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Mean-Field Maximum Principle for Optimal Control of Forward–Backward Stochastic Systems with Jumps and its Application to Mean-Variance Portfolio Problem

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Abstract We study mean-field type optimal stochastic control problem for systems governed by mean-field controlled forward–backward stochastic differential equations with jump processes, in which the coefficients depend on the marginal law of the state process through its expected value. The control variable is allowed to enter both diffusion and jump coefficients. Moreover, the cost functional is also of mean-field type. Necessary conditions for optimal control for these systems in the form of maximum principle are established by means of convex perturbation techniques. As an application, time-inconsistent mean-variance portfolio selection mixed with a recursive utility functional optimization problem is discussed to illustrate the theoretical results.

Keywords Mean-field forward–backward stochastic differential equation with jumps · Optimal stochastic control · Mean-field maximum principle · Mean-variance portfolio selection with recursive utility functional · Time-inconsistent control problem

Mathematics Subject Classification 93E20 · 60H10 · 49K45 · 60H30

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1 Introduction

In this paper, we consider stochastic optimal control for systems governed by nonlinear mean-field controlled forward–backward stochastic differential equations with Poisson jump processes (FBSDEJs) of the form

$$\begin{cases} \mathrm{d}x(t) = f\left(t, x(t), E(x(t)), u(t)\right)\mathrm{d}t + \sigma\left(t, x(t), E(x(t)), u(t)\right)\mathrm{d}W(t) \\ \qquad + \int_\Theta c(t, x(t_-), E(x(t_-)), u(t), \theta)N\left(\mathrm{d}\theta, \mathrm{dt}\right), \\ \mathrm{d}y(t) = -\int_\Theta g(t, x(t), E(x(t)), y(t), E(y(t)), z(t), E(z(t)), r\left(t,\theta\right), \\ \qquad u(t))\mu\left(\mathrm{d}\theta\right)\mathrm{d}t + z(t)\mathrm{d}W(t) + \int_\Theta r\left(t,\theta\right)N\left(\mathrm{d}\theta, \mathrm{d}t\right), \\ x(0) = \zeta, \; y(T) = h\left(x(T), E\left(x(T)\right)\right), \end{cases} \tag{1.1}$$

where f, σ, c, g, h are given maps and the initial condition ζ is an $\mathcal{F}_0$-measurable random variable. The mean-field FBSDEJs-(1.1) called McKean–Vlasov systems are obtained as the mean square limit of an interacting particle system of the form

$$\begin{cases} \mathrm{d}x_n^j(t) = f(t, x_n^j(t), \frac{1}{n}\sum_{i=1}^n x_n^i(t), u(t))\mathrm{d}t \\ \qquad +\sigma(t, x_n^j(t), \frac{1}{n}\sum_{i=1}^n x_n^i(t), u(t))\mathrm{d}W^j(t) \\ \qquad + \int_\Theta c(t, x_n^j(t_-), \frac{1}{n}\sum_{i=1}^n x_n^i(t_-), u(t), \theta)N^j\left(\mathrm{d}\theta, \mathrm{dt}\right), \\ \mathrm{d}y_n^j(t) = -\int_\Theta g(t, x_n^j(t), \frac{1}{n}\sum_{i=1}^n x_n^i(t), y_n^j(t), \frac{1}{n}\sum_{i=1}^n y_n^i(t), z_n^j(t), \\ \qquad \frac{1}{n}\sum_{i=1}^n z_n^i(t), r(t,\theta), u(t))\mu\left(\mathrm{d}\theta\right)\mathrm{d}t \\ \qquad +z_n^j(t)\mathrm{d}W^j(t) + \int_\Theta r\left(t,\theta\right)N^j\left(\mathrm{d}\theta, \mathrm{d}t\right), \end{cases}$$

where $(W^j(\cdot){:}j{\geq}1)$ is a collection of independent Brownian motions and $(N^j(\cdot,\cdot) : j \geq 1)$ is a collection of independent Poisson martingale measure. Noting that mean-field FBSDEJs-(1.1) occur naturally in the probabilistic analysis of financial optimization problems and the optimal control of dynamics of the McKean–Vlasov type. Moreover, the above mathematical mean-field approaches play an important role in different fields of economics, finance, physics, chemistry and game theory.

The expected cost to be minimized over the class of admissible control has the form

$$J\left(u(\cdot)\right) = E\left[\int_0^T \int_\Theta \ell(t, x(t), E(x(t)), y(t), E(y(t)), z(t), E(z(t)), \right. \\ \left. r\left(t,\theta\right), u(t))\,\mu\left(d\theta\right)dt + \phi\left(x(T), E(x(T))\right) + \varphi\left(y(0), E\left(y(0)\right)\right)\right], \tag{1.2}$$

where ℓ, ϕ, φ is an appropriate functions. This cost functional is also of mean-field type, as the functions ℓ, ϕ, φ depend on the marginal law of the state process through its expected value. It is worth mentioning that since the cost functional J is possibly a nonlinear function of the expected value stands in contrast to the standard formulation of a control problem. This leads to the so-called time-inconsistent control problem where the Bellman dynamic programming does not hold. The reason for this is that one cannot apply the law of iterated expectations on the cost functional.

