Commun. Math. Stat. (2016) 4:449–458
DOI 10.1007/s40304-016-0094-1

On m-Embedded Primary Subgroups of Finite Groups

Jia Zhang[1] · Long Miao[1] · Baojun Li[2]

Received: 23 August 2016 / Revised: 12 October 2016 / Accepted: 21 October 2016 /
Published online: 25 November 2016

Abstract Suppose that A is a subgroup of a group G. A is called to be m-embedded in G if G has a subnormal subgroup T and a $\{1 \leq G\}$-embedded subgroup C such that $G = AT$ and $A \cap T \leq C \leq A$. In this paper, we shall investigate the structure of finite groups by using m-embedded subgroups and obtain some new characterization about p-supersolvability and generalized hypercentre of finite groups. Some results in Guo and Shum (Arch Math 80:561–569, 2003) , Ramadan et al. (Arch Math 85:203–210, 2005), Tang and Miao (Turk J Math 39:501–506, 2015), and Xu and Zhang (Can Math Bull 57(4):884–889, 2014) are generalized.

Keywords m-Embedded subgroup · Sylow subgroup · p-Supersolvability · Generalized hypercentre

Mathematics Subject Classification 20D10 · 20D20

1 Introduction

Every group considered in this paper is finite. Most of the notation is standard and can be found in [3,10]. Let $|G|$ denote the order of a group G, $|G|_p$ denote the p-part of $|G|$ and $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $A \rtimes B$ denote the semidirect product of groups A and B, where B is an operator group of A. Let $\mathcal{F}$ be a class of

✉ Long Miao
lmiao@yzu.edu.cn

1 School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, People's Republic of China

2 School of Mathematics, Chengdu University of Information Technology, Chengdu 610225, People's Republic of China

groups and I/K be a chief factor of a group G. I/K is called Frattini provided that $I/K \leq \Phi(G/K)$. Moreover, I/K is called $\mathcal{F}$-central if $I/K \rtimes (G/C_G(I/K)) \in \mathcal{F}$. Otherwise, I/K is called $\mathcal{F}$-eccentric. The symbol $Z_{\mathcal{F}}(G)(Z_{p\mathcal{F}}(G))$ denotes the $\mathcal{F}$-hypercentre($p\mathcal{F}$-hypercentre) of a group G, which is the product of all such normal subgroups H of G whose G-chief factors(whose G-chief factors of order divisible by p)are $\mathcal{F}$-central. In addition, $\mathcal{U}$ and $p\mathcal{U}$ denote the class of all supersolvable groups and all p-supersolvable groups, respectively.

Suppose that A is a subgroup of G, $K \leq H \leq G$. (1) if $AH = AK$, then A covers the pair (K, H); (2) if $A \cap H = A \cap K$, then A avoids (K, H). In 1939, Ore [11] introduced the notion of quasinormal subgroups. Furthermore, if E is a quasinormal subgroup of G, then for every maximal pair of G, that is, a pair (K, H), where K is a maximal subgroup of H, E either covers or avoids (K, H). In 1992, Doerk and Hawkes [3] gave the definition of CAP-subgroups, that is, a subgroup A of G is called a CAP-subgroup if A either covers or avoids each pair (K, H), where H/K is a chief factor of G. Based on the definitions and observations above, Guo and Skiba introduced new concepts as follows:

Definition 1.1 [6, Definition 1.1] Let $\Sigma = \{G_0 \leq G_1 \leq \ldots \leq G_n\}$ be some subgroup series of G and A be a subgroup of G. Then A is Σ-embedded in G if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$, for some i.

In [6], quasinormal subgroups, CAP-subgroups, and partial CAP-subgroups [1] (or a semi cover-avoiding subgroups [8]) are Σ-embedded subgroups.

Definition 1.2 [6, Definition 2.7] Let A be a subgroup of G. Then A is m-embedded in G if G has a subnormal subgroup T and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = AT$ and $A \cap T \leq C \leq A$.

In [6, Example 2.8], every c-normal subgroup of G is also m-embedded in G.

