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Convergence of the Generalized Kähler-Ricci Flow

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Abstract In this paper, we consider the convergence of the generalized Kähler-Ricci flow with semi-positive twisted form θ on Kähler manifold M. We give detailed proofs of the uniform Sobolev inequality and some uniform estimates for the metric potential and the generalized Ricci potential along the flow. Then assuming that there exists a generalized Kähler-Einstein metric, if the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, we prove that the generalized Kähler-Ricci flow must converge in C^∞ topology to a generalized Kähler-Einstein metric exponentially fast, where we get the exponential decay without using the Futaki invariant.

Keywords Complex Monge-Ampère equation · Generalized Kähler-Einstein metric · Sobolev inequality · Moser-Trudinger type inequality

Mathematics Subject Classification 53C55 · 32W20

1 Introduction

Let M be a compact complex manifold with Kähler class $[\omega_0]$. To find a Kähler-Einstein metric in a given Kähler class is an important problem in Kähler geometry,

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that is, when $2\pi c_1(M) = k[\omega_0]$, whether there exists a unique Kähler metric $\omega \in [\omega_0]$, such that $Ric(\omega) = k\omega$. One approach to this problem is the continuous method, see the works of Aubin and Yau [1,26]. The other approach is the Kähler-Ricci flow. The Kähler-Ricci flow was first used by Cao in [3] to give a parabolic proof of Calabi-Yau theorem. In recent years, the convergence of Kähler-Ricci flow has become the main object of geometry analysis. H.D. Cao, B.L. Chen, X.X. Chen, S. Donaldson, G. Perelman, D.H. Phong, J. Song, G. Tian, X.H. Zhu, X.P. Zhu, and others have done substantial work on this problem (see [4,5,7,12,15,17,18,22,24] etc.).

When the Kähler class is not proportional to the first Chern class, i.e., $2\pi c_1(M) - k[\omega_0] = [\alpha] \neq 0$. Fixing a closed $(1, 1)$-form $\theta \in [\alpha]$, we can consider the following generalized Kähler-Einstein equation

$$Ric(\omega) = k\omega + \theta, \tag{1.1}$$

which was introduced by Song and Tian in [22]. The generalized Kähler-Ricci flow

$$\frac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) + k\omega(t) + \theta \tag{1.2}$$

with initial metric ω_0 is studied, respectively, by the first author in [14], Collins and Székelyhidi in [8]. When the constant $k \leq 0$, by Cao's arguments for the parabolic Complex Monge-Ampère equation in [3], the long-time solution of the generalized Kähler-Ricci flow must converge to a generalized Käler-Einstein metric. When $k > 0$, there again exist obstructions. In the Kähler case, Perelman claims that, when a Kähler-Einstein metric exists, the Kähler-Ricci flow must converge to it. Tian and Zhu have proved this claim in [24] and [25]. In [14], we have obtained uniform Perelman's estimates, noncollapsing theorem and the asymptotic behavior of the generalized Ricci potential along the generalized Kähler-Ricci flow. Here, we study the convergence of the generalized Kähler-Ricci flow in the case of $k > 0$ and $\theta \geq 0$. Collins and Székelyhidi considered this problem in [8] builds on the work of Tian and Zhu in [25]. In this paper, we consider this problem by different methods, which making use of the Moser-Trudinger type inequality to get the C^0 estimate for the metric potential. Then we prove that the generalized Kähler-Ricci flow must converge in C^∞ topology to a generalized Kähler-Einstein metric exponentially fast, we get the exponential decay without using the Futaki invariant. For convenience, we let $g(t)$ be the metric corresponding to form $\omega(t)$, and denote $\frac{\omega^n(t)}{n!}$ by dV_t. Without loss of generality, we assume $k = 1$ in the rest of this paper.

The long-time existence of the flow (1.1) follows from the standard arguments for the parabolic complex Monge-Ampére equation. Since the generalized Kähler-Ricci flow preserves the Kähler class, we can write this flow as the parabolic Monge-Ampére equation on the potentials:

$$\frac{\partial \phi}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega_0^n} + \phi + u(0), \tag{1.3}$$

where $u(0)$ is the generalized Ricci potential with respect to the metric ω_0. Let $u(t)$ be the generalized Ricci potential with normalization $\frac{1}{V}\int_M e^{-u(t)}dV_t = 1$, then we have

$$\sqrt{-1}\partial\overline{\partial}\dot{\phi} = -Ric(\omega(t)) + \omega(t) + \theta = \sqrt{-1}\partial\overline{\partial}u.$$

From this equation we see ϕ evolves by

$$\dot{\phi} = u(t) + c(t) \tag{1.4}$$

for $c(t)$ depending only on time. We can use the function $c(t)$ to adjust the initial valve $\phi(0)$. We shall assume that

$$\phi(0) = c(0) = \int_0^{+\infty} e^{-t} \parallel \nabla\dot{\phi} \parallel_{L^2}^2 \mathrm{dt} - \frac{1}{V}\int_M u(0)\mathrm{d}V_0. \tag{1.5}$$

We note that the quantity $\int_0^{+\infty} e^{-t} \parallel \nabla\dot{\phi} \parallel_{L^2}^2 \mathrm{dt}$ (first written down in [6]; see also [13]) is finite and independent of the choice of initial condition. The constant $c(0)$ plays an important role in proving convergence of the flow. By this constant, a uniform bound of $\dot{\phi}$ can be obtained, see [16]. In this paper, we get the following theorem.

Theorem 1.1 *Let (M, ω_0) be a compact Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - [\omega_0]$ be a real closed semipositive $(1, 1)$-form. Assuming that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, if there exists a generalized Kähler-Einstein metric in $[\omega_0]$, then the generalized Kähler-Ricci flow (1.2) converges to a generalized Kähler-Einstein metric exponentially fast.*

Remark 1.2 Here, by the assumption in Theorem 1.1, we can have a Moser-Trudinger type inequality from the arguments in [29], and then we get the C^∞ estimates of the metric potential along the flow (1.3) proved in the following sections. Hence we know that the first eigenvalue of $\overline{\partial}$ on TM is bounded from 0 uniformly. When $\theta = 0$, Phone and Sturm have proved that $\omega(t)$ converge to a Kähler-Einstein metric exponentially fast in [18] by the fact that the Futaki invariant vanishing when $\mathcal{V}_{\theta,\omega_0}$ is bounded from below. But when $\theta \neq 0$, we can not get the corresponding Futaki invariant, so we need another method to get the convergence of $\omega(t)$ along the flow (1.2).

In fact, this paper is my master's thesis [14] of follow-up, we want to generalize the Futaki invariant to the generalized Kähler-Ricci flow case so that we can get the exponential convergence of the metric $\omega(t)$ along the flow (1.2) by the vanishing Futaki invariant when $\mathcal{V}_{\theta,\omega_0}$ is bounded from below, but we did not get a suitable generalized Futaki invariant, so this paper lasting for a while until we get the method used in Sect. 5.

Remark 1.3 In fact, the generalized Kähler-Einstein metric obtained from the convergence of flow (1.2) is the given one. In [21], Stoppa discussed the twisted cscK equation, that is, finding a metric $\omega \in [\omega_0]$ such that

$$R(\omega) - tr_{\omega}\theta = \tilde{S}_{\theta}, \tag{1.6}$$

where θ is a real closed semipositive $(1, 1)$-form and $\tilde{S}_{\theta}$ is a constant. By the definition of the twisted Mabuchi $\mathcal{K}$-energy, it is easy to check that the second derivative along a path $\varphi_t \in \mathcal{H}_{\omega_0}$ is given by

$$\frac{V}{n!}\frac{\mathrm{d}^2}{\mathrm{dt}^2}\mathcal{V}_{\theta,\omega_0}(\varphi_t) = \|\bar{\partial}\nabla^{1,0}_{\omega_{\varphi_t}}\dot{\varphi}_t\|^2_{\omega_{\varphi_t}} + (\partial\dot{\varphi}_t \wedge \bar{\partial}\dot{\varphi}_t, \theta)_{\omega_{\varphi_t}}$$
$$- \int_M \left(\ddot{\varphi}_t - \frac{1}{2}\|\nabla^{1,0}_{\omega_{\varphi_t}}\dot{\varphi}_t\|^2_{\omega_{\varphi_t}}\right)\left(R(\omega_t) - tr_{\omega_{\varphi_t}}\theta - \bar{S}_{\theta}\right)dV_{\varphi_t}.$$

If either the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, $\mathcal{V}_{\theta,\omega_0}$ is strictly convex along geodesics in $\mathcal{H}_{\omega_0}$. Then, the results of Chen and Tian [6] on the regularity of weak geodesics imply uniqueness of solution of the twisted cscK equation. The above facts were pointed out by Stoppa in [21].

The outline of this paper is as follows. In Sect. 2, by the heat kernel theory and the arguments in Ye [27] and Zhang [28], we prove the uniform Sobolev inequality along the generalized Kähler-Ricci flow (1.2), which will play an important role when we get the C^0 estimate for the metric potential. Then, we get the regularity estimates for the flow (1.3) with the initial value (1.5) in Sect. 3, such as Laplacian C^2 estimate and Calabi's C^3 estimate by the maximum principle. In Sect. 4, we prove the $W^{k,2}$ estimates for the generalized Ricci potential which builds on the work of [18]. At last, we prove the convergence of the generalized Kähler-Ricci flow with exponential delay under the assumption in Theorem 1.1.

