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Near-Relaxed Control Problem of Fully Coupled Forward–Backward Doubly System

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Abstract In this paper, we are concerned with an optimal control problem where the system is driven by a fully coupled forward–backward doubly stochastic differential equation. We study the relaxed model for which an optimal solution exists. This is an extension of initial control problem, where admissible controls are measure valued processes. We establish necessary as well as sufficient optimality conditions to the relaxed one.

Keywords Fully coupled forward–backward doubly stochastic differential equation · Relaxed control · Maximum principle · Adjoint equation · Variational principle

Mathematics Subject Classification 93 E20 · 60 H30 · 60H10 · 91B28

1 Introduction

Nonlinear backward doubly stochastic differential equations have been introduced by Pardoux and Peng [13] who have considered a new kind of BSDE, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic Itô integral $\mathrm{d}W_t$ and a backward stochastic Itô integral $\mathrm{d}B_t$. More precisely, they dealt with the following BDSDE

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$$\begin{cases} dY_t = f(t, Y_t, Z_t)dt + g(t, Y_t, Z_t)d\overleftarrow{B}_t - Z_t dW_t, \\ Y_T = \xi. \end{cases} \tag{1.1}$$

They proved that if f and g are uniform Lipschitz, then (1.1), for any square integrable terminal value ξ, has a unique solution (Y_t, Z_t) in the interval $[0, T]$. They also showed that BDSDEs can produce a probabilistic representation for solutions to some quasi-linear stochastic partial differential equations. Since this first existence and uniqueness result, many papers have been devoted to existence and/or uniqueness result under weaker assumptions. Among these papers, we can distinguish two different classes: scalar BDSDEs and multidimensional BDSDEs. In the first case, one can take advantage of the comparison theorem: we refer to Shi et al. [16]. They weakened the uniform Lipschitz assumptions to linear growth and continuous conditions by virtue of a comparison theorem introduced by themselves. They obtained the existence of solutions to BDSDEs, but without uniqueness. In this spirit, let us mention the contributions of N'zi and Owo [10], which dealt with discontinuous coefficients. For multidimensional BDSDE, there is no comparison theorem, and to overcome this difficulty, a monotonicity assumption on the generator f in the variable y is used. This appears in the works of Peng and Shi [15] who have introduced a class of forward–backward doubly stochastic differential equations, under the Lipschitz condition and monotonicity assumptions. Unfortunately, the uniform Lipschitz condition cannot be satisfied in many applications. More recently, N'zi and Owo [11] established existence and uniqueness result under non-Lipschitz assumptions.

In this paper, we study a stochastic control problem where the system is governed by a nonlinear fully coupled forward–backward doubly stochastic differential equation (fully coupled FBDSDE) of the type

$$\begin{cases} dx_t^v = b(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t)dt + \sigma(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t)dW_t - z_t dB_t, \\ dy_t^v = -f(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t)dt - g(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t)dB_t + Z_t^v dW_t, \\ x_0^v = \xi, \quad y_T^v = h\left(x_T^v\right), \end{cases} \tag{1.2}$$

where $B = (B_t)_{t\geq 0}$ and $W = (W_t)_{t\geq 0}$ are standard $d-$dimensional standard Brownian motions, defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^{(B,W)})_{t\geq 0}, \mathbb{P})$. The control variable $v = (v_t)$, called strict control, is process with values in some set U of $\mathbb{R}^k$. We denote by $\mathcal{U}$ the class of all strict controls.

The criteria to be minimized, over the set $\mathcal{U}$, have the form

$$J(v) = \mathbb{E}\left[\Phi\left(x_T^v\right) + \Psi\left(y^v(0)\right) + \int_0^T l\left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t\right) dt\right], \tag{1.3}$$

where Φ, Ψ and l are given maps, and (x_t, y_t, z_t, Z_t) is the trajectory of the system controlled by v.

A control $u \in \mathcal{U}$ is called optimal if it satisfies

$$J(u) = \inf_{v\in\mathcal{U}} J(v). \tag{1.4}$$

Stochastic control problems for the forward–backward system have been studied by many authors. The first contribution of control problem of the forward–backward system is made by Peng [14] who obtained the maximum principle with the control domain being convex. Xu [19] established the maximum principle for this kind of problem in the case where the control domain is not necessary convex, with uncontrolled diffusion coefficient and a restricted functional cost. The work of Peng (convex control domain) is generalized by Wu [18], where the system is governed by a fully coupled FBSDE. Shi and Wu [17] extended the result of Xu [19] to the fully coupled FBSDE with convex control domain and uncontrolled diffusion coefficient. Ji and Zhou [9] used the Ekeland variational principle and established a maximum principle of controlled FBSDE systems, while the forward state is constrained in a convex set at the terminal time, and apply the result to state constrained stochastic linear-quadratic control models and a recursive utility optimization problem are investigated. All the cited previous works on stochastic control of FBSDE are obtained by introducing two adjoint equations. The common fact in most of these works is that an optimal control in the class of admissible controls may fail to exist, in the absence of the Fillipov convexity conditions. For this class of problems, the resulting necessary conditions give rise to an extremal control, which is not necessarily optimal. To handle this problem of existence without imposing the Fillipov conditions, the idea is then to embed the class $\mathcal{U}$ of ordinary controls into a wider class $\mathcal{R}$ of relaxed controls in which the controller chooses at time t a probability measure $\mu_t(da)$ on the control set U rather than an element $u_t \in U$. Existence of an optimal control has been obtained by using the compactification method in Ref [6].

Our objective in this paper is to establish necessary, as well as sufficient optimality conditions, of the Pontryagin maximum principle type for relaxed models.

To go deeper in this kind of problem, the proof is based on the approximation of the relaxed optimal control by a sequence of ordinary controls, which are nearly optimal by the so-called Chattering lemma. Ekeland's variational principle is then applied to establish the existence of a sequence of ε-optimal controls which satisfy necessary conditions for near optimality. This intermediate result is of independent interest, in the sense that an ε-optimal controls exists always and is sufficient in most practical situation. The relaxed stochastic maximum principle, which is the main result of the second part, is derived by using a stability property of the corresponding fully coupled forward–backward doubly stochastic differential equation and adjoint equation with respect to the control variable.

We note that necessary optimality conditions for strict controls, where the systems are governed by a Backward doubly stochastic differential equation, were studied only by Bahlali and Gherbel [5], and Han et al. [8]. Also, we note that necessary optimality conditions for relaxed controls, where the systems are governed by a stochastic differential equation, were studied by Bahlali and Chala [1,2], Chala [3,4].

This paper is organized as follows. In Sect. 2, we give the precise formulation of the problem and introduce the relaxed model. We formulate the problem and give the various assumptions used throughout this paper. In Sect. 3, we give our first main result, the necessary optimality conditions for near control problem and under additional hypothesis. Finally, in the last Section, in this paper we derive our second main result, necessary, as well as sufficient conditions of optimality for relaxed controls.

Along this paper, we denote by C some positive constant, $\mathcal{M}_{n\times d}(\mathbb{R})$ the space of $n \times d$ real matrix, and $\mathcal{M}^d_{n\times d}(\mathbb{R})$ the linear space of vectors $M = (M_1, M_2, \ldots, M_d)$ where $M_i \in \mathcal{M}_{n\times d}(\mathbb{R})$. We use the standard calculus of inner and matrix product.