On the other hand, In 2014, Xu and Zhang [14] investigated p-nilpotency of a group by using m-embedded property of primary subgroups. In 2015, Tang and Miao [13] obtained some results about p-supersolvability of finite groups by using m-embedded primary subgroups. They proved the following theorem:

Theorem 1.3 [13, Theorem 1.3] *Let G be a p-solvable group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. If every maximal subgroup of P is m-embedded in G, then G is p-supersolvable.*

It is clear that the p-solvability in [13, Theorem 1.3] is essential. Naturally, the question is that

What is the structure of a group if the p-solvability is removed in [13, Theorem 1.3]?

Along the clue, we obtained the following result:

Theorem 1.4 *Let E be a normal subgroup of G and P be a Sylow p-subgroup of E where p is an odd prime divisor of $|E|$. If every maximal subgroup of P is m-embedded in G, then every G-chief factor A/B below E satisfies one of the following conditions*:

(1) $A/B \leq \Phi(G/B)$; (2) A/B *is a* $p' -$ *group*; (3) $|A/B|_p = p$.

With the further consideration, we will study the p-supersolvability and generalized hypercentre of a finite group by using m-embedded subgroups, where p is a prime. Some theorems in [9,12–14] are corollaries of our results.

2 Preliminaries

For the sake of convenience, we first list here some known results which will be useful in the sequel.

Lemma 2.1 [6, Lemma 2.13] *Let K and H be subgroups of G. Suppose that K is m-embedded in G and H is normal in G. Then*

(1) *If $H \leq K$, then K/H is m-embedded in G/H.*
(2) *If $K \leq E \leq G$, then K is m-embedded in E.*
(3) *If $(|H|, |K|) = 1$, then KH/H is m-embedded in G/H.*

Lemma 2.2 [6, Lemma 2.14] *Let P be a normal non-identity p-subgroup of G with $|P| = p^n$ and $P \cap \Phi(G) = 1$. Suppose that there is an integer k such that $1 \leq k < n$ and the subgroups of P of order p^k are m-embedded in G, then some maximal subgroup of P is normal in G.*

Lemma 2.3 [6, Lemma 2.5] *Every $\{1 \leq G\}$-embedded subgroup of G is subnormal in G.*

Lemma 2.4 [5, Lemma 2.8] *Let G be a p-supersolvable group. If $O_{p'}(G) = 1$, then G is supersolvable.*

3 Main Results

Theorem 3.1 *Let p be an odd prime divisor of $|G|$ and P be a normal p-subgroup of G. If every minimal subgroup of P is $\{1 \leq G\}$-embedded in G, then $P \leq Z_{\mathcal{U}}(G)$.*

Proof Assume that the assertion is false and choose (G, P) to be a counterexample of minimal order.

(1) G has a unique normal subgroup N such that P/N is a chief factor of G, $N \leq Z_{\mathcal{U}}(G)$ and $|P/N| > p$.
Let P/N be a chief factor of G. Then, clearly, (G, N) satisfies the hypothesis of the theorem. The choice of (G, P) implies that $N \leq Z_{\mathcal{U}}(G)$. If $|P/N| = p$, then $P/N \leq Z_{\mathcal{U}}(G/N)$ and so $P \leq Z_{\mathcal{U}}(G)$, a contradiction. Hence $|P/N| > p$. Assume that P/L is a chief factor of G with $P/N \neq P/L$. With the same discussion as above, we have that $L \leq Z_{\mathcal{U}}(G)$. Then $P/N = NL/N \leq NZ_{\mathcal{U}}(G)/N \leq Z_{\mathcal{U}}(G/N)$. It follows from $N \leq Z_{\mathcal{U}}(G)$ that $P \leq Z_{\mathcal{U}}(G)$, a contradiction.
(2) The exponent of P is p.
Let C be a Thompson critical subgroup of P. If $\Omega(C) < P$, then $\Omega(C) \leq N \leq Z_{\mathcal{U}}(G)$ by (1), so $P \leq Z_{\mathcal{U}}(G)$ by [7, Lemma 4.4], which is impossible. Hence $P = C = \Omega(C)$. Then by [7, Lemma 4.3], the exponent of P is p.

(3) P is a minimal normal subgroup of G.
If not, then $N \neq 1$. Let H/N be a minimal subgroup of P/N. Then there exists an element $x \in H \backslash N$, $H = \langle x \rangle N$ and $|\langle x \rangle| = p$ by (2). By hypothesis and [6, Lemma 2.3], H/N is $\{1 \leq G/N\}$-embedded in G/N. Then $P/N \leq Z_{\mathcal{U}}(G/N)$ by the choice of (G, P). Hence $|P/N| = p$, a contradiction. Hence $N = 1$ and (3) holds.