2 The Uniform Sobolev Inequality

In this section, we prove a uniform Sobolev inequality along the generalized Kähler-Ricci flow. It based on the work of [27] and [28].

Proposition 2.1 *Let (M, ω_0) be a compact Kähler manifold with complex dimension $n \geq 2$ and $g(t)$ be a solution of the generalized Kähler-Ricci flow. Then there exist positive constants A and B depending only on the initial metric $g(0)$ and θ, such that, for all $v \in W^{1,2}(M, g(t))$ and $t > 0$, it holds*

$$\left(\int_M v^{\frac{2n}{n-1}}\mathrm{d}V_t\right)^{\frac{n-1}{n}} \leq A\int_M \left(|\nabla v|^2 + \frac{1}{4}(R - tr_{g(t)}\theta)v^2\right)\mathrm{d}V_t + B\int_M v^2\mathrm{d}V_t.$$

Then by Perelman's estimates for the flow (1.2) in [14], we have

$$\left(\int_M v^{\frac{2n}{n-1}}\mathrm{d}V_t\right)^{\frac{n-1}{n}} \leq C\int_M |\nabla v|^2\,\mathrm{d}V_t + C\int_M v^2\mathrm{d}V_t, \tag{2.1}$$

where C depending only on $g(0)$ and θ.

We first recall the generalized $\mathcal{W}$ functional

$$\mathcal{W}_\theta(g, f, \tau) = \int_M e^{-f}\tau^{-n}\left(\tau\left(R - tr_g\theta + |\nabla f|^2\right) + f\right)\mathrm{d}V_g \tag{2.2}$$

and μ functional

$$\mu_\theta(g, \tau) = \inf\left\{\mathcal{W}_\theta(g, f, \tau)|f \in C^\infty(M), \frac{1}{V}\int_M e^{-f}\tau^{-n}\mathrm{d}V_g = 1\right\} \tag{2.3}$$

in [14], *where g is a Kähler metric, τ is a positive scale parameter and n is the complex dimension of the Kähler manifold.*

Let $v = e^{-\frac{f}{2}}\tau^{-\frac{n}{2}}V^{-\frac{1}{2}}$, we have $\|v\|^2_{2(g)} = \int_M v^2\mathrm{d}V_g = 1$, then the generalized $\mathcal{W}$ functional and μ functional can be written as

$$\mathcal{W}_\theta = V\int_M\left(\tau((R - tr_g\theta)v^2 + 4|\nabla v|^2) - v^2\log v^2\right)\mathrm{d}V_g$$
$$-V\log V - nV\log\tau, \tag{2.4}$$
$$\mu_\theta(g, \tau) = \inf_{\|v\|_{2(g)}=1} V\int_M\left(\tau((R - tr_g\theta)v^2 + 4|\nabla v|^2) - v^2\log v^2\right)\mathrm{d}V_g$$
$$-V\log V - nV\log\tau. \tag{2.5}$$

Lemma 2.2 (Ye [27] or Zhang [28]) *Assume that (M, g) is a compact Riemann manifold with dimension n, then for each $\alpha > 0$ and all $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$, there holds*

$$\int_M u^2\log u^2\mathrm{d}V \le \frac{n\alpha\mathcal{C}_S(M, g)^2}{2}\int_M|\nabla u|^2\mathrm{d}V - \frac{n}{2}\log\alpha + \frac{n}{2}(\log 2 + \alpha V^{-\frac{2}{n}} - 1),$$

where $\mathcal{C}_S(M, g)$ is the Sobolev constant of (M, g).

Lemma 2.3 (Liu [14]) *If τ satisfies the following equality*

$$\frac{\partial\tau}{\partial t} = \tau - 1.$$

Then $\mu_\theta(g, \tau)$ is nondecreasing along the generalized Kähler-Ricci flow.

Then we prove the Sobolev inequality by using the heat kernel.

Proof of Proposition 2.1 Firstly, for any t, we let $\tau(s) = 1 - e^{-t}(1-\varepsilon^2)e^s$, $\varepsilon \in (0, 1)$. By Lemma 2.3, $\mu_\theta(g(t), \tau(t)) \ge \mu_\theta(g(0), \tau(0))$, that is

$$\inf_{\|v\|_{2(g(t))}=1}\int_M\left(\varepsilon^2((R_{g(t)}-tr_{g(t)}\theta)v^2+4|\nabla v|^2_{g(t)})-v^2\log v^2\right)\mathrm{d}V_t-2n\log\varepsilon$$
$$\geq\inf_{\|v_0\|_{2(g(0))}=1}\int_M\left(\sigma(t)\left((R_{g(0)}-tr_{g(0)}\theta)v_0^2+4|\nabla v_0|^2_{g(0)}\right)\right.$$
$$\left.-v_0^2\log v_0^2\right)\mathrm{d}V_0-n\log\sigma(t),$$

where $\sigma(t)=1-e^{-t}(1-\varepsilon^2)$. Since $\varepsilon^2\leq\sigma(t)\leq 1$,

$$\int_M\sigma(t)((R_{g(0)}-tr_{g(0)}\theta)v_0^2\mathrm{d}V_0\geq-\max(R-tr\theta)^-(\cdot,0).$$

From Lemma 2.2, let α satisfying $4\sigma(t)=n\alpha\mathcal{C}_S(M,g)^2$, then

$$\int_M 4\sigma(t)|\nabla v_0|^2_{g(0)}\mathrm{d}V_0-\int_M v_0^2\log v_0^2\mathrm{d}V_0\geq n\log\sigma(t)-L_0,$$

where $L_0=n\log n\mathcal{C}_S(M,g(0))^2-n\log 2-n+\frac{4}{\mathcal{C}_S(M,g(0))^2}V^{-\frac{1}{n}}$. Combining the above three inequalities, we have

$$\int_M v^2\log v^2\mathrm{d}V_t\leq\int_M\varepsilon^2\left(\left(R_{g(t)}-tr_{g(t)}\theta\right)v^2+4|\nabla v|^2_{g(t)}\right)\mathrm{d}V_t$$
$$-2n\log\varepsilon+L_0+\max(R-tr\theta)^-(\cdot,0). \tag{2.6}$$

Second, fix a time t_0 during the generalized K$\ddot{a}$hler-Ricci flow. We shall show the upper bound of short time heat kernel for the fundamental solution of equation

$$\triangle u(x,t)-\frac{1}{4}(R_{g(t_0)}-tr_{g(t_0)}\theta)u(x,t)-\frac{\partial}{\partial t}u(x,t)=0 \tag{2.7}$$

under the fixed metric $g(t_0)$.

Let u be a positive solution of Eq. (2.7). Given $T\in(0,1]$ and $t\in(0,T]$, we take $p(t)=\frac{T}{T-t}$, so that $p(0)=1$ and $p(T)=\infty$. By direct computation

$$\frac{\partial}{\partial t}\|u\|_{p(t)}=-\frac{-p'(t)}{p^2(t)}\|u\|_{p(t)}\log\int_M u^{p(t)}(x,t)\mathrm{d}V_{t_0}$$
$$+\frac{1}{p(t)}\left(\int_M u^{p(t)}(x,t)\mathrm{d}V_{t_0}\right)^{\frac{1}{p(t)}-1}$$
$$\times\left[\int_M u^{p(t)}(\log u)p'(t)\mathrm{d}V_{t_0}\right.$$
$$\left.+p(t)\int_M u^{p(t)-1}\left(\Delta u-\frac{1}{4}(R_{g(t_0)}-tr_{g(t_0)}\theta)u\right)\mathrm{d}V_{t_0}\right].$$

Integrating by parts and multiplying both side by $p^2(t)\|u\|_{p(t)}^{p(t)}$,

$$\begin{aligned} p^2(t)\|u\|_{p(t)}^{p(t)}\frac{\partial}{\partial t}\|u\|_{p(t)} &= -p'(t)\|u\|_{p(t)}^{p(t)+1}\log\int_M u^{p(t)}\mathrm{d}V_{t_0} \\ &+ p(t)p'(t)\|u\|_{p(t)}\int_M u^{p(t)}\log u\mathrm{d}V_{t_0} \\ &- p^2(t)(p(t)-1)\|u\|_{p(t)}\int_M u^{p(t)-2}|\nabla u|^2\mathrm{d}V_{t_0} \\ &- p^2(t)\|u\|_{p(t)}\int_M \frac{1}{4}(R_{g(t_0)} - tr_{g(t_0)}\theta)u^{p(t)}\mathrm{d}V_{t_0}. \end{aligned}$$

Dividing both sides by $\|u\|_{p(t)}$, we obtain

$$\begin{aligned} p^2(t)\|u\|_{p(t)}^{p(t)}\frac{\partial}{\partial t}\log\|u\|_{p(t)} &= -p'(t)\|u\|_{p(t)}^{p(t)}\log\int_M u^{p(t)}\mathrm{d}V_{t_0} \\ &+ p(t)p'(t)\int_M u^{p(t)}\log u\mathrm{d}V_{t_0} \\ &- p^2(t)\int_M \frac{1}{4}(R_{g(t_0)} - tr_{g(t_0)}\theta)(u^{\frac{p(t)}{2}})^2\mathrm{d}V_{t_0} \\ &- 4(p(t)-1)\int_M |\nabla u^{\frac{p(t)}{2}}|^2\mathrm{d}V_{t_0}. \end{aligned}$$

