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Estimates and Monotonicity of the First Eigenvalues Under the Ricci Flow on Closed Surfaces

Shouwen Fang[1] · **Liang Zhao**[2] · **Peng Zhu**[3]

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Abstract In the paper we first derive the evolution equation for eigenvalues of geometric operator $-\Delta_\phi + cR$ under the Ricci flow and the normalized Ricci flow on a closed Riemannian manifold M, where Δ_ϕ is the Witten–Laplacian operator, $\phi \in C^\infty(M)$, and R is the scalar curvature. We then prove that the first eigenvalue of the geometric operator is nondecreasing along the Ricci flow on closed surfaces with certain curvature conditions when $0 < c \leq \frac{1}{2}$. As an application, we obtain some monotonicity formulae and estimates for the first eigenvalue on closed surfaces.

Keywords First eigenvalue · Witten–Laplacian · Ricci flow

Mathematics Subject Classification 53C21 · 53C44

1 Introduction

The eigenvalues of geometric operators have always been an active subject in the study of geometry and analysis of manifolds. Recently, there has been increasing attentions on the eigenvalue problems under various geometric flows. In [13], Perelman introduced the so-called $\mathcal{F}$-functional and proved that it is nondecreasing along the

✉ Shouwen Fang
shwfang@163.com

1 School of Mathematical Science, Yangzhou University, Yangzhou 225002, China

2 Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

3 School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, Jiangsu, China

Ricci flow coupled to a backward heat equation. The nondecreasing of the functional implies the monotonicity of the first eigenvalue of $-4\Delta + R$ along the Ricci flow. As an application, he was able to rule out nontrivial steady or expanding breathers on closed manifolds. In [12], Ma obtained the monotonicity of the first eigenvalue of the Laplace–Beltrami operator on a domain with Dirichlet boundary condition along the Ricci flow. Cao [1] showed that the eigenvalues of $-\Delta + \frac{R}{2}$ are nondecreasing under the Ricci flow on manifolds with nonnegative curvature operator. Li [8] introduced families of functionals of the type of $\mathcal{F}$-functional and $\mathcal{W}$-functional of Perelman and proved that they are nondecreasing under the Ricci flow. In particular, he got the monotonicity of eigenvalues of $-4\Delta + kR$ and ruled out compact steady Ricci breathers. Later, Cao [2] also improved his own previous results and proved that the first eigenvalues of $-\Delta + cR\ \left(c \geq \frac{1}{4}\right)$ are nondecreasing under the Ricci flow on manifolds without curvature assumption. Ling [10] showed a Faber–Krahn type of comparison theorem, gave a sharp bound for the first eigenvalue of the Laplace–Beltrami operator under the normalized Ricci flow, and constructed a class of monotonic quantities on closed n-dimensional manifolds [11]. Cao et al. [3] derived various monotonicity formulae and estimates for the first eigenvalue of $-\Delta + cR\ \left(0 < c \leq \frac{1}{2}\right)$ on closed surfaces. Moreover, Zhao got some monotonic quantities for the first eigenvalue of the Laplace–Beltrami operator under the Yamabe flow [15], and proved that the first eigenvalue of the p-Laplacian operator is increasing and the differentiable almost everywhere along powers of the mth mean curvature flow [16] and the H^k-flow [17]. Guo and his collaborators [7] derived an explicit formula for the evolution of the lowest eigenvalue of the Laplace–Beltrami operator with potential in abstract geometric flows. The first author, Xu and Zhu [4] proved the monotonicity of eigenvalues of $-\Delta_\phi + cR\ \left(c > \frac{1}{4}\right)$ along the system of Ricci flow coupled to a heat equation. Not long ago the first author, Yang and Zhu also generalized Cao's result [1] to the Witten–Laplacian operator and obtained evolution equations and monotonicity of eigenvalues of $-\Delta_\phi + \frac{R}{2}$ along the Yamabe flow [5] and the Ricci flow [6].