(4) $P \leq \Phi(G)$.
If not, then $P \nleq \Phi(G)$. By (1), we may choose a minimal subgroup H of P such that $G = HM = PM$ and $P \cap M = 1$. Since $|P : P \cap M| = p$, $|P| = p$ and $P \leq Z_{\mathcal{U}}(G)$, a contradiction.

(5) The final contradiction.
By [6, Lemma 2.3], every minimal subgroup of P is $\{1 \leq M\}$-embedded in M. Then $P \leq Z_{\mathcal{U}}(M)$ by the choice of (G, P), for every maximal subgroup M of G. We assert that $C_G(P) \leq \Phi(G)$. If not, then $C_G(P) \nleq \Phi(G)$ and $G = C_G(P)M_1$ for some maximal subgroup M_1 of G. Next, we choose a minimal normal subgroup N of M_1 contained in P. Then $|N| = p$. Further, $N^G = N^{C_G(P)M_1} = N^{M_1} = N$ and $N \trianglelefteq G$. Then $|N| = |P| = p$, a contradiction. Set $Z = \cap(C_M(X/Y))$, where X/Y is an M-chief factor below P for every maximal subgroup M of G. Then M/Z is an abelian group of exponent dividing $p - 1$ and $O^p(Z) \leq C_G(P) \leq \Phi(G)$. Hence $M/\Phi(G)$ is a strictly p-closed group and $M/\Phi(G)$ is supersolvable by [15, Theorem 1.9]. Then $G/\Phi(G)$ is minimal non-supersolvable and G is solvable by [15, Theorem 2.3]. Further, we have $F(G) \leq C_G(P) \leq \Phi(G) < F(G)$, a contradiction.

The final contradiction completes our proof. □

Theorem 3.2 *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If every minimal subgroup of P is m-embedded in G, then G is p-supersolvable.*

Proof Assume that the assertion is false and choose G to be a counterexample of minimal order. Furthermore, we have that

(1) $O_{p'}(G) = 1$.
Assume that $T = O_{p'}(G) \neq 1$. By Lemma 2.1(3), G/T satisfies the conditions of the theorem, and the minimal choice of G implies that G/T is p-supersolvable. Hence G is p-supersolvable, a contradiction.

(2) $P \cap E \trianglelefteq E$, where E is a proper normal subgroup of G.
Assume that E is a proper normal subgroup of G. By (1), $P \cap E \neq 1$. By Lemma 2.1 and the choice of G, E is p-supersolvable. Hence E is supersolvable by (1) and Lemma 2.4. Then $P \cap E \trianglelefteq E$.

(3) There exists a minimal subgroup H of P such that H has a normal complement in G.
Otherwise, all minimal subgroups of P are $\{1 \leq G\}$-embedded in G. Then all minimal subgroups of P are contained in $O_p(G)$ by Lemma 2.3. Further, $O_p(G) \leq Z_{\mathcal{U}}(G)$. Hence G is p-supersolvable by [2, Theorem 6], a contradiction.

(4) The final contradiction.
By (3) and hypothesis, $G = HM$, $M \trianglelefteq G$, $H \cap M = 1$. Then $P \cap M \trianglelefteq M$ by (2). If every minimal subgroup of $P \cap M$ is $\{1 \leq G\}$-embedded in G, then $P \cap M \leq Z_{\mathcal{U}}(G)$ by Theorem 3.1 and $1 \trianglelefteq P \cap M \trianglelefteq M \trianglelefteq G$ a normal subgroup series of G such that every G-chief factor either cyclic of order p or p'-group. Hence G is p-supersolvable, a contradiction.

Now we assume that there exists a minimal subgroup H_1 of $P \cap M$ such that H_1 has a normal complement in G. Further, $G = H_1 M_1$, $M_1 \trianglelefteq G$, $H_1 \cap M_1 = 1$. Similar to the previous discussion, we consider $P \cap M \cap M_1$. Now, we set $G_0 = G$, $G_1 = M$, $G_2 = M \cap M_1$. Repeat above discussion, we have $1 = G_{s+1} \trianglelefteq G_s \trianglelefteq \cdots \trianglelefteq G_3 \trianglelefteq G_2 \trianglelefteq G_1 \trianglelefteq G_0 = G$ a normal subgroup series of G such that every G-chief factor either cyclic of order p or p'-group. Then G is p-supersolvable, a contradiction.