Let $v = \frac{u^{\frac{p(t)}{2}}}{\|u^{\frac{p(t)}{2}}\|_2}$, after dividing by $\|u\|_{p(t)}^{p(t)} = \|u^{\frac{p(t)}{2}}\|_2^2$, we get

$$\begin{aligned} p^2(t)\frac{\partial}{\partial t}\log\|u\|_{p(t)} &= p'(t)\int_M v^2\log v^2\mathrm{d}V_{t_0} - 4(p(t)-1) \\ &\times\int_M\left(|\nabla v|^2 + \frac{1}{4}(R_{g(t_0)} - tr_{g(t_0)}\theta)v^2\right)\mathrm{d}V_{t_0} \\ &+ \Big(4(p(t)-1) - p^2(t)\Big)\int_M \frac{1}{4}(R_{g(t_0)} - tr_{g(t_0)}\theta)v^2\mathrm{d}V_{t_0}. \end{aligned}$$

Through computing,

$$p'(t) = \frac{T}{(T-t)^2}, \qquad \frac{4(p(t)-1)}{p'(t)} = \frac{4t(T-t)}{T} \le T \le 1,$$
$$-T \le \frac{4(p(t)-1) - p^2(t)}{p'(t)} = \frac{4t(T-t) - T^2}{T} \le 0.$$

We conclude that

$$p^2(t)\frac{\partial}{\partial t}\log\|u\|_{p(t)} \le p'(t)\left(-\frac{4(p(t)-1)}{p'(t)}\int_M\left(|\nabla v|^2+\frac{1}{4}(R_{g(t_0)}-tr_{g(t_0)}\theta)v^2\right)\mathrm{d}V_{t_0}\right.$$
$$\left.+\int_M v^2\log v^2\mathrm{d}V_{t_0}+T\max\left(R_{g(t_0)}-tr_{g(t_0)}\theta\right)^-\right)$$
$$\le p'(t)\left(-2n\log\left(\frac{p(t)-1}{p'(t)}\right)^{\frac{1}{2}}+L_0+2\max\left(R_{g(0)}-tr_{g(0)}\theta\right)^-\right),$$

where we take ε satisfying $\varepsilon^2=\frac{p(t)-1}{p'(t)}\le\frac{T}{4}\le\frac{1}{4}$ and use the inequality (2.6). Note that $\frac{p'(t)}{p^2(t)}=\frac{1}{T}$ and $\frac{4(p(t)-1)}{p'(t)}=\frac{4t(T-t)}{T}$, hence

$$\frac{\partial}{\partial t}\log\|u\|_{p(t)}\le\frac{1}{T}\left(-n\log\frac{t(T-t)}{T}+L_0+2\max\left(R_{g(0)}-tr_{g(0)}\theta\right)^-\right).$$

Integrating from $t=0$ to $t=T$, we get

$$\log\frac{\|u(\cdot,T)\|_\infty}{\|u(\cdot,0)\|_1}\le -n\log T+L+2\max\left(R_{g(0)}-tr_{g(0)}\theta\right)^-,$$

where $L=L_0+2n$.

Since $u(x,T)=\int_M P(x,y,T)u(y,0)dV_{t_0}(y)$, where P is the heat kernel of equation (2.7), so

$$P(x,y,T)\le\frac{\exp\left(L+2\max\left(R_{g(0)}-tr_{g(0)}\theta\right)^-\right)}{T^n}:=\frac{\Lambda}{T^n}. \tag{2.8}$$

At last, we show that the upper bound of short time heat kernel implies the uniform Sobolev inequality. Let t_0 be a fixed time during the generalized Kähler-Ricci flow, $F=\max\left(R_{g(0)}-tr_{g(0)}\theta\right)^-$, P_F be the heat kernel of operator $-\triangle+\Psi$, where $\Psi=\frac{1}{4}(R_{g(t_0)}-tr_{g(t_0)}\theta)+F+1\ge 1$.

We are now extending estimate (2.8) for every positive time t. For $y\in M$, consider $f(x,t)=P_F(x,y,1+t)$. We have $\left(\frac{\partial}{\partial t}-\triangle\right)f(x,t)=-\Psi f(x,t)$ and $f(x,0)\le\Lambda$. So we conclude that $f(x,y,t)\le e^{-t}\Lambda$ by the maximum principle, hence $P_F(x,y,1+t)\le e^{-t}\Lambda\le\frac{\Lambda e(n+1)!}{(t+1)^n}$. So

$$P_F(x,y,t)\le Ct^{-n}, \tag{2.9}$$

for every time t, where C depends only on $g(0)$, θ and n.

By Hölder inequality, for any $f \in L^2(M)$,we have

$$\left|\int_M P_F(x,y,t)f(y)dV_{t_0}(y)\right| \le \left(\int_M P_F^2(x,y,t)dV_{t_0}(y)\right)^{\frac{1}{2}} \|f\|_2$$
$$\le \left(\int_M P_F(x,y,t)dV_{t_0}(y)Ct^{-n}\right)^{\frac{1}{2}} \|f\|_2$$
$$\le C^{\frac{1}{2}}t^{-\frac{n}{2}}\|f\|_2.$$

Then the uniform Sobolev inequality follows from the following theorem:

Lemma 2.4 (Davies [9]) *Let $\Psi \ge 0$, $\mu > 2$ and $\tilde{C} > 0$. Assume that the inequality*

$$\|e^{-tH}u\|_\infty \le \tilde{C}t^{-\frac{\mu}{4}}\|u\|_2$$

holds true for each $t > 0$ and all $u \in L^2(M)$. Then the Sobolev inequality

$$\|u\|^2_{\frac{2\mu}{\mu-2}} \le C(\mu,\tilde{C})Q(u)$$

holds true for all $u \in W^{1,2}(M)$, where $Q(u) = \int_M(|\nabla u|^2 + \Psi u^2)\mathrm{d}V$ and the positive constant $C(\mu,\tilde{C})$ can be bounded from above in terms of upper bounds for $\tilde{C}$, μ and $\frac{1}{\mu-2}$.

Taking $\mu = 2n$, then Proposition 2.1 follows from the arbitrarity of t_0. □

3 The Prior Estimates for Metric Potential

In this section, we generalize the Laplacian C^2 estimate and Calabi's C^3 estimate along the flow (1.3) with initial value (1.5). For simplification, the geometric quantities at time $t = 0$ will be signed by symbol hat. In the following arguments, all indices are raised and lowered with respect to the metric $g(t)$ unless indicated explicitly otherwise. Adopting the arguments as that in [16], we have

Proposition 3.1 *Let M be a compact Kähler manifold with Kähler metric ω_0 satisfying $\omega_0 \in 2\pi c_1(M) - [\theta]$. For the generalized Kähler-Ricci flow defined as (1.3) with initial value (1.5), we have the priori estimates*

$$\sup_M \parallel \phi \parallel_{C^0} \le C_0 \Longleftrightarrow \sup_M \parallel \phi \parallel_{C^k} \le C_k, k \in \mathbb{N}^+. \tag{3.1}$$

Proof First, by the similar arguments in [16], we have

$$\sup_M \|\dot{\phi}\|_{C^0} \le C < \infty, \tag{3.2}$$

where constant C depends only on $g(0)$.

Let $h = h^{\alpha}{}_{\beta}$ defined by $h^{\alpha}{}_{\beta} = \hat{g}^{\alpha\bar{\lambda}} g_{\bar{\lambda}\beta}$ and $G = \log \frac{\omega_{\phi}^n}{\omega_0^n}$.

We first prove the Laplacian C^2 estimates. Through computing, we have

$$\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right) \log trh = \frac{1}{trh}\left\{\hat{\triangle}(G - \dot{\phi}) - \hat{R}\right\} - \frac{1}{trh} g^{p\bar{q}} g_{\bar{m}j} \hat{g}^{r\bar{m}} \hat{R}^{j}{}_{r\bar{q}p} + \left\{\frac{\hat{g}^{\gamma\bar{s}} \phi_{\gamma}{}^{t}{}_{p} \phi_{\bar{s}t}{}^{p}}{trh} - \frac{g^{\delta\bar{k}} \partial_{\bar{k}} trh \partial_{\delta} trh}{(trh)^2}\right\}. \tag{3.3}$$

Now, we let ϕ evolve by the generalized K$\ddot{a}$hler-Ricci flow (1.3). When we choose a locally holomorphic basis, which such that $(\hat{g}_{i\bar{i}})$ be identity and $(g_{i\bar{i}})$ be diagonal matrix. Since $(g_{i\bar{i}})$ is positive definite, so we have $g_{i\bar{i}} = 1 + \phi_{i\bar{i}} > 0$. Through computing respectively,

$$\frac{\hat{g}^{\gamma\bar{s}} \phi_{\gamma}{}^{t}{}_{p} \phi_{\bar{s}t}{}^{p}}{trh} - \frac{g^{\delta\bar{k}} \partial_{\bar{k}} trh \partial_{\delta} trh}{(trh)^2} \geq 0, \tag{3.4}$$

$$-g^{p\bar{q}} g_{\bar{m}j} \hat{g}^{r\bar{m}} \hat{R}^{j}{}_{r\bar{q}p} = -\sum_{i,j=1}^{n} \frac{1 + \phi_{i\bar{i}}}{1 + \phi_{j\bar{j}}} \hat{R}_{\bar{i}i\bar{j}j} \geq -C_0(trh) \sum_{j} \frac{1}{1 + \phi_{j\bar{j}}}, \tag{3.5}$$

$$\triangle\phi = \sum_{j=1}^{n} \frac{\phi_{j\bar{j}}}{1 + \phi_{j\bar{j}}} = n - \sum_{j=1}^{n} \frac{1}{1 + \phi_{j\bar{j}}}. \tag{3.6}$$

Putting (3.4), (3.5) and (3.6) into (3.3), we have

$$\begin{aligned} \left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right)(\log trh - A\phi) &\geq A\dot{\phi} - (1 + An) + (A - C_2) \sum_{j} \frac{1}{1 + \phi_{j\bar{j}}} \\ &\geq -C_3 + C_4 \sum_{j} \frac{1}{1 + \phi_{j\bar{j}}}, \end{aligned} \tag{3.7}$$

where we choose A sufficiently large such that $A - C_2 > 0$. C_3 and C_4 depend only on $g(0)$.

