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Quantization of the Blow-Up Value for the Liouville Equation with Exponential Neumann Boundary Condition

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Abstract In this paper, we analyze the asymptotic behavior of solution sequences of the Liouville-type equation with Neumann boundary condition. In particular, we will obtain a sharp mass quantization result for the solution sequences at a blow-up point.

Keywords Neumann problem · Concentration–compactness phenomena · Blow-up behaviors · Mass quantization

Mathematics Subject Classification 35B40 · 35J65

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$. The so-called Liouville equation is

$$-\Delta u = V(x)e^{2u} \quad \text{in } \Omega, \tag{1.1}$$

which was first studied by Liouville in 1853 in [14]. In 1991, Brezis and Merle [1] initiated the study of the blow-up analysis for the Liouville equation. Under the finite

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energy condition, they first showed that any solution belongs to L^∞, and further, they analyzed the convergence of a sequence of solutions and obtained a concentration–compactness type result. Their results initiate many works on the asymptotic behavior of blow-up solutions, for there are many applications in geometric and physical problems, for example, in the problem of prescribing Gaussian curvature [3,4,7], in the theory of the mean field equation [5,8,9], and in the Chern–Simons theory [10,17–20]. See also the reference therein.

In particular, in the celebrated paper by Li and Shafrir [15], they initiated to evaluate the blow-up value at the blow-up point. They showed at the each blow-up point the blow-up value is quantized, i.e., there is no contribution of mass outside the m disjoint balls (whose radii is going to zero) which contain a contribution of $4\pi m$ mass for some positive integer m. Concerning the mean field equation, this kind of mass quantization leads to the crucial compactness property of solutions. Then, the existence issues can be attacked by variational methods; see [8,9].

The aim of the present paper is to generalize the blow-up analysis for (1.1) to a Liouville-type equation with Neumann boundary condition. In other words, we consider the following Neumann boundary problem:

$$\begin{cases} -\Delta u = V(x)e^{2u} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = h(x)e^{u} & \text{on } L. \end{cases} \tag{1.2}$$

Here L is a proper subset of $\partial\Omega$, and $V(x)$ and $h(x)$ are nonnegative functions. This problem plays a very important role in the study of the construction of prescribed Gaussian curvature surfaces with prescribed geodesic curvature on their boundary.

Guo and Liu [11] have analyzed the asymptotic behavior of solutions in the case $V(x) \equiv 0$. Their problem is

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta = h(x)e^{u} & \text{on } \partial\Omega. \end{cases}$$

They obtained a Brezis–Merle type concentration–compactness phenomena and a Li-Shafrir type energy quantization result.

In this paper, we pursue this line of investigation on more general class of system (1.2). Now we introduce some notions firstly. Define

$$B_R^+(x_0) = \left\{ x \in \mathbb{R}^2 | x - x_0 = (s,t), s^2 + t^2 < R^2, t > 0 \right\},$$
$$L_R(x_0) = \left\{ x \in \mathbb{R}^2 | x - x_0 = (s,0), |s| < R \right\} = \partial B_R^+(x_0) \cap \partial\mathbb{R}^2_+,$$
$$S_R^+(x_0) = \left\{ x \in \mathbb{R}^2 | x - x_0 = (s,t), s^2 + t^2 = R^2, t > 0 \right\} = \partial B_R^+(x_0) \cap \mathbb{R}^2_+.$$

In addition, we use the notions B_R^+, L_R and S_R^+ for $B_R^+(0)$, $L_R(0)$ and $S_R^+(0)$, respectively. For simplicity, we consider the following Neumann boundary value problem of Liouville equation in B_R^+:

$$\begin{cases} -\Delta u = V(x)e^{2u} & \text{in } B_R^+, \\ \dfrac{\partial u}{\partial n} = h(x)e^{u} & \text{on } L_R. \end{cases} \tag{1.3}$$

From [2], we have the following Brezis–Merle type concentration–compactness theorem.

Theorem 1.1 *Assume that* $\{u_n\}$ *is a sequence of solution for the following problem*

$$\begin{cases} -\Delta u_n = V_n(x)e^{2u_n} & \text{in } B_R^+, \\ \dfrac{\partial u_n}{\partial n} = h_n(x)e^{u_n} & \text{on } L_R, \end{cases} \tag{1.4}$$

with the energy conditions

$$\int_{B_R^+} e^{2u_n}\mathrm{d}x \le C; \qquad \int_{L_R} e^{u_n}\mathrm{d}s \le C. \tag{1.5}$$

Here $V_n(x)$ *and* $h_n(x)$ *satisfy*

$$0 < a < V_n(x) < C, \forall x \in B_R^+; \qquad 0 < b < h_n(x) < C, \forall x \in L_R, \tag{1.6}$$

for positive numbers a, b *and* C.

Define the blow-up set as

$$S = \left\{x \in B_R^+ \cup L_R,\ \text{there is a sequence } y_n \to x,\ \text{such that } u_n(y_n) \to +\infty\right\}.$$

Then, there exists a subsequence, denoted still by $\{u_n\}$, *satisfying one of the following alternatives*:

(*i*) $\{u_n\}$ *is bounded in* $L^\infty_{loc}(B_R^+ \cup L_R)$,
(*ii*) $\{u_n\} \to -\infty$ *uniformly on compacts of* $B_R^+ \cup L_R$,
(*iii*) *there exists a finite blow-up set* $S = \{p_1, p_2, \ldots, p_m\} \subset B_R^+ \cup L_R$. *Moreover,* $u_n(x) \to -\infty$ *uniformly on compact subsets of* $(B_R^+ \cup L_R)\backslash S$, *and*

$$\int_{B_R^+} V_n e^{2u_n}\phi\mathrm{d}x + \int_{L_R} h_n e^{u_n}\phi\mathrm{d}s \to \sum_{i=1}^m \alpha_i\phi(p_i),$$

for every $\phi \in C_o^\infty(B_R^+ \cup L_R)$ *with* $\alpha_i \ge \pi$.

From Theorem 1.1, the blow-up set S is nonempty if u_n is blow-up. We can define the blow-up value at each blow-up point. For $p \in S \cap B_R^+$, we define the blow-up value at point p as:

$$m(p) = \lim_{r\to 0}\lim_{n\to\infty}\int_{B_r(p)\cap B_R^+} V_n e^{2u_n}\mathrm{d}x;$$

For $p \in S \cap L_R$, we define the blow-up value at point p as:

$$m(p) = \lim_{r\to 0}\lim_{n\to\infty}\left(\int_{B_r(p)\cap B_R^+} V_n e^{2u_n}\mathrm{d}x + \int_{B_r(p)\cap L_R} h_n e^{u_n}\mathrm{d}s\right).$$

Li and Shafrir [15] have shown that $m(p) = 4m\pi$ for any $p \in S \cap B_R^+$. In this paper, we want further to show that $m(p) = 2m\pi$ for any $p \in S \cap L_R$. Our main theorem is the following Li-Shafrir type energy quantization theorem.

Theorem 1.2 *For $R > 0$, let $\{V_n\}$ and $\{h_n\}$ be two sequences of functions satisfying*

$$0 < a \le V_n \to V \in C^0(\overline{B}_R^+), \quad 0 < b \le h_n \to h \in C^0(L_R). \tag{1.7}$$

Let $\{u_n\}$ be a sequence of solutions of (1.4), (1.5) with the following properties:

$$u_n(x_n) = \max_{\bar{B}_R^+} u_n \to +\infty, \tag{1.8}$$

$$\max_{\bar{B}_R^+\setminus B_r^+(0)} u_n \to -\infty, \quad \text{for } 0 < r < R. \tag{1.9}$$

Then,

$$\alpha := \lim_{n\to\infty}\left(\int_{B_R^+} V_n e^{2u_n} + \int_{L_R} h_n e^{u_n}\right) = 2\pi m \tag{1.10}$$

for some positive integer m.