The final contradiction completes our proof. □

Theorem 3.3 *Let E be a normal subgroup of G and P be a Sylow p-subgroup of E where p is an odd prime divisor of $|E|$. If every minimal subgroup of P is m-embedded in G, then $E \leq Z_{p\mathcal{U}}(G)$.*

Proof Assume that the assertion is false and choose (G, E) to be a counterexample of minimal order. Furthermore, we have that

(1) $O_{p'}(E) = 1$.
If $K = O_{p'}(E) \neq 1$, then we consider G/K. $(G/K, E/K)$ satisfies the hypothesis of the theorem by Lemma 2.1(3). The minimal choice of (G, E) implies that $E/K \leq Z_{p\mathcal{U}}(G/K)$, and so $E \leq Z_{p\mathcal{U}}(G)$, a contradiction.

(2) $P \trianglelefteq G$.
By Theorem 3.2, E is p-supersolvable. By (1) and Lemma 2.4, E is supersolvable. Then $P \trianglelefteq G$.

(3) There exists a minimal subgroup H of P such that H has a normal complement in G.
Otherwise, all minimal subgroups of P are $\{1 \leq G\}$-embedded in G. By Theorem 3.1, $P \leq Z_{\mathcal{U}}(G)$. Hence $E \leq Z_{p\mathcal{U}}(G)$, a contradiction.

(4) The final contradiction.
By (3) and hypothesis, $G = HM$, $M \trianglelefteq \trianglelefteq G$, $H \cap M = 1$. Then $M \trianglelefteq G$, $|P : P \cap M| = p$ and $P \cap M \trianglelefteq G$. By the choose of (G, E), we have that $P \cap M \leq Z_{p\mathcal{U}}(G)$. Hence $P \leq Z_{p\mathcal{U}}(G)$ and $E \leq Z_{p\mathcal{U}}(G)$, a contradiction.

The final contradiction completes our proof. □

Corollary 3.4 *Let E be a normal subgroup of G such that G/E is p-supersolvable and P be a Sylow p-subgroup of E where p is an odd prime divisor of $|E|$. If every minimal subgroup of P is m-embedded in G, then G is p-supersolvable.*

By Theorem 3.3, it is easy to prove the following corollaries:

Corollary 3.5 [9, Theorem 3.8] *Let p be an odd prime number dividing the order of a group G and $\mathcal{F}$ a saturated formation containing the class $p\mathcal{U}$ of all p-supersolvable groups. Also let N be a normal subgroup of G such that $G/N \in \mathcal{F}$. If P is a Sylow p-subgroup of N and every minimal subgroup of P is c-normal in G, then $G \in \mathcal{F}$.*

Theorem 3.6 *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. If every maximal subgroup of P is m-embedded in G, then every chief factor A/B of G satisfies one of the following conditions*:

(1) $A/B \leq \Phi(G/B)$; (2) A/B *is a* p'-*group*; (3) $|A/B|_p = p$.

Proof Assume that the theorem is false and let G be a counterexample of minimal order.

(1) $O_{p'}(G) = 1$.
Assume that $O_{p'}(G) \neq 1$. The hypothesis also holds for $G/O_{p'}(G)$ by Lemma 2.1, and for G, a contradiction.
(2) If $O_p(G) \neq 1$, then $O_p(G) \cap \Phi(G) = 1$.
Assume that $O_p(G) \cap \Phi(G) \neq 1$. We may choose a minimal normal subgroup L of G such that $L \leq O_p(G) \cap \Phi(G)$. By induction, G/L holds and so G holds, a contradiction.
(3) $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. By (2) and [4, Theorem 1.8.17], $O_p(G) = L_1 \times L_2 \times \cdots \times L_t$ where L_i are the minimal normal subgroups of G, $i = 1, 2, \ldots, t$. For every $L \in \{L_i\}$ and we consider G/L. Clearly, G/L holds by the choice of G. Further, we assert that $O_p(G)$ is a minimal normal subgroup of G. Otherwise, there exists two different minimal normal subgroups L_1 and L_2 such that G/L_j satisfies the hypothesis of theorem and so every G/L_j-chief factor holds by the choice of G where $j = 1, 2$. If $L_1L_2/L_2 \leq \Phi(G/L_2)$, then $L_1L_2 \leq \Phi(G)L_2$ by [3, A. Lemma 9.11]. Since $L_1L_2 \leq O_p(G)$, $L_1L_2 \leq O_p(G) \cap \Phi(G)L_2 = (O_p(G) \cap \Phi(G))L_2 = L_2$ by (2), a contradiction. Hence $L_1 \cong L_1L_2/L_2$ satisfies the condition (2) or (3), then every G-chief factor holds, a contradiction.