An admissible control $u(\cdot)$ is an $\mathcal{F}_t$-adapted and square-integrable process with values in a nonempty convex subset $\mathcal{A}$ of $\Re$. We denote by $\mathcal{U}([0,T])$ the set of all admissible controls. Any admissible control $u^*(\cdot) \in \mathcal{U}([0,T])$ satisfying

$$J\left(u^*(\cdot)\right) = \inf_{u(\cdot)\in\mathcal{U}([0,T])} J\left(u(\cdot)\right), \tag{1.3}$$

is called an optimal control.

The mean-field stochastic differential equation was introduced by Kac [1] as a stochastic model for the Vlasov kinetic equation of plasma and the study of this model was initiated by McKean [2]. Since then, many authors made contributions on mean-field stochastic problems and their applications, see for instance [3–23]. In a recent paper, mean-field games for large population multi-agent systems with Markov jump parameters have been investigated in Wang and Zhang [3]. Decentralized tracking-type games for large population multi-agent systems with mean-field coupling have been studied in Li and Zhang [4]. Discrete-time indefinite mean-field linear-quadratic optimal control problem has been investigated in Ni et al. [5]. Discrete time mean-field stochastic linear-quadratic optimal control problems with applications have been derived by Elliott et al. [6]. In Buckdahn, Li and Peng [7], a general notion of mean-field BSDE associated with a mean-field SDE was obtained in a natural way as a limit of some high-dimensional system of FBSDEs governed by a d-dimensional Brownian motion, and influenced by positions of a large number of other particles. In Buckdahn et al. [8], a general maximum principle was introduced for a class of stochastic control problems involving SDEs of mean-field type. However, sufficient conditions of optimality for mean-field SDE have been established by Shi [9]. In Meyer-Brandis, Øksendal and Zhou [10], a stochastic maximum principle of optimality for systems governed by controlled Itô-Levy process of mean-field type was proved using Malliavin calculus. Mean-field singular stochastic control problems have been investigated in Hafayed and Abbas [11]. More interestingly, mean-field type stochastic maximum principle for optimal singular control has been studied in Hafayed [12], in which convex perturbations used for both absolutely continuous and singular components. The maximum principle for optimal control of mean-field FBSDEJs with uncontrolled diffusion has been studied in Hafayed [13]. The necessary and sufficient conditions for near-optimality of mean-field jump diffusions with applications have been derived by Hafayed et al. [14]. Singular optimal control for mean-field forward–backward stochastic systems and applications to finance have been investigated in Hafayed [15]. Second-order necessary conditions for optimal control of mean-field jump diffusion have been obtained by Hafayed and Abbas [16]. Under partial information, mean-field type stochastic maximum principle for optimal control has been investigated in Wang, Zhang and

Zhang [17]. Under the condition that the control domain is convex, Andersson and Djehiche [18] and Li [19] investigated problems for two types of more general controlled SDEs and cost functionals, respectively. The linear-quadratic optimal control problem for mean-field SDEs has been studied by Yong [20] and Shi [9]. The mean-field stochastic maximum principle for jump diffusions with applications has been investigated in Shen and Siu [21] Recently, maximum principle for mean-field jump diffusions stochastic delay differential equations and its applications to finance have been derived by Yang, Meng and Shi [22]. Mean-field optimal control for backward stochastic evolution equations in Hilbert spaces has been investigated in Xu and Wu [23].

The optimal control problems for stochastic systems described by Brownian motions and Poisson jumps have been investigated by many authors including [24,25,27–30]. The necessary and sufficient conditions of optimality for FBSDEJs were obtained by Shi and Wu [24]. General maximum principle for fully coupled FBSDEJs has been obtained in Shi [25], where the author generalized Yong's maximum principle [26] to jump case.

In this paper, our main goal is to derive a maximum principle for optimal stochastic control of mean-field FBSDEJs, where the coefficient depends not only on the state process but also its marginal law of the state process through its expected value. The cost functional is also of mean-field type. Our mean-field control problem is not simple extension from the mathematical point of view, but also provide interesting models in many applications such as mathematical finance; (mean-variance portfolio selection problems), optimal control for mean-field systems. The proof of our result is based on convex perturbation method. These necessary conditions are described in terms of two adjoint processes, corresponding to the mean-field forward and backward components with jumps and a maximum conditions on the Hamiltonian. In the end, as an application to finance, a mean-variance portfolio selection mixed with a recursive utility optimization problem is given, where explicit expression of the optimal portfolio selection strategy is obtained in feedback form involving both state process and its marginal distribution, via the solutions of Riccati ordinary differential equations. To streamline the presentation of this paper, we only study the 1-dimensional case.

The rest of this paper is structured as follows. In Sect. 2, we formulate the mean-field stochastic control problem and describe the assumptions of the model. Section 3 is devoted to prove our mean-field stochastic maximum principle. As an illustration, using these results, a mean-variance portfolio selection mixed problem with recursive utility (time-inconsistent solution) is discussed in the last Sect. 4.

