence of (1.1).

Proof of local existence From $u_0 \in H^k$, $\phi_0 \in H_Q^{k+1}$ for $k \ge 2$, by the density theorem of Sobolev spaces and Lemma 2.3 we may choose a sequence (u_{i0}, ϕ_{i0}) in $H^k \times H_Q^{k+1}$ satisfying $u_{i0} \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $\phi_{i0} - Q \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^3)$ such that

$$(u_{i0}, \phi_{i0}) \to (u_0, \phi_0)\ in\ H^k(\mathbb{R}^d) \times H_Q^{k+1}(\mathbb{R}^d),\ as\ i \to \infty. \tag{3.23}$$

For a section V of $\phi^\star T\mathbb{S}^2$, we have the relation between $\nabla_\alpha V$ and $D_\alpha V$:

$$\nabla_\alpha V = D_\alpha V + A(\phi)(D\phi, V),$$

where A is the second fundamental form of $\mathbb{S}^2$ in $\mathbb{R}^3$. Thus, there are multi-linear vector valued functions B_i on $\mathbb{R}^3$ such that

$$D_{\mathbf{a}}\phi = \nabla_{\mathbf{a}}\phi + \sum_{\sigma} B_{\sigma(\mathbf{a})}(\phi)(\nabla_{\mathbf{a}_1}\phi, \ldots, \nabla_{\mathbf{a}_s}\phi), \tag{3.24}$$

where $|\mathbf{a}| \geq 2$ and the sum is over all multi-indices $\mathbf{a}_1, \ldots, \mathbf{a}_s$ such that $|\mathbf{a}_i| \geq 1$ for all i and $(\mathbf{a}_1, \ldots, \mathbf{a}_s) = \sigma(\mathbf{a})$ is a permutation of $\mathbf{a}$. By (3.23) and (3.24), we can obtain

$$\|D\phi_{i0}\|_{\mathbf{H}^k} \to \|D\phi_0\|_{\mathbf{H}^k}, \ as\ i \to \infty.$$

Let Ω_i be the support of $(u_{i0}, \phi_{i0} - Q)$; there exists R_i sufficiently large such that $\Omega_i \subset\subset [-R_i, R_i]^{2d}$. Then, (u_{i0}, ϕ_{i0}) can be regarded as a function defined on a flat torus $\mathbb{T}_i^{2d} = \mathbb{R}^{2d}/(2R_i \cdot \mathbb{Z})^{2d}$, and hence, we consider the following Cauchy problem:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u - \operatorname{div}(\nabla\phi \odot \nabla\phi), & on\ \mathbb{T}_i^d \times (0, T], \\ \operatorname{div} u = 0, \\ \partial_t \phi + u \cdot \nabla\phi = \phi \times \Delta\phi, & on\ \mathbb{T}_i^d \times (0, T], \\ (u, \phi)(0) = (u_{i0}, \phi_{i0}) : \mathbb{T}_i^d \times \mathbb{T}_i^d \to \mathbb{R}^d \times \mathbb{S}^2. \end{cases} \tag{3.25}$$

By Proposition 2.1 and Lemma 3.1, we obtain that there exists $T > 0$, which does not depend on i, such that (3.25) admits a smooth solution (u_i, ϕ_i) on $\mathbb{T}_i^{2d} \times [0, T]$. Moreover, the following bound holds uniformly with respect to i:

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|u_i\|_{H^k(\mathbb{T}_i^d)} + \left(\int_0^t \|\nabla u_i\|^2_{H^k(\mathbb{T}_i^d)} \mathrm{d}s \right)^{1/2} + \|\nabla\phi_i\|_{\mathbf{H}^k(\mathbb{T}_i^d)} \right) \\ &\leq C \left(T, \|u_0\|_{H^k(\mathbb{T}_i^d)}, \|\nabla\phi_0\|_{\mathbf{H}^k(\mathbb{T}_i^d)} \right). \end{aligned}$$

Combining this and Proposition 2.2, we may further obtain

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|u_i\|_{H^k(\mathbb{T}_i^d)} + \left(\int_0^t \|\nabla u_i\|^2_{H^k(\mathbb{T}_i^d)} \mathrm{d}s \right)^{1/2} + \|\nabla\phi_i\|_{H_Q^k(\mathbb{T}_i^d)} \right) \\ &\leq \tilde{C} \left(T, \|u_0\|_{H^k(\mathbb{T}_i^d)}, \|\nabla\phi_0\|_{H_Q^k(\mathbb{T}_i^d)} \right). \end{aligned} \tag{3.26}$$