Hence $O_p(G)$ is a minimal normal subgroup of G and $\Phi(G) = 1$. Then there exists a maximal subgroup M of G such that $G = O_p(G)M$. We assert that $O_p(G) < P$. If not, by Lemma 2.2, $|P| = p$ and so G holds, a contradiction. Hence we may choose a maximal subgroup P_1 of P such that $M_p \leq P_1$ and $O_p(G) \nleq P_1$. By hypothesis, P_1 is m-embedded in G, there exists a subnormal subgroup T in G and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = P_1T$ and $P_1 \cap T \leq C \leq P_1$. We assert that $C = 1$. Otherwise, $C \neq 1$. If $C < O_p(G)$ by Lemma 2.3, then we obtain C neither covers nor avoids maximal pair (M, G) since $O_p(G) \cap M = 1$, a contradiction. Hence we may assume that $C = O_p(G)$ by Lemma 2.3, that is, $O_p(G) \leq P_1$, a contradiction. Then we have $|T_p| = p$.

If $O_p(G) \cap T \neq 1$, then $O_p(G) \cap T = T_p \trianglelefteq T$ and so T is p-solvable. Furthermore, $T_{p'}$ is a Hall p'-subgroup of G. By [4, Theorem 1.8.19] and (1), $C_T(T_p) = T_p$. Hence $N_T(T_p)/C_T(T_p) = T/C_T(T_p) \hookrightarrow C_{p-1}$. By Schur–Zassenhaus Theorem, $T_{p'}$ is cyclic and T is supersolvable. We assert that p is the largest prime divisor of $|G|$. Otherwise, if $q \neq p$ is the largest prime divisor of $|G|$, then $Q \trianglelefteq G$, which contradicts (1). Then we assume that $p_1 < p_2 < \cdots < p_n = p$, where $\pi(G) = \{p_1, p_2, \ldots, p_n = p\}$. Since $T_{p'}$ is cyclic, $G_{p'}$ is cyclic and G_{p_1} is cyclic where G_{p_1} a Sylow p_1-subgroup of G. By Burnside Theorem, G is p_1-nilpotent and $G_{p_1'} \trianglelefteq G$. Next, we consider $G_{p_1'}$. Similar to the previous discussion, $G_{\{p_1,p_2\}'} \trianglelefteq G_{p_1'}$. Repeat above discussion, we get a normal subgroup series of G: $1 \trianglelefteq P \trianglelefteq \ldots \trianglelefteq G_{\{p_1,p_2\}'} \trianglelefteq G_{p_1'} \trianglelefteq G$. Hence G has supersolvable type Sylow tower and $P \trianglelefteq G$, a contradiction.

If $O_p(G) \cap T = 1$, then $O_p(G) \leq N_G(T)$ by [3, Lemma A.14.3] and $O_p(G)T = O_p(G) \times T$. Since $O_p(G) \cap Z(P) \neq 1$, we may pick a minimal subgroup H of $O_p(G) \cap Z(P)$ and $H \trianglelefteq G$. Clearly, G/H satisfies the hypothesis of Theorem and so G/H holds by the choice of G. Then every chief factor of G satisfies one of the three conditions in the conclusion of Theorem, a contradiction.

(4) The final contradiction.

Let P_2 be a maximal subgroup of P. By hypothesis and (3), P_2 is m-embedded in G, we may choose a subnormal subgroup K_2 of G such that $G = P_2K_2$ and $P_2 \cap K_2 = 1$. Hence there exists a maximal normal subgroup K such that $|G : K| = p$. Clearly, K_p is m-embedded in G, where K_p is a Sylow p-subgroup of K, we may choose a subnormal subgroup K_3 of G such that $G = K_pK_3 = KK_3$ and $K_p \cap K_3 = 1$. Since $|K_pK_3| = |KK_3|$, we have $|K \cap K_3| = \frac{|K|}{|K_p|}$. If $K \cap K_3 \neq 1$, then $K \cap K_3$ is a p'-group, which contradicts (1). Hence $K \cap K_3 = 1$ and K is a normal p-subgroup, which contradicts (3).