In this paper, we consider an n-dimensional closed Riemannian manifold M with a time-dependent Riemannian metric $g(t)$. $(M, g(t))$ is a smooth solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t), \tag{1.1}$$

or the normalized Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2r}{n} g_{ij}, \tag{1.2}$$

where $r = \dfrac{\int_M R \mathrm{d}\nu}{\int_M \mathrm{d}\nu}$ is the average scalar curvature. Let ∇ be the Levi–Civita connection on (M, g), Δ the Laplace–Beltrami operator, $d\nu$ the Riemannian volume measure, and $\mathrm{d}\mu$ the weight volume measure on (M, g), i.e.,

$$\mathrm{d}\mu = e^{-\phi(x)}\mathrm{d}\nu,$$

where $\phi \in C^{\infty}(M)$. Then the Witten–Laplacian (also called symmetric diffusion operator)

$$\Delta_{\phi} = \Delta - \nabla\phi \cdot \nabla$$

is a symmetric operator on $L^2(M, \mu)$. When ϕ is a constant function, the Witten–Laplacian operator is just the Laplace–Beltrami operator. As an extension of the Laplace–Beltrami operator, many classical results in Riemannian geometry asserted in terms of the Laplace–Beltrami operator have been extended to the analogous ones on the Witten–Laplacian operator. For example, we can refer to [4–6,9,14]. Inspired by Cao et al. [3], we study the eigenvalues of geometric operator $-\Delta_{\phi} + cR$ under the Ricci flow and the normalized Ricci flow. The purpose of this paper is to obtain the monotonicity and some bounds for the first eigenvalue of the operator along the Ricci flow on closed surfaces under some curvature assumptions for the case $0 < c \leq \frac{1}{2}$.

The rest of this paper is organized as follows. In Sect. 2, we will derive the evolution equations of eigenvalues under the Ricci flow and the normalized Ricci flow. In Sect. 3, we consider the first eigenvalue of the geometric operator $-\Delta_{\phi} + cR$ $\left(0 < c \leq \frac{1}{2}\right)$ on closed surfaces. We will show that the first eigenvalue is nondecreasing along the Ricci flow with some curvature conditions. In Sect. 4, we will obtain some monotonicity formulae and lower bounds for the first eigenvalue on closed surfaces.

Throughout this paper, we use Einstein convention, i.e., repeated index implies summation.

2 Evolution Equations of Eigenvalues

In this section, we establish the evolution of eigenvalues of geometric operator $-\Delta_{\phi} + cR$ under the Ricci flow and the normalized Ricci flow, respectively.

Let $(M, g(t))$ be a closed Riemannian manifold, and $(M, g(t))$, $t \in [0, T)$ be a smooth solution to the Ricci flow Eq. (1.1). Let λ be an eigenvalue of the operator $-\Delta_{\phi} + cR$ at time t_0 where $0 \leq t_0 < T$, and f the corresponding eigenfunction, i.e.,

$$-\Delta_{\phi} f + cRf = \lambda f,$$

with the normalization

$$\int_M f^2 \mathrm{d}\mu = 1. \tag{2.1}$$

Assume that $f(x, t)$ is a C^1-family of smooth functions on M, and satisfies the normalization condition for any t. We need to use the following functional

$$\lambda(f, t) = \int_M \left(-f\Delta_{\phi} f + cRf^2\right) \mathrm{d}\mu = \int_M \left(-\Delta_{\phi} f + cRf\right) f \mathrm{d}\mu,$$

where f satisfies the equality (2.1). At time t, if f is the eigenfunction of λ, then

$$\lambda(f, t) = \lambda(t).$$

Let us first recall the evolution equation of the above functional under the general geometric flow, which had been proved in [4].