The proof of Theorem 1.2 follows closely the idea of Li and Shafrir in [15] where they proved the quantization of the blow-up value for the Liouville equation. The approach in [15] is based on a classification result of bubbling equation $-\triangle u = e^{2u}$ in $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} e^{2u} < \infty$ and a "sup + inf" type inequality $u(0) + C_1 \inf_{B_1} u \le C_2$ for equation $-\triangle u = Ve^{2u}$ in B_1. For our problem, we need the corresponding results. On the one hand, besides of the above bubbling equation, there exists the other kind of bubbling equation, i.e.,

$$\begin{cases} -\Delta u = V(0)e^{2u} & \text{in } \mathbb{R}^2 \cap \{t > -\Lambda\}, \\ \dfrac{\partial u}{\partial n} = h(0)e^{u} & \text{on } \mathbb{R}^2 \cap \{t = -\Lambda\}, \end{cases}$$

with the energy conditions

$$\int_{\mathbb{R}^2\cap\{t>-\Lambda\}} V(0)e^{2u} \le C, \quad \int_{\mathbb{R}^2\cap\{t=-\Lambda\}} h(0)e^{u} \le C.$$

We will use the classification result proved in [16] to handle our problem. On the other hand, we need to prove a "sup + inf" type inequality under Neumann boundary value condition.

The paper is organized as follows. In Introduction, we state the problem and the main theorems. In Sect. 2, we prove a "sup + inf" type inequality and other auxiliary results. In Sect. 3, we complete the proof of Theorem 1.2.

2 A sup + inf Inequality Under Neumann Boundary Value Condition

In this section, we establish a "sup + inf" inequality under Neumann boundary value condition. We start to show some auxiliary lemmas.

Lemma 2.1 *Under the hypotheses of Theorem 1.2, we have* $\alpha \geq 2\pi$.

Proof Let $x_n = (s_n, t_n)$, $\delta_n = e^{-u_n(x_n)}$, then $\delta_n \to 0$ and $x_n \to 0$. By letting $\tilde{u}_n(x) = u_n(\delta_n x + x_n) + \log \delta_n$, we see that

$$\begin{cases} -\Delta \tilde{u}_n = V_n(\delta_n x + x_n) e^{2\tilde{u}_n} & \text{in } B_{\frac{R}{2\delta_n}} \cap \{t > -\frac{t_n}{\delta_n}\}, \\ \dfrac{\partial \tilde{u}_n}{\partial n} = h_n(\delta_n x + x_n) e^{\tilde{u}_n} & \text{on } B_{\frac{R}{2\delta_n}} \cap \{t = -\frac{t_n}{\delta_n}\}, \\ \tilde{u}_n(x) \leq \tilde{u}_n(0) = 0 & \text{in } B_{\frac{R}{2\delta_n}} \cap \{t \geq -\frac{t_n}{\delta_n}\}, \end{cases}$$

with the energy conditions

$$\int_{B_{\frac{R}{2\delta_n}} \cap \{t > -\frac{t_n}{\delta_n}\}} V_n(\delta_n x + x_n) e^{2\tilde{u}_n} \leq C, \qquad \int_{B_{\frac{R}{2\delta_n}} \cap \{t = -\frac{t_n}{\delta_n}\}} h_n(\delta_n x + x_n) e^{\tilde{u}_n} \leq C.$$

Now we distinguish two cases.

Case (1) $\frac{t_n}{\delta_n} \to \Lambda < +\infty$.

In this case, by Theorem 1.1, $\{\tilde{u}_n\}$ admits a subsequence converging to $\tilde{u}$ in $C^{1,\alpha}_{loc}(\mathbb{R}^2 \cap \{t \geq -\Lambda\})$, which satisfies

$$\begin{cases} -\Delta \tilde{u} = V(0) e^{2\tilde{u}} & \text{in } \mathbb{R}^2 \cap \{t > -\Lambda\}, \\ \dfrac{\partial \tilde{u}}{\partial n} = h(0) e^{\tilde{u}} & \text{on } \mathbb{R}^2 \cap \{t = -\Lambda\}, \\ \tilde{u}(x) \leq \tilde{u}(0) = 0 & \text{in } \mathbb{R}^2 \cap \{t \geq -\Lambda\}, \end{cases}$$

with the energy conditions

$$\int_{\mathbb{R}^2 \cap \{t > -\Lambda\}} V(0) e^{2\tilde{u}} \leq C, \quad \int_{\mathbb{R}^2 \cap \{t = -\Lambda\}} h(0) e^{\tilde{u}} \leq C.$$

It follows from the classification results in [16] that

$$\tilde{u}(s,t) = \log \frac{2\lambda}{\sqrt{V(0)}(\lambda^2 + (s-s_0)^2 + (t + \Lambda + \frac{h(0)\lambda}{\sqrt{V(0)}})^2)}, \quad \lambda > 0, s_0 \in \mathbb{R},$$

and

$$\int_{\mathbb{R}^2 \cap \{t > -\Lambda\}} V(0) e^{2\tilde{u}} \mathrm{d}x + \int_{\mathbb{R}^2 \cap \{t = -\Lambda\}} h(0) e^{\tilde{u}} \mathrm{d}s = 2\pi.$$

Therefore,

$$\begin{aligned}
\alpha &\geq \lim_{n\to\infty} \left(\int_{B^+_{R\delta_n}(x_n)} V_n e^{2u_n} + \int_{L_{R\delta_n}(x_n)} h_n e^{u_n} \right) \\
&= \lim_{n\to\infty} \left(\int_{B_R \cap \{t > -\frac{t_n}{\delta_n}\}} V_n\left(\delta_n x + x_n\right) e^{2\tilde{u}_n} + \int_{B_R \cap \{t = -\frac{t_n}{\delta_n}\}} h_n\left(\delta_n x + x_n\right) e^{\tilde{u}_n} \right) \\
&= \int_{B_R \cap \{t > -\Lambda\}} V(0) e^{2\tilde{u}} + \int_{B_R \cap \{t = -\Lambda\}} h(0) e^{\tilde{u}} = 2\pi + o_R(1).
\end{aligned}$$

Let $R \to \infty$, we get that $\alpha \geq 2\pi$.
Case (2) $\frac{t_n}{\delta_n} \to +\infty$.
In this case, also by Theorem 1.1, $\{\tilde{u}_n\}$ admits a subsequence converging to $\tilde{u}$ in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$, which satisfies

$$\begin{cases} -\Delta \tilde{u} = V(0) e^{2\tilde{u}} & \text{in } \mathbb{R}^2, \\ \tilde{u}(x) \leq \tilde{u}(0) = 0 & \text{in } \mathbb{R}^2, \end{cases}$$

with the energy condition

$$\int_{\mathbb{R}^2} V(0) e^{2\tilde{u}} \leq C.$$

It follows from [6] that

$$\tilde{u}(s,t) = \log \frac{\lambda}{1 + \frac{V(0)}{4}\lambda^2((s-s_0)^2 + (t-t_0)^2)}, \quad \lambda > 0, s_0, t_0 \in \mathbb{R},$$

and

$$\int_{\mathbb{R}^2} V(0) e^{2\tilde{u}} \mathrm{d}x = 4\pi.$$

Therefore,

$$\alpha \geq \lim_{n\to\infty} \left(\int_{B^+_{R\delta_n}(x_n)} V_n e^{2u_n} + \int_{L_{R\delta_n}(x_n)} h_n e^{u_n} \right)$$

$$= \lim_{n\to\infty}\left(\int_{B_R\cap\{t>-\frac{t_n}{\delta_n}\}} V_n\left(\delta_n x + x_n\right) e^{2\tilde{u}_n} + \int_{B_R\cap\{t=-\frac{t_n}{\delta_n}\}} h_n\left(\delta_n x + x_n\right) e^{\tilde{u}_n}\right)$$
$$= \int_{B_R} V(0)e^{2\tilde{u}} = 4\pi + o_R(1).$$

Let $R \to \infty$, we get that $\alpha \geq 4\pi$. □

Corollary 2.2 *Let u_n be a sequence of solutions of (1.4) with $u_n(x_n) = \max_{\bar{B}_R^+} u_n$, $\forall n$. Assume $\lim_{n\to\infty}(\int_{B_R^+} V_n e^{2u_n} + \int_{L_R} h_n(x)e^{u_n}) = \alpha < 2\pi$, then we have $u_n(x_n) \leq C$, where the constant C depends only on α and R.*

The "sup + inf" inequality and the Harnark inequality for

$$-\Delta u = Ve^u \tag{2.1}$$

are shown in [15], which is as following lemmas:

Lemma 2.3 [15] *Let $V \in L^\infty(B_R)$ satisfy $a \leq V(x) \leq b$, $\forall x \in B_R$, where a, b are positive constants. Suppose that u is a solution to (2.1) in B_R. Then*

$$u(0) + C_1 \inf_{B_R^+} u + 2(C_1 + 1)\log R \leq C_2,$$

where $C_1 \geq 1$ and C_2 are constants depending only on a and b.