The final contradiction completes our proof. □

From Theorem 3.6, it is easy to prove the following corollaries:

Corollary 3.7 [14, Theorem 3.1] *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. If every maximal subgroup P_1 of P is m-embedded in G and $N_G(P_1)$ is p-nilpotent, then G is p-nilpotent.*

Proof Clearly, G is not a non-abelian simple group and $O_{p'}(G) = 1$. Then we may pick a minimal normal subgroup L of G. Further, L satisfies one of the three conditions in Theorem 3.6. We only need to consider the condition that $|L|_p = p$. Then we consider the group $N_G(L_p)$ where L_p is a Sylow p-subgroup of L. Next, we prove that $|L| = p$. If $N_G(L_p) < G$, then $N_G(L_p)$ is p-nilpotent since $P \leq N_G(L_p)$. Further, $N_L(L_p) = C_L(L_p)$ and so L is p-nilpotent by Burnside Theorem. Then $|L| = p$. If $N_G(L_p) = G$, then $|L| = p$.

Since G/L is p-nilpotent, G is p-supersolvable and so G is supersolvable by Lemma 2.4. Hence $P \trianglelefteq G$, p is the largest prime divisor of $|G|$. Since $G = L \rtimes M$, $P \cap M$ is a maximal subgroup of P and $P \cap M \trianglelefteq G$. Then $G = N_G(P \cap M)$ is p-nilpotent. □

Corollary 3.8 [13, Theorem 1.2] *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. If every maximal subgroup of P is m-embedded in G and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Proof See the proof of Corollary 3.7. □

Corollary 3.9 [9, Theorem 3.1] *Let p be an odd prime dividing the order of a group G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 3.10 [13, Theorem 1.3] *Let G be a p-solvable group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. If every maximal subgroup of P is m-embedded in G, then G is p-supersolvable.*

Proof Clearly, G is not a non-abelian simple group. Then we may pick a minimal normal subgroup L of G. Further, L satisfies one of the three conditions in Theorem 3.6. Since G/L satisfies the hypothesis of Theorem 3.6, G/L is p-supersolvable by induction. Then G is p-supersolvable. □

Theorem 3.11 *Let E be a normal subgroup of G and P be a Sylow p-subgroup of E where p is an odd prime divisor of $|E|$. If every maximal subgroup of P is m-embedded in G, then every G-chief factor A/B below E satisfies one of the following conditions*:

(1) $A/B \leq \Phi(G/B)$; (2) A/B *is a* p'-*group*; (3) $|A/B|_p = p$.

Proof Assume that the theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal.

(1) $O_{p'}(E) = 1$.
Assume that $O_{p'}(E) \neq 1$. The hypothesis also holds for $(G/O_{p'}(E), E/O_{p'}(E))$ by Lemma 2.1, and for (G, E). Then every G-chief factor below E holds, a contradiction.

(2) If $O_p(E) \neq 1$, then $O_p(E) \cap \Phi(G) = 1$.
Assume that $O_p(E) \cap \Phi(G) \neq 1$. We may choose a minimal normal subgroup L of G such that $L \leq O_p(E) \cap \Phi(G)$. By induction, $(G/L, E/L)$ satisfies the hypothesis of theorem and so every G/L-chief factor below E/L holds. Then every G-chief factor below E holds, a contradiction.

(3) $O_p(E) = 1$.