2 Formulation of the Problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t^{(B,W)})_{t\geq 0}, \mathbb{P})$ be a probability space, where $d-$dimensional Brownian motions $W = (W_t : 0 \leq t \leq T)$ and $B = (B_t : 0 \leq t \leq T)$ are defined. We assume that $(\mathcal{F}_t^{(B,W)})$ defined by $\forall t \geq 0$, $\mathcal{F}_t^{(B,W)} = \sigma[W(r) - W(0); 0 \leq r \leq t] \vee \sigma[B(r) - B(t); t \leq r \leq T] \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of ν-null sets and $\sigma_1 \vee \sigma_2$ denotes the σ-fields generated by $\sigma_1 \cup \sigma_2$. Note that the collection $(\mathcal{F}_t^{(B,W)})$ is neither increasing nor decreasing, and it does not constitute a classical filtration.

For any $n \in \mathbb{N}$, let $\mathcal{M}^2(0, T; \mathbb{R}^n)$ denote the set of n-dimensional jointly measurable random processes $\{\varphi_t, t \in [0, T]\}$ which satisfy:

$(i) : \mathbb{E}\left[\int_0^T |\varphi_t|^2 \,\mathrm{d}t\right] < \infty$, $(ii) : \varphi_t$ is $\left(\mathcal{F}_t^{(B,W)}\right)$ measurable, for $a.e.\ t \in [0, T]$.

We denote similarly by $\mathcal{S}^2([0, T]; \mathbb{R}^n)$—the set of continuous n dimensional random processes which satisfy

$(i) : \mathbb{E}\left[\sup_{0\leq t\leq T} |\varphi_t|^2\right] < \infty$, $(ii) : \varphi_t$ is $\left(\mathcal{F}_t^{(B,W)}\right)$ measurable, for any $t \in [0, T]$.

Let T be a strictly positive real number and U is a nonempty subset of $\mathbb{R}^k$.

For any $v \in \mathcal{U}$, we consider the following fully coupled forward–backward doubly system:

$$\begin{cases} \mathrm{d}x_t^v = b\big(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t\big)\mathrm{d}t + \sigma\big(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t\big)\mathrm{d}W_t - z_t^v \mathrm{d}B_t, \\ \mathrm{d}y_t^v = -f\left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t\right)\mathrm{d}t - g\big(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t\big)\mathrm{d}B_t + Z_t^v \mathrm{d}W_t, \\ x_0^v = \xi, \quad y_T^v = h\left(x_T^v\right), \end{cases}$$

where $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times k} \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times k} \times U \to \mathcal{M}_{n\times d}(\mathbb{R})$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d} \times U \to \mathbb{R}^m$, $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d} \times U \to \mathcal{M}_{n\times k}(\mathbb{R})$, $h : \mathbb{R}^n \to \mathbb{R}^m$.

We defined the criterion to be minimized, with initial and final cost, as follows:

$$J(v) = \mathbb{E}\left[\Phi\left(x_T^v\right) + \Psi\left(y^v(0)\right) + \int_0^T l\left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t\right)\mathrm{d}t\right],$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$, $\Psi : \mathbb{R}^m \to \mathbb{R}$, $l : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d} \times U \to \mathbb{R}$.

The control problem is to minimize the functional J over $\mathcal{U}$ if $u \in \mathcal{U}$ is an optimal solution, that is $J(u) = \inf_{v\in\mathcal{U}} J(v)$.

Definition 2.1 An admissible control u is process with valued in U such that $\sup_{0\leq t\leq T} |u_t|^2 < \infty$, We denote by $\mathcal{U}$ the set of all admissible controls.

Next, we will give some notations $\xi = (x_t, y_t, z_t, Z_t)^*$, $A(t, \xi) = \begin{pmatrix} b(t) \\ \sigma(t) \\ -f(t) \\ -g(t) \end{pmatrix}(t, \xi)$.

We use the usual inner product $\langle ., \cdot . \rangle$ and Euclidean norm $|.|$ in $\mathbb{R}$, and $\mathbb{R}$ into $\mathbb{R}$. All the equalities and inequalities, mentioned in this paper, are in the sense of $dt \times dP$ almost surely on $[0, T] \times \Omega$. We assume that

$\mathbf{H}_1$ For each $\xi \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d}$, $A(t, \xi)$ is an $\mathcal{F}_t-$measurable process defined on $[0, T]$ with $A(t, \xi) \in \mathcal{M}^2\left(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d}\right)$.

$\mathbf{H}_2$ $A(t,)$ and $h(y)$ satisfy Lipschitz conditions: there exists a constant $k > 0$, such that $\left|A(t, \xi) - A(t, \xi')\right| \leq k\left|\xi - \xi'\right| \forall \xi, \xi' \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d}, \forall t \in [0, T]$, $\left|h(y) - h(y')\right| \leq k\left|y - y'\right|$, $\forall y, y' \in \mathbb{R}^m$.

The following monotonic conditions introduced in [15] are the main assumptions in this paper

$$\mathbf{H}_3 \begin{cases} \left\langle A(t, \xi) - A(t, \xi'), \xi - \xi \right\rangle \leq \beta \left|\xi - \xi'\right|^2, \\ \forall \xi = (x, y, z, Z)^*; \xi' = \left(x', y', z', Z'\right)^* \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d}, \forall t \in [0, T] \\ \left\langle h(y) - h(y'), y - y' \right\rangle \geq 0, \end{cases}$$

or

$$\mathbf{H}_3' \begin{cases} \left\langle A(t, \xi) - A(t, \xi'), \xi - \xi \right\rangle \geq -\beta \left|\xi - \xi'\right|^2, \\ \forall \xi = (x, y, z, Z)^*; \xi' = \left(x', y', z', Z'\right)^* \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d}, \forall t \in [0, T] \\ \left\langle h(y) - h(y'), y - y' \right\rangle \leq 0, \end{cases}$$

where β is a positive constant.

Proposition 2.2 *For any given admissible control $v(.)$, we assume* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) (*or* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_3'}$)') *hold. Then the fully coupled FBDSDEs* (1.2) *has the unique solution*:

$$(x_t, y_t, z_t, Z_t) \in \mathcal{M}^2\left(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times d}\right).$$

The proof can be seen in [15]. The proof under the assumptions ($\mathbf{H_1}$), ($\mathbf{H_2}$), and ($\mathbf{H_3'}$) is similar.

We also assume that

$\mathbf{H}_4$ (i) b, σ, f, g, h, Φ, and Ψ are continuously differentiable with respect to (x, y, z, Z). (ii) All the derivatives of b, σ, f, g, and h are bounded by $C(1 + |x| + |y| + |z| + |Z| + |v|)$. (iii) The derivatives of Φ, Ψ are bounded by $C(1 + |x|)$ and $C(1 + |y|)$, respectively.

Under the above assumption, for every $v \in \mathcal{U}$ Eq. (1.2) has unique strong solution and the function cost J is well defined from $\mathcal{U}$ into $\mathbb{R}$.

Lastly, we need the following extension of Itô's formula (for more details see [13]).