Lemma 2.4 [15] *For $R > 0$, $0 < R_0 \leq R/4$, we set $\tilde{\Omega} = \{x \in \mathbb{R}^2 | R_0 < |x| < R\}$. Let u be a solution to (2.1) in $\tilde{\Omega}$ with $\|V\|_{L^\infty(\tilde{\Omega})} \leq C_1$ and $u(x)+\log|x| \leq C_2$, $\forall x \in \tilde{\Omega}$ for some positive constants C_1 and C_2. Then, there exists constant $\beta \in (0, 1)$ and C_3 depending only on C_1 and C_2 such that*

$$\sup_{\partial B_r} u \leq C_3 + \beta \inf_{\partial B_r} u + 2(\beta - 1)\log r, \quad \forall 2R_0 \leq r \leq R/2.$$

Now we establish the "sup + inf" inequality and the Harnark inequality for Neumann boundary value problem.

Lemma 2.5 *Let $\{V_n\}$ and $\{h_n\}$ be two sequences of functions satisfying (1.7). Let $\{u_n\}$ be a sequence of solutions of (1.4), (1.5) satisfying (1.8)–(1.9). Then, for each $C_1 > 1$, there exists C_2 such that*

$$u_n(x_n) + \log r + C_1\left(\inf_{B_r^+(x_n)} u_n + \log r\right) \leq C_2 \tag{2.2}$$

for $0 < r \leq R_0 \leq R$ and for sufficiently large n provided $B_{R_0}^+(x_n) \subset B_R^+$.

Proof Let $x_n=(s_n,t_n)$, then $x_n\to 0$. Clearly u_n satisfies

$$\begin{cases} -\Delta u_n = V_n e^{2u_n} & \text{in } B^+_{R_0}(x_n),\\ \dfrac{\partial u_n}{\partial n} = h_n e^{u_n} & \text{on } L_{R_0}(x_n). \end{cases} \tag{2.3}$$

We first use a slightly modified version of the arguments provided in [15]. For $\forall 0<r\le R_0$ and $\forall n\in N$, we define

$$\psi(r)=u_n(x_n)+\log r + C_1\left(\frac{1}{|S^+_r(x_n)|}\int_{S^+_r(x_n)} u_n+\log r\right).$$

We have

$$\begin{aligned}\psi'(r)&=\frac{1+C_1}{r}+\frac{C_1}{|S^+_r(x_n)|}\int_{S^+_r(x_n)}\frac{\partial u_n}{\partial n}\\ &=\frac{1+C_1}{r}-\frac{C_1}{|S^+_r(x_n)|}\left(\int_{B^+_r(x_n)}V_n e^{2u_n}+\int_{L_r(x_n)}\frac{\partial u_n}{\partial n}\right)\\ &=\frac{1+C_1}{r}-\frac{C_1}{|S^+_r(x_n)|}\left(\int_{B^+_r(x_n)}V_n e^{2u_n}+\int_{L_r(x_n)}h_n e^{u_n}\right).\end{aligned}$$

So that $\psi'(r)=0$ if $\int_{B^+_r(x_n)}V_n e^{2u_n}+\int_{L_r(x_n)}h_n e^{u_n}=\frac{(1+C_1)|S^+_r(x_n)|}{rC_1}$. Note that $\frac{(1+C_1)|S^+_r(x_n)|}{rC_1}$ is independent of r. At this point, either $\int_{B^+_{R_0}(x_n)}V_n e^{2u_n}+\int_{L_{R_0}(x_n)}h_n e^{u_n}\le\frac{(1+C_1)|S^+_r(x_n)|}{rC_1}$, and then, we take $r_n=R_0$, or $\int_{B^+_{R_0}(x_n)}V_n e^{2u_n}+\int_{L_{R_0}(x_n)}h_n e^{u_n}>\frac{(1+C_1)|S^+_r(x_n)|}{rC_1}$ and we may choose $r_n\in(0,R_0)$, such that $\int_{B^+_{r_n}(x_n)}V_n e^{2u_n}+\int_{L_{r_n}(x_n)}h_n e^{u_n}=\frac{(1+C_1)|S^+_r(x_n)|}{rC_1}$. In any case, $\forall n\in N$, we have $\psi(r)\le\psi(r_n)$, $0<r\le R_0$.

For $\forall x\in B_1\cap\{t\ge -\frac{t_n}{r_n}\}$, we define $\omega_n(x)=u_n(x_n+r_nx)+\log r_n$. We see that $\omega_n(x)$ satisfies

$$\begin{cases} -\Delta\omega_n = V_n(r_nx+x_n)e^{2\omega_n} & \text{in } B_1\cap\{t>-\frac{t_n}{r_n}\},\\ \dfrac{\partial\omega_n}{\partial n}=h_n(r_nx+x_n)e^{\omega_n} & \text{on } B_1\cap\{t=-\frac{t_n}{r_n}\}. \end{cases}$$

Now we argue by contradiction and assume that $\omega_n(0)=u_n(x_n)+\log r_n\to+\infty$. Set $\rho_n=e^{-\omega_n(0)}$, then $\rho_n\to 0$. Consider the sequence of functions $\widetilde{\omega}_n(x)=\omega_n(\rho_nx)+\log\rho_n$. Then $\widetilde{\omega}_n$ satisfies

$$\begin{cases} -\Delta\widetilde{\omega}_n = V_n\left(r_n\rho_n x + x_n\right)e^{2\widetilde{\omega}_n} & \text{in } B_{\frac{1}{\rho_n}} \cap \{t > -\frac{t_n}{r_n\rho_n}\}, \\ \dfrac{\partial\widetilde{\omega}_n}{\partial n} = h_n\left(r_n\rho_n x + x_n\right)e^{\widetilde{\omega}_n} & \text{on } B_{\frac{1}{\rho_n}} \cap \{t = -\frac{t_n}{r_n\rho_n}\}, \\ \widetilde{\omega}_n(x) \le \widetilde{\omega}_n(0) = 0 \end{cases}$$

with the energy conditions

$$\int_{B_{\frac{1}{\rho_n}}\cap\{t>-\frac{t_n}{r_n\rho_n}\}} V_n\left(r_n\rho_n x + x_n\right)e^{2\tilde{\omega}_n} \le C, \quad \int_{B_{\frac{1}{\rho_n}}\cap\{t=-\frac{t_n}{r_n\rho_n}\}} h_n(r_n\rho_n x + x_n)e^{\tilde{\omega}_n} \le C.$$

Then, as in Lemma 2.1, we have to analyze the following two situations:

Case (1) : $\frac{t_n}{r_n\rho_n} \to \Lambda < +\infty$.
Case (2) : $\frac{t_n}{r_n\rho_n} \to +\infty$.