Assume that $O_p(E) \neq 1$. By (2) and [4, Theorem 1.8.17], $O_p(E) = L_1 \times L_2 \times \cdots \times L_t$ where L_i are the minimal normal subgroups of G, $i = 1, 2, \ldots, t$. We assert that $O_p(E)$ is a minimal normal subgroup of G. Otherwise, there exists two minimal normal subgroups L_1 and L_2, then we consider $(G/L_1, E/L_1)$ and $(G/L_2, E/L_2)$. Clearly, $(G/L_j, E/L_j)$ satisfies the hypothesis of theorem and so every G/L_j-chief factor below E/L_j holds by the choice of (G, E) where $j = 1, 2$. If $L_1L_2/L_2 \leq \Phi(G/L_2)$, then $L_1L_2 \leq \Phi(G)L_2$ by [3, A. Lemma 9.11]. Since $L_1L_2 \leq O_p(E)$, $L_1L_2 \leq O_p(E) \cap \Phi(G)L_2 = (O_p(E) \cap \Phi(G))L_2 = L_2$ by (2), a contradiction. Hence $L_1 \cong L_1L_2/L_2$ satisfies the condition (2) or (3), then every G-chief factor holds, a contradiction. Clearly, $O_p(E) \nleq \Phi(G) = 1$. Then there exists a maximal subgroup M of G such that $G = O_p(E)M$. Then $P = O_p(E)(P \cap M)$. We assert that $O_p(E) < P$. If not, by Lemma 2.2, $|P| = p$ and so (G, E) holds, a contradiction. Hence we may choose a maximal subgroup P_1 of P such that $P \cap M \leq P_1$ and $O_p(E) \nleq P_1$. By hypothesis, P_1 is m-embedded in G, there exists a subnormal subgroup T in G and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = P_1T$ and $P_1 \cap T \leq C \leq P_1$. We assert that $C = 1$. Otherwise, $C \neq 1$. If $C < O_p(E)$ by Lemma 2.3, then we obtain C neither covers nor avoids maximal pair (M, G) since $O_p(E) \cap M = 1$, a contradiction. Hence we may assume that $C = O_p(E)$ by

Lemma 2.3, i.e., $O_p(E) \le P_1$ and so $P \le O_p(E)(P \cap M) \le P_1 < P$, a contradiction. Then we have $|T_p| = p$.

If $O_p(E) \cap T \ne 1$, then $O_p(E) \cap T = T_p \trianglelefteq T$ and so T is p-solvable. Furthermore, $T_{p'}$ is a Hall p'-subgroup of G. By [4, Theorem 1.8.19] and (1), $C_T(T_p) = T_p$. Hence $N_T(T_p)/C_T(T_p) = T/C_T(T_p) \hookrightarrow C_{p-1}$. By Schur–Zassenhaus Theorem, $T_{p'}$ is cyclic and T is supersolvable. Then $E \cap T$ is supersolvable. We assert that p is the largest prime divisor of $|E|$. Otherwise, if $q \ne p$ is the largest prime divisor of $|E|$, then $Q \le O_q(E) = 1$ where Q is a Sylow q-subgroup of $E \cap T$, which contradicts (1). Then we assume that $p_1 < p_2 < \cdots < p_n = p$, where $\pi(E) = \{p_1, p_2, \ldots, p_n = p\}$. Since $T_{p'}$ is cyclic, $E_{p'} = E \cap T_{p'}$ is cyclic and E_{p_1} is cyclic where E_{p_1} a Sylow p_1-subgroup of E. By Burnside Theorem, E is p_1-nilpotent and $E_{p_1'} \trianglelefteq E$. Next, we consider $E_{p_1'}$. Similar to the previous discussion, $E_{\{p_1,p_2\}'} \trianglelefteq E_{p_1'}$. Repeat above discussion, we get a normal subgroup series of E: $1 \trianglelefteq P \trianglelefteq \cdots \trianglelefteq E_{\{p_1,p_2\}'} \trianglelefteq E_{p_1'} \trianglelefteq E$. Hence E has supersolvable type Sylow tower and $P \trianglelefteq E$, a contradiction.

If $O_p(E) \cap T = 1$, then $O_p(E) \le N_G(T)$ by [3, Lemma A.14.3] and $O_p(E)T = O_p(E) \times T$. Since $O_p(E) \cap Z(G_p) \ne 1$ where G_p is a Sylow p-subgroup of G, we may pick a minimal subgroup H of $O_p(E) \cap Z(G_p)$ and $H \trianglelefteq G$. Clearly, $(G/H, E/H)$ satisfies the hypothesis of Theorem and so $(G/H, E/H)$ holds by the choice of (G, E). Then every G-chief factor below E holds, a contradiction.

(4) The final contradiction.