2 Problem Statement and Preliminaries

We consider stochastic optimal control problem of mean-field type of the following kind. Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a fixed filtered probability space equipped with a P-completed right continuous filtration on which a 1-dimensional Brownian motion $W = (W(t))_{t\in[0,T]}$ is defined. Let η be a homogeneous $(\mathcal{F}_t)$-Poisson point process independent of W. We denote by $\widetilde{N}(\mathrm{d}\theta, \mathrm{d}t)$ the random counting measure induced by η, defined on $\Theta \times \Re_+$, where Θ is a fixed nonempty subset of $\Re$ with its Borel σ-field $\mathcal{B}(\Theta)$. Further, let $\mu(\mathrm{d}\theta)$ be the local characteristic measure of η, i.e., $\mu(\mathrm{d}\theta)$ is a σ-finite measure on $(\Theta, \mathcal{B}(\Theta))$ with $\mu(\Theta) < +\infty$. We then define $N(d\theta, dt) := \widetilde{N}(\mathrm{d}\theta, \mathrm{d}t) - \mu(\mathrm{d}\theta)\,\mathrm{d}t$, where $N(\cdot,\cdot)$ is Poisson martingale measure on $\mathcal{B}(\Theta) \times \mathcal{B}(\Re_+)$ with local characteristics $\mu(d\theta)\,dt$. We assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is P-augmentation of the natural filtration $(\mathcal{F}_t^{(W,N)})_{t\in[0,T]}$ defined as follows

$$\mathcal{F}_t^{(W,N)} := \sigma(W(s) : s \in [0,t]) \vee \sigma\left(\int_0^s \int_B N(\mathrm{d}\theta, \mathrm{d}r) : s \in [0,t],\, B \in \mathcal{B}(\Theta)\right) \vee \mathcal{G}_0,$$

where $\mathcal{G}_0$ denotes the totality of P-null sets, and $\sigma_1 \vee \sigma_2$ denotes the σ-field generated by $\sigma_1 \cup \sigma_2$.

In the sequel, $L^2_{\mathcal{F}}([0,T];\Re)$ denotes the Hilbert space of $\mathcal{F}_t$-adapted processes $(X(t))_{t\in[0,T]}$ such that $E\int_0^T |X(t)|^2\,\mathrm{d}t < +\infty$ and $\mathcal{M}^2_{\mathcal{F}}([0,T];\Re)$ denote the Hilbert space of $\mathcal{F}_t$-predictable processes $(\psi(t,\theta))_{t\in[0,T]}$ defined on $[0,T] \times \Theta$ such that $E\int_0^T \int_\Theta |\psi(t,\theta)|^2\,\mu(\theta)\mathrm{d}t < +\infty$. In what follows, C represents a generic constants, which can be different from line to line. For simplicity of notation, we still use $f_x(t) = \frac{\partial f}{\partial x}(t, x^*(\cdot), E(x^*(\cdot)), u^*(\cdot))$, etc.

Throughout this paper, we also assume that the functions $f, \sigma : [0,T] \times \Re \times \Re \times \mathcal{A} \to \Re$, $c : [0,T] \times \Re \times \mathcal{A} \times \Theta \to \Re$, $g, \ell : [0,T] \times \Re \times \Re \times \Re \times \Re \times \Re \times \Re \times \Re \times \mathcal{A} \to \Re$ and $h, \phi, \varphi : \Re \times \Re \to \Re$ satisfy the following standing assumptions:

Assumption (H1) 1. The functions f, σ and c are global Lipschitz in $(x, \widetilde{x}, u)$ and g is global Lipschitz in $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, r, u)$.

2. The functions $f, \sigma, \ell, c, g, h, \phi$ and φ are continuously differentiable in their variables including $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, r, u)$.

Assumption (H2) 1. The derivatives of f, σ, g, ϕ with respect to their variables including $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, r, u)$ are bounded, and

$$\int_\Theta (|c_x(t, x, \widetilde{x}, u, \theta)|^2 + |c_{\widetilde{x}}(t, x, \widetilde{x}, u, \theta)|^2 + |c_u(t, x, \widetilde{x}, u, \theta)|^2)\mu(\mathrm{d}\theta) < +\infty.$$

2. The derivatives b_ρ are bounded by $C(1 + |x| + |\widetilde{x}| + |y| + |\widetilde{y}| + |z| + |\widetilde{y}| + |r| + |u|)$ for $\rho = x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, r, u$ and $b = f, \sigma, g, c, \ell$. Moreover, $\varphi_y, \varphi_{\widetilde{y}}$ are bounded by $C(1 + |y| + |\widetilde{y}|)$ and $h_x, h_{\widetilde{x}}$ are bounded by $C(1 + |x| + |\widetilde{x}|)$.

3. For all $t \in [0,T]$, $f(t,0,0,0)$, $g(t,0,0,0,0,0,0,0,0) \in L^2_{\mathcal{F}}([0,T];\Re)$, $\sigma(t,0,0,0) \in L^2_{\mathcal{F}}([0,T];\Re \times \Re)$ and $c(t,0,0,0,\cdot) \in \mathcal{M}^2_{\mathcal{F}}([0,T];\Re)$.