Arguing as in Lemma 2.1, we can drive either

$$\lim_{n\to\infty}\left(\int_{B_{\frac{1}{\rho_n}}\cap\{t>-\frac{t_n}{r_n\rho_n}\}} V_n(r_n\rho_n x + x_n)e^{2\tilde{\omega}_n} + \int_{B_{\frac{1}{\rho_n}}\cap\{t=-\frac{t_n}{r_n\rho_n}\}} V_n(r_n\rho_n x + x_n)e^{\tilde{\omega}_n}\right) = 2\pi,$$

or

$$\lim_{n\to\infty}\left(\int_{B_{\frac{1}{\rho_n}}\cap\{t>-\frac{t_n}{r_n\rho_n}\}} V_n(r_n\rho_n x + x_n\right)e^{2\tilde{\omega}_n} = 4\pi.$$

But

$$\begin{aligned} &\int_{B_{\frac{1}{\rho_n}}\cap\{t>-\frac{t_n}{r_n\rho_n}\}} V_n(r_n\rho_n x + x_n)e^{2\tilde{\omega}_n} + \int_{B_{\frac{1}{\rho_n}}\cap\{t=-\frac{t_n}{r_n\rho_n}\}} V_n(r_n\rho_n x + x_n)e^{\tilde{\omega}_n} \\ &= \int_{B_1\cap\{t>-\frac{t_n}{r_n}\}} V_n(r_n x + x_n)e^{2\omega_n} + \int_{B_1\cap\{t=-\frac{t_n}{r_n}\}} h_n(r_n x + x_n)e^{\omega_n} \\ &= \int_{B^+_{r_n}(x_n)} V_n e^{2u_n} + \int_{L_{r_n}(x_n)} h_n e^{u_n} \\ &= \frac{(1+C_1)|S^+_r(x_n)|}{rC_1} < 2\pi, \end{aligned}$$

for n sufficiently large, which is the desired contradiction. So there is a constant C such that $\omega_n(0) = u_n(x_n) + \log r_n \le C$. Consequently, we have

$$\psi(r) = u_n(x_n) + \log r + C_1\left(\frac{1}{|S^+_r(x_n)|}\int_{S^+_r(x_n)} u_n + \log r\right) \le C(1 + C_1) = C_2.$$

Notice that u_n is superharmonic and $\frac{\partial u_n}{\partial n} \geq 0$ from (2.3), we have $\inf\limits_{B_r^+(x_n)} u_n = \inf\limits_{S_r^+(x_n)} u_n \leq \frac{1}{|S_r^+(x_n)|} \int_{S_r^+(x_n)} u_n$. Then, we derive the desired inequality. □

Lemma 2.6 *For $R > 0, 0 < R_0 \leq R/4$, we define $T = \{x \in \mathbb{R}_+^2 | R_0 < |x - x_0| < R\}$. Assume that $||V||_{L^\infty(T)} \leq C_1$ and $||h||_{L^\infty(\partial T \cap \partial \mathbb{R}_+^2)} \leq C_1$. Let u be a solution of*

$$\begin{cases} -\Delta u = Ve^{2u} & in\ T, \\ \dfrac{\partial u}{\partial n} = he^u & on\ \partial T \cap \partial \mathbb{R}_+^2, \end{cases}$$

with $u(x) + \log|x - x_0| \leq C_2, \forall x \in \overline{T}$. Then, there exists constant $\beta \in (0, 1)$ and C_3 such that

$$\sup_{S_r^+(x_0)} u \leq C_3 + \beta \inf_{S_r^+(x_0)} u + (\beta - 1)\log r, \quad \forall 2R_0 \leq r \leq R/2.$$

Here β and C_3 are dependent only on C_1 and C_2.

Proof Without loss of generality, we assume that $x_0 = 0$. For $2R_0 \leq r \leq R/2$, by letting $\tilde{u}(x) = u(rx) + \log r$, then $\tilde{u}(x)$ satisfies

$$\begin{cases} -\Delta \tilde{u} = V(rx)e^{2\tilde{u}} & \text{in } B_2^+ \backslash B_{\frac{1}{2}}^+, \\ \dfrac{\partial \tilde{u}}{\partial n} = h(rx)e^{\tilde{u}} & \text{on } L_2 \backslash L_{\frac{1}{2}}. \end{cases}$$

For $\frac{1}{2} \leq |x| \leq 2$, by the given assumptions we have $\tilde{u}(x) = u(rx) + \log(r|x|) - \log|x| \leq C_2 + \log 2$. It follows that $|V(rx)|e^{2\tilde{u}(x)} \leq C$ on $B_2^+ \backslash B_{\frac{1}{2}}^+$ and $|h(rx)|e^{\tilde{u}(x)} \leq C$ on $L_2 \backslash L_{\frac{1}{2}}$. Define $\omega(x) = \frac{1}{\pi} \int_{B_2^+ \backslash B_{\frac{1}{2}}^+} \log \frac{4}{|x-y|} V(ry)e^{2\tilde{u}(y)} + \frac{1}{\pi} \int_{L_2 \backslash L_{\frac{1}{2}}} \log \frac{4}{|x-y|} h(ry)e^{\tilde{u}(y)}$. Then, $\omega(x)$ is bounded in $B_2^+ \backslash B_{\frac{1}{2}}^+$ and satisfies

$$\begin{cases} -\Delta \omega = V(rx)e^{2\tilde{u}} & \text{in } B_2^+ \backslash B_{\frac{1}{2}}^+, \\ \dfrac{\partial \omega}{\partial n} = h(rx)e^{\tilde{u}} & \text{on } L_2 \backslash L_{\frac{1}{2}}. \end{cases}$$

Let $g = \omega - \tilde{u}$. Then, we have

$$\begin{cases} -\Delta g = 0 & \text{in } B_2^+ \backslash B_{\frac{1}{2}}^+, \\ \dfrac{\partial g}{\partial n} = 0 & \text{on } L_2 \backslash L_{\frac{1}{2}}. \end{cases}$$

We conclude that g is bounded below. Then, by the Harnack inequality we get

$$\sup_{S_1^+}(g + C) \leq \beta^{-1} \inf_{S_1^+}(g + C),$$

for some constants C and $\beta \in (0, 1)$. Then, returning to the original u we obtain the desired estimates. □

3 Proof of Theorem 1.2

In this section, we prove the main theorem.

Proof of Theorem 1.2 We divide the proof into two steps.

Step 1. In this step, we want to show: After passing to a subsequence, there exist m sequences of points $\{x_n^{(j)} = (s_n^{(j)}, t_n^{(j)})\}_{j=0}^{m-1}$ in $\bar{B}_R^+$ and m sequences of positive numbers $\{k_n^{(j)}\}_{j=0}^{m-1}$ with $\lim_{n\to\infty} x_n^{(j)} = 0$ and $\lim_{n\to\infty} k_n^{(j)} = \infty (0 \le j \le m-1)$ such that

(a) For any $0 \le j \le m-1$, $u_n(x_n^{(j)}) = \max_{x \in \bar{B}^+_{k_n^{(j)}\delta_n^{(j)}}(x_n^{(j)})} u_n(x) \to \infty$;

(b) For any $0 \le j \le m-1$, $\frac{|x_n^{(i)} - x_n^{(j)}|}{k_n^{(j)}\delta_n^{(j)}} \to \infty$, $\forall i \ne j$, where $\delta_n^{(j)} = e^{-u_n(x_n^{(j)})}$;

(c) For any $0 \le j \le m-1$,

$$\begin{aligned}\beta_j &:= \lim_{n\to\infty}\left(\int_{B^+_{k_n^{(j)}\delta_n^{(j)}}(x_n^{(j)})} V_n e^{2u_n} + \int_{L_{k_n^{(j)}\delta_n^{(j)}}(x_n^{(j)})} h_n e^{u_n}\right)\\ &= \lim_{n\to\infty}\left(\int_{B^+_{2k_n^{(j)}\delta_n^{(j)}}(x_n^{(j)})} V_n e^{2u_n} + \int_{L_{2k_n^{(j)}\delta_n^{(j)}}(x_n^{(j)})} h_n e^{u_n}\right).\end{aligned}$$

Further, when $\frac{t_n^{(j)}}{\delta_n^{(j)}} \to \Lambda < \infty$, $\beta_j = 2\pi$; And when $\frac{t_n^{(j)}}{\delta_n^{(j)}} \to \infty$, $\beta_j = 4\pi$.

(d) $\max_{x \in \bar{B}_R^+} \{u_n(x) + \log \min_{0 \le j \le m-1} |x - x_n^{(j)}|\} \le C, \forall n$.