Proposition 2.3 *Let* $\alpha \in \mathcal{S}^2([0,T];\mathbb{R}^n)$, $\beta \in \mathcal{M}^2(0,T;\mathbb{R}^n)$, $\gamma \in \mathcal{M}^2\left(0,T;\mathbb{R}^{n\times k}\right)$ *and* $\delta \in \mathcal{M}^2(0,T;\mathbb{R}^n)$ *be such that*

$$\alpha_t = \xi + \int_t^T \beta(s)\,\mathrm{d}s + \int_t^T \gamma(s)\,\overleftarrow{\mathrm{d}B}_s - \int_t^T \delta_s \mathrm{d}W_s, \quad 0 \le t \le T.$$

Then

$$|\alpha_t|^2 = |\xi|^2 + 2\int_0^t (\alpha_s, \beta_s)\,\mathrm{d}s + 2\int_0^t \left(\alpha_s, \gamma(s)\,\overleftarrow{\mathrm{d}B}_s\right) + 2\int_0^t (\alpha_s, \delta_s \mathrm{d}W_s) - \int_0^t \|\gamma_s\|^2\,\mathrm{d}s + \int_0^t \|\delta_s\|^2\,\mathrm{d}s, \tag{2.1}$$

$$\mathbb{E}|\alpha_t|^2 = \mathbb{E}|\alpha_0|^2 + 2\mathbb{E}\int_0^t (\alpha_s, \beta_s)\,\mathrm{d}s - \mathbb{E}\int_0^t \|\gamma_s\|^2\,\mathrm{d}s + \mathbb{E}\int_0^t \|\delta_s\|^2\,\mathrm{d}s. \tag{2.2}$$

More generally, if $\phi \in C^2(\mathbb{R})$,

$$\begin{aligned}\phi(\alpha_t) = \phi(\alpha_0) &+ \int_0^t \left(\phi'(\alpha_s), \beta(s)\right)\mathrm{d}s + \int_0^t \left(\phi'(\alpha_s), \gamma(s)\right)\overleftarrow{\mathrm{d}B}_s \\ &- \int_0^t \left(\phi'(\alpha_s), \delta_s\right)\mathrm{d}W_s - \frac{1}{2}\int_0^t Tr\left[\phi''(\alpha_s)\gamma(s)\gamma^*(s)\right]\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t Tr\left[\phi''(\alpha_s)\delta(s)\delta^*(s)\right]\mathrm{d}s.\end{aligned} \tag{2.3}$$

Here $S^2\left(0,T;\mathbb{R}^k\right)$ denotes the space of (classes of $dP \otimes dt$ a.e. equal) all $\mathcal{F}_t-$progressively measurable $k-$dimensional processes ν with $\mathbb{E}\left(\sup_{0\le t\le T}|\nu(t)|^2\right) < \infty$.

In this section, we generalize the result of strict control problem obtained in [15] to a relaxed control problem. The idea is to replace the strict control v_t by a $\mathbb{P}(U)$-valued process η_t, where $\mathbb{P}(U)$ denotes the space of probability measures equipped with the topology of weak convergence. Our main goal in this paper is to establish a necessary conditions of optimality for relaxed controls.

Definition 2.4 A relaxed control $(\eta_t)_t$ is a $\mathbb{P}(U)$-value process, progressively measurable with respect to $(\mathcal{F}_t)_t$, and such that for each t, $\mathbf{1}_{[0,t]}\cdot\eta$ is $\mathcal{F}_t-$measurable. We denote by $\mathcal{R}$ the set of all relaxed controls.

Remark 2.5 The set of strict controls is embedded into the set of relaxed controls by the mapping $f : v \longmapsto f_v(\mathrm{d}t, \mathrm{d}a) = \mathrm{d}t\delta_{v_t}(\mathrm{d}a)$,where δ_{v_t} is the atomic measure concentrated at a single point v_t.

For any $\eta \in \mathcal{R}$, we consider the following relaxed fully coupled FBSDE doubly

$$\begin{cases} \mathrm{d}x_t^\eta = \int_U b\left(t, x_t^\eta, y_t^\eta, z_t^\eta, Z_t^\eta, a\right) \eta\left(\mathrm{d}a\right) \mathrm{d}t + \int_U \sigma\left(t, x_t^\eta, y_t^\eta, z_t^\eta, Z_t^\eta, a\right) \eta\left(\mathrm{d}a\right) \mathrm{d}W_t \\ \qquad -z_t^\eta \mathrm{d}B_t, \\ \mathrm{d}y_t^\eta = -\int_U f\left(t, x_t^\eta, y_t^\eta, z_t^\eta, Z_t^\eta, a\right) \eta\left(\mathrm{d}a\right) \mathrm{d}t - \int_U g\left(t, x_t^\eta, y_t^\eta, z_t^\eta, Z_t^\eta, a\right) \eta\left(\mathrm{d}a\right) \mathrm{d}B_t \\ \qquad +Z_t^\eta \mathrm{d}W_t, \\ x_0^\eta = \xi, \quad y_T^\eta = h\left(x_T^\eta\right). \end{cases} \tag{2.4}$$

We define the criterion to be minimized, with initial and final cost, in relaxed model, as follows:

$$\mathcal{J}\left(\eta\right) = \mathbb{E}\left[\Phi\left(x_T^\eta\right) + \Psi\left(y^\eta\left(0\right)\right) + \int_0^T \int_U l\left(t, x_t^\eta, y_t^\eta, z_t^\eta, Z_t^\eta, a\right) \eta\left(\mathrm{d}a\right) \mathrm{d}t\right]. \tag{2.5}$$

Our objective is to minimize the functional J over $\mathcal{R}$. If $\mu \in \mathcal{R}$ is an optimal relaxed control, that is

$$\mathcal{J}\left(\mu\right) = \inf_{\eta \in \mathcal{R}} \mathcal{J}\left(\eta\right). \tag{2.6}$$

Throughout this section, we suppose that U is compact, and b, f, g and σ are bounded: $b_x, b_y, b_z, b_Z, \sigma_x, \sigma_y, \sigma_z, \sigma_Z, f_x, f_y, f_z, f_Z, g_x, g_y, g_z$, and g_Z are Lipschitz continuous in (x, y, z, Z).

2.1 Approximation of the Trajectory

We equipped the set $\mathbb{P}\left(U\right)$ of probability measures on U with the topology of stable convergence. Since U is compact, then, with this topology $\mathbb{P}\left(U\right)$ is a compact metrizable space. The stable convergence is required for measurable and bounded function $F\left(t, a\right)$ such that for each fixed $t \in [0, T]$, $F\left(t, .\right)$ is continuous (instead of functions continuous with respect to the pair (t, a) for the stable convergence). The space $\mathbb{P}\left(U\right)$ is equipped with its Borel σ-field, which is the smallest $\sigma-$field such that the mapping $\mu \longmapsto \int F\left(t, a\right) \mu_t\left(\mathrm{d}a\right)$ are measurable for any bounded measurable function F, continuous with respect to a. For more details see El Karoui and et al. [6]. The next Lemma, know as the Chattering Lemma, tells us that any relaxed control is a weak limit of a sequence of strict controls. This Lemma was first proved for deterministic measures and then extended to random measures in [7].

Lemma 2.6 (Chattering Lemma) *Let μ_t be a predictable process with values in the space of probability measures on U. Then there exists a sequence of predictable processes $(u^n)_n = (u^n)_n$ with values in U such that*

$$\mathrm{d}t\mu_t^n\left(\mathrm{d}a\right) = \mathrm{d}t\delta_{u_t^n}\left(\mathrm{d}a\right) \to \mathrm{d}t\mu_t\left(\mathrm{d}a\right) \text{ weakly, } P\text{-as.} \tag{2.7}$$

Proof See Fleming [7]. □

Notation 2.7 *We denote by* $\Gamma_t^\mu = \left(x_t^\mu, y_t^\mu, z_t^\mu, Z_t^\mu\right)$.

The next Lemma gives the stability of controlled stochastic differential equation with respect to the control variable.