Lemma 2.1 *Suppose that λ is an eigenvalue of the operator $-\Delta_\phi + cR$, f is the eigenfunction of λ at time t_0, and the metric $g(t)$ evolves by*

$$\frac{\partial}{\partial t} g_{ij} = v_{ij},$$

where v_{ij} is a symmetric two-tensor. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(f,t)|_{t=t_0} &= \int_M \left(v_{ij} f_{ij} - v_{ij}\phi_i f_j + c\frac{\partial R}{\partial t} f \right) f \mathrm{d}\mu \\ &\quad + \int_M \left(v_{ij,i} - \frac{1}{2} V_j \right) f_j f \mathrm{d}\mu,\end{aligned} \tag{2.2}$$

where $V = Tr(v)$.

Remark 2.2 In fact, Lemma 2.1 can also give us the evolution of eigenvalues. By the eigenvalue perturbation theory, we may assume that there is a C^1-family of smooth eigenvalues and eigenfunctions. Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = \frac{\mathrm{d}}{\mathrm{d}t}\lambda(f, t) \tag{2.3}$$

for any time t, when f is the eigenfunction of λ at time t. In particular, we further assume that the first eigenvalue and first eigenfunction are smooth in the time along the Ricci flow (for example, see [3]).

Now we can calculate the evolution equation of eigenvalues of the geometric operator under the Ricci flow. In Lemma 2.1, when the symmetric two-tensor $v_{ij} = -2R_{ij}$, we get the following result.

Theorem 2.3 *Let $g(t), t \in [0, T)$, be a solution to the Ricci flow (1.1) on a closed manifold M^n. Assume that there is a C^1-family of smooth functions $f(x,t) > 0$, which satisfy*

$$-\Delta_\phi f(x,t) + cRf(x,t) = \lambda(t) f(x,t),$$

and the normalization

$$\int_M f(x,t)^2 \mathrm{d}\mu = 1.$$

Then the eigenvalue $\lambda(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = (4c-2)\int_M R_{ij}(f_{ij}-\phi_i f_j)f\mathrm{d}\mu + 2c\int_M R_{ij}f_i f_j\mathrm{d}\mu + 2c\int_M |Rc|^2 f^2\mathrm{d}\mu$$
$$+ 2c\int_M R_{ij}(f\phi_i - f_i)(f\phi_j - f_j)\mathrm{d}\mu - 2c\int_M R_{ij}\phi_{ij}f^2\mathrm{d}\mu. \quad (2.4)$$

Proof The proof also follows from a direct computation. Note that the evolution of scalar curvature is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2$$

and

$$\mathbf{div}Rc = \frac{1}{2}\nabla R.$$

Using (2.3) and plugging $v_{ij} = -2R_{ij}$ into the equality (2.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = \int_M \left(-2R_{ij}f_{ij}f + 2R_{ij}\phi_i f_j f + c\Delta R f^2 + 2c|Rc|^2 f^2\right)\mathrm{d}\mu. \quad (2.5)$$

From integration by parts and $\frac{1}{2}\Delta R = \mathbf{div}(\mathbf{div}Rc)$, we get

$$\frac{1}{2}\int_M \Delta R f^2\mathrm{d}\mu$$
$$= \int_M \left(2R_{ij}f_i f_j + 2R_{ij}f_{ij}f - 4R_{ij}\phi_i f_j f + R_{ij}\phi_i\phi_j f^2 - R_{ij}\phi_{ij}f^2\right)\mathrm{d}\mu. \quad (2.6)$$

Substituting (2.6) into (2.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = (4c-2)\int_M R_{ij}f_{ij}f\mathrm{d}\mu + 4c\int_M R_{ij}f_i f_j\mathrm{d}\mu + 2c\int_M |Rc|^2 f^2\mathrm{d}\mu$$
$$+ (2-8c)\int_M R_{ij}\phi_i f_j f\mathrm{d}\mu + 2c\int_M R_{ij}\phi_i\phi_j f^2\mathrm{d}\mu - 2c\int_M R_{ij}\phi_{ij}f^2\mathrm{d}\mu$$
$$= (4c-2)\int_M R_{ij}(f_{ij}-\phi_i f_j)f\mathrm{d}\mu + 2c\int_M R_{ij}f_i f_j\mathrm{d}\mu + 2c\int_M |Rc|^2 f^2\mathrm{d}\mu$$
$$+ 2c\int_M R_{ij}(f\phi_i - f_i)(f\phi_j - f_j)\mathrm{d}\mu - 2c\int_M R_{ij}\phi_{ij}f^2\mathrm{d}\mu.$$