Proof Let $x_n^{(0)} = x_n = (s_n^{(0)}, t_n^{(0)})$, $\delta_n^{(0)} = e^{-u_n(x_n^{(0)})}$. By letting $\tilde{u}_n^{(0)}(x) = u_n(\delta_n^{(0)} x + x_n^{(0)}) + \log \delta_n^{(0)}$, we see that

$$\begin{cases} -\Delta \tilde{u}_n^{(0)} = V_n(\delta_n^{(0)} x + x_n^{(0)}) e^{2\tilde{u}_n^{(0)}} & \text{in } B_{\frac{R}{\frac{\delta_n}{2}}} \cap \left\{t > -\frac{t_n^{(0)}}{\delta_n^{(0)}}\right\},\\ \dfrac{\partial \tilde{u}_n^{(0)}}{\partial n} = h_n(\delta_n^{(0)} x + x_n^{(0)}) e^{\tilde{u}_n^{(0)}} & \text{on } B_{\frac{R}{\frac{\delta_n}{2}}} \cap \left\{t = -\frac{t_n^{(0)}}{\delta_n^{(0)}}\right\},\\ \tilde{u}_n^{(0)}(x) \le \tilde{u}_n^{(0)}(0) = 0 & \text{in } B_{\frac{R}{\frac{\delta_n}{2}}} \cap \left\{t \ge -\frac{t_n^{(0)}}{\delta_n^{(0)}}\right\}, \end{cases}$$

with the energy conditions

$$\int_{B_{\frac{R}{\frac{\delta_n}{2}}} \cap \left\{t > -\frac{t_n^{(0)}}{\delta_n^{(0)}}\right\}} V_n\left(\delta_n^{(0)} x + x_n^{(0)}\right) e^{2\tilde{u}_n^{(0)}} \le C, \quad \int_{B_{\frac{R}{\frac{\delta_n}{2}}} \cap \left\{t = -\frac{t_n^{(0)}}{\delta_n^{(0)}}\right\}} h_n\left(\delta_n x + x_n\right) e^{\tilde{u}_n^{(0)}} \le C.$$

As in Lemma 2.1, we distinguish two cases.
Case (i) $\frac{t_n^{(0)}}{\delta_n^{(0)}} \to \Lambda < \infty$. In this case, $\{\tilde{u}_n^{(0)}\}$ admits a subsequence converging to $\tilde{u}^{(0)}$ in $C^{1,\alpha}_{loc}(\mathbb{R}^2 \cap \{t \geq -\Lambda\})$. Then, we may select $k_n^{(0)} \to \infty$, such that

$$\lim_{n\to\infty}\left(\int_{B^+_{k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} h_n e^{u_n}\right)$$
$$= \lim_{n\to\infty}\left(\int_{B_{k_n^{(0)}}\cap\{t>-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} V_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{2\tilde{u}_n^{(0)}} + \int_{B_{k_n^{(0)}}\cap\{t=-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} h_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{\tilde{u}_n^{(0)}}\right)$$
$$\to 2\pi.$$

and

$$\lim_{n\to\infty}\left(\int_{B^+_{2k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{2k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} h_n e^{u_n}\right)$$
$$= \lim_{n\to\infty}\left(\int_{B_{2k_n^{(0)}}\cap\{t>-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} V_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{2\tilde{u}_n^{(0)}} + \int_{B_{2k_n^{(0)}}\cap\{t=-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} h_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{\tilde{u}_n^{(0)}}\right)$$
$$\to 2\pi.$$

Case (ii) $\frac{t_n^{(0)}}{\delta_n^{(0)}} \to \infty$. Similarly, $\{\tilde{u}_n^{(0)}\}$ admits a subsequence converging to $\tilde{u}^{(0)}$ in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$. And also we may select $k_n^{(0)} \to \infty$, such that

$$\lim_{n\to\infty}\left(\int_{B^+_{k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} h_n e^{u_n}\right)$$
$$= \lim_{n\to\infty}\left(\int_{B_{k_n^{(0)}}\cap\{t>-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} V_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{2\tilde{u}_n^{(0)}} + \int_{B_{k_n^{(0)}}\cap\{t=-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} h_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{\tilde{u}_n^{(0)}}\right)$$
$$\to 4\pi$$

and

$$\lim_{n\to\infty}\left(\int_{B^+_{2k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{2k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} h_n e^{u_n}\right)$$
$$= \lim_{n\to\infty}\left(\int_{B_{2k_n^{(0)}}\cap\{t>-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} V_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{2\tilde{u}_n^{(0)}} + \int_{B_{2k_n^{(0)}}\cap\{t=-\frac{t_n^{(0)}}{\delta_n^{(0)}}\}} h_n\left(\delta_n^{(0)}x + x_n^{(0)}\right)e^{\tilde{u}_n^{(0)}}\right)$$
$$\to 4\pi.$$

Next we suppose that we have selected l sequences $\{x_n^{(j)}\}_{j=0}^{l-1}$, $\{k_n^{(j)}\}_{j=0}^{l-1}(l \geq 1)$ satisfying a), b) and c) for $m = l$. At this point, either $M_n = \max_{x\in\bar{B}_R^+}\{u_n(x) +$

$\log \min_{0\le j\le l-1}|x - x_n^{(j)}|\} \le C, \forall n \in N$, and then, we stop and define $m = l$, or $M_n \to \infty$. We define $\bar{x}_n^{(l)}$ as a point where M_n is attained. So we have $u_n(\bar{x}_n^{(l)}) \to \infty$. Letting $\bar{\delta}_n^{(l)} = e^{-u_n(\bar{x}_n^{(l)})}$, $M_n \to \infty$ reads as $\min_{0\le j\le l-1}|\bar{x}_n^{(l)} - x_n^{(j)}|/\bar{\delta}_n^{(l)} \to \infty$. First we see that for all $|x| \le \frac{1}{2}\min\limits_{0\le j\le l-1}|\bar{x}_n^{(l)} - x_n^{(j)}|/\bar{\delta}_n^{(l)}$, we have

$$\begin{aligned}\min_{0\le j\le l-1}|\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)}x - x_n^{(j)}| &\ge \min_{0\le j\le l-1}|\bar{x}_n^{(l)} - x_n^{(j)}| - \bar{\delta}_n^{(l)}|x| \\ &\ge \frac{1}{2}\min_{0\le j\le l-1}|\bar{x}_n^{(l)} - x_n^{(j)}|.\end{aligned}$$

Define $\tilde{u}_n(x) = u_n(\bar{\delta}_n^{(l)}x + \bar{x}_n^{(l)}) + \log \bar{\delta}_n^{(l)}$. Then, $\tilde{u}_n$ satisfies

$$\begin{cases} -\Delta\tilde{u}_n = V_n(\bar{\delta}_n x + \bar{x}_n^{(l)})e^{2\tilde{u}_n}, & |x| \le \frac{1}{2}\min\limits_{0\le j\le l-1}|\bar{x}_n^{(l)} - x_n^{(j)}|/\bar{\delta}_n^{(l)}, \quad \bar{\delta}_n x + \bar{x}_n^{(l)} \in B_R^+, \\ \frac{\partial\tilde{u}_n}{\partial n} = h_n(\bar{\delta}_n x + \bar{x}_n^{(l)})e^{\tilde{u}_n}, & |x| \le \frac{1}{2}\min\limits_{0\le j\le l-1}|\bar{x}_n^{(l)} - x_n^{(j)}|/\bar{\delta}_n^{(l)}, \quad \bar{\delta}_n x + \bar{x}_n^{(l)} \in L_R, \\ \tilde{u}_n(0) = 0, & \\ \tilde{u}_n(x) \le 2\log 2, & |x| \le \frac{1}{2}\min\limits_{0\le j\le l-1}|\bar{x}_n^{(l)} - x_n^{(j)}|/\bar{\delta}_n^{(l)}, \quad \bar{\delta}_n x + \bar{x}_n^{(l)} \in \bar{B}_R^+. \end{cases}$$

Let $\bar{x}_n^{(l)} = (s_n^{(l)}, t_n^{(l)})$, we distinguish two cases.