Let $[0, T]$ be any time interval, and (z_0, t_0) be a point in $M \times [0, T]$, where the function $\log trh - A\phi$ attains its maximum. If this point is not at time $t = 0$, then the left handside of (3.7) is non-positive, and we obtain the estimates

$$\frac{1}{1 + \phi_{i\bar{i}}} \leq C_5, \quad 1 \leq i \leq n. \tag{3.8}$$

Since, at point (z_0, t_0), we have

$$\begin{aligned} trh &= trh\left(\frac{\det \hat{g}_{i\bar{j}}}{\det g_{i\bar{j}}}\right) \exp(\dot{\phi} - \phi - \hat{u}) \\ &= \exp(\dot{\phi} - \phi - \hat{u}) \sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{1 + \phi_{j\bar{j}}} \leq C_6, \end{aligned} \tag{3.9}$$

we get the estimate

$$\sup_{M\times[0,T]} trh \le e^{A\|\phi\|_{C^0}} \exp(\log trh - A\phi)(z_0,t_0) \le C_7. \tag{3.10}$$

Since T is arbitrary, this establishes the boundness of the trace of $g_{\bar{k}j}$, combining the matrix is positive, the proof of Laplacian C^2 estimate is completed. At the same time, by the estimates (3.8), we know that there exist constants A and B, such that

$$0 < A \le 1 + \phi_{i\bar{i}} \le B, \quad 1 \le i \le n, \tag{3.11}$$

which implies that the metrics g and $g(0)$ are uniform equivalent.

Next, we prove the Calabi's C^3 estimate. Let S be defined by

$$S = g^{j\bar{r}} g^{s\bar{k}} g^{m\bar{t}} \phi_{j\bar{k}m}\phi_{\bar{r}s\bar{t}}. \tag{3.12}$$

By direct calculation, we have

$$S = g^{m\bar{\gamma}} g_{\bar{\mu}\beta} g^{l\bar{\alpha}} (\nabla_m hh^{-1})^{\beta}_{l} \overline{(\nabla_\gamma hh^{-1})^{\mu}_{\alpha}} = |\nabla hh^{-1}|^2. \tag{3.13}$$

As computing in [16], we have

$$\begin{aligned}\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right) S = {} & |\nabla hh^{-1}|^2 + g^{m\bar{\gamma}} g_{\bar{\mu}\beta} g^{l\bar{\alpha}} \big((-\nabla^{\bar{q}} \hat{R}^{\beta}_{l\bar{q}m} - \theta^{\beta}_{l,m}) \overline{(\nabla_\gamma hh^{-1})^{\mu}_{\alpha}} \\ & + (\nabla_m hh^{-1})^{\beta}_{l} \overline{(-\nabla^{\bar{q}} \hat{R}^{\mu}_{\alpha\bar{q}\gamma} - \theta^{\mu}_{\alpha,j})}\big) + |\nabla(\nabla hh^{-1})|^2 + |\bar{\nabla}(\nabla hh^{-1})|^2 \\ & + (\nabla_m hh^{-1})^{\beta}_{l} \overline{(\nabla_\gamma hh^{-1})^{\mu}_{\alpha}} (\theta^{m\bar{\gamma}} g_{\bar{\mu}\beta} g^{l\bar{\alpha}} - g^{m\bar{\gamma}} \theta_{\bar{\mu}\beta} g^{l\bar{\alpha}} + g^{m\bar{\gamma}} g_{\bar{\mu}\beta} \theta^{l\bar{\alpha}}).\end{aligned} \tag{3.14}$$

Because $\hat{R}^{\beta}_{l\bar{q}m}$ is a fixed tensor and θ is a fixed form, we obtain immediately

$$\nabla_m \theta_{l\bar{q}} = \hat{\nabla}_m \theta_{l\bar{q}} - X^{s}_{ml} \theta_{s\bar{q}}, \tag{3.15}$$

$$\nabla_p \hat{R}^{\beta}_{l\bar{q}m} = \hat{\nabla}_p \hat{R}^{\beta}_{l\bar{q}m} + X^{\beta}_{ps} \hat{R}^{s}_{l\bar{q}m} - X^{s}_{pl} \hat{R}^{\beta}_{s\bar{q}m} - X^{s}_{pm} \hat{R}^{\beta}_{l\bar{q}s}, \tag{3.16}$$

where we denote $X^{k}_{il} = (\nabla_i hh^{-1})^{k}_{l}$. Hence we have

$$\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right) S \ge -C_1 S - C_2. \tag{3.17}$$

From the Laplacian C^2 estimate, we have

$$\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right) (S + Ltrh) \ge (LA - C_1)S - (LC_3 + C_2). \tag{3.18}$$

When choosing L large enough such that $LA - C_1 > 0$, we conclude that

$$\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right)(S + Ltrh) \geq C_4 S - C_5. \tag{3.19}$$

Then by maximal principle, we get the uniform bound of S.

In order to apply the standard parabolic estimates to obtain the higher order estimates, we require a derivative bound of $g(t)$ in the t-direction. Given the estimates proved so far, it is sufficient to bound $|Ric(g(t))|$. The evolution of the Ricci curvature along the generalized Kähler-Ricci flow is given by

$$\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right)(R_{k\bar{l}}) = -R_{p\bar{q}}R_{k\bar{l}}{}^{p\bar{q}} + R_{k\bar{q}}R^{\bar{q}}_{\bar{l}} + \nabla_{\bar{l}}\nabla_k\theta_{i\bar{j}}g^{i\bar{j}}. \tag{3.20}$$

Then given the estimates on trh and S, we have

$$\begin{aligned}\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right)|Ric| &= \frac{1}{|Ric|}\Big(-R_{p\bar{q}}R_{i\bar{j}}{}^{p\bar{q}}R^{i\bar{j}} + g^{i\bar{q}}g^{p\bar{l}}g^{k\bar{j}}\theta_{p\bar{q}}R_{k\bar{l}}R_{i\bar{j}}\\ &\quad + \nabla_{\bar{l}}\nabla_k\theta_{m\bar{n}}g^{m\bar{n}}R_{i\bar{j}}g^{i\bar{l}}g^{k\bar{j}} + |\nabla Ric|^2 + |Ric|^2 - |\nabla|Ric||^2\Big)\\ &\geq -C_6(|R_m|^2 + 1).\end{aligned} \tag{3.21}$$

At the same time, from the computations above for the evolution of S, there exist uniform constants C_7 and C_8 such that

$$\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right)S \geq C_7|R_m|^2 - C_8. \tag{3.22}$$

Then by applying the maximum principle to the quantity $|Ric| + \frac{C_6+1}{C_7}S$ we obtain the desired upper bound on $|Ric|$.

The remaining part of the proof is standard: Differentiating the equation (1.3) with respect to z^k, we get

$$\left(\triangle - \frac{\mathrm{d}}{\mathrm{dt}}\right)\left(\frac{\partial\phi}{\partial z^k}\right) = \hat{g}^{i\bar{j}}\frac{\partial\hat{g}_{i\bar{j}}}{\partial z^k} - g^{i\bar{j}}\frac{\partial\hat{g}_{i\bar{j}}}{\partial z^k} - k\frac{\partial\phi}{\partial z^k} - \frac{\partial\hat{u}}{\partial z^k}. \tag{3.23}$$

Then we know that the coefficients of operator $\triangle - \frac{\mathrm{d}}{\mathrm{dt}}$ are bounded in parabolic $C^{0,\alpha}$ norm and the right hand side of (3.23) also has estimates in parabolic $C^{0,\alpha}$ norm for all $0 < \alpha < 1$. Then the general theory of parabolic P.D.E. can be applied to give the uniform C^k estimates for all $k \in \mathbb{N}^+$. □

4 The $W^{k,2}$ Estimates for the Generalized Ricci Potential

In this section, we prove the $W^{k,2}$ estimates for the generalized Ricci potential, which plays an important role in the exponential convergence of the Kähler-Ricci flow.