Under the assumptions (H1) and (H2), the FBSDEJ-(1.1) has a unique solution $(x(t), y(t), z(t), r(t,\cdot)) \in L^2_{\mathcal{F}}([0,T];\Re) \times L^2_{\mathcal{F}}([0,T];\Re) \times L^2_{\mathcal{F}}([0,T];\Re) \times L^2_{\mathcal{F}}([0,T];\Re)$. (See [21, Theorem 3.1], for mean-field BSDE with jumps)

For any $u(\cdot) \in \mathcal{U}([0,T])$ with its corresponding state trajectories $(x(\cdot), y(\cdot), z(\cdot), r(\cdot,\cdot))$ we introduce the following adjoint equations:

$$\begin{cases} \mathrm{d}\Psi(t) = -\{f_x(t)\Psi(t) + E(f_{\widetilde{x}}(t)\Psi(t)) + \sigma_x(t)Q(t) + E(\sigma_{\widetilde{x}}(t)Q(t)) \\ \qquad + \int_\Theta [g_x(t,\theta)K(t) + E(g_{\widetilde{x}}(t,\theta)K(t)) + c_x(t,\theta)R(t,\theta) \\ \qquad + E(c_{\widetilde{x}}(t,\theta)R(t,\theta)) + \ell_x(t,\theta) + E(\ell_{\widetilde{x}}(t,\theta))]\mu(\mathrm{d}\theta)\}\mathrm{d}t \\ \qquad + Q(t)dW(t) + \int_\Theta R_t(\theta)N(\mathrm{d}\theta,\mathrm{d}t), \\ \Psi(T) = -[h_x(x(T),E(x(T)))K(T) + E(h_{\widetilde{x}}(x(T),E(x(T)))K(T))] \\ \qquad + \phi_x(x(T),E(x(T))) + E(\phi_{\widetilde{x}}(x(T),E(x(T)))), \\ \mathrm{d}K(t) = \int_\Theta [g_y(t,\theta)K(t) + E(g_{\widetilde{y}}(t,\theta)K(t)) - \ell_y(t,\theta) - E(\ell_{\widetilde{y}}(t,\theta))]\mu(\mathrm{d}\theta)\,\mathrm{d}t \\ \qquad + \int_\Theta [g_z(t,\theta)K(t) + E(g_{\widetilde{z}}(t,\theta)K(t)) - \ell_z(t,\theta) - E(\ell_{\widetilde{z}}(t,\theta))]\mu(\mathrm{d}\theta)\,\mathrm{d}W(t) \\ \qquad + \int_\Theta (g_r(t,\theta)K(t) - \ell_r(t,\theta))N(\mathrm{d}\theta,\mathrm{d}t), \\ K(0) = -(\varphi_y(0) + E(\varphi_{\widetilde{y}}(0))). \end{cases} \tag{2.1}$$

Note that the first adjoint equation (backward) corresponding to the forward component turns out to be a linear mean-field backward SDE with jumps, and the second adjoint equation (forward) corresponding to the backward component turns out to be a linear mean-field (forward) SDE with jump processes. Further, we define the Hamiltonian function

$$H : [0,T] \times \Re \times \Re \times \Re \times \Re \times \Re \times \Re \times \Re \times \mathcal{A} \times \Re \times \Re \times \Re \times \Re \to \Re,$$

associated with the stochastic control problems (1.1) and (1.2) as follows

$$\begin{aligned} H(t,x,\widetilde{x},y,\widetilde{y},z,\widetilde{z},r,u,\Psi,Q,K,R) &:= \Psi(t)f(t,x,\widetilde{x},u) + Q(t)\sigma(t,x,\widetilde{x},u) \\ &- \int_\Theta [K(t)g(t,x,\widetilde{x},y,\widetilde{y},z,\widetilde{z},r,u) + R(t,\theta)c(t,x,\widetilde{x},u,\theta)) \\ &+ \ell(t,x,\widetilde{x},y,\widetilde{y},z,\widetilde{z},r,u)]\,\mu(\mathrm{d}\theta). \end{aligned} \tag{2.2}$$

If we denote by

$$H(t) := H(t,x(t),\widetilde{x}(t),y(t),\widetilde{y}(t),z(t),\widetilde{z}(t),r(t,\cdot),u(t),\Psi(t),Q(t),K(t),R(t,\cdot)),$$

then the adjoint equation (2.1) can be rewritten as the following stochastic Hamiltonian system's type

$$\begin{cases} -\mathrm{d}\Psi(t) = (H_x(t) + E(H_{\widetilde{x}}(t)))dt - Q(t)dW(t) - \int_\Theta R(t,\theta)N(d\theta,dt), \\ \Psi(T) = -[h_x(x(T),E(x(T)))K(T) + E(h_{\widetilde{x}}(x(T),E(x_T(t)))K(T))] \\ \qquad + \phi_x(x(T),E(x(T))) + E(\phi_{\widetilde{x}}(x(T),E(x(T)))). \\ -dK(t) = (H_y(t) + E(H_{\widetilde{y}}(t)))dt + (H_z(t) + E(H_{\widetilde{z}}(t)))dW(t) \\ \qquad + \int_\Theta H_r(t,\theta)N(d\theta,dt) \\ K(0) = -(\varphi_y(0) + E(\varphi_{\widetilde{y}}(0))). \end{cases} \tag{2.3}$$

Thanks to Lemma 3.1 in Shen and Siu [21], under assumptions (H1) and (H2), the adjoint equations (2.1) admit a unique solution $(\Psi(t), Q(t), K(t), R(t,\cdot))$ such that

$$(\Psi(t), Q(t), K(t), R(t,\cdot))$$
$$\in L^2_{\mathcal{F}}([0,T];\Re)\times L^2_{\mathcal{F}}([0,T];\Re)\times L^2_{\mathcal{F}}([0,T];\Re)\times \mathcal{M}^2_{\mathcal{F}}([0,T];\Re).$$