Let P_2 be a maximal subgroup of P. By hypothesis and (3), P_2 is m-embedded in E, we may choose a subnormal subgroup K_2 of E such that $E = P_2K_2$ and $P_2 \cap K_2 = 1$. Hence there exists a maximal normal subgroup K such that $|E : K| = p$. Clearly, K_p is m-embedded in E, where K_p is a Sylow p-subgroup of K, we may choose a subnormal subgroup K_3 of E such that $E = K_pK_3 = KK_3$ and $K_p \cap K_3 = 1$. Since $|K_pK_3| = |KK_3|$, we have $|K \cap K_3| = \frac{|K|}{|K_p|}$. If $K \cap K_3 \ne 1$, then $K \cap K_3$ is a p'-group, which contradicts (1). Hence $K \cap K_3 = 1$ and K is a normal p-subgroup, which contradicts (3).

The final contradiction completes our proof. □

Corollary 3.12 *Let E be a normal subgroup of G and P be a Sylow p-subgroup of E where p is an odd prime divisor of $|E|$. Suppose that $G/E = \overline{G}$ and every chief factor $\overline{A}/\overline{B}$ of $\overline{G}$ satisfies one of the following conditions*:

$$(1)\ \overline{A}/\overline{B} \le \Phi(\overline{G}/\overline{B});\qquad (2)\ \overline{A}/\overline{B} \text{ is a } p'\text{-group};\qquad (3)\ |\overline{A}/\overline{B}|_p = p.$$

If every maximal subgroup of P is m-embedded in G, then every chief factor A/B of G satisfies one of the following conditions:

$$(1)\ A/B \le \Phi(G/B);\qquad (2)\ A/B \text{ is a } p'\text{-group};\qquad (3)\ |A/B|_p = p.$$

Corollary 3.13 [12, Theorem 3.1] *Let p be a prime, G be a p-solvable group and let H be a normal subgroup of G such that $G/H \in p\mathcal{U}$, $p\mathcal{U}$ is the class of all p-supersolvable groups. If the maximal subgroups of the Sylow p-subgroups of H are c-normal in G, then $G \in p\mathcal{U}$.*

Acknowledgements This research is supported by NSFC (Grant #11271016) and NSFC (Grant #11471055) and Qing Lan project of Jiangsu Province and High-level personnel of support program of Yangzhou University and 333 high-level personnel training project in Jiangsu Province.

References

1. Ballester-Bolinches, A., Ezquerro, L.M., Skiba, A.N.: Local embeddings of some families of subgroups of finite groups. Acta. Math. Sin. (Engl. Ser.) **6**, 869–882 (2009)
2. Ballester-Bolinches, A., Pedraza-Aguilera, M.C.: On minimal subgroups of finite groups. Acta. Math. Hung. **73**(4), 335–342 (1996)
3. Doerk, K., Hawkes, T.: Finite Solvable Groups. Walter de Gruyter, Berlin (1992)
4. Guo, W.: The Theory of Classes of Groups. Kluwer Academic Publishers, Dordrecht (2000)
5. Guo, W., Shum, K.P., Skiba, A.N.: Criterions of supersolvability for products supersolvable groups. Publ. Math. Debr. **68**, 433–449 (2006)
6. Guo, W., Skiba, A.N.: Finite groups with systems of Σ-embedded subgroups. Sci. China Math. **9**, 1909–1926 (2011)
7. Guo, W., Skiba, A.N.: Finite groups with generalized Ore supplement conditions for primary subgroups. J. Algebra **432**, 205–227 (2015)
8. Guo, X., Wang, J., Shum, K.P.: On semi-cover-avoiding maximal subgroups and solvability of finite groups. Commun. Algebra **34**, 3235–3244 (2006)
9. Guo, X., Shum, K.P.: On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups. Arch. Math. **80**, 561–569 (2003)
10. Huppert, B., Blackburn, N.: Finite Groups III. Springer-Verlag, Berlin (1982)
11. Ore, O.: Contributions in the theory of groups of finite order. Duke Math. J. **5**, 431–460 (1939)
12. Ramadan, M., Ezzat Mohamed, M., Heliel, A.A.: On c-normality of certain subgroups of prime power order of finite groups. Arch. Math. **85**, 203–210 (2005)
13. Tang, J., Miao, L.: A note on m-embedded subgroups of finite groups. Turk. J. Math. **39**, 501–506 (2015)
14. Xu, Y., Zhang, X.: m-embedded subgroups and p-nilpotency of finite groups. Can. Math. Bull. **57**(4), 884–889 (2014)
15. Weinstein, M.: Between Nilpotent and Solvable. Polynal Publishing House, Passaic (1982)