□

By Lemma 2.1 we can also get the evolution equation of eigenvalues of the geometric operator under the normalized Ricci flow. We have the following theorem.

Theorem 2.4 *Let $g(t)$, $t \in [0, T)$, be a solution to the normalized Ricci flow (1.2) on a closed manifold M^n. Assume that there is a C^1-family of smooth functions $f(x,t) > 0$, which satisfy*

$$-\Delta_\phi f(x,t) + cRf(x,t) = \lambda(t) f(x,t), \tag{2.7}$$

and the normalization

$$\int_M f(x,t)^2 \mathrm{d}\mu = 1.$$

Then the eigenvalue $\lambda(t)$ satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = & -\frac{2r\lambda}{n} + (1-2c)\lambda \int_M R f^2 \mathrm{d}\mu + c(2c-1)\int_M R^2 f^2 \mathrm{d}\mu \\ & + (2c-1)\int_M R|\nabla f|^2 \mathrm{d}\mu \\ & + 2\int_M R_{ij} f_i f_j \mathrm{d}\mu + c\int_M R(-2ff_i\phi_i - f^2\Delta\phi + f^2|\nabla\phi|^2)\mathrm{d}\mu \\ & + 2c\int_M |Rc|^2 f^2 \mathrm{d}\mu. \end{aligned} \tag{2.8}$$

Proof We note that the evolution of scalar curvature under the normalized Ricci flow is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2 - \frac{2r}{n} R$$

and

$$v_{ij} = -2R_{ij} + \frac{2r}{n} g_{ij}.$$

By Lemma 2.1 it is easy to get the extra term $-\frac{2r\lambda}{n}$ than (2.5), i.e., we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = -\frac{2r\lambda}{n} + \int_M \left(-2R_{ij} f_{ij} f + 2R_{ij}\phi_i f_j f + c\Delta R f^2 + 2c|Rc|^2 f^2\right) \mathrm{d}\mu. \tag{2.9}$$

By the contracted second Bianchi identity $\nabla_i R_{ij} = \frac{1}{2}\nabla_j R$ and integration by parts, it follows that

$$\begin{aligned} -2\int_M R_{ij} f_{ij} f \mathrm{d}\mu = & 2\int_M R_{ij} f_i f_j \mathrm{d}\mu - 2\int_M R_{ij}\phi_i f_j f \mathrm{d}\mu \\ & - \int_M R|\nabla f|^2 \mathrm{d}\mu - \int_M R f \Delta_\phi f \mathrm{d}\mu. \end{aligned} \tag{2.10}$$

Integrating by parts again, we derive that

$$\begin{aligned}\int_M c\Delta R f^2 \mathrm{d}\mu &= c\int_M R(2f\Delta f + 2|\nabla f|^2 - 4ff_i\phi_i - f^2\Delta\phi + f^2|\nabla\phi|^2)\mathrm{d}\mu \\ &= c\int_M R(2f\Delta_\phi f + 2|\nabla f|^2 - 2ff_i\phi_i - f^2\Delta\phi + f^2|\nabla\phi|^2)\mathrm{d}\mu.\end{aligned} \tag{2.11}$$