Moreover, since the derivatives of $f, \sigma, c, g, h, \varphi, \phi$ with respect to $x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, r$ are bounded, we deduce from standard arguments that there exists a constant $C > 0$ such that

$$E\left\{\sup_{t\in[0,T]}|\Psi(t)|^2+\sup_{t\in[0,T]}|K(t)|^2+\int_0^T|Q(t)|^2\,\mathrm{d}t+\int_0^T\int_\Theta|R(t,\theta)|^2\,\mu(\mathrm{d}\theta)\,\mathrm{d}t\right\}<C. \tag{2.4}$$

3 Mean-Field Type Necessary Conditions for Optimal Control of FBSDEJs

In this section, we establish a set of necessary conditions of Pontraygin's type for a stochastic control to be optimal where the system evolves according to nonlinear controlled mean-field FBSDEJs. Convex perturbation techniques are applied to prove our mean-field stochastic maximum principle.

The following theorem constitutes the main contribution of this paper.

Let $(x^*(\cdot), y^*(\cdot), z^*(\cdot), r^*(\cdot,\cdot))$ be the trajectory of the mean-field FBSDEJ-(1.1) corresponding to the optimal control $u^*(\cdot)$, and $(\Psi^*(\cdot), Q^*(\cdot), K^*(\cdot), R^*(\cdot,\cdot))$ be the solution of adjoint equation (2.1) corresponding to $u^*(\cdot)$.

Theorem 3.1 (Maximum principle for mean-field FBSDEJs) *Let Assumptions* (H1) *and* (H2) *hold. If* $(u^*(\cdot), x^*(\cdot), y^*(\cdot), z^*(\cdot), r^*(\cdot,\cdot))$ *is an optimal solution of the mean-field control* problems (1.1) and (1.2)*. Then the maximum principle holds, that is* $\forall u\in\mathcal{A}$

$$H_u\left(t, \lambda^*(t,\theta), u^*, \Lambda^*(t,\theta)\right)(u-u^*(t))\geq 0,\ P-a.s., a.e.,\ t\in[0,T], \tag{3.1}$$

where $\lambda^*(t,\theta) = (x^*(t), E(x^*(t)), y^*(t), E(y^*(t)), z^*(t), E(z^*(t)), r^*(t,\theta))$ *and* $\Lambda^*(t,\theta) = (\Psi^*(t), Q^*(t), K^*(t), R^*(t,\theta))$.

We derive the variational inequality (3.1) in several steps, from the fact that

$$J\left(u^\varepsilon(\cdot)\right)\geq J\left(u^*(\cdot)\right). \tag{3.2}$$

Since the control domain $\mathcal{A}$ is convex and for any given admissible control $u(\cdot)\in\mathcal{U}([0,T])$ the following perturbed control process

$$u^\varepsilon(t)=u^*(t)+\varepsilon\left(u(t)-u^*(t)\right),$$

is also an element of $\mathcal{U}([0,T])$.

Let $\lambda^\varepsilon(t,\theta)=(x^\varepsilon(t), y^\varepsilon(t), z^\varepsilon(t), r^\varepsilon(t,\theta))$ be the solution of state equation (1.1) and $\Lambda^\varepsilon(t,\theta)=(\Psi^\varepsilon(t), Q^\varepsilon(t), K^\varepsilon(t), R^\varepsilon(t,\theta))$ be the solution of the adjoint equation (2.1) corresponding to perturbed control $u^\varepsilon(\cdot)$.

Variational equations. We introduce the following variational equations which have a mean-field type. Let $\left(x_1^\varepsilon(\cdot), y_1^\varepsilon(\cdot), z_1^\varepsilon(\cdot), r_1^\varepsilon(\cdot,\cdot)\right)$ be the solution of the following forward–backward stochastic system described by Brownian motions and Poisson jumps of mean-field type

$$
\begin{cases}
\mathrm{d}x_1^\varepsilon(t) = \left\{f_x(t)x_1^\varepsilon(t) + f_{\widetilde{x}}(t)E(x_1^\varepsilon(t)) + f_u(t)u(t)\right\}\mathrm{d}t \\
\qquad + \left\{\sigma_x(t)x_1^\varepsilon(t) + \sigma_{\widetilde{x}}(t)E\left(x_1^\varepsilon(t)\right) + \sigma_u(t)u(t)\right\}\mathrm{d}W(t) \\
\qquad + \int_\Theta [c_x(t,\theta)x_1^\varepsilon(t) + c_{\widetilde{x}}(t,\theta)E(x_1^\varepsilon(t)) + c_u(t,\theta)u(t)]N(d\theta, dt), \\
x_1^\varepsilon(0) = 0, \\
dy_1^\varepsilon(t) = -\int_\Theta \{g_x(t,\theta)x_1^\varepsilon(t) + g_{\widetilde{x}}(t,\theta)E(x_1^\varepsilon(t)) + g_y(t,\theta)y_1^\varepsilon(t) \\
\qquad + g_{\widetilde{y}}(t,\theta)E(y_1^\varepsilon(t)) + g_z(t,\theta)z_1^\varepsilon(t) + g_{\widetilde{z}}(t,\theta)E(z_1^\varepsilon(t)) + g_r(t,\theta)r_1^\varepsilon(t,\theta) \\
\qquad + g_u(t,\theta)u(t)\}\mu(d\theta)dt + z_1^\varepsilon(t)dW(t) + \int_\Theta r_1^\varepsilon(t,\theta)N(d\theta, dt), \\
y_1^\varepsilon(T) = -\left[h_x(T) + E(h_{\widetilde{x}}(T))\right]x_1^\varepsilon(T).
\end{cases}
\tag{3.3}
$$

Duality relations. Our first Lemma below deals with the duality relations between $\Psi^*(t), x_1^\varepsilon(t)$ and $K^*(t), y_1^\varepsilon(t)$. This Lemma is very important for the proof of Theorem 3.1.