Lemma 2.8 *Let* $\mu_t \in \mathcal{R}$ *be a optimal relaxed control and* $(x^\mu, y^\mu, z^\mu, Z^\mu)$ *the corresponding trajectory. Then there exists a sequence* $(u^n)_n \subset U$ *such that*

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_t \left|x_t^n - x_t^\mu\right|^2\right] = 0, \tag{2.8}$$

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_t \left|y_t^n - y_t^\mu\right|^2\right] = 0, \tag{2.9}$$

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T \left|z_t^n - z_t^\mu\right|^2 dt\right] = 0, \tag{2.10}$$

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T \left|Z_t^n - Z_t^\mu\right|^2 dt\right] = 0, \tag{2.11}$$

$$\lim_{n\to+\infty} J\left(u^n\right) = \mathcal{J}\left(\mu\right), \tag{2.12}$$

where (x^n, y^n, z^n, Z^n) *denotes the solution of Eq.* (1.2) *associated with* u^n.

Proof We need to prove that $\left(\widetilde{x}_t^n, \widetilde{y}_t^n, \widetilde{z}_t^n, \widetilde{Z}_t^n\right) = \left(x_t^n - x_t^\mu, y_t^n - y_t^\mu, z_t^n - z_t^\mu, Z_t^n - Z_t^\mu\right)$ converge to 0 in $\mathcal{M}^2(0,T)$ as n tends to infinity. Applying Itô's formula to $\left\langle G\left(x_t^n - x_t^\mu\right), \left(y_t^n - y_t^\mu\right)\right\rangle$ on $[0,T]$ and using (2.7) and the Notation 2.7. It follows that

$$\begin{aligned}
&\mathbb{E}\left\langle \widetilde{x}_T^n, h_x\left(x_T\right)\widetilde{x}_T^n\right\rangle \\
&\quad = \mathbb{E}\int_0^T \left\langle \widetilde{y}_t^n, b_x^n\left(t,\Gamma_t^n\right)\widetilde{x}_t^n + b_y^n\left(t,\Gamma_t^n\right)\widetilde{y}_t^n + b_z^n\left(t,\Gamma_t^n\right)\widetilde{z}_t^n + b_Z^n\left(t,\Gamma_t^n\right)\widetilde{Z}_t^n\right\rangle \mathrm{d}t \\
&\qquad + \mathbb{E}\int_0^T \left\langle \widetilde{z}_t^n, g_x^n\left(t,\Gamma_t^n\right)\widetilde{x}_t^n + g_y^n\left(t,\Gamma_t^n\right)\widetilde{y}_t^n + g_z^n\left(t,\Gamma_t^n\right)\widetilde{z}_t^n + g_Z^n\left(t,\Gamma_t^n\right)\widetilde{Z}_t^n\right\rangle \mathrm{d}t \\
&\qquad + \mathbb{E}\int_0^T \left\langle \widetilde{Z}_t^n, \sigma_x^n\left(t,\Gamma_t^n\right)\widetilde{x}_t^n + \sigma_y^n\left(t,\Gamma_t^n\right)\widetilde{y}_t^n + \sigma_z^n\left(t,\Gamma_t^n\right)\widetilde{z}_t^n + \sigma_Z^n\left(t,\Gamma_t^n\right)\widetilde{Z}_t^n\right\rangle \mathrm{d}t \\
&\qquad + \mathbb{E}\int_0^T \left\langle \widetilde{x}_t^n, f_x^n\left(t,\Gamma_t^n\right)\widetilde{x}_t^n + f_y^n\left(t,\Gamma_t^n\right)\widetilde{y}_t^n + f_z^n\left(t,\Gamma_t^n\right)\widetilde{z}_t^n + f_Z^n\left(t,\Gamma_t^n\right)\widetilde{Z}_t^n\right\rangle \mathrm{d}t \\
&\qquad + \mathbb{E}\int_0^T \left\langle \widetilde{x}_t^n, f\left(t,\Gamma_t^n,u_t^n\right) - \int_U f\left(t,\Gamma_t^\mu,\mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t \\
&\qquad + \mathbb{E}\int_0^T \left\langle \widetilde{y}_t^n, b\left(t,\Gamma_t^n,u_t^n\right) - \int_U b\left(t,\Gamma_t^\mu,\mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^T \left\langle \widetilde{z}_t^n, g\left(t, \Gamma_t^n, u_t^n\right) - \int_U g\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t \\
&+\mathbb{E}\int_0^T \left\langle \widetilde{Z}_t^n, \sigma\left(t, \Gamma_t^n, u_t^n\right) - \int_U \sigma\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t.
\end{aligned}
$$

We get

$$
\begin{aligned}
&\mathbb{E}\left\langle \widetilde{x}_T^n, h_x\left(x_T\right)\widetilde{x}_T^n\right\rangle + \beta\mathbb{E}\int_0^T \left(\left|\widetilde{x}_t^n\right|^2 + \left|\widetilde{y}_t^n\right|^2 + \left|\widetilde{z}_t^n\right|^2 + \left|\widetilde{Z}_t^n\right|^2\right)\mathrm{d}t \\
&\le \mathbb{E}\int_0^T \left\langle \widetilde{x}_t^n, f_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U f_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t \\
&\quad +\mathbb{E}\int_0^T \left\langle \widetilde{y}_t^n, b_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U b_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t \\
&\quad +\mathbb{E}\int_0^T \left\langle \widetilde{z}_t^n, g_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U g_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t \\
&\quad +\mathbb{E}\int_0^T \left\langle \widetilde{Z}_t^n, \sigma_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U \sigma_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right\rangle \mathrm{d}t.
\end{aligned}
$$

By using the above hypothesis, we could write the last inequality as follows:

$$
\begin{aligned}
&\beta\mathbb{E}\int_0^T \left(\left|\widetilde{x}_t^n\right|^2 + \left|\widetilde{y}_t^n\right|^2 + \left|\widetilde{z}_t^n\right|^2 + \left|\widetilde{Z}_t^n\right|^2\right)\mathrm{d}t \\
&\quad \le \frac{\beta}{4}\mathbb{E}\int_0^T \left(\left|\widetilde{x}_t^n\right|^2 + \left|\widetilde{y}_t^n\right|^2 + \left|\widetilde{z}_t^n\right|^2 + \left|\widetilde{Z}_t^n\right|^2\right)\mathrm{d}t + \frac{1}{\beta}\Pi_t^n,
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_t^n = \mathbb{E}\int_0^T &\left[\left|f_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U f_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right|^2\right. \\
&\left. + \left|b_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U b_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right|^2\right]\mathrm{d}t \\
&+\mathbb{E}\int_0^T \left[\left|g_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U g_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right|^2\right. \\
&\left. + \left|\sigma_x\left(t, \Gamma_t^n, u_t^n\right) - \int_U \sigma_x\left(t, \Gamma_t^\mu, \mu\right)\mu\left(\mathrm{d}a\right)\right|^2\right]\mathrm{d}t.
\end{aligned}
$$

Thus, we get

$$
\mathbb{E}\int_0^T \left(\left|\widetilde{x}_t^n\right|^2 + \left|\widetilde{y}_t^n\right|^2 + \left|\widetilde{z}_t^n\right|^2 + \left|\widetilde{Z}_t^n\right|^2\right)\mathrm{d}t \le \frac{4}{3\beta^2}\Pi_t^n.
$$

Then, we have $\left(\widetilde{x}_t^n, \widetilde{y}_t^n, \widetilde{z}_t^n, \widetilde{Z}_t^n\right)$ converge to 0 in $\mathcal{M}^2(0,T)$ as n tends to infinity. Applying the Buckholder–Davis–Gundy inequality, we get

$$\lim_{n\to+\infty}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}\left|x_t^n - x_t^\mu\right|^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\left|y_t^n - y_t^\mu\right|^2\right]\right.$$
$$\left. + \mathbb{E}\int_0^T\left|z_t^n - z_t^\mu\right|^2 \mathrm{d}t + \mathbb{E}\int_0^T\left|Z_t^n - Z_t^\mu\right|^2 \mathrm{d}t\right\} = 0.$$

The proof is completed.