Let's denote $\mathcal{H}_{\omega_0}$ to be the set of all smooth strictly ω_0-plurisubharmonic functions, i.e.,

$$\mathcal{H}_{\omega_0} = \left\{ \varphi \in C^\infty(M) : \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \right\}.$$

Through the work of Bando and Mabuchi [2], Ding [10], Donaldson [11],Tian [23] and others, it is well known that the Mabuchi $\mathcal{K}$-energy is very useful in Kähler geometry. Let's recall the following twisted Mabuchi $\mathcal{K}$-energy which was first introduced by Song and Tian in [22].

Definition 4.1 For every $(\varphi_0, \varphi_1) \in \mathcal{H}_{\omega_0} \times \mathcal{H}_{\omega_0}$, we define

$$\mathcal{M}_\theta(\varphi_0, \varphi_1) = -\frac{n!}{V} \int_0^1 \int_M \dot{\varphi}_t (R(\omega_{\varphi_t}) - tr_{\omega_{\varphi_t}}\theta - \bar{S}_\theta) \mathrm{d}V_t \mathrm{d}t, \tag{4.1}$$

where $\{\varphi_t | 0 \le t \le 1\}$ is an arbitrary piecewise smooth path in $\mathcal{H}_{\omega_0}$ such that $\varphi_t|_{t=0} = \varphi_0$ and $\varphi_t|_{t=1} = \varphi_1$, $R(\omega_{\varphi_t})$ is the scalar curvature of ω_{φ_t} , and $\bar{S}_\theta = \frac{1}{V}\int_M n(2\pi c_1(M) - [\theta]) \cup [\omega_0]^{n-1}$. For every $\varphi \in \mathcal{H}_{\omega_0}$, we define

$$\mathcal{V}_{\theta,\omega_0}(\varphi) = \mathcal{M}_\theta(0, \varphi). \tag{4.2}$$

Now we state and proof the $W^{k,2}$ estimates for the generalized Ricci potential, our arguments resemble those in [18].

Proposition 4.2 *Let M be a compact Kähler manifold with complex dimension n and $g(t)$ be a solution of the generalized Kähler-Ricci flow. Assuming that all covariant derivatives of fixed order (including order 0) of Riemann curvature and θ with respect to the metric $g(t)$ are uniformly bounded. If the twisted Mabuchi $\mathcal{K}$-energy is bounded from below on $\mathcal{H}_{\omega_0}$, then we have for any $s \ge 0$,*

$$\lim_{t\to+\infty} \| R_{i\bar{j}} - g_{i\bar{j}} - \theta_{i\bar{j}} \|_{(s)} = 0, \tag{4.3}$$

where $\|\cdot\|_{(s)}$ denotes the Sobolev norm of order s with respect to the metric $g(t)$.

Proof Define $Y(t) = \int_M |\nabla u|^2 \,\mathrm{d}V_t$, where u is the generalized Ricci potential with respect to metric $g(t)$. From Lemme 4.4 in [14], we know that $\int_0^{+\infty} Y(t)\mathrm{d}t < +\infty$, $\lim_{t\to+\infty} Y(t) = 0$, and

$$\begin{aligned}
&\int_0^{+\infty} \mathrm{d}t \int_M |\bar{\nabla}\nabla u|^2 \,\mathrm{d}V_t + \int_0^{+\infty} \mathrm{d}t \int_M |\nabla\nabla u|^2 \,\mathrm{d}V_t \\
&= Y(0) + (n+1)\int_0^{+\infty} Y(t)\mathrm{d}t - \int_0^{+\infty} \mathrm{d}t \int_M |\nabla u|^2 \left(R - tr_{g(t)}\theta\right) \mathrm{d}V_t \\
&\quad -\frac{1}{2}\int_0^{+\infty} \mathrm{d}t \int_M \theta(grad\, u, \mathcal{J} grad\, u)\mathrm{d}V_t,
\end{aligned}$$

so we can conclude that

$$\int_0^{+\infty} \mathrm{dt} \int_M |\bar{\nabla}\nabla u|^2 \,\mathrm{d}V_t + \int_0^{+\infty} \mathrm{dt} \int_M |\nabla\nabla u|^2 \,\mathrm{d}V_t < +\infty. \tag{4.4}$$

Instead of Bochner-Kodaira formulas for the complex Laplacian $\nabla^{\bar{p}}\nabla_{\bar{p}}$, it is simpler to use the real Laplacian $\triangle_{\mathbb{R}} = \triangle + \bar{\triangle}$, which gives at once

$$\begin{aligned}\triangle |\nabla^s\bar{\nabla}^r u|^2 = {} & \frac{1}{2}\triangle_{\mathbb{R}} u_{\bar{K}J} u_{K\bar{J}} + u_{\bar{K}J}\overline{\frac{1}{2}\triangle_{\mathbb{R}} u_{\bar{K}J}} \\ & + |\nabla u_{\bar{K}J}|^2 + |\bar{\nabla} u_{\bar{K}J}|^2,\end{aligned} \tag{4.5}$$

where $u_{\bar{K}J} = \nabla_{j_s}\cdots\nabla_{j_1}\nabla_{\bar{k}_r}\cdots\nabla_{\bar{k}_1}u$.

The time evolution of $|\nabla^s\bar{\nabla}^r u|^2$ is given by

$$\begin{aligned}\left(|\nabla^s\bar{\nabla}^r u|^2\right)^{\cdot} = {} & \sum_{\alpha=1}^{r}(R^{l_\alpha\bar{k}_\alpha} - \theta^{l_\alpha\bar{k}_\alpha})u_{\bar{K}J}u^{\bar{k}_1\cdots\ \cdots\bar{k}_r J}{}_{l_\alpha} - (u_{\bar{K}J})^{\cdot}u_{K\bar{J}} - u_{\bar{K}J}(u_{K\bar{J}})^{\cdot} \\ & + \sum_{\beta=1}^{s}(R^{j_\beta\bar{m}_\beta} - \theta^{j_\beta\bar{m}_\beta})u_{\bar{K}J}u^{\bar{K}j_1\cdots\ \cdots j_s}{}_{\bar{m}_\beta} - (r+s)|\nabla^s\bar{\nabla}^r u|^2 .\end{aligned} \tag{4.6}$$

By induction, we find that

$$\begin{aligned}(u_{\bar{K}J})^{\cdot} = {} & \frac{1}{2}\triangle_{\mathbb{R}}u_{\bar{K}J} + u_{\bar{K}J} - \frac{1}{2}\sum_{\alpha=1}^{r}R_{\bar{k}_\alpha}{}^{\bar{m}_\alpha}u_{\bar{k}_1\cdots\bar{m}_\alpha\cdots\bar{k}_r J} - \frac{1}{2}\sum_{\beta=1}^{s}R^{m_\beta}{}_{j_\beta}u_{\bar{K}j_1\cdots m_\beta\cdots j_s} \\ & + \sum_{1\le\alpha<\beta\le s}R^{m_\alpha}{}_{j_\alpha}{}^{m_\beta}{}_{j_\beta}u_{\bar{K}j_1\cdots m_\alpha\cdots m_\beta\cdots j_s} + \sum_{1\le\alpha<\beta\le r}R_{\bar{k}_\alpha}{}^{\bar{m}_\alpha}{}_{\bar{k}_\beta}{}^{\bar{m}_\beta}u_{\bar{k}_1\cdots\bar{m}_\alpha\cdots\bar{m}_\beta\cdots\bar{k}_r J} \\ & + \sum_{\alpha=1}^{r}\sum_{\beta=1}^{s}R_{\bar{k}_\alpha j_\beta}{}^{\bar{m}_\alpha n_\beta}u_{\bar{k}_1\cdots\bar{m}_\alpha\cdots\bar{k}_r j_1\cdots n_\beta\cdots j_s} + \sum_{j=1}^{r+s-1}D^j R_m \times D^{r+s-j}u \\ & + \sum_{j=1}^{r+s-1}D^j\theta \times D^{r+s-j}u.\end{aligned}$$

Here D denotes covariant differentiation in either j or $\bar{j}$ indices, and $D^k R_m$, $D^k\theta$ and $D^k u$ denote all tensors obtained by k covariant differentiations of R_m, θ and u respectively. The $A \times B$ denotes linear combination of the tensors A and B contracted with respect to the metric $g(t)$. The last two terms in the above equation are lower order terms which is actually absent when $r = s = 1$. Combining the above equations, we have

$$\begin{aligned}\left(|\nabla^s\bar{\nabla}^r u|^2\right)^{\cdot} = {} & \frac{1}{2}\triangle_{\mathbb{R}}|\nabla^s\bar{\nabla}^r u|^2 - |\nabla u_{\bar{K}J}|^2 - |\bar{\nabla}u_{\bar{K}J}|^2 - (r+s-2)|\nabla^s\bar{\nabla}^r u|^2 \\ & + 2\sum_{\alpha=1}^{r}\sum_{\beta=1}^{s}R_{\bar{k}_\alpha j_\beta}{}^{\bar{m}_\alpha m_\beta}u_{\bar{k}_1\cdots\bar{m}_\alpha\cdots\bar{k}_r j_1\cdots m_\beta\cdots j_s}u_{K\bar{J}}\end{aligned}$$

$$
\begin{aligned}
&+2 \sum_{1 \leq \alpha<\beta \leq s} R^{m_\alpha}{}_{j_\alpha}{}^{m_\beta}{}_{j_\beta} u_{\bar{K} j_1 \cdots m_\alpha \cdots m_\beta \cdots j_s} u_{K \bar{J}} \\
&+2 \sum_{1 \leq \alpha<\beta \leq r} R_{\bar{k}_\alpha}{}^{\bar{m}_\alpha}{}_{\bar{k}_\beta}{}^{\bar{m}_\beta} u_{\bar{k}_1 \cdots \bar{m}_\alpha \cdots \bar{m}_\beta \cdots \bar{k}_r J} u_{K \bar{J}} \\
&+2 \sum_{j=1}^{r+s-1} D^j R_m \times D^{r+s-j} u \times u_{K \bar{J}}+2 \sum_{j=1}^{r+s-1} D^j \theta \times D^{r+s-j} u \times u_{K \bar{J}} \\
&-\sum_{\alpha=1}^r \theta^{l_\alpha \bar{k}_\alpha} u_{\bar{K} J} u^{\bar{k}_1 \cdots \cdots \bar{k}_r J}{}_{l_\alpha}-\sum_{\beta=1}^s \theta^{j_\beta \bar{m}_\beta} u_{\bar{K} J} u^{\bar{K} j_1 \cdots \cdots j_s}{}_{\bar{m}_\beta} .
\end{aligned}
$$