Lemma 3.2 *We have*

$$
\begin{aligned}
E\left(\Psi^*(T)x_1^\varepsilon(T)\right) = E\int_0^T &[\Psi^*(t)f_u(t)u(t) + Q^*(t)\sigma_u(t)u(t) \\
&+ \int_\Theta R^*(t,\theta)c_u(t,\theta)u(t)\mu(\mathrm{d}\theta)]\mathrm{d}t \\
&- E\int_0^T\int_\Theta \left\{x_1^\varepsilon(t)g_x(t,\theta)K(t)\right. \\
&+ x_1^\varepsilon(t)E(g_{\widetilde{x}}(t,\theta)K(t)) + x_1^\varepsilon(t)\ell_x(t,\theta) \\
&+ \left. x_1^\varepsilon(t)E(\ell_{\widetilde{x}}(t,\theta))\right\}\mu(\mathrm{d}\theta)\mathrm{d}t,
\end{aligned}
\tag{3.4}
$$

similarly, we get

$$
\begin{aligned}
E\left(K^*(T)y_1^\varepsilon(T)\right) = &-E\left\{[\varphi_y(y(0), E(y(0))) + E(\varphi_{\widetilde{y}}(y(0), E(y(0))))]y_1^\varepsilon(0)\right\} \\
&+ E\int_0^T\int_\Theta \left\{K^*(t)g_x(t,\theta)x_1^\varepsilon(t) + K^*(t)g_{\widetilde{x}}(t,\theta)E\left(x_1^\varepsilon(t)\right)\right. \\
&- K^*(t)g_u(t,\theta)u(t) - y_1^\varepsilon(t)\ell_y(t,\theta) - y_1^\varepsilon(t)E(\ell_{\widetilde{y}}(t,\theta)) \\
&- \left. z_1^\varepsilon(t)\ell_z(t,\theta) - z_1^\varepsilon(t)E(\ell_{\widetilde{z}}(t,\theta)) - r_1^\varepsilon(t,\theta)\ell_r(t,\theta)\right\}\mu(d\theta)dt,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&E\left\{[\phi_x(x(T), E(x(T))) + E(\phi_{\widetilde{x}}(x(T), E(x(T))))]x_1^\varepsilon(T)\right\} \\
&\quad + E\left\{[\varphi_y(y(0), E(y(0))) + E(\varphi_{\widetilde{y}}(y(0), E(y(0))))]y_1^\varepsilon(0)\right\}
\end{aligned}
$$

$$= E\int_0^T \int_\Theta \big\{x_1^\varepsilon(t)\ell_x(t,\theta) + x_1^\varepsilon(t)E(\ell_{\widetilde{x}}(t,\theta)) - y_1^\varepsilon(t)\ell_y(t,\theta)$$
$$-y_1^\varepsilon(t)E(\ell_{\widetilde{y}}(t,\theta)) - z_1^\varepsilon(t)\ell_z(t,\theta) - z_1^\varepsilon(t)E(\ell_{\widetilde{z}}(t,\theta))$$
$$- r_1^\varepsilon(t,\theta)\ell_r(t,\theta) - \ell_u(t,\theta)u(t)\big\}\mu(\mathrm{d}\theta)\mathrm{d}t + E\int_0^T H_u(t)u(t)\mathrm{d}t. \quad (3.6)$$

Proof By applying integration by parts formula for jump processes (see Lemma 6.1) to $\Psi^*(t)x_1^\varepsilon(t)$, we get

$$\begin{aligned}E\left(\Psi^*(T)x_1^\varepsilon(T)\right) &= E\int_0^T \Psi^*(t)\mathrm{d}x_1^\varepsilon(t) + E\int_0^T x_1^\varepsilon(t)\mathrm{d}\Psi^*(t)\\
&+E\int_0^T Q^*(t)\left[\sigma_x(t)x_1^\varepsilon(t) + \sigma_{\widetilde{x}}(t)E\left(x_1^\varepsilon(t)\right) + \sigma_u(t)u(t)\right]\mathrm{d}t\\
&+E\int_0^T\int_\Theta \left[c_x(t,\theta)x_1^\varepsilon(t) + c_{\widetilde{x}}(t,\theta)E(x_1^\varepsilon(t))\right.\\
&\left.+ c_u(t,\theta)u(t)\right]R(t,\theta)\mu(\mathrm{d}\theta)\mathrm{d}t\\
&= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon. \end{aligned} \quad (3.7)$$