Let us prove (2.12).

Since Φ, Ψ are Lipschitz continuous in (x, y), then by using the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
&\left|J\left(u^n\right) - \mathcal{J}(\mu)\right| \\
&\le C\left(\mathbb{E}\left|x_T^n - x_T^\mu\right|^2\right)^{1/2} + C\left(\mathbb{E}\left|y^n(0) - y^\mu(0)\right|^2\right)^{1/2} \\
&+ C\left(\int_0^T \mathbb{E}\left|x_t^n - x_t^\mu\right|^2 \mathrm{d}t\right)^{1/2} + C\left(\int_0^T \mathbb{E}\left|y_t^n - y_t^\mu\right|^2 \mathrm{d}t\right)^{1/2} \\
&+ C\left(\int_0^T \mathbb{E}\left|z_t^n - z_t^\mu\right|^2 \mathrm{d}t\right)^{1/2} + C\left(\int_0^T \mathbb{E}\left|Z_t^n - Z_t^\mu\right|^2 \mathrm{d}t\right)^{1/2} \\
&+ \left(\mathbb{E}\left|\int_0^T l\left(t, x_t^\mu, y_t^\mu, z_t^\mu, Z_t^\mu, u_t^n\right)\mathrm{d}t\right.\right. \\
&\left.\left. - \int_0^T\int_U l\left(t, x_t^\mu, y_t^\mu, z_t^\mu, Z_t^\mu, a\right)\mu_t(\mathrm{d}a)\,\mathrm{d}t\right|^2\right)^{1/2}.
\end{aligned}$$

From (2.8)–(2.11), the first, the second, the third, the fourth, the fifth and sixth terms in the right-hand side converge to zero. Since l is continuous and bounded, then from (2.7) and by using the dominated convergence theorem, the seventh term in the right-hand side tends to zero. The Lemma is proved. □

Remark 2.9 As a sequence, it is easy to see that the strict and relaxed optimal control problems have the same value function.

3 Necessary Optimality Conditions for Near Optimality

In this section, we derive a necessary condition of optimality for near controls. This result is based on Ekeland‘s variational principle which is given by the following.

Lemma 3.1 (Ekeland's variational principle) *Let (E, d) be a complete metric space and $f : E \to \overline{\mathbb{R}}$ be lower-semicontinuous and bounded from below. Given $\varepsilon > 0$, suppose $u^\varepsilon \in E$ satisfies $f(u^\varepsilon) \le \inf(f) + \varepsilon$. Then for any $\lambda > 0$, there exists $v \in E$ such that*

1. $f(v) \leq f(u^{\varepsilon})$.
2. $d(u^{\varepsilon}, v) \leq \lambda$.
3. $f(v) < f(w) + \frac{\varepsilon}{\lambda} d(v, w), \quad \forall w \neq v$.

To apply Ekeland's variational principle, we have to endow the set $\mathcal{U}$ of strict controls with an appropriate metric. For any $u, v \in \mathcal{U}$, we set $d(u, v) = P \otimes dt\{(w, t) \in \Omega \times [0, T], u(t, w) \neq v(t, w)\}$, where $P \otimes dt$ is the product measure of P with the Lebesgue measure dt.

Let us summarize some of the proprieties satisfied by d.

Lemma 3.2 1. *($\mathcal{U}$,d) is a complete metric space.*
2. *The cost functional J is continuous from $\mathcal{U}$ into $\mathbb{R}$.*

Proof See Mezerdi [12]. □

Now let $\mu \in \mathcal{R}$ be an optimal relaxed control and denote by $(x^{\mu}, y^{\mu}, z^{\mu}, Z^{\mu})$ the trajectory of the system controlled by μ. From Lemmas 2.6 and 2.8, there exists a sequence $(u^n)_n$ of strict controls such that $dt\mu_t^n(\mathrm{d}a) = \mathrm{d}t\delta_{u_t^n}(\mathrm{d}a) \underset{n\to\infty}{\longrightarrow} \mathrm{d}t\mu_t(\mathrm{d}a)$ weakly, $\mathbb{P} - a.s$, $\lim_{n\to\infty} \mathbb{E}\left[\sup_t \left|x_t^n - x_t^{\mu}\right|^2\right] = 0$, $\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left|y_t^n - y_t^{\mu}\right|^2\right] = 0$, $\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left|z_t^n - z_t^{\mu}\right|^2 \mathrm{d}t\right] = 0$, $\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left|Z_t^n - Z_t^{\mu}\right|^2 \mathrm{d}t\right] = 0$, where $\left(x_t^n, y_t^n, z_t^n, Z_t^n\right)$ is the solution of Eq. (1.2) controlled by u^n.

Introduce the following adjoint equation in the strict form:

$$\begin{cases} \mathrm{d}p_t^n = -\left[l_x\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) + b_x\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) p_t^n\right]\mathrm{d}t \\ \qquad\quad +\left[g_x\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) R_t^n + f_x\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) q_t^n\right]\mathrm{d}t \\ \qquad\quad -\left[l_z\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) + b_z\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) p_t^n\right]\mathrm{d}B_t \\ \qquad\quad +\left[g_z\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) R_t^n + f_z\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) q_t^n\right]\mathrm{d}B_t \\ \qquad\quad -\sigma_x\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) P_t^n\mathrm{d}t - \sigma_z\left(t, x_t^n, y_t^n, z_t^n, Z_t, u_t^n\right) P_t^n\mathrm{d}B_t \\ \qquad\quad +P_t^n\mathrm{d}W_t, \\ p_T^n = \Phi_x\left(x_T^n\right) - h_x\left(x_T^n\right) q_T^n, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \mathrm{d}q_t^n = -\left[l_y\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) + b_y\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) p_t^n\right]\mathrm{d}t_t \\ \qquad\quad +\left[f_y\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) q_t^n + g_y\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) R_t^n\right]\mathrm{d}t_t \\ \qquad\quad -\left[l_Z\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) + b_Z\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) p_t^n\right]\mathrm{d}W_t \\ \qquad\quad +\left[f_Z\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) q_t^n + g_Z\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) R_t^n\right]\mathrm{d}W_t \\ \qquad\quad -\sigma_x\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) P_t^n\mathrm{d}t - \sigma_Z\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, u_t^n\right) P_t^n d W_t \\ \qquad\quad -R_t^n\mathrm{d}B_t, \\ q_0^n = \Psi_y\left(y^n(0)\right), \end{cases} \tag{3.2}$$

where $(p^n, P^n, q^n, R^n) \in \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n\times d}\right) \times \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^m) \times \mathcal{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n\times d}\right)$, and it is easy to check that fully coupled FBDSDEs (3.1), (3.2) satisfy **(H1)**, **(H2)** and **(H′3)**, so it has a unique solution (p^n, P^n, q^n, R^n).