Case (1) $\frac{t_n^{(l)}}{\bar{\delta}_n^{(l)}} \to t_0 < +\infty$. As before, we conclude that $\tilde{u}_n$ converges in $C_{loc}^{1,\alpha}(\mathbb{R}^2 \cap \{t \ge -t_0\})$ to a function $\tilde{u}$ satisfying

$$\begin{cases} -\Delta\tilde{u} = V(0)e^{2\tilde{u}}, & \text{in } \mathbb{R}^2 \cap \{t > -t_0\}, \\ \frac{\partial\tilde{u}}{\partial n} = h(0)e^{\tilde{u}}, & \text{on } \mathbb{R}^2 \cap \{t = -t_0\}, \\ \tilde{u}(x) \le 2\log 2, & \text{on } \mathbb{R}^2 \cap \{t \ge -t_0\}, \\ \tilde{u}(0) = 0, & \end{cases}$$

with the energy conditions

$$\int_{\mathbb{R}^2\cap\{t>-t_0\}} V(0)e^{2\tilde{u}} \le C, \quad \int_{\mathbb{R}^2\cap\{t=-t_0\}} h(0)e^{\tilde{u}} \le C.$$

It follows from the classification results in [16] that

$$\tilde{u}(s,t) = \log\frac{2\lambda}{\sqrt{V(0)}(\lambda^2 + (s-s_0)^2 + (t + t_0 + \frac{h(0)\lambda}{\sqrt{V(0)}})^2)}, \quad t \ge -t_0.$$

Since $\tilde{u}(0) = 0$ and $\tilde{u}(x) \le 2\log 2$, we have

$$\frac{\sqrt{V(0)}}{2(V(0) + h^2(0))} \le \lambda \le \frac{2}{\sqrt{V(0)}}, \quad 0 \le |s_0|, t_0 \le \frac{2}{\sqrt{V(0)}}.$$

We see that $\tilde{u}$ attains its maximum at $\bar{x} = (s_0, -t_0)$. We choose γ satisfying $|\bar{x}| \le \frac{1}{2\gamma}$. Then for given $k > \frac{2}{\gamma}$ and any $\frac{1}{\gamma} \le |x| \le k$, we have $\tilde{u}(x) < \tilde{u}(\bar{x})$. This implies that, when n is sufficiently large, $u_n(\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} x) < u_n(\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} \bar{x})$ for any $\frac{1}{\gamma} \le |x| \le k$. On the other hand, we may find $y_n^{(l)}$ such that $y_n^{(l)} \le \frac{1}{\gamma}$ and $u_n(\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} x) \le u_n(\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} y_n^{(l)})$, $|x| \le k$. Set $x_n^{(l)} = \bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} y_n^{(l)}$. Thus, for all $|x - x_n^{(l)}| \le \frac{1}{2} k \bar{\delta}_n^{(l)}$, it follows that

$$|x - \bar{x}_n^{(l)}|/\bar{\delta}_n^{(l)} \le |x - x_n^{(l)}|/\bar{\delta}_n^{(l)} + |x_n^{(l)} - \bar{x}_n^{(l)}|/\bar{\delta}_n^{(l)} \le \frac{1}{2}k + \frac{1}{\gamma} \le k,$$

where we have used $k > \frac{2}{\gamma}$. Hence, we have $u_n(x) = u_n(\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} \frac{x - \bar{x}_n^{(l)}}{\bar{\delta}_n^{(l)}}) \le u_n(\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} y_n^{(l)}) = u_n(x_n^{(l)})$. Now we set $\delta_n^{(l)} = e^{-u_n(x_n^{(l)})}$, we can also obtain:

$$\delta_n^{(l)} \le \bar{\delta}_n^{(l)} \le 2\delta_n^{(l)}. \tag{3.1}$$

By (3.1) we can choose $k_n^{(l)} \to +\infty$ such that

$$\begin{aligned} u_n(x_n^{(l)}) &= \max_{x \in \bar{B}^+_{k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} u_n(x), \\ \lim_{n\to\infty} &\left(\int_{B^+_{k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} V_n e^{2u_n} + \int_{L_{k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} h_n e^{u_n} \right) \\ &= \lim_{n\to\infty} \left(\int_{B^+_{2k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} V_n e^{2u_n} + \int_{L_{2k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} h_n e^{u_n} \right) = 2\pi. \end{aligned} \tag{3.2}$$

In addition, since $\delta_n^{(l)} \le \bar{\delta}_n^{(l)}$, we have

$$\frac{|x_n^{(l)} - x_n^{(i)}|}{\delta_n^{(l)}} \ge \frac{|\bar{x}_n^{(l)} - x_n^{(i)}|}{\bar{\delta}_n^{(l)}} - \frac{|x_n^{(l)} - \bar{x}_n^{(l)}|}{\bar{\delta}_n^{(l)}} \to +\infty, 0 \le i \le l-1. \tag{3.3}$$

We are left to prove that

$$\frac{|x_n^{(l)} - x_n^{(i)}|}{\delta_n^{(i)}} \to \infty, 0 \le i \le l-1.$$

We argue by contradiction and assume $\frac{x_n^{(l)} - x_n^{(i)}}{\delta_n^{(i)}} \to \tilde{x}$. Then, we have

$$\begin{aligned} \log \delta_n^{(i)} - \log \delta_n^{(l)} &= u_n(x_n^{(l)}) + \log \delta_n^{(i)} \\ &= u_n(x_n^{(i)} + \delta_n^{(i)} \frac{x_n^{(l)} - x_n^{(i)}}{\delta_n^{(i)}}) + \log \delta_n^{(i)} \\ &\to \tilde{u}(\tilde{x}), \end{aligned}$$

which clearly contradicts to (3.3). Thus, we have proved (b) for $m = l$ and $\frac{|x_n^{(i)} - x_n^{(j)}|}{\delta_n^{(i)}} \to \infty, \forall i \neq j, 0 \le i, j \le l$. It is clear that $B^+_{k_n^{(j)} \delta_n^{(j)}}(x_n^{(j)}), 0 \le j \le l$ do not intersect. By further reducing $\{k_n^{(j)}\}$, we can assume that c) holds.

Case (2) $\frac{t_n^{(l)}}{\bar{\delta}_n^{(l)}} \to +\infty$. As before $\tilde{u}_n$ converges in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$ to a function $\tilde{u}$ satisfying

$$\begin{cases} -\Delta \tilde{u} = V(0)e^{2\tilde{u}} & \text{in } \mathbb{R}^2, \\ \tilde{u}(x) \le 2\log 2 & \text{on } \mathbb{R}^2, \\ \tilde{u}(0) = 0, \end{cases}$$

with the energy condition

$$\int_{\mathbb{R}^2} V(0)e^{2\tilde{u}} \le C.$$

The proof of Case (2) is similar to the Case 1. We note that the difference with case (1) is the following

$$\begin{aligned} &\lim_{n\to\infty} \left(\int_{B^+_{k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} V_n e^{2u_n} + \int_{L_{k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} h_n e^{u_n} \right) \\ &= \lim_{n\to\infty} \left(\int_{B^+_{2k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} V_n e^{2u_n} + \int_{L_{2k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} h_n e^{u_n} \right) = 4\pi. \end{aligned} \tag{3.4}$$

We omit the proof. So under the two cases, we can obtain b) and c).

We continue in this manner until d) holds. We must stop after a finite step since each time we find a mass of 2π or 4π near $x_n^{(j)}$. □

Step 2 In this step, we show that the mass contribution outside the chosen neighborhoods of the m centers $x_n^{(0)}, \ldots, x_n^{(m-1)}$ tends to zero. Namely,

$$\lim_{n\to\infty} \left(\int_{B_R^+ \setminus \cup_{l=0}^{m-1} B^+_{k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} V_n e^{2u_n} + \int_{L_R \setminus \cup_{l=0}^{m-1} L_{k_n^{(l)} \delta_n^{(l)}}(x_n^{(l)})} V_n e^{2u_n} h_n e^{u_n} \right) = 0.$$

To prove this result, we deal with a slightly more general situation that ours.