Combining (2.9)-(2.11), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) &= -\frac{2r\lambda}{n} + (2c-1)\int_M Rf\Delta_\phi f\mathrm{d}\mu + (2c-1)\int_M R|\nabla f|^2\mathrm{d}\mu \\ &\quad + 2\int_M R_{ij}f_if_j\mathrm{d}\mu \\ &\quad + c\int_M R(-2ff_i\phi_i - f^2\Delta\phi + f^2|\nabla\phi|^2)\mathrm{d}\mu + 2c\int_M |Rc|^2f^2\mathrm{d}\mu \\ &= -\frac{2r\lambda}{n} + (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + c(2c-1)\int_M R^2f^2\mathrm{d}\mu \\ &\quad + (2c-1)\int_M R|\nabla f|^2\mathrm{d}\mu \\ &\quad + 2\int_M R_{ij}f_if_j\mathrm{d}\mu + c\int_M R(-2ff_i\phi_i - f^2\Delta\phi + f^2|\nabla\phi|^2)\mathrm{d}\mu \\ &\quad + 2c\int_M |Rc|^2f^2\mathrm{d}\mu.\end{aligned}$$

In the above formula the last equality holds because of (2.7). We complete the proof of the theorem. □

Remark 2.5 In Theorem 2.4 if we let ϕ be a constant function on M, our theorem reduce to Cao et al.'s Lemma 3.1 in [3]. So our result is an extension version of Cao, Hou, and Ling's.

3 Monotonicity of the First Eigenvalues

In this section, we consider the first eigenvalues of geometric operators $-\Delta_\phi + cR$ $\left(0 < c \le \frac{1}{2}\right)$ on closed surfaces with nonnegative scalar curvature. We will obtain evolution equations and monotonicity of the first eigenvalues along the Ricci flow using the results in Sect. 2.

Let $\lambda(t)$ be the first eigenvalue of the operator $-\Delta_\phi + cR$, and $f(x,t) > 0$ be the corresponding eigenfunction, which satisfies

$$-\Delta_\phi f(x,t) + cRf(x,t) = \lambda(t)f(x,t), \tag{3.1}$$

and the normalization

$$\int_M f(x,t)^2 \mathrm{d}\mu = 1.$$

From Theorem 2.3 in Sect. 2, we can easily get the evolution equation of the first eigenvalue under the Ricci flow.

Theorem 3.1 *Let $g(t), t \in [0,T)$, be a solution to the Ricci flow (1.1) on a closed surface M^2. Assume that $\lambda(t)$ and $f(x,t)$ are defined as above. Then we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) =&(1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + 2c^2\int_M R^2 f^2\mathrm{d}\mu + c\int_M R|\nabla f|^2\mathrm{d}\mu \\ &+ c\int_M R|f\nabla\phi - \nabla f|^2\mathrm{d}\mu - c\int_M R\Delta\phi f^2\mathrm{d}\mu. \end{aligned} \tag{3.2}$$

Proof When M is a 2-dimensional manifold, the Ricci curvature satisfies $R_{ij} = \frac{1}{2}Rg_{ij}$. Plugging it into (2.4), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) &= (2c-1)\int_M R\Delta_\phi f f\mathrm{d}\mu + c\int_M R|\nabla f|^2\mathrm{d}\mu + c\int_M R^2 f^2\mathrm{d}\mu \\ &\quad + c\int_M Rg_{ij}(f\phi_i - f_i)(f\phi_j - f_j)\mathrm{d}\mu - c\int_M R\Delta\phi f^2\mathrm{d}\mu \\ &= (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + 2c^2\int_M R^2 f^2\mathrm{d}\mu + c\int_M R|\nabla f|^2\mathrm{d}\mu \\ &\quad + c\int_M R|f\nabla\phi - \nabla f|^2\mathrm{d}\mu - c\int_M R\Delta\phi f^2\mathrm{d}\mu. \end{aligned}$$

The last equality follows from (3.1). We complete the proof of the theorem. □

It is obvious now that the first eigenvalue of the geometric operator is monotonic along the Ricci flow as a consequence of Theorem 3.1.