A simple computation shows that

$$\begin{aligned}I_1^\varepsilon &= E\int_0^T \Psi^*(t)\mathrm{d}x_1^\varepsilon(t)\\
&= E\int_0^T \left\{\Psi^*(t)f_x(t)x_1^\varepsilon(t) + \Psi^*(t)f_{\widetilde{x}}(t)E\left(x_1^\varepsilon(t)\right) + \Psi^*(t)f_u(t)u(t)\right\}\mathrm{d}t, \end{aligned} \quad (3.8)$$

and

$$\begin{aligned}I_2^\varepsilon &= E\int_0^T x_1^\varepsilon(t)\mathrm{d}\Psi^*(t) = -E\int_0^T \big\{x_1^\varepsilon(t)f_x(t)\Psi^*(t) + x_1^\varepsilon(t)E\left(f_{\widetilde{x}}(t)\Psi^*(t)\right)\\
&+x_1^\varepsilon(t)\sigma_x(t)Q^*(t) + x_1^\varepsilon(t)E\left(\sigma_{\widetilde{x}}(t)Q^*(t)\right)\\
&+\int_\Theta \left[x_1^\varepsilon(t)g_x(t,\theta)K^*(t) + x_1^\varepsilon(t)E\left(g_{\widetilde{x}}(t,\theta)K^*(t)\right)\right.\\
&+x_1^\varepsilon(t)c_x(t,\theta)R(t,\theta) + x_1^\varepsilon(t)E(c_{\widetilde{x}}(t,\theta)R(t,\theta))\\
&\left.+ x_1^\varepsilon(t)\ell_x(t,\theta) + x_1^\varepsilon(t)E(\ell_{\widetilde{x}}(t,\theta))\right]\mu(\mathrm{d}\theta)\big\}\mathrm{d}t. \end{aligned} \quad (3.9)$$

From (3.7), we get

$$\begin{aligned}I_3^\varepsilon &= E\int_0^T Q^*(t)\left[\sigma_x(t)x_1^\varepsilon(t) + \sigma_{\widetilde{x}}(t)E\left(x_1^\varepsilon(t)\right) + \sigma_u(t)u(t)\right]\mathrm{d}t\\
&= E\int_0^T Q^*(t)\sigma_x(t)x_1^\varepsilon(t)\mathrm{d}t + E\int_0^T Q^*(t)\sigma_{\widetilde{x}}(t)E\left(x_1^\varepsilon(t)\right)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+E\int_0^T Q^*(t)\sigma_u(t)u(t)\mathrm{d}t\\
I_4^\varepsilon &= E\int_0^T\int_\Theta [c_x(t,\theta)x_1^\varepsilon(t)+c_{\widetilde{x}}(t,\theta)E(x_1^\varepsilon(t))\\
&\quad +c_u(t,\theta)u(t)]R(t,\theta)\mu(\mathrm{d}\theta)\mathrm{d}t\\
&= E\int_0^T\int_\Theta c_x(t,\theta)x_1^\varepsilon(t)R(t,\theta)\mu(\mathrm{d}\theta)\mathrm{d}t\\
&\quad +E\int_0^T\int_\Theta c_{\widetilde{x}}(t,\theta)E(x_1^\varepsilon(t))R(t,\theta)\mu(\mathrm{d}\theta)\mathrm{d}t\\
&\quad +E\int_0^T\int_\Theta c_u(t,\theta)u(t)R(t,\theta)\mu(\mathrm{d}\theta)\mathrm{d}t. \qquad (3.10)
\end{aligned}
$$

The duality relation (3.4) follows immediately from combining (3.8)–(3.10) and (3.7).

Let us turn to second duality relation (3.5). By applying integration by parts formula for jump process (Lemma 6.1) to $K^*(t)y_1^\varepsilon(t)$, we get

$$
\begin{aligned}
E\left(K^*(T)y_1^\varepsilon(T)\right) &= E\left(K^*(0)y_1^\varepsilon(0)\right)+E\int_0^T\left(K^*(t)\mathrm{d}y_1^\varepsilon(t)\right)+E\int_0^T\left(y_1^\varepsilon(t)\mathrm{d}K^*(t)\right)\\
&\quad +E\int_0^T\int_\Theta z_1^\varepsilon(t)[g_z(t,\theta)K^*(t)+E\left(g_{\widetilde{z}}(t,\theta)K^*(t)\right)\\
&\quad -\ell_z(t,\theta)-E\left(\ell_{\widetilde{z}}(t,\theta)\right)]\mu\left(\mathrm{d}\theta\right)\mathrm{d}t\\
&\quad +E\int_0^T\int_\Theta\left[r_1^\varepsilon(t,\theta)(g_r(t,\theta)K^*(t)-\ell_r(t,\theta))\right]\mu(\mathrm{d}\theta)\mathrm{d}t.\\
&= I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon+I_4^\varepsilon+I_5^\varepsilon. \qquad (3.11)
\end{aligned}
$$