We define the Hamiltonian function H as follows:

$$\begin{aligned} &H\left(t, x_t, y_t, z_t, Z_t, p_t, P_t, q_t, R_t, u_t\right) \\ &\quad = b\left(t, x_t, y_t, z_t, Z_t, u_t\right) p_t - f\left(t, x_t, y_t, z_t, Z_t, u_t\right) q_t \\ &\qquad - g\left(t, x_t, y_t, z_t, Z_t, u_t\right) R_t + \sigma\left(t, x_t, y_t, z_t, Z_t, u_t\right) P_t \\ &\qquad + l\left(t, x_t, y_t, z_t, Z_t, u_t\right). \end{aligned}$$

Fully coupled FBDSDEs (3.1), (3.2) can be rewritten as

$$\begin{cases} \mathrm{d}p_t^n = -H_x^n(t)\,dt - H_z^n(t)\,\mathrm{d}B_t + P_t^n \mathrm{d}W_t, \\ p_T^n = \Phi_x\left(x_T^n\right) - h_x\left(x_T^n\right) q_T^n, \end{cases}$$
$$\begin{cases} \mathrm{d}q_t^n = -H_y^n(t)\,\mathrm{d}t - H_Z^n(t)\,\mathrm{d}W_t - R_t^n \mathrm{d}B_t, \\ q_0^n = \Psi_y\left(y^n(0)\right). \end{cases}$$

According to the optimality of μ_t and Lemma 3.1, there exists a sequence $(\varepsilon_n)_n$ of positive real numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$J\left(u^n\right) = J\left(\mu^n\right) \leq J(\mu) + \varepsilon_n.$$

A suitable version of lemma 3.1 implies that, given any $\varepsilon_n > 0$, there exists $(u^n)_n \subset \mathcal{U}$ such that

$$\begin{aligned} &J\left(u^n\right) \leq \inf_{u\in\mathcal{U}} J(u) + \varepsilon_n, \\ &J\left(u^n\right) \leq J(u) + \varepsilon_n d\left(u^n, u\right); \quad u \in \mathcal{U}. \end{aligned} \tag{3.3}$$

Let us define the perturbation

$$u_t^{n,\theta} = \begin{cases} v & \text{if } \ [\tau, \tau+\theta], \\ u_t^n & \text{Otherwise.} \end{cases}$$

From (3.3), we have

$$0 \leq J\left(u_t^{n,\theta}\right) - J\left(u^n\right) + \varepsilon_n \mathrm{d}\left(u_t^{n,\theta}, u^n\right).$$

From the definition of the metric d, we obtain

$$0 \leq J\left(u_t^{n,\theta}\right) - J\left(u^n\right) + \varepsilon_n C\theta. \tag{3.4}$$

From these above inequalities, we shall establish the necessary conditions of optimality for near controls.

Theorem 3.3 (The necessary condition of optimality for near controls) *Let u be an optimal control minimizing the functional J over $\mathcal{U}$ and (x, y, z, Z) denotes corresponding optimal trajectory. Then, there are three unique adapted processes*

$(p^n, P^n, q^n, R^n) \in \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^{n\times d}) \times \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^m) \times \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^{n\times d})$, which are, respectively, solution of stochastic differential equations (3.1) and (3.2) such that a.e; as we have

$$0 \leq \left[H\left(t, x^n_t, y^n_t, z^n_t, Z^n_t, p^n_t, P^n_t, q^n_t, R^n_t, u^n_t\right) - H\left(t, x^n_t, y^n_t, z^n_t, Z^n_t, p^n_t, P^n_t, q^n_t, R^n_t, v\right)\right] + C\varepsilon_n.$$

Proof From inequalities (3.4), we use the same method as in [17] with index n. □

4 Necessary Conditions of Optimality for Relaxed Controls

In this section, we will state and prove the necessary conditions of optimality for relaxed controls. For this end, let us summarize and prove some of lemmas that we will use thereafter.

And for short notation let use the Notation 2.7.

Introduce the following adjoint equation in the relaxed form:

$$\begin{cases} dp^\mu_t = -\left[\int_U l_x\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a) + \int_U b_x\left(t, \Gamma^\mu_t, a\right)\mu(\mathrm{d}a)\, p^\mu_t\right]\mathrm{d}t \\ \qquad + \left[\int_U g_x\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, R^\mu_t + \int_U f_x\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, q^\mu_t\right]\mathrm{d}t \\ \qquad - \left[\int_U l_z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a) + \int_U b_z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, p^\mu_t\right]\mathrm{d}B_t \\ \qquad + \left[\int_U g_z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, R^\mu_t + \int_U f_z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, q^\mu_t\right]\mathrm{d}B_t \\ \qquad - \int_U \sigma_x\left(t, \Gamma^\mu_t, a\right)\mu(\mathrm{d}a)\, P^\mu_t\mathrm{d}t - \int_U \sigma_z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, P^\mu_t\mathrm{d}B_t \\ \qquad + P^\mu_t\mathrm{d}W_t, \\ p^\mu_T = \Phi_x(x^\mu_T) - h_x(x^\mu_T)q^\mu_T \end{cases} \tag{4.1}$$

and

$$\begin{cases} \mathrm{d}q^\mu_t = -\left[\int_U ly\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a) + \int_U b_y\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, p^\mu_t\right]\mathrm{d}t \\ \qquad + \left[\int_U f_y\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, q^\mu_t + \int_U g_y\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, R^\mu_t\right]\mathrm{d}t \\ \qquad - \left[\int_U l_Z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a) + \int_U b_Z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, p^\mu_t +\right]\mathrm{d}W_t \\ \qquad + \left[\int_U f_Z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, q^\mu_t + \int_U g_Z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, R^\mu_t\right]\mathrm{d}W_t \\ \qquad - \int_U \sigma_x\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, P^\mu_t\mathrm{d}t - \int_U \sigma_Z\left(t, \Gamma^\mu_t, a\right)\mu_t(\mathrm{d}a)\, P^\mu_t\mathrm{d}W_t \\ \qquad - R^\mu_t\mathrm{d}B_t, \\ q^\mu_0 = \Psi_y\left(y^\mu(0)\right). \end{cases} \tag{4.2}$$

It is easy to check that fully coupled FBDSDEs (4.1) , (4.2) satisfy (H1), (H2) and (H′3). So, it has a unique solution $(p^{\mu}, P^{\mu}, q^{\mu}, R^{\mu}) \in \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n\times d}\right) \times \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^m) \times \mathcal{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n\times d}\right)$.

Using the Notation 2.7, we define the Hamiltonian function $\mathcal{H}$ as follows:

$$\begin{aligned}\mathcal{H}\left(t,\Gamma^{\mu}_t,p_t,P_t,q_t,R_t,\mu\right) = & \int_U b\left(t,\Gamma^{\mu}_t,a\right)\mu_t(\mathrm{d}a)\,p^{\mu}_t \\ & -\int_U f\left(t,\Gamma^{\mu}_t,a\right)\mu_t(\mathrm{d}a)\,q^{\mu}_t \\ & -\int_U g\left(t,\Gamma^{\mu}_t,a\right)\mu_t(\mathrm{d}a)\,R^{\mu}_t \\ & +\int_U \sigma\left(t,\Gamma^{\mu}_t,a\right)\mu_t(\mathrm{d}a)\,P^{\mu}_t+\int_U l\left(t,\Gamma^{\mu}_t,a\right)\mu_t(\mathrm{d}a)\,.\end{aligned}$$

Fully coupled FBDSDEs (4.1), (4.2), can be rewritten as

$$\begin{cases} \mathrm{d}p^{\mu}_t = -\mathcal{H}^{\mu}_x(t)\,\mathrm{d}t - \mathcal{H}^{\mu}_z(t)\,\mathrm{d}B_t + P^{\mu}_t\mathrm{d}W_t, \\ p^{\mu}_T = \Phi_x\left(x^{\mu}_T\right) - h_x\left(x^{\mu}_T\right)q^{\mu}_T, \end{cases}$$
$$\begin{cases} \mathrm{d}q^{\mu}_t = -\mathcal{H}^{\mu}_y(t)\,\mathrm{d}t - \mathcal{H}^{\mu}_Z(t)\,\mathrm{d}W_t - R^{\mu}_t\mathrm{d}B_t, \\ q^{\mu}_0 = \Psi_y\left(y^{\mu}(0)\right), \end{cases}$$

where $(p^{\mu}, P^{\mu}, q^{\mu}, R^{\mu}) \in \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n\times d}\right) \times \mathcal{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^m) \times \mathcal{L}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n\times d}\right)$.