Set $Y_{r,s}(t)=\int_M \mid \nabla^s \bar{\nabla}^r u \mid^2 \mathrm{d} V_t$, the time evolution of $Y_{r,s}(t)$ is given by

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} Y_{r,s}(t)= & \int_M \frac{\mathrm{d}}{\mathrm{dt}} \mid \nabla^s \bar{\nabla}^r u \mid^2 \mathrm{d} V_t+\int_M \mid \nabla^s \bar{\nabla}^r u \mid^2 \triangle \dot{\phi} d V_t \\
\leq & \int_M \frac{1}{2} \triangle_{\mathbb{R}} \mid \nabla^s \bar{\nabla}^r u \mid^2 \mathrm{d} V_t+C Y_{r,s}(t) \\
& -\int_M \mid \nabla^{s+1} \bar{\nabla}^r u \mid^2 \mathrm{d} V_t-\int_M \mid \bar{\nabla} \nabla^s \bar{\nabla}^r u \mid^2 \mathrm{d} V_t \\
& +C \sum_{j=1}^{r+s-1}\left(\int_M \mid D^{r+s-j} u \mid^2 \mathrm{d} V_t\right)^{\frac{1}{2}} Y_{r,s}^{\frac{1}{2}}(t) .
\end{aligned} \tag{4.7}
$$

For any $a \geq b$, integrating both sides of the above inequality form b to a, we have

$$
\begin{aligned}
Y_{r,s}(a)-Y_{r,s}(b) \leq & C \int_b^a Y_{r,s}(t) \mathrm{d} t+C \sum_{j=1}^{r+s-1} \int_b^a \\
& \times\left(\left(\int_M \mid D^{r+s-j} u \mid^2 \mathrm{d} V_t\right)^{\frac{1}{2}} Y_{r,s}^{\frac{1}{2}}(t)\right) \mathrm{d} t \\
\leq & C \sum_{j=1}^{r+s-1}\left(\int_b^{+\infty} \int_M \mid D^{r+s-j} u \mid^2 \mathrm{d} V_t d t\right)^{\frac{1}{2}}\left(\int_b^{+\infty} Y_{r,s}(t) \mathrm{d t}\right)^{\frac{1}{2}} \\
& +C \int_b^{+\infty} Y_{r,s}(t) \mathrm{dt} .
\end{aligned} \tag{4.8}
$$

We argue now by induction. When $r+s=2$, we have

$$
\int_0^{+\infty} \mathrm{dt} \int_M \mid \bar{\nabla} \nabla u \mid^2 \mathrm{d} V_t+\int_0^{+\infty} \mathrm{dt} \int_M \mid \nabla \nabla u \mid^2 \mathrm{d} V_t<+\infty .
$$
$Y(t) \rightarrow 0$.

Assume that

$$\int_0^{+\infty} \mathrm{d}t \int_M \mid D^j u \mid^2 \mathrm{d}V_t < +\infty, \quad j \leq r+s, \tag{4.9}$$

$$\int_M \mid D^j u \mid^2 \mathrm{d}V_t \to 0, \qquad j < r+s. \tag{4.10}$$

Then $Y_{r,s}(t)$ is integrable on $[0, +\infty)$, so there exists $b_m \in [m, m+1)$, such that $Y_{r,s}(b_m) \to 0$. By (4.8), we have $Y_{r,s}(t) \to 0$ as $t \to +\infty$. Since any covariant derivative of u of order $r+s$ differs from $\nabla^s \bar{\nabla}^r u$ by $D^j R_m \times D^{r+s-2-j} u$ $(j \geq 0)$ and all covariant derivatives of Riemann curvature tensors are bounded, we conclude that $\int_M \mid D^{r+s} u \mid^2 \mathrm{d}V_t \to 0$ from the second induction hypothesis and the fact $Y_{r,s}(t) \to 0$. To establish the first induction hypothesis at order $r+s+1$, we return to (4.7). Integrating from 0 to $+\infty$,

$$\begin{aligned}
&\int_0^{+\infty} \mathrm{d}t \int_M \mid \nabla^{s+1} \bar{\nabla}^r u \mid^2 \mathrm{d}V_t + \int_0^{+\infty} \mathrm{d}t \int_M \mid \bar{\nabla} \nabla^s \bar{\nabla}^r u \mid^2 \mathrm{d}V_t \\
&\quad \leq Y_{r,s}(0) + C \sum_{j=1}^{r+s-1} \left(\int_0^{+\infty} \int_M \mid D^{r+s-j} u \mid^2 \mathrm{d}V_t \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^{+\infty} Y_{r,s}(t) \mathrm{d}t \right)^{\frac{1}{2}} \\
&\qquad + C \int_0^{+\infty} Y_{r,s}(t) \mathrm{d}t.
\end{aligned}$$

This implies the L^2 norm of $\nabla^{s+1} \bar{\nabla}^r u$ and $\bar{\nabla} \nabla^s \bar{\nabla}^r u$ are integrable on $[0, +\infty)$. Using the first induction hypothesis and again the uniform boundness of the Riemann curvature tensors and all its covariant derivatives, we can deduce the conclusion that $\int_M \mid D^{r+s+1} u \mid^2 \mathrm{d}V_t$ is integrable on $[0, +\infty)$. So we prove the assumption (4.10) for all $j \in \mathbb{N}$. Since $R_{i\bar{j}} - g_{i\bar{j}} - \theta_{i\bar{j}} = -u_{i\bar{j}}$, the L^2 norm convergence to 0 of all covariant derivatives of u implies that the convergence of all Sobolev norms of $R_{i\bar{j}} - g_{i\bar{j}} - \theta_{i\bar{j}}$. The proof of Proposition 4.2 is completed. □

Remark 4.3 In fact, we should only assume that the Riemann curvature and all covariant derivatives of θ are bounded in the assumption of Proposition 4.2. Through computing by induction, we have

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t} - \triangle \right) \mid \nabla^r \bar{\nabla}^s R_m \mid^2 = &- \mid \nabla^{r+1} \bar{\nabla}^s R_m \mid^2 - \mid \bar{\nabla} \nabla^r \bar{\nabla}^s R_m \mid^2 \\
&- (r+s+2) \mid \nabla^r \bar{\nabla}^s R_m \mid^2 \\
&+ \sum_{\substack{i+j=s \\ k+l=r}} \nabla^k \bar{\nabla}^i (R_m + \theta) \times \nabla^l \bar{\nabla}^j R_m \times \nabla^r \bar{\nabla}^s R_m \\
&+ \sum_{\substack{i+j=s \\ k+l=r}} \bar{\nabla}^k \nabla^i (R_m + \theta) \times \bar{\nabla}^l \nabla^j R_m \times \bar{\nabla}^r \nabla^s R_m \\
&+ \langle \nabla^r \bar{\nabla}^{s+1} \nabla \theta, \nabla^r \bar{\nabla}^s R_m \rangle + \langle \bar{\nabla}^r \nabla^{s+1} \bar{\nabla} \theta, \bar{\nabla}^r \nabla^s R_m \rangle.
\end{aligned}$$

We have known that the Riemann curvature is bounded uniformly, then we assume that $\mid D^j R_m \mid \leq C$, for $j < r + s$. We consider the function

$$F = \mid \nabla^r \bar{\nabla}^s R_m \mid^2 + A \mid \nabla^{r-1} \bar{\nabla}^s R_m \mid^2. \tag{4.11}$$

If $r = 0$, then $s \geq 1$, we consider $F = \mid \bar{\nabla}^s R_m \mid^2 + A \mid \bar{\nabla}^{s-1} R_m \mid^2$,

$$\left(\frac{\mathrm{d}}{\mathrm{dt}} - \triangle \right) F \leq -A \mid \nabla^r \bar{\nabla}^s R_m \mid^2 + C \mid \nabla^r \bar{\nabla}^s R_m \mid^2 + C. \tag{4.12}$$

By maximum principle, $\mid \nabla^r \bar{\nabla}^s R_m \mid^2$ is uniformly bounded. Since $D^{r+s} R_m$ differs from $\nabla^r \bar{\nabla}^s R_m$ by $D^k R_m \cdot D^{r+s-2-k} R_m (0 \leqslant k \leqslant r+s-2)$, by induction hypothesis, we get the uniform bound of $|D^{r+s} R_m|$.