If we regard each (u_i, ϕ_i) as a function from $[-R_i, R_i]^d \times [-R_i, R_i]^d$ into $\mathbb{R}^d \times \mathbb{S}^2$, then there exists a $(u, \phi) \in L^\infty([0, T]; H^k(\mathbb{R}^d) \times H_Q^{k+1}(\mathbb{R}^d))$ and a subsequence which is still denoted by (u_i, ϕ_i) such that for any compact domain $\mathcal{X}_1, \mathcal{X}_2 \subset \mathbb{R}^d$

$$\begin{aligned} &(u_i, \phi_i) \to (u, \phi)\ \ weakly^\star \\ &in\ L^\infty([0, T]; H^k(\mathcal{X}_1)) \cap L^2([0, T]; H^{k+1}(\mathcal{X}_1)) \times L^\infty([0, T]; H_Q^{k+1}(\mathcal{X}_2)), \end{aligned}$$

and hence, we easily obtain (u, ϕ) which is a strong solution to the Cauchy problem (1.1). This completes the proof of local existence.

□

4 Uniqueness

In this section, we prove the uniqueness of (1.1) using the ideas of McGahagan [17] and Song-Wang [21].

Assume that $(u_1, \phi_1), (u_2, \phi_2) \in H^2 \times H^3_Q$ are two solutions to the system (1.1) with the same initial map $(u_0, \phi_0) \in H^2 \times H^3_Q$.

By $\mathbb{S}^2 \subset \mathbb{R}^3$ and (1.1), we have for $\lambda = 1, 2$

$$\begin{aligned}\|\phi_\lambda(t,x) - \phi_0(x)\|_{L^2} &\le \|\int_0^t \partial_s \phi_\lambda(s,x)ds\|_{L^2}\\ &\le Ct\|u_\lambda \cdot \nabla\phi_\lambda - \phi_\lambda \times \Delta\phi_\lambda\|_{L^2} \le Ct.\end{aligned}$$

This, together with Gagliardo–Nirenberg interpolation inequality, implies

$$\|\phi_\lambda - \phi_0\|_{L^\infty} \le C\|\phi_\lambda - \phi_0\|_{L^2}^{1-d/4}\|\Delta(\phi_\lambda - \phi_0)\|_{L^2}^{d/4} \le Ct^{1-d/4}.$$

From this, for any $\delta_0 > 0$ sufficiently small, there exists $T' > 0$ such that $|\phi_1 - \phi_2| < \delta_0$ for any $(t, x) \in [0, T'] \times \mathbb{R}^d$. And hence, there exists a unique minimizing geodesic $\gamma_{(t,x)}(s) : [0, l] \to \mathbb{S}^2$ such that $\gamma_{(t,x)}(0) = \phi_1(t, x)$ and $\gamma_{(t,x)}(l) = \phi_2(t, x)$, where l is the length of the geodesic γ. Let (t, x) vary; the family of geodesics gives rise to a map $U : [0, 1] \times [0, T'] \times \mathbb{R}^d \to \mathbb{S}^2$ connecting ϕ_1 and ϕ_2, where $U(s, t, x) = \gamma_{(t,x)}(s)$. Therefore, we can define a global bundle morphism $\mathcal{P}(s) : \phi_1^*T\mathbb{S}^2 = \gamma(0)^*T\mathbb{S}^2 \to \gamma(s)^*T\mathbb{S}^2$ for any $s \in [0, l]$ by the parallel transportation along each geodesic.

Using the similar argument to [17, Lemma 4.3], we have the following lemma.

Lemma 4.1 *We have the following inequalities for derivatives of the geodesics γ and their lengths l:*

$$\begin{aligned}&|\partial_k l| \le |\mathcal{P}\nabla\phi_2 - \nabla\phi_1|,\\ &|\partial_k \gamma| \lesssim |\nabla\phi_1| + |\nabla\phi_2|,\\ &|\partial_t \gamma| \lesssim |\nabla_k\partial_k\phi_1| + |\nabla_k\partial_k\phi_2| + |u_1 \cdot \nabla\phi_1| + |u_2 \cdot \nabla\phi_2|,\\ &|D_j\partial_k\gamma| \lesssim |\nabla_j\partial_k\phi_1| + |\nabla_j\partial_k\phi_2| + (|\partial_j\phi_1| + |\partial_j\phi_2|)(|\partial_k\phi_1| + |\partial_k\phi_2|).\end{aligned}$$

Next, in order to obtain the uniqueness, it suffices to prove

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u_1 - u_2\|_{L^2}^2 + \|\phi_1 - \phi_2\|_{L^2}^2 + \|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2)\\ &\quad \le C(\|u_1 - u_2\|_{L^2}^2 + \|\phi_1 - \phi_2\|_{L^2}^2 + \|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2), \qquad (4.1)\end{aligned}$$

where the constant C depends on $\|u_\lambda\|_{H^2}$ and $\|\nabla\phi_\lambda\|_{H^2}$ for $\lambda = 1, 2$.