Corollary 3.2 *Let $g(t), t \in [0,T)$, be a solution to the Ricci flow (1.1) on a closed surface M^2 with nonnegative scalar curvature. Assume that $0 < c \leq \frac{1}{2}$ and $R \geq \frac{1}{2c}\Delta\phi, \forall t \in [0,T)$, then the first eigenvalue of the operator $-\Delta_\phi + cR$ is nondecreasing under the Ricci flow.*

Remark 3.3 For $c = \frac{1}{2}$ the same result was obtained by the first author, Yang and Zhu in [5,6]. Moreover, for $c > \frac{1}{4}$ the first author, Xu and Zhu proved a similar monotonicity along the system of Ricci flow coupled to a heat equation in [4].

4 Some Monotonic Quantities and Estimates

In the last section, we derive some monotonic quantities and lower bounds for the first eigenvalues on closed surfaces using the previous results.

Now let us come to the normalized Ricci flow. By Theorem 2.4 we can get the evolution equation of the first eigenvalue under the normalized Ricci flow.

Theorem 4.1 *Let $g(t), t \in [0, T)$, be a solution to the normalized Ricci flow (1.2) on a closed surface M^2. Assume that $\lambda(t)$ and $f(x,t)$ are defined as above (3.1). Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = -r\lambda + (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + 2c^2\int_M R^2f^2\mathrm{d}\mu + c\int_M R|\nabla f|^2\mathrm{d}\mu + c\int_M R|f\nabla\phi - \nabla f|^2\mathrm{d}\mu - c\int_M R\Delta\phi f^2\mathrm{d}\mu. \tag{4.1}$$

Proof The same computation as Theorem 3.1, using $R_{ij} = \frac{1}{2}Rg_{ij}$ and (2.8) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) &= -r\lambda + (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + c(2c-1)\int_M R^2f^2\mathrm{d}\mu \\
&\quad + (2c-1)\int_M R|\nabla f|^2\mathrm{d}\mu \\
&\quad + \int_M R|\nabla f|^2\mathrm{d}\mu + c\int_M R(-2ff_i\phi_i - f^2\Delta\phi + f^2|\nabla\phi|^2)\mathrm{d}\mu \\
&\quad + c\int_M R^2f^2\mathrm{d}\mu \\
&= -r\lambda + (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + 2c^2\int_M R^2f^2\mathrm{d}\mu + 2c\int_M R|\nabla f|^2\mathrm{d}\mu \\
&\quad + c\int_M R(-2ff_i\phi_i - f^2\Delta\phi + f^2|\nabla\phi|^2)\mathrm{d}\mu \\
&= -r\lambda + (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + 2c^2\int_M R^2f^2\mathrm{d}\mu + c\int_M R|\nabla f|^2 d\mu \\
&\quad + c\int_M R|f\nabla\phi - \nabla f|^2\mathrm{d}\mu - c\int_M R\Delta\phi f^2\mathrm{d}\mu.
\end{aligned}$$

□

Therefore, from Theorem 4.1 it is easy to see that the following monotonicity holds under the normalized Ricci flow.

Corollary 4.2 *Let $g(t), t \in [0, T)$, be a solution to the normalized Ricci flow (1.2) on a closed surface M^2 with nonnegative scalar curvature, and $\lambda(t)$ be the first eigenvalue of the operator $-\Delta_\phi + cR$. Assume that $0 < c \le \frac{1}{2}$ and $R \ge \frac{1}{2c}\Delta\phi, \forall t \in [0, T)$, then $e^{rt}\lambda(t)$ is nondecreasing under the normalized Ricci flow. In particular, the first eigenvalue has a time-dependent lower bound*

$$\lambda(t) \ge \lambda(0)e^{-rt}.$$

Remark 4.3 When ϕ is a constant function on the surface M^2, the same result had been given in [3].