Lemma 4.1 *Let* (p^n, q^n) *and* (p^{μ}, q^{μ}) *the solution of* (3.1), (3.2), *and* (4.1), (4.2), *then we have*

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\left|p^n_t - p^{\mu}_t\right|^2 + \int_t^T \left|P^n_s - P^{\mu}_s\right|^2\mathrm{d}s\right] = 0, \tag{4.3}$$

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\left|q^n_t - q^{\mu}_t\right|^2 + \int_t^T \left|R^n_s - R^{\mu}_s\right|^2\mathrm{d}s\right] = 0. \tag{4.4}$$

Proof We need to prove that $\left(\widetilde{p}^n_t, \widetilde{q}^n_t, \widetilde{R}^n_t, \widetilde{P}^n_t\right) = \left(p^n_t - p^{\mu}_t, q^n_t - q^{\mu}_t, R^n_t - R^{\mu}_t, P^n_t - P^{\mu}_t\right)$ converge to 0 in $\mathcal{M}^2(0,T)$ as n tends to infinity. Applying Itô's formula to $\left\langle G\left(p^n_t - p^{\mu}_t\right), \left(q^n_t - q^{\mu}_t\right)\right\rangle$ on $[0,T]$ and using (2.7), it follows that

$$\begin{aligned}&\mathbb{E}\left[(q_T - q_T)\left(\Phi_x\left(x^n_T\right) - \Phi_x\left(x^{\mu}_T\right)\right)\right] - \mathbb{E}\left[(q_T - q_T)\left(h_x\left(x^n_T\right)q^n_T - h_x\left(x^{\mu}_T\right)q^{\mu}_T\right)\right] \\ &\quad +\mathbb{E}\left[(p_0 - p_0)\left(\Psi_y\left(y^n(0)\right) - \Psi_y\left(y^{\mu}(0)\right)\right)\right] \\ &= \mathbb{E}\int_0^T \left\langle\left(q^n_t - q^{\mu}_t\right), G\left(H^{\mu}_x(t) - H^n_x(t)\right)\right\rangle\mathrm{d}t \\ &\quad +\mathbb{E}\int_0^T \left\langle G\left(p^n_t - p^{\mu}_t\right), \left(H^{\mu}_y(t) - H^n_y(t)\right)\right\rangle\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&-\mathbb{E}\int_0^T \left\langle \left(R_t^n - R_t^{\mu}\right), G\left(H_z^{\mu}(t) - H_z^n(t)\right)\right\rangle \mathrm{d}t \\
&-\mathbb{E}\int_0^T \left\langle G\left(P_t^n - P_t^{\mu}\right), \left(H_Z^{\mu}(t) - H_Z^n(t)\right)\right\rangle \mathrm{d}t \\
&= \mathbb{E}\int_0^T \left\langle A\left(t,\xi^n\right) - A\left(t,\xi^{\mu}\right), \xi^n - \xi^{\mu}\right\rangle \mathrm{d}t,
\end{aligned}
$$

where $\xi^n = \left(p_t^n, q_t^n, R_t^n, P_t^n\right)^*$, $\xi^{\mu} = \left(p_t^{\mu}, q_t^{\mu}, R_t^{\mu}, P_t^{\mu}\right)^*$, and

$$
A\left(t,\xi^n\right) = \begin{pmatrix} H_y^n(t) \\ H_x^n(t) \\ -H_z^n(t) \\ -H_Z^n(t) \end{pmatrix}, \quad A\left(t,\xi^{\mu}\right) = \begin{pmatrix} H_y^{\mu}(t) \\ H_x^{\mu}(t) \\ -H_z^{\mu}(t) \\ -H_Z^{\mu}(t) \end{pmatrix}.
$$

By applying the monotonicity conditions to above criteria, we get

$$
E\int_0^T \left(\left|p_t^n - p_t^{\mu}\right|^2 + \left|q_t^n - q_t^{\mu}\right|^2 + \left|R_t^n - R_t^{\mu}\right|^2 + \left|P_t^n - P_t^{\mu}\right|^2\right) \le 0.
$$

In Gronwall's lemma and Bukholder–Davis–Gundy inequality, we have the desired result in (4.3) and (4.4). □

Theorem 4.2 (Necessary conditions of optimality for relaxed controls) *Let μ be an optimal relaxed control minimizing the cost $\mathcal{J}$ over $\mathcal{R}$ and $\left(x_t^{\mu}, y_t^{\mu}, z_t^{\mu}, Z_t^{\mu}\right)$ the corresponding optimal trajectory. Then there exist three unique adapted processes* $(p^{\mu}, P^{\mu}, q^{\mu}, R^{\mu}) \in \mathcal{L}_{\mathcal{F}}^2([0,T];\mathbb{R}^n) \times \mathcal{L}_{\mathcal{F}}^2\left([0,T];\mathbb{R}^{n\times d}\right) \times \mathcal{L}_{\mathcal{F}}^2([0,T];\mathbb{R}^m) \times \mathcal{L}_{\mathcal{F}}^2\left([0,T];\mathbb{R}^{n\times d}\right)$, *solution of the stochastic forward–backward doubly differential equations* (4.1) *and* (4.2), *such that for all $\eta \in \mathcal{R}$, we have*

$$
\begin{aligned}
&\mathcal{H}^{\eta}\left(t, x_t^{\mu}, y_t^{\mu}, z_t^{\mu}, Z_t^{\mu}, p^{\mu}, P^{\mu}, q^{\mu}, R^{\mu}, \mu_t\right) \\
&\quad = \max_{\eta\in\mathcal{R}} \mathcal{H}\left(t, x_t^{\mu}, y_t^{\mu}, z_t^{\mu}, Z_t^{\mu}, p^{\mu}, P^{\mu}, q^{\mu}, R^{\mu}, \eta\right).
\end{aligned} \tag{4.5}
$$

Proof Let μ be an optimal relaxed control, from Theorem 3.3, there exists a sequence $(u^n)_u \subset \mathcal{U}$ such that for all $v \in \mathcal{U}$

$$
\begin{aligned}
0 \le &\left[\mathcal{H}\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, p_t^n, P_t^n, q_t^n, R_t^n, u_t^n\right)\right. \\
&\left.-\mathcal{H}\left(t, x_t^n, y_t^n, z_t^n, Z_t^n, p_t^n, P_t^n, q_t^n, R_t^n, v\right)\right] + C\varepsilon_n,
\end{aligned}
$$

where $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

According to (2.8)–(2.10), (2.12), (4.4) and (4.5), the result follows immediately by letting n going to infinity in the last inequality. □

Theorem 4.3 (The sufficient conditions of optimality for relaxed controls) *Assume that for every $\eta \in \mathcal{R}$ and for all $t \in [0,T]$. The functions Φ and Ψ are convex. The* *function* $(x^\mu, y^\mu, z^\mu, Z^\mu) \to \mathcal{H}\left(t, x_t^\mu, y_t^\mu, z_t^\mu, Z^\mu, p_t^\mu, P_t^\mu, q_t^\mu, R_t^\mu, \mu_t\right)$, *is concave, the functional* $(x^\mu, y^\mu, z^\mu, Z^\mu) \to \mathcal{H}\left(t, x_t^\mu, y_t^\mu, z_t^\mu, Z^\mu, p_t^\mu, P_t^\mu, q_t^\mu, R_t^\mu, \mu_t\right)$ *is linear in* μ. *Then* μ *is an optimal control of the problem* (2.4)–(2.6) *if it satisfies* (4.5).