First, by the similar computations to [17, P393, (ii)] and (1.1), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}^2\\
&\quad\lesssim C(\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}+\|u_1-u_2\|_{L^2})\\
&\qquad\cdot\Big\{\|\nabla u_1-\nabla u_2\|_{L^2}+\|u_1-u_2\|_{L^4}\|D\partial\phi_2\|_{L^4}+\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}\\
&\qquad+\|l\sup_{s\in[0,l]}|\partial_t\gamma||\nabla\phi_1|\|_{L^2}+\|l\sup_{s\in[0,l]}|\partial\gamma||\partial_t\phi_1|\|_{L^2}\\
&\qquad+\|\nabla l\|_{L^2}\|\nabla\phi_1\|_{L^\infty}\|\nabla\phi_2\|_{L^\infty}+\|l\sup_{s\in[0,l]}|DR(\partial\gamma,\partial_s\gamma)\mathcal{P}(s)\nabla\phi_1|\|_{L^2}\\
&\qquad+\|l^2\sup_{s\in[0,l]}|\partial\gamma||\nabla\phi_1|^2\|_{L^2}\Big\}.
\end{aligned}\tag{4.2}$$

For $d=3$, we estimate each term with a factor of l by taking l in $L^{\frac{2d}{d-2}}$ and the rest of the term in L^d. By Sobolev embedding and Lemma 4.1, (4.2) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}^2 &\le C(\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}+\|u_1-u_2\|_{L^2})\cdot\big[\|\nabla u_1-\nabla u_2\|_{L^2}\\
&\quad+\|u_1-u_2\|_{L^2}^{1-d/4}\|\nabla u_1-\nabla u_2\|_{L^2}^{d/4}+\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}\\
&\quad+\|\nabla l\|_{L^2}C(u_\lambda,\phi_\lambda)\big]\\
&\le\frac{1}{4}\|\nabla u_1-\nabla u_2\|_{L^2}^2+C(\|\mathcal{P}\nabla\phi_1\\
&\quad-\nabla\phi_2\|_{L^2}+\|u_1-u_2\|_{L^2})^2\\
&\quad+C\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}^2C(u_\lambda,\phi_\lambda),
\end{aligned}$$

where

$$\begin{aligned}
C(u_\lambda,\phi_\lambda)&=\|(|\nabla_k\partial_k\phi_1|+|\nabla_k\partial_k\phi_2|+|u_1\cdot\nabla\phi_1|+|u_2\cdot\nabla\phi_2|)|\nabla\phi_1|\|_{L^d}\\
&\quad+\|(|\nabla\phi_1|+|\nabla\phi_2|)(|u_1\cdot\nabla\phi_1|+|D\nabla\phi_1|)\|_{L^d}+\|\nabla\phi_1\|_{L^\infty}\|\nabla\phi_2\|_{L^\infty}\\
&\quad+\|(|D\nabla\phi_1|+|D\nabla\phi_2|)|\nabla\phi_1|(|\nabla\phi_1|+|\nabla\phi_2|)\|_{L^d}\\
&\quad+\|(|\nabla\phi_1|+|\nabla\phi_2|)^2|\nabla\phi_1|(1+|\nabla\phi_1|+|\nabla\phi_2|)\|_{L^d}\\
&\quad+\|l(|\nabla\phi_1|+|\nabla\phi_2|)|\nabla\phi_1|^2\|_{L^d}.
\end{aligned}$$

For $d=2$, we bound l in L^∞. Apply a theorem due to Brezis and Wainger [6]:

$$\begin{aligned}
\|l\|_{L^\infty}\lesssim\|\phi_1-\phi_2\|_{L^\infty}&\lesssim\|\phi_1-\phi_2\|_{H^1}\\
&\quad\big(1+\log^{1/2}(1+\|\partial^2(\phi_1-\phi_2)\|_{L^2})\big)\lesssim\|\phi_1-\phi_2\|_{H^1}.
\end{aligned}$$