Since dimension $n = 2$, the average scalar curvature r is a constant. Now we assume that $r \neq 0$. Notice that on a surface the scalar curvature under the normalized Ricci flow evolves by

$$\frac{\partial R}{\partial t} = \Delta R + R^2 - rR.$$

Let $\rho(t)$ and $\sigma(t)$ be two solutions to the ODE $y' = y^2 - ry$ with initial value respectively

$$\rho(0) = \max_{x \in M} R(0) \qquad \text{and} \qquad \sigma(0) = \min_{x \in M} R(0).$$

By the maximum principle, we have

$$R(t) \leq \rho(t) = \frac{r}{1 + (\frac{r}{\rho(0)} - 1)e^{rt}} \tag{4.2}$$

and

$$R(t) \geq \sigma(t) = \frac{r}{1 + (\frac{r}{\sigma(0)} - 1)e^{rt}}. \tag{4.3}$$

Thus, we can also get the following monotonic quantity on a closed surface from Theorem 4.1.

Theorem 4.4 *Let $g(t)$, $t \in [0, T)$, be a solution to the normalized Ricci flow (1.2) on a closed surface M^2 with positive scalar curvature, and $\lambda(t)$ be the first eigenvalue of the operator $-\Delta_\phi + cR$. If $\forall c > 0$ and $R \geq \frac{1}{2c}\Delta\phi$, $\forall t \in [0, T)$, then $\dfrac{\rho(t)^{2c}}{\sigma(t)} e^{2crt}\lambda(t)$ is nondecreasing under the normalized Ricci flow. In particular, the first eigenvalue has a time-dependent lower bound*

$$\lambda(t) \geq \lambda(0)\frac{\sigma(t)\rho(0)^{2c}}{\sigma(0)\rho(t)^{2c}} e^{-2crt}.$$

Proof The positive scalar curvature on M^2 implies that $\sigma(0) > 0$. According to (4.1)–(4.3), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) =& -r\lambda + (1-2c)\lambda \int_M Rf^2 \mathrm{d}\mu + 2c^2 \int_M R^2 f^2 \mathrm{d}\mu + c\int_M R|\nabla f|^2 \mathrm{d}\mu \\
&+ c\int_M R|f\nabla\phi - \nabla f|^2 \mathrm{d}\mu - c\int_M R\Delta\phi f^2 \mathrm{d}\mu \\
\geq& -r\lambda + (1-2c)\lambda \int_M Rf^2 \mathrm{d}\mu \\
\geq& -r\lambda + \lambda[\sigma(t) - 2c\rho(t)].
\end{aligned}$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\rho(t)^{2c}}{\sigma(t)}e^{2crt}\lambda(t)\right] \geq 0.$$

The lower bound follows from the monotonicity directly. $\square$

Finally, if we change some conditions in Theorem 4.4, there is also another monotonic quantity on a closed surface.

Theorem 4.5 *Let $g(t)$, $t \in [0, T)$, be a solution to the normalized Ricci flow (1.2) on a closed surface M^2 with positive scalar curvature, and $\lambda(t)$ be the first eigenvalue of the operator $-\Delta_\phi + cR$. If $0 < c \leq \frac{1}{2}$ and $R \geq \frac{1}{c}\Delta\phi$, $\forall t \in [0, T)$, then $\dfrac{e^{crt}}{\sigma(t)^{1-c}}\lambda(t)$ is nondecreasing under the normalized Ricci flow. In particular, the first eigenvalue has a time-dependent lower bound*

$$\lambda(t) \geq \lambda(0)\left(\frac{\sigma(t)}{\sigma(0)}\right)^{1-c} e^{-crt}.$$

Proof By the divergence theorem and (3.1), we have

$$\int_M |\nabla f|^2 \mathrm{d}\mu = -\int_M f\Delta_\phi f \mathrm{d}\mu = \lambda - c\int_M Rf^2 \mathrm{d}\mu. \tag{4.4}$$