Lemma 3.1 *For $R > 0$, Let $\{V_n\}$ and $\{h_n\}$ be two sequences of functions satisfying (1.7). Let $\{u_n\}$ be a sequence of solutions of (1.4) and (1.5) satisfying (1.8)–(1.9). Assume that $\{x_n^{(j)}\}_{j=0}^{m-1}$ are $m (m \ge 1)$ sequences of points, $\{r_n^{(j)}\}_{j=0}^{m-1}$ are m sequences of positive numbers which satisfy*

$$u_n(x_n^{(j)}) = \max_{x \in \bar{B}^+_{r_n^{(j)}}(x_n^{(j)})} u_n(x) \to \infty, x_n^{(j)} = (s_n^{(j)}, t_n^{(j)}), \quad \forall 0 \le j \le m-1, \tag{3.5}$$

$$\lim_{n\to\infty}\frac{r_n^{(j)}}{\delta_n^{(j)}}=\infty,\quad \forall 0\le j\le m-1, \tag{3.6}$$

where $\delta_n^{(j)}=e^{-u_n(x_n^{(i)})}$,

$$\frac{|x_n^{(i)}-x_n^{(j)}|}{r_n^{(j)}}\to\infty,\quad \forall i\ne j, 0\le i,j\le m-1, \tag{3.7}$$

$$\max_{x\in\bar{B}_R^+\setminus\cup_{j=0}^{m-1}B^+_{r_n^{(j)}}(x_n^{(j)})}\{u_n(x)+\log\min_{0\le j\le m-1}|x-x_n^{(j)}|\}\le C, \forall n, \tag{3.8}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\left(\int_{B^+_{r_n^{(j)}}(x_n^{(j)})}V_ne^{2u_n}+\int_{L_{r_n^{(j)}}(x_n^{(j)})}h_ne^{u_n}\right)\\&=\lim_{n\to\infty}\left(\int_{B^+_{2r_n^{(j)}}(x_n^{(j)})}V_ne^{2u_n}+\int_{L_{2r_n^{(j)}}(x_n^{(j)})}h_ne^{u_n}\right)=\beta_j,\end{aligned} \tag{3.9}$$

where $\beta_j=2\pi$ *when* $\frac{t_n^{(j)}}{\delta_n^{(j)}}\to\Lambda<\infty$, *and* $\beta_j=4\pi$ *when* $\frac{t_n^{(j)}}{\delta_n^{(j)}}\to\infty$ *for all* $0\le j\le m-1$. *Then,*

$$\lim_{n\to\infty}\left(\int_{B_R^+}V_ne^{2u_n}+\int_{L_R}h_ne^{u_n}\right)=\sum_{j=1}^m\beta_j.$$

Proof We will follow the approach of [15] to prove the lemma by induction on m. First we prove the lemma for $m=1$. We also distinguish two cases.
Case (1) $\frac{t_n}{\delta_n^{(0)}}\to\Lambda<\infty$.

In this case, we can assume that $\lim_{n\to\infty}r_n^{(0)}=0$, since otherwise the lemma obviously holds due to (1.9). We also assume that $B^+_{\frac{R}{2}}(x_n)\subset B_R^+$.

By using Lemma 2.6, we obtain that

$$\sup_{S_r^+(x_n)}u_n\le C+\beta\inf_{S_r^+(x_n)}u_n+(\beta-1)\log r,\quad \forall 2r_n^{(0)}\le r\le\frac{R}{2}.$$

By using Lemma 2.5, we obtain that

$$\inf_{S_r^+(x_n)}u_n\le C-\frac{1}{C_1}u_n(x_n)-(1+\frac{1}{C_1})\log r,\quad \forall 0<r<R.$$

It follows that

$$\sup_{S_r^+(x_n)} u_n \le C - \frac{\beta}{C_1} u_n(x_n) - (1+\frac{\beta}{C_1})\log r, \quad \forall 2r_n^{(0)} \le r \le \frac{R}{2},$$

namely

$$e^{u_n(x)} \le C(\delta_n^{(0)})^{\beta/C_1} |x-x_n|^{-(1+\frac{\beta}{C_1})}, \forall x \in B^+_{\frac{R}{2}}(x_n) \backslash B^+_{2r_n^{(0)}}(x_n).$$

Therefore, we have

$$\begin{aligned}
&\int_{B^+_{R/2}(x_n)\backslash B^+_{2r_n^{(0)}}(x_n)} V_n e^{2u_n} + \int_{L_{R/2}(x_n)\backslash L_{2r_n^{(0)}}(x_n)} h_n e^{u_n} \\
&\le C(\delta_n^{(0)})^{2\beta/C_1} \int_{2r_n^{(0)}}^{\infty} r^{-2(1+\frac{\beta}{C_1})} r\mathrm{d}r + C(\delta_n^{(0)})^{\beta/C_1} \int_{2r_n^{(0)}}^{\infty} s^{-(1+\frac{\beta}{C_1})}\mathrm{d}s \\
&= C(\frac{\delta_n^{(0)}}{2r_n^{(0)}})^{2\beta/C_1} + C(\frac{\delta_n^{(0)}}{2r_n^{(0)}})^{\beta/C_1} \to 0.
\end{aligned}$$

By (1.9), (3.9) and above formula, we obtain that $\lim_{n\to\infty}\left(\int_{B_R^+} V_n e^{2u_n} + \int_{L_R} h_n e^{u_n}\right) = \beta_0 = 2\pi$.

Case (2) $\frac{t_n}{\delta_n^{(0)}} \to +\infty$. Note that for n sufficiently large, $\overline{B}^+_{r_n^{(j)}}(x_n^{(j)})$ is contained in the interior of B_R^+. The proof is very similar with case (1). We can use Lemmas 2.3, 2.4, 2.5 and 2.6 to obtain $\lim_{n\to\infty}\left(\int_{B_R^+} V_n e^{2u_n} + \int_{L_R} h_n e^{u_n}\right) = \beta_0 = 4\pi$.

Next we proceed the proof by induction. Suppose that lemma holds for $1, 2, \ldots, m-1 (m \ge 2)$, we prove that it holds for m. Without loss of generality, we assume that

$$d_n = |x_n^{(0)} - x_n^{(1)}| = \min\{|x_n^{(i)} - x_n^{(j)}|, i \ne j, 0 \le i, j \le m-1\}$$

and $x_n^{(0)} = 0$. There exist two cases.

Case 1 For some constant A, we have

$$|x_n^{(i)} - x_n^{(j)}| \le Ad_n, \quad 0 \le i, j \le m-1.$$

In this case, we will establish

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_{B^+_{4Ad_n}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{4Ad_n}(x_n^{(0)})} h_n e^{u_n}\right) \\
&= \lim_{n\to\infty}\left(\int_{B^+_{2Ad_n}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{2Ad_n}(x_n^{(0)})} h_n e^{u_n}\right) = \sum_{j=0}^{m-1}\beta_j. \qquad (3.10)
\end{aligned}$$

Once (3.10) is established, we introduce $x_n'^{(0)} = x_n^{(0)}$, $r_n'^{(0)} = 2Ad_n$ and $\beta_0' = \sum_{j=0}^{m-1} \beta_j$. And then we can apply Lemma 3.1 for $m = 1$ to obtain

$$\lim_{n\to\infty} \left(\int_{B_R^+} V_n e^{2u_n} + \int_{L_R} h_n e^{u_n} \right) = \beta_0'.$$

We adopt the method applied in [15]. For $x \in \bar{B}^+_{R/d_n}$, define $\tilde{u}_n(x) = u_n(d_n x) + \log d_n$. Denote

$$\begin{aligned}
\tilde{V}_n(x) &= V_n(d_n x), \quad x \in \bar{B}^+_{R/d_n}, \\
\tilde{h}_n(x) &= h_n(d_n x), \quad x \in \bar{B}^+_{R/d_n}, \\
\tilde{x}_n^{(j)} &= x_n^{(j)}/d_n, \quad 0 \le j \le m-1, \\
\tilde{\delta}_n^{(j)} &= e^{-\tilde{u}_n(\tilde{x}_n^{(j)})} = \delta_n^{(j)}/d_n, \quad 0 \le j \le m-1, \\
\tilde{r}_n^{(j)} &= r_n^{(j)}/d_n, \quad 0 \le j \le m-1.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\tilde{u}_n(\tilde{x}_n^{(j)}) = \max_{x \in \bar{B}^+_{\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \tilde{u}_n(x) \to \infty, \quad 0 \le j \le m-1, \\
&\lim_{n\to\infty} \frac{\tilde{r}_n^{(j)}}{\tilde{\delta}_n^{(j)}} = \infty, \quad 0 \le j \le m-1, \\
&\tilde{r}_n^{(j)} \to 0, \quad 0 \le j \le m-1, \\
&\max_{x \in \bar{B}^+_{R/d_n} \setminus \cup_{j=0}^{m-1} B^+_{\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \{\tilde{u}_n(x) + \log \min_{0\le j\le m-1} |x - \tilde{x}_n^{(j)}|\} \le C, \forall n, \\
&\lim_{n\to\infty} \left(\int_{B^+_{\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \tilde{V}_n e^{2\tilde{u}_n} + \int_{L_{\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \tilde{h}_n e^{\tilde{u}_n} \right) \\
&= \lim_{n\to\infty} \left(\int_{B^+_{2\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \tilde{V}_n e^{2\tilde{u}_n} + \int_{L_{2\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \tilde{h}_n e^{\tilde{u}_n} \right) = \beta_j, \quad 0 \le j \le m-1.
\end{aligned}$$