5 Convergence of the Generalized Kähler-Ricci Flow

In this section, we argue the convergence of the the generalized Kähler-Ricci flow, which is similar to the Kähler case in [19]. First, we introduce Aubin's functionals on $\mathcal{H}_{\omega_0}$.

$$\begin{aligned}
I_{\omega_0}(\phi) &= \frac{n!}{V} \int_M \phi (\mathrm{d}V_0 - \mathrm{d}V_t), \\
J_{\omega_0}(\phi) &= \frac{n!}{V} \int_0^1 \int_M \dot{\phi}_t (\mathrm{d}V_0 - \mathrm{d}V_t) \mathrm{dt} \\
&= \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \omega_0^i \wedge w_\phi^{n-i-1},
\end{aligned}$$

where ϕ_t is a path with $\phi_0 = c$, $\phi_1 = \phi$.

$$\begin{aligned}
F^0_{\omega_0}(\phi) &= J_{\omega_0}(\phi) - \frac{n!}{V} \int_M \phi \mathrm{d}V_0 \\
F_{\omega_0}(\phi) &= J_{\omega_0}(\phi) - \frac{n!}{V} \int_M \phi \mathrm{d}V_0 - \log \left(\frac{n!}{V} \int_M e^{-u(0)-\phi} \mathrm{d}V_0 \right).
\end{aligned}$$

Through computing, we conclude that $\frac{1}{n} J_{\omega_0} \leq \frac{1}{n+1} I_{\omega_0} \leq J_{\omega_0}$ and the time derivatives of I_{ω_0} and J_{ω_0} along any path ϕ_t can be written as follows:

$$\begin{aligned}
\frac{\partial}{\partial t} I_{\omega_0}(\phi_t) &= \frac{n!}{V} \int_M \dot{\phi}_t (\mathrm{d}V_0 - \mathrm{d}V_t) - \frac{n!}{V} \int_M \phi_t \Delta \dot{\phi}_t \mathrm{d}V_t, \\
\frac{\partial}{\partial t} J_{\omega_0}(\phi_t) &= \frac{n!}{V} \int_M \dot{\phi}_t (\mathrm{d}V_0 - \mathrm{d}V_t).
\end{aligned}$$

Now, we use the Sobolev inequality and Moser-Trudinger inequality stated in Proposition 2.1 and Lemma 5.5 respectively to show a uniform C^0 estimate for potential ϕ along the generalized Kähler-Ricci flow. We first establish some relations between the above functionals along the flow. Following the Kähler case in [19], we have

Lemma 5.1 *There exists C depending only on $g(0)$ and θ, such that ϕ which evolves along the flow* (1.3) *satisfies*:

(i) $\mathcal{V}_{\theta,\omega_0}(\phi) - F^0_{\omega_0}(\phi) - \frac{n!}{V}\int_M \dot{\phi} dV_t = C$,

(ii) $|F_{\omega_0}(\phi) - \mathcal{V}_{\theta,\omega_0}| + |F^0_{\omega_0}(\phi) - \mathcal{V}_{\theta,\omega_0}| \leq C$,

(iii) $\frac{(n-1)!}{V}\int_M (-\phi) dV_t - C \leq J_{\omega_0}(\phi) \leq \frac{n!}{V}\int_M \phi dV_0 + C$,

(iv) $\frac{n!}{V}\int_M \phi \mathrm{d}V_0 \leq \frac{n \cdot n!}{V}\int_M (-\phi)\mathrm{d}V_t - (n+1)\mathcal{V}_{\theta,\omega_0} + C$.

Lemma 5.2 *Let (M, ω_0) be a compact Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - [\omega_0]$ be a real closed semipositive $(1,1)$-form. Assume that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If ω is a generalized Kähler-Einstein metric, then the first eigenvalue of $\triangle_\omega$ satisfies $\lambda_1 > 1$.*

Proof Recall that we can characterize the first eigenvalue variationally by

$$\lambda_1(\triangle_\omega) = \inf_{\substack{\int_M \phi \mathrm{d}V_\omega = 0 \\ \phi \neq 0}} \frac{\int_M |\nabla \phi|^2_\omega \mathrm{d}V_\omega}{\int_M \phi^2 \mathrm{d}V_\omega}, \tag{5.1}$$

and it is well known that there exists an eigenfunction u such that $\triangle_\omega u = -\lambda_1 u$. Through computating, we have

$$\begin{aligned} \lambda_1^2 \int_M u^2 \mathrm{d}V_\omega &= \int_M (\triangle u)^2 \mathrm{d}V_\omega \\ &= \int_M |\nabla\nabla u|^2_\omega \mathrm{d}V_\omega + \int_M |\nabla u|^2_\omega \mathrm{d}V_\omega + \int_M \theta^{i\bar{j}} u_i u_{\bar{j}} \mathrm{d}V_\omega \\ &\geqslant \lambda_1 \int_M u^2 \mathrm{d}V_\omega, \end{aligned} \tag{5.2}$$

where we can conclude that $\lambda_1 \geqslant 1$. In fact, we can prove that $\lambda_1 > 1$.

- In the case that M admits no nontrivial Hamiltonian holomorphic vector field: if $\lambda_1 = 1$, we have $\int_M |\nabla\nabla u|^2_\omega \mathrm{d}V_\omega = 0$, so $u_{\bar{i}\bar{j}} = 0$ for any $i,\ j$, then it implies that $X = g^{i\bar{j}} u_{\bar{j}} \frac{\partial}{\partial z^i}$ is a nontrivial Hamiltonian holomorphic vector field, which contradicts the assumption.
- In the case that the twisting form θ is strictly positive at a point: if $\lambda_1 = 1$, we know $X = g^{i\bar{j}} u_{\bar{j}} \frac{\partial}{\partial z^i}$ is a nontrivial Hamiltonian holomorphic vector field from the above case. Since θ is strictly positive at a point, then there exists a neighborhood

U such that $\theta > 0$ on U. Since

$$\int_M \theta^{i\bar{j}} u_i u_{\bar{j}} \mathrm{d}V_\omega = \int_M \theta(X, \overline{X}) \mathrm{d}V_\omega = 0, \tag{5.3}$$

we imply that $X = 0$ on U, combining with X is holomorphic, we get that $X = 0$ on M, hence we have $u = C$, which we also get a contradiction. □

By the Sobolev inequality (2.1) and the Poincaré inequality (see [14, Lemma 4.2]) along the generalized Kähler-Ricci flow, we obtain the following lemma by following the arguments in [19] or [20]:

Lemma 5.3 *We have the following estimate along the generalized Kähler-Ricci flow*

$$osc(\phi) \leq \frac{A}{V} \int_M \phi \mathrm{d}V_0 + B, \tag{5.4}$$

where the constants A and B depend only on $g(0)$.

Lemma 5.4 *Let (M, ω_0) be a compact Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - [\omega_0]$ be a real closed semipositive $(1, 1)$-form. Assume that ϕ evolves along the flow (1.3) with initial value (1.5) and the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If*

$$\sup_{t \geq 0} \frac{1}{V} \int_M \phi \mathrm{d}V_0 \leq C < +\infty, \tag{5.5}$$

then the generalized Kähler-Ricci flow converges in C^∞ to a generalized Kähler-Einstein metric exponentially fast.

Proof By the arguments in the Kähler case, we get the uniform bound of $\| \phi \|_{C^0}$, see [19]. Then by Proposition 3.1, we obtain the uniform C^k norm of ϕ for any $k \in \mathbb{N}^+$. So the metric $g(t)$ are uniform equivalent and bounded in C^∞. Then we have $\lambda_1(t) \geqslant C_0 > 0$, where $\lambda_1(t)$ is the first eigenvalue of $\triangle_{g(t)}$ and C_0 is a uniform constant.

Step 1. We prove that $\varphi(t) = \dot{\phi} - \frac{1}{V} \int_M \dot{\phi} \mathrm{d}V_t$ converge to 0 in C^∞, by Poincaré equality, we have

$$C_0 \int_M \varphi^2 \mathrm{d}V_t \leqslant \lambda_1(t) \int_M \varphi^2 \mathrm{d}V_t \leqslant \int_M |\nabla \varphi|^2 \mathrm{d}V_t = Y(t) \to 0. \tag{5.6}$$

when $t \to +\infty$. Since the metrics are all uniform equivalent along the flow, so $\int_M \varphi^2 \mathrm{d}V_0 \to 0$, which implies that $\varphi \to 0$ in C^∞. If not, there exist r, ε_0, and a time sequence $\{t_i\}$ with $\|\varphi(t_i) - 0\|_{C^r} \geqslant \varepsilon_0$. But $\varphi(t_i)$ are bounded in C^∞ topology, so there is a subsequence denoted it also by $\varphi(t_i)$, such that $\varphi(t_i)$ converge in C^∞ to a smooth function $\varphi_\infty \neq 0$. This is a contradiction since $\varphi(t_i)$ do converge to 0 in L^2 norm.