Then, (4.2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}^2\le\frac{1}{4}\|\nabla u_1-\nabla u_2\|_{L^2}^2+C(\|\mathcal{P}\nabla\phi_1-\nabla\phi_2\|_{L^2}+\|u_1-u_2\|_{L^2})^2$$

$$
\begin{aligned}
&+ C\|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}\|\phi_1 - \phi_2\|_{H^1} C(u_\lambda, \phi_\lambda) \\
&\leq \frac{1}{4}\|\nabla u_1 - \nabla u_2\|_{L^2}^2 + C(\|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2} + \|u_1 - u_2\|_{L^2})^2 \\
&\quad + C(\|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2 + \|\phi_1 - \phi_2\|_{L^2}^2) C(u_\lambda, \phi_\lambda).
\end{aligned}
$$

By Sobolev embedding, we can bound $C(u_\lambda, \phi_\lambda)$ by

$$
C(u_\lambda, \phi_\lambda) \lesssim (1 + \|\nabla\phi_1\|_{H^2} + \|\nabla\phi_2\|_{H^2})^4 (1 + \|u_1\|_{H^2} + \|u_2\|_{H^2}).
$$

Then, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2 \leq \quad & \frac{1}{4}\|\nabla u_1 - \nabla u_2\|_{L^2}^2 + C\Big(\|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2 \\
& + \|\phi_1 - \phi_2\|_{L^2}^2 + \|u_1 - u_2\|_{L^2}^2\Big).
\end{aligned} \tag{4.3}
$$

Second, by ϕ-equation and Sobolev embedding, we easily obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_1 - \phi_2\|_{L^2}^2 \\
&\quad \leq \|u_1 - u_2\|_{L^2}\|\phi_1 - \phi_2\|_{L^2}\|\nabla\phi_1\|_{H^2} \\
&\qquad + \|\phi_1 - \phi_2\|_{L^2}\|\nabla\phi_1 - \nabla\phi_2\|_{L^2}(\|u_2\|_{H^2} + \|\nabla\phi_1\|_{H^2} + \|\nabla\phi_2\|_{H^2}) \\
&\qquad + \|\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2\|\phi_2\|_{L^\infty} \\
&\quad \leq C(\|u_1 - u_2\|_{L^2}^2 + \|\phi_1 - \phi_2\|_{L^2}^2 + \|\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2).
\end{aligned} \tag{4.4}
$$

Using properties of the parallel transport, we have

$$
\begin{aligned}
\|\nabla\phi_1 - \nabla\phi_2\|_{L^2} &\lesssim \|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2} + \|l\|_{L^2} \\
&\lesssim \|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2} + \|\phi_1 - \phi_2\|_{L^2}.
\end{aligned} \tag{4.5}
$$

Then, (4.4) becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_1 - \phi_2\|_{L^2}^2 \leq C(\|u_1 - u_2\|_{L^2}^2 + \|\phi_1 - \phi_2\|_{L^2}^2 + \|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2). \tag{4.6}
$$

Finally, by u-equation, $\nabla \cdot u_\lambda = 0$, Sobolev embedding and (4.5), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1 - u_2\|_{L^2}^2 + \|\nabla(u_1 - u_2)\|_{L^2}^2 \\
&\quad \leq C\|\nabla(u_1 - u_2)\|_{L^2}(\|u_1 - u_2\|_{L^2} + \|\nabla(\phi_1 - \phi_2)\|_{L^2}) \\
&\quad \leq \frac{1}{4}\|\nabla(u_1 - u_2)\|_{L^2}^2 + C(\|u_1 - u_2\|_{L^2}^2 + \|\nabla(\phi_1 - \phi_2)\|_{L^2}^2) \\
&\quad \leq \frac{1}{4}\|\nabla(u_1 - u_2)\|_{L^2}^2 + C(\|u_1 - u_2\|_{L^2}^2 + \|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2}^2).
\end{aligned} \tag{4.7}
$$

Hence, from (4.3), (4.6) and (4.7), the bound (4.1) follows. Since $\|u_1-u_2\|_{L^2} = \|\phi_1-\phi_2\|_{L^2} = \|\mathcal{P}\nabla\phi_1 - \nabla\phi_2\|_{L^2} = 0$ at initial time, we then obtain $(u_1, \phi_1) = (u_2, \phi_2)$ on $[0, T']$ by (4.1) and Gronwall's inequality. By repeating the above argument, we can prove $(u_1, \phi_1) = (u_2, \phi_2)$ on the whole interval $[0, T]$ and finish the proof of the uniqueness.

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