Note that the scalar curvature on M^2 is positive and $R \geq \dfrac{1}{c}\Delta\phi$. Substituting (4.3) and (4.4) into (4.1), we arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) =& -r\lambda + (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + 2c^2\int_M R^2 f^2 \mathrm{d}\mu + c\int_M R|\nabla f|^2 \mathrm{d}\mu \\
&+ c\int_M R|f\nabla\phi - \nabla f|^2\mathrm{d}\mu - c\int_M R\Delta\phi f^2 \mathrm{d}\mu \\
\geq& -r\lambda + (1-2c)\lambda\int_M Rf^2\mathrm{d}\mu + c^2\int_M R^2 f^2\mathrm{d}\mu + c\int_M R|\nabla f|^2\mathrm{d}\mu \\
\geq& -r\lambda + (1-2c)\sigma(t)\lambda + c^2\int_M R^2 f^2\mathrm{d}\mu + c\sigma(t)\lambda - c^2\int_M \sigma(t)Rf^2\mathrm{d}\mu \\
\geq& -r\lambda + (1-c)\sigma(t)\lambda.
\end{aligned}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{e^{crt}}{\sigma(t)^{1-c}}\lambda(t)\right] \geq 0.$$

Hence, it follows that the time-dependent lower bound holds now. $\square$

Remark 4.6 Here we just consider the first eigenvalues on closed surfaces with positive Euler characteristic class. For the other cases the similar monotonic formulae and estimates can also be obtained when both the first and second derivatives of ϕ along the normalized Ricci flow are bounded.

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References

1. Cao, X.D.: Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator. Math. Ann. **337**(2), 435–441 (2007)
2. Cao, X.D.: First eigenvalues geometric operators under the Ricci flow. Proc. Am. Math. Soc. **136**(11), 4075–4078 (2008)
3. Cao, X.D., Hou, S.B., Ling, J.: Estimate and monotonicity of the first eigenvalue under the Ricci flow. Math. Ann. **354**(2), 451–463 (2012)
4. Fang, S.W., Xu, H.F., Zhu, P.: Evolution and monotonicity of eigenvalues under the Ricci flow, Sci. China Math. **58**. doi:10.1007/s11425-014-4943-7 (2015)
5. Fang, S.W., Yang, F.: First eigenvalues of geometric operators under the Yamabe flow, preprint
6. Fang, S.W., Yang, F., Zhu, P.: Eigenvalues of geometric operators related to the Witten–Laplacian under the Ricci flow, preprint
7. Guo, H.X., Philipowski, R., Thalmaier, A.: Entropy and lowest eigenvalue on evolving manifolds. Pacific J. Math. **264**(1), 61–82 (2013)
8. Li, J.F.: Eigenvalues and energy functionals with monotonicity formulae under Ricci flow. Math. Ann. **338**(4), 927–946 (2007)
9. Li, X.D.: Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds. J. Math. Pures Appl. **54**, 1295–1361 (2005)
10. Ling, J.: A comparison theorem and a sharp bound via the Ricci flow, http://arxiv.org/abs/0710.2574
11. Ling, J.: A class of monotonic quantities along the Ricci flow, http://arxiv.org/abs/0710.4291
12. Ma, L.: Eigenvalue monotonicity for the Ricci-Hamilton flow. Ann. Global Anal. Geom. **29**(3), 287–292 (2006)
13. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications, http://arxiv.org/abs/math/0211159
14. Xia, C.Y., Xu, H.W.: Inequalities for eigenvalues of the drifting Laplacian on Riemannian manifolds. Ann. Global Anal. Geom. **45**(3), 155–166 (2014)
15. Zhao, L.: The first eigenvalue of the Laplace operator under Yamabe flow. Math. Appl. **24**(2), 274–278 (2011)
16. Zhao, L.: The first eigenvalue of the p-Laplace operator under powers of the mth mean curvature flow. Results. Math. **63**, 937–948 (2013)
17. Zhao, L.: The first eigenvalue of the p-Laplace operator under powers of mean curvature flow. Math. Method. Appl. Sci. **37**(5), 744–751 (2014)