Step 2. We prove $\dot{\phi} \to 0$ in C^∞ topology as $t \to +\infty$.

Let $\alpha(t)=\frac{1}{V}\int_M \dot{\phi}\mathrm{d}V_t$. From the course of getting a uniform bound of $\dot{\phi}$, see [16], we conclude that

$$0\leq \alpha(t)=\int_t^{+\infty} e^{t-s}\|\nabla\dot{\phi}\|_{L^2}^2\mathrm{ds}. \tag{5.7}$$

Since $\|\nabla\dot{\phi}\|_{L^2}^2$ converges to 0 as $t\to+\infty$ proved in [14], we conclude that $\alpha(t)$ converges to 0 as $t\to+\infty$. Combining Step 1, we obtain $\dot{\phi}\to 0$ in C^∞ topology as $t\to+\infty$.

Step 3. We prove the generalized Kähler-Ricci flow converges in C^∞ to a generalized Kähler-Einstein metric exponentially fast.

The evolution of $\varphi(t)$ as follows:

$$\frac{\partial\varphi}{\partial t}=\triangle\frac{\partial\phi}{\partial t}+\varphi-\frac{1}{V}\int_M\frac{\partial\phi}{\partial t}\triangle\frac{\partial\phi}{\partial t}\mathrm{d}V_t. \tag{5.8}$$

Define energy $E=\frac{1}{2}\int_M\varphi^2\mathrm{d}V_t$, when t is large, by Step 1 we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}E&=\int_M(-1-\varphi)|\nabla\varphi|^2_{g(t)}\mathrm{d}V_t+\int_M\varphi^2\mathrm{d}V_t\\&\leq(-1+\varepsilon)\int_M|\nabla\varphi|^2_{g(t)}\mathrm{d}V_t+\int_M\varphi^2\mathrm{d}V_t.\end{aligned}$$

Now we claim that there exists a uniform constant $\delta>0$, such that $\lambda_1(t)\geqslant 1+\delta$. We first prove that $\lambda_1(g(t))\geq 1$, if not, there exists a time sequence $t_i\to+\infty$, such that $\lambda_1(t_i)<1$. But $g(t_i)$ are bounded in C^∞ topology so there exists a subsequence denoted also by $g(t_i)$, such that $g(t_i)$ converge in C^∞ to a generalized Kähler-Einstein metric g_∞ by the fact that $\dot{\phi}\to 0$ in C^∞ topology, and $\lambda_1(t_i)\to\lambda_1(g_\infty)$, so $\lambda_1(g_\infty)\leq 1$, which contradicts $\lambda_1(g_\infty)>1$ by Lemma 5.2.

Then we start to prove the claim. If the claim is not true, then there exists a time sequence $\{t_i\}$, such that $\lambda_1(t_i)\to 1$. But $g(t_i)$ are bounded in C^∞ topology so there is a subsequence denoted also by $g(t_i)$, such that $g(t_i)$ converge in C^∞ to a Kähler metric g_∞ and $\lambda_1(t_i)\to\lambda_1(g_\infty)$, since $\dot{\phi}\to 0$ in C^∞ topology, g_∞ is actually a generalized Kähler-Einstein and $\lambda_1(g_\infty)=1$, which also contradicts $\lambda_1(g_\infty)>1$ by Lemma 5.2. By this claim, we imply that

$$\frac{\mathrm{d}}{\mathrm{dt}}E\leqslant((-1+\varepsilon)(1+\delta)+1)\int_M\varphi^2\mathrm{d}V_t,$$

where ε is small enough such that ε satisfies $(1-\varepsilon)(1+\delta)>1$. So we have

$$E\leqslant C_0e^{-2C_1t} \tag{5.9}$$

for $t\gg 1$.

Since $\varphi(t)=\dot{\phi}-\frac{1}{V}\displaystyle\int_M\dot{\phi}\mathrm{d}V_t$, we have the following equality

$$\begin{aligned}\frac{1}{nV}\int_M |\partial\dot{\phi}|^2 \mathrm{d}V_t &= \frac{1}{nV}\int_M |\partial\varphi|^2 \mathrm{d}V_t \\ &= -\frac{1}{nV}\int_M \varphi\triangle\varphi \mathrm{d}V_t \\ &\leqslant C_5\int_M |\varphi|\mathrm{d}V_t \\ &\leqslant C_6 E^{\frac{1}{2}}. \end{aligned} \tag{5.10}$$

By (5.9), we know that $Y(t) = \int_M \mid \nabla u \mid^2 \mathrm{d}V_t$ converge to 0 exponentially fast. In the proof of Proposition 4.2, by inductive hypotheses, we can deduce the exponential decay of the L^2 norms $Y_{r,s}$ of $\nabla^s\bar{\nabla}^r u$. From (4.7), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}Y_{r,s}(t) \le& -2cY_{r,s}(t) + CY_{r,s}(t) \\ &- \int_M \mid \nabla^{s+1}\bar{\nabla}^r u \mid^2 \mathrm{d}V_t - \int_M \mid \bar{\nabla}\nabla^s\bar{\nabla}^r u \mid^2 \mathrm{d}V_t \\ &+ C\sum_{j=1}^{r+s-1}\int_M \mid D^{r+s-j}u \mid^2 \mathrm{d}V_t. \end{aligned} \tag{5.11}$$

By using the interpolation formula, for any $\varepsilon > 0$, we have

$$\begin{aligned}Y_{r,s}(t) \le& \varepsilon\left(\int_M \mid \nabla^{s+1}\bar{\nabla}^r u \mid^2 \mathrm{d}V_t + \int_M \mid \bar{\nabla}\nabla^s\bar{\nabla}^r u \mid^2 \mathrm{d}V_t\right) \\ &+ C(\varepsilon)\int_M \mid D^{r+s-1}u \mid^2 \mathrm{d}V_t, \end{aligned} \tag{5.12}$$

where $C(\varepsilon)$ independent of t. Taking ε sufficiently small and putting (5.12) into (5.11), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}Y_{r,s}(t) &\le -2cY_{r,s}(t) + C\sum_{j=1}^{r+s-1}\int_M \mid D^{r+s-j}u \mid^2 \mathrm{d}V_t \\ &\leqslant -2cY_{r,s}(t) + Ce^{-ct}, \end{aligned} \tag{5.13}$$

where in the last inequality, we have assumed by induction that all L^2 norms of $D^{r+s-j}u$ decay exponentially. Integrating between t and 0, we see that $Y_{r,s}(t)$ decays exponentially. Hence we know that the Sobolev norm $\parallel R_{i\bar{j}} - g_{i\bar{j}} - \theta_{i\bar{j}} \parallel_{(s)}$ with respect to the metric $g(t)$ decay exponentially to 0. Then Sobolev imbedding theorem yields the exponential decay of $\|u\|_{C^k}$ for any k. By the fact that the metrics along the generalized Kähler-Ricci flow are all uniformly equivalent, we conclude that with respect to the metric $g(0)$, $\|u\|_{C^k(g(0))}$ also decay exponentially to 0 for any k. Hence $\|\dot{g}_{i\bar{j}}(t)\|_{C^k(g(0))} = \|R_{i\bar{j}}(t) - g_{i\bar{j}}(t) - \theta_{i\bar{j}}\|_{C^k(g(0))}$ decays exponentially to 0.

From the above argument, we prove that the generalized Kähler-Ricci flow converges in C^∞ topology to a generalized Kähler-Einstein metric. □

At last, we proof the Theorem 1.1 by using the following Moser-Trudinger type inequality which proved by Zhang and Zhang in [29].

Lemma 5.5 *Let* (M, ω_0) *be a Kähler manifold, and* $\theta \in [\alpha] = 2\pi c_1(M) - [\omega_0]$ *is a real closed semipositive* $(1, 1)$*-form. Assuming that the twisting form* θ *is strictly positive at a point or* M *admits no nontrivial Hamiltonian holomorphic vector field. If there exists a generalized Kähler-Einstein metric in* $[\omega_0]$*. Then, for any Kähler metric* $\omega \in [\omega_0]$ *there exist uniform positive constants* $\{C_i\}_{i=1}^4$ *depending only on k and the geometry of* (M, ω)*, such that*

$$F_\omega(\varphi) \geq C_1 J_\omega(\varphi) - C_2, \tag{5.14}$$

and

$$\mathcal{V}_{\theta,\omega}(\varphi) \geq C_3 J_\omega(\varphi) - C_4, \tag{5.15}$$

for all $\varphi \in \mathcal{H}_\omega$*.*

Proof of Theorem 1.1 From assumption, the Moser-Trudinger inequality holds along the generalized Kähler-Ricci flow by Lemma 5.5. Since $\mathcal{V}_{\theta,\omega_0}$ decreases along the flow. It follows that $J_{\omega_0}(\phi)$ is uniformly bounded from above. Thus by Lemma 5.1 (iii), we have

$$\int_M (-\phi) \mathrm{d}V_t \leq C. \tag{5.16}$$

Since $J_{\omega_0} \geq 0$, applying Lemma 5.5 we know that the twisted Mabuchi $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega_0}$ is uniformly bounded from below. Then by Lemma 5.1 (iv), we have

$$\int_M \phi \mathrm{d}V_0 \leq C. \tag{5.17}$$

So Theorem 1.1 follows from Lemma 5.4. □

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