Proof Let μ be an arbitrary relaxed control (candidate to be optimal). For any admissible relaxed control η, we have

$$\begin{aligned}&\mathcal{J}(\mu) - \mathcal{J}(\eta)\\ &\quad = \mathbb{E}\left[\Phi\left(x_T^\mu\right) - \Phi\left(x_T^\eta\right)\right] + \mathbb{E}\left[\Psi\left(y_0^\mu\right) - \Psi\left(y_0^\eta\right)\right]\\ &\qquad + \mathbb{E}\int_0^T \left[\int_U l\left(x^\mu, y^\mu, z^\mu, Z^\mu, a\right)\mu(\mathrm{d}a) - \int_U l\left(t, x^\eta, y^\eta, z_t^\eta, Z_t^\eta, a\right)\eta(da)\right]\mathrm{d}t.\end{aligned}$$

Since Φ and Ψ are convex, we have

$$\begin{aligned}&\mathcal{J}(\mu) - \mathcal{J}(\eta)\\ &\quad \le \mathbb{E}\left[p_T^\mu\left(x_T^\mu - x_T^\eta\right)\right] + \mathbb{E}\left[q_0^\mu\left(y_0^\mu - y_0^\eta\right)\right]\\ &\qquad + \mathbb{E}\int_0^T \left[\int_U l\left(t, x^\mu, y^\mu, z_t^\mu, Z_t^\mu, a\right)\mu(\mathrm{d}a) - \int_U l\left(t, x^\eta, y^\eta, z_t^\eta, Z_t^\eta, a\right)\eta(\mathrm{d}a)\right]\mathrm{d}t.\end{aligned}$$

By applying Itô's formula respectively to $p_t^\mu\left(x_t^\mu - x_t^\eta\right)$ and $q_t^\mu\left(y_t^\mu - y_t^\eta\right)$, and using the Notation 2.7, by take expectation, we have

$$\begin{aligned}\mathcal{J}(\mu) - \mathcal{J}(\eta) \le\ & \mathbb{E}\int_0^T \left[\int_U l\left(t, \Gamma_t^\mu, a\right)\mu(\mathrm{d}a) - \int_U l\left(t, \Gamma_t^\eta, a\right)\eta(\mathrm{d}a)\right]\mathrm{d}t\\ &+ \mathbb{E}\int_0^T p_t^\mu\left[\int_U b\left(t, \Gamma_t^\mu, a\right)\mu(\mathrm{d}a) - \int_U b\left(t, \Gamma_t^\eta, a\right)\eta(\mathrm{d}a)\right]\mathrm{d}t\\ &+ \mathbb{E}\int_0^T P_t^\mu\left[\int_U \sigma\left(t, \Gamma_t^\mu, a\right)\mu(\mathrm{d}a) - \int_U \sigma\left(t, \Gamma_t^\eta, a\right)\eta(\mathrm{d}a)\right]\mathrm{d}t\\ &+ \mathbb{E}\int_0^T q_t^\mu\left[\int_U f\left(t, \Gamma_t^\mu, a\right)\mu(\mathrm{d}a) - \int_U f\left(t, \Gamma_t^\eta, a\right)\eta(\mathrm{d}a)\right]\mathrm{d}t\\ &+ \mathbb{E}\int_0^T R_t^\mu\left[\int_U g\left(t, \Gamma_t^\mu, a\right)\mu(\mathrm{d}a) - \int_U g\left(t, \Gamma_t^\eta, a\right)\eta(\mathrm{d}a)\right]\mathrm{d}t\\ &- \mathbb{E}\int_0^T \int_U \mathcal{H}_x\left(t, \Gamma_t^\mu, p_t^\mu, P_t^\mu, q_t^\mu, R_t^\mu, a\right)\mu(\mathrm{d}a)\left(x_t^\mu - x_t^\eta\right)\mathrm{d}t\\ &- \mathbb{E}\int_0^T \int_U \mathcal{H}_y\left(t, \Gamma_t^\mu, p_t^\mu, P_t^\mu, q_t^\mu, R_t^\mu, a\right)\mu(\mathrm{d}a)\left(y_t^\mu - y_t^\eta\right)\mathrm{d}t\\ &- \mathbb{E}\int_0^T \int_U \mathcal{H}_z\left(t, \Gamma_t^\mu, p_t^\mu, P_t^\mu, q_t^\mu, R_t^\mu, a\right)\mu(\mathrm{d}a)\left(z_t^\mu - z_t^\eta\right)\mathrm{d}t\\ &- \mathbb{E}\int_0^T \int_U \mathcal{H}_Z\left(t, \Gamma_t^\mu, p_t^\mu, P_t^\mu, q_t^\mu, R_t^\mu, a\right)\mu(\mathrm{d}a)\left(Z_t^\mu - z_t^\eta\right)\mathrm{d}t,\end{aligned}$$

replacing $\mathcal{H}$ by its value defined above, we get

$$\begin{aligned}\mathcal{J}(\mu)-\mathcal{J}(\eta) \leq \mathbb{E}\int_0^T \Big[\int_U \mathcal{H}^\mu\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a) \\ -\int_U \mathcal{H}^\eta\left(t,\Gamma_t^\eta,p_t^\eta,P_t^\eta,q_t^\eta,R_t^\eta,a\right)\eta(\mathrm{d}a)\Big]\mathrm{d}t \\ -\mathbb{E}\int_0^T\int_U \mathcal{H}_x\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(x_t^\mu-x_t^\eta\right)\mathrm{d}t \\ -\mathbb{E}\int_0^T\int_U \mathcal{H}_y\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(y_t^\mu-y_t^\eta\right)\mathrm{d}t \\ -\mathbb{E}\int_0^T\int_U \mathcal{H}_z\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(z_t^\mu-z_t^\eta\right)\mathrm{d}t \\ -\mathbb{E}\int_0^T\int_U \mathcal{H}_Z\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(Z_t^\mu-Z_t^\eta\right)\mathrm{d}t.\end{aligned}$$

By the concavity of $\mathcal{H}$ in $(x; y, z, Z)$, that

$$\begin{aligned}&\Big[\int_U \mathcal{H}^\mu\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a) \\ &\quad -\int_U \mathcal{H}^\eta\left(t,\Gamma_t^\eta,p_t^\eta,P_t^\eta,q_t^\eta,R_t^\eta,a\right)\eta(\mathrm{d}a)\Big] \\ &\leq \int_U \mathcal{H}_x\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(x_t^\mu-x_t^\eta\right) \\ &\quad +\int_U \mathcal{H}_y\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(y_t^\mu-y_t^\eta\right) \\ &\quad +\mathbb{E}\int_0^T\int_U \mathcal{H}_z\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(z_t^\mu-z_t^\eta\right) \\ &\quad +\mathbb{E}\int_0^T\int_U \mathcal{H}_Z\left(t,\Gamma_t^\mu,p_t^\mu,P_t^\mu,q_t^\mu,R_t^\mu,a\right)\mu(\mathrm{d}a)\left(Z_t^\mu-Z_t^\eta\right).\end{aligned}$$

This implies that

$$\mathcal{J}(\mu)-\mathcal{J}(\eta) \leq 0.$$

The theorem is proved.. □

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