We assume that $\tilde{x}_n^{(j)} \to \tilde{x}^{(j)}$ for $0 \le j \le m-1$. Set $S = \{\tilde{x}^{(j)}, \quad 0 \le j \le m-1\}$. Note that

$$1 \le |\tilde{x}^{(i)} - \tilde{x}^{(j)}| \le A.$$

Hence we know that the set of blow-up points of $\tilde{u}_n$ in $\bar{B}^+_{4A}(x_n^{(0)})$ is S. Then, it follows from Theorem 1.1 that $u_n \to -\infty$ uniformly on any compact sets of $\bar{B}^+_{4A}(x_n^{(0)}) \setminus S$. Now we apply the case $m = 1$ of Lemma 3.1 to conclude

$\int_{B^+_{\frac{1}{2}}(\tilde{x}^{(j)})} \tilde{V}_n e^{2\tilde{u}_n} + \int_{L_{\frac{1}{2}}(\tilde{x}^{(j)})} \tilde{h}_n e^{\tilde{u}_n} \to \beta_j$. Consequently,

$$\lim_{n\to\infty} \left(\int_{B^+_{4A}(x_n^{(0)})} \tilde{V}_n e^{2\tilde{u}_n} + \int_{L_{4A}(x_n^{(0)})} \tilde{h}_n e^{\tilde{u}_n} \right) = \lim_{n\to\infty} \left(\int_{B^+_{2A}(x_n^{(0)})} \tilde{V}_n e^{2\tilde{u}_n} + \int_{L_{2A}(x_n^{(0)})} \tilde{h}_n e^{\tilde{u}_n} \right) = \sum_{j=0}^{m-1} \beta_j .$$

A simple change of variables leads to (3.10). Then, we derive the desired conclusion.
Case 2 A proper subset J of $\{0, 1, 2, \dots, m-1\}$ containing $\{0, 1\}$ and a constant A satisfy

$$|x_n^{(j)} - x_n^{(0)}| \le A d_n, \forall j \in J; \lim_{n\to\infty} |x_n^{(j)} - x_n^{(0)}|/d_n = \infty, \quad \forall j \notin J.$$

Without loss of generality, we assume that $J = \{0, 1, 2, ..., k-1\}$. In this case, we consider $\tilde{u}_n(x) = u_n(d_n x) + \log d_n$ in $\bar{B}^+_{4A}$. Arguing as in case 1, we obtain:

$$\lim_{n\to\infty} \left(\int_{B^+_{4Ad_n}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{4Ad_n}(x_n^{(0)})} h_n e^{u_n} \right) = \lim_{n\to\infty} \left(\int_{B^+_{2Ad_n}(x_n^{(0)})} V_n e^{2u_n} + \int_{L_{2Ad_n}(x_n^{(0)})} h_n e^{u_n} \right) = \sum_{j=0}^{k-1} \beta_j := \beta'_0.$$

We set $r_n'^{(0)} = A d_n$ and $x_n'^{(0)} = x_n^{(0)}$. If the $m-k+1$ sequences $x_n'^{(0)}$, $\{x_n^{(j)}\}_{j=k}^{m-1}$ with the radius $r_n'^{(0)}$, $\{r_n^{(j)}\}_{j=k}^{m-1}$ and the mass β'_0, $\{\beta_j\}_{j=k}^{m-1}$ satisfy (3.5)–(3.9), we may apply the case $m-k+1$ of Lemma 3.1. Now we need to verify (3.5)–(3.9). We only need to show (3.7) since others are obvious. Note that

$$\frac{|x_n'^{(0)} - x_n^{(j)}|}{r_n'^{(0)}} = \frac{|x_n^{(j)} - x_n^{(0)})|}{A d_n} \to \infty, \quad \forall j \notin J.$$

Therefore, we obtain:

$$\int_{B^+_R} V_n e^{2u_n} + \int_{L_R} h_n e^{u_n} \to \beta'_0 + \sum_{j=k}^{m-1} \beta_j = \sum_{j=0}^{m-1} \beta_j .$$

□

References

1. Brezis, H., Merle, F.: Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. Commun. Partial Differ. Equ. **16**, 1223–1253 (1991)

2. Bao, J., Wang, L.H., Zhou, C.Q.: Blow up analysis for solutions to Neumann boundary value problem. J. Math. Anal. Appl. **418**, 142–162 (2014)
3. Chen, W., Ding, W.: Scalar curvatures on S^2. Trans. Am. Math. Soc. **303**, 365–382 (1987)
4. Chang, S.Y.A., Gursky, M.J., Yang, P.C.: The scalar curvature equation on 2- and 3-spheres. Calc. Var. Partial Differ. Equ. **1**, 205–229 (1993)
5. Chen, C.C., Lin, C.S.: Topological degree for a mean field equation on Riemann surfaces. Commun. Pure Appl. Math. **56**, 1667–1727 (2003)
6. Chen, W., Li, C.: Classification of solutions of some nonlinear elliptic equations. Duke Math. J. **63**, 615–623 (1991)
7. Chang, S.Y.A., Yang, P.C.: Prescribing Gaussian curvature on S^2. Acta Math. **159**, 215–259 (1987)
8. Djadli, Z.: Existence result for the mean field problem on Riemann surfaces of all genuses. Commun. Contemp. Math. **10**, 205–220 (2008)
9. Ding, W., Jost, J., Li, J., Wang, G.: Existence results for mean field equations. Ann. Inst. H. Poincare Anal. Non Lineaire **16**, 653–666 (1999)
10. Ding, W., Jost, J., Li, J., Wang, G.: An analysis of the two-cotex case in the Chern–Simons–Higgs model. Calc. Var. Partial Differ. Equ. **7**, 87–97 (1998)
11. Guo, Y.X., Liu, J.Q.: Blow-up analysis for solutions of the Laplacian equation with exponential Neumann boundary condition in dimension two. Commun. Contemp. Math. **8**, 737–761 (2006)
12. Jost, J., Wang, G., Zhou, C.Q.: Metrics of constant curvature on a Riemann surface with two corners on the boundary. Ann. Inst. H. Poincare Anal. Non Lineaire **26**, 437–456 (2009)
13. Jost, J., Wang, G., Zhou, C.Q., Zhu, M.M.: The boundary value problem for the super-Liouville equation. Ann. Inst. H. Poincare Anal. Non Lineaire **31**, 685–706 (2014)
14. Liouville, J.: Sur l'équation aux différences partielles $\frac{d^2}{dudv}\log\lambda \pm \frac{\lambda}{2a^2} = 0$. J. Math. Pures Appl. **18**, 71 (1853)
15. Li, Y.Y., Shafrir, I.: Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. Indiana Univ. Math. J. **43**, 1255–1270 (1994)
16. Li, Y.Y., Zhu, M.J.: Uniqueness theorems through the method of moving spheres. Duke Math. J. **80**, 383–417 (1995)
17. Nolasco, M., Tarantello, G.: Double vortex condensates in the Chern-Simons-Higgs theory. Calc. Var. Partial Differ. Equ. **9**, 31–94 (1999)
18. Struwe, M., Tarantello, G.: On the multivortex solutions in the Chern–Simons gauge theory. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **1**, 109–121 (1998)
19. Spruck, J., Yang, Y.: On multivortices in the electroweak theory I: existence of periodic solutions. Commun. Math. Phys. **144**, 1–16 (1992)
20. Tarantello, G.: Multiple condensate solutions for the Chern–Simons–Higgs theory. J. Math. Phys. **37**, 3769–3796 (1996)