From (3.4), we obtain

$$
\begin{aligned}
I_2^\varepsilon &= E\int_0^T K^*(t)\mathrm{d}y_1^\varepsilon(t)\\
&= -E\int_0^T\int_\Theta\left\{K^*(t)g_x(t,\theta)x_1^\varepsilon(t)+K^*(t)g_{\widetilde{x}}(t,\theta)E\left(x_1^\varepsilon(t)\right)\right.\\
&\quad +K^*(t)g_y(t,\theta)y_1^\varepsilon(t)+K^*(t)g_{\widetilde{y}}(t,\theta)E\left(y_1^\varepsilon(t)\right)+K^*(t)g_z(t,\theta)z_1^\varepsilon(t)\\
&\quad +K^*(t)g_{\widetilde{z}}(t,\theta)E\left(z_1^\varepsilon(t)\right)+K^*(t)g_r(t,\theta)r_1^\varepsilon(t,\theta)\\
&\quad \left.+K^*(t)g_u(t,\theta)u(t)\right\}\mu(\mathrm{d}\theta)\mathrm{d}t, \qquad (3.12)
\end{aligned}
$$

from (2.1), we obtain

$$
\begin{aligned}
I_3^\varepsilon &= E\int_0^T y_1^\varepsilon(t)\mathrm{d}K^*(t)=E\int_0^T\int_\Theta\{y_1^\varepsilon(t)g_y(t,\theta)K^*(t)+y_1^\varepsilon(t)E(g_{\widetilde{y}}(t,\theta)K^*(t))\\
&\quad -y_1^\varepsilon(t)\ell_y(t,\theta)-y_1^\varepsilon(t)E(\ell_{\widetilde{y}}(t,\theta))\}\mu(\mathrm{d}\theta)\mathrm{d}t, \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4^{\varepsilon} = E \int_0^T \int_{\Theta} & \left[z_1^{\varepsilon}(t) g_z(t,\theta) K^*(t) + z_1^{\varepsilon}(t) E\left(g_{\widetilde{z}}(t,\theta) K^*(t)\right) \right. \\
& \left. - z_1^{\varepsilon}(t) \ell_z(t,\theta) - z_1^{\varepsilon}(t) E\left(\ell_{\widetilde{z}}(t,\theta)\right)\right] \mu(\mathrm{d}\theta)\mathrm{d}t, \\
I_5^{\varepsilon} = E \int_0^T \int_{\Theta} & \left\{ r_1^{\varepsilon}(t,\theta) g_r(t,\theta) K^*(t) - r_1^{\varepsilon}(t,\theta)\ell_r(t,\theta)\right\} \mu(\mathrm{d}\theta)\mathrm{d}t. \qquad (3.14)
\end{aligned}
$$

Since

$$
\begin{aligned}
I_1^{\varepsilon} &= E\left(K^*(0) y_1^{\varepsilon}(0)\right) \\
&= -E\left\{ [\varphi_y(y(0), E(y(0))) + E(\varphi_{\widetilde{y}}(y(0), E(y(0))))] y_1^{\varepsilon}(0)\right\},
\end{aligned}
$$

the duality relation (3.5) follows immediately by combining (3.12)–(3.14) and (3.11). Let us turn to (3.6). Combining (3.4) and (3.5) we get

$$
\begin{aligned}
& E\left(\Psi^*(T) x_1^{\varepsilon}(T)\right) + E\left(K^*(T) y_1^{\varepsilon}(T)\right) \\
& \quad = -E[\varphi_y(y(0), E(y(0))) + E(\varphi_{\widetilde{y}}(y(0), E(y(0)))] y_1^{\varepsilon}(0) \\
& \qquad + E \int_0^T \int_{\Theta} \left\{ x_1^{\varepsilon}(t)\ell_x(t,\theta) + x_1^{\varepsilon}(t) E(\ell_{\widetilde{x}}(t,\theta)) - \ell_y(t,\theta) - E(\ell_{\widetilde{y}}(t,\theta)) \right. \\
& \qquad \left. - \ell_u(t,\theta)u(t) - \ell_z(t,\theta) - E(\ell_{\widetilde{z}}(t,\theta)) - r_1^{\varepsilon}(t,\theta)\ell_r(t,\theta)\right\} \mu(\mathrm{d}\theta)\mathrm{d}t \\
& \qquad + E\int_0^T H_u(t) u(t)\mathrm{d}t.
\end{aligned}
$$

From (2.3) and (3.3), we get

$$
\begin{aligned}
& E\left(\Psi^*(T) x_1^{\varepsilon}(T)\right) + E\left(K^*(T) y_1^{\varepsilon}(T)\right) \\
& \quad = [\phi_x(x(T), E(x(T))) + E(\phi_{\widetilde{x}}(x(T), E(x(T)))] x_1^{\varepsilon}(T).
\end{aligned}
$$

Using (2.2), we obtain

$$
\begin{aligned}
E\int_0^T & \{\Psi(t) f_u(t) u(t) + Q(t)\sigma_u(t) u(t) \\
& + \int_{\Theta} [-K(t) g_u(t) u(t) + R(t,\theta) c_u(t,\theta) u(t)) \\
& + \ell_u(t,\theta) u(t)] \mu(\mathrm{d}\theta)\}\mathrm{d}t = E\int_0^T H_u(t) u(t)\mathrm{d}t,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& E\left\{ [\phi_x(x(T), E(x(T))) + E(\phi_{\widetilde{x}}(x(T), E(x(T))))] x_1^{\varepsilon}(T)\right\} \\
& \quad + E\left\{ [\varphi_y(y(0), E(y(0))) + E(\varphi_{\widetilde{y}}(y(0), E(y(0))))] y_1^{\varepsilon}(0)\right\}
\end{aligned}
$$

$$= E\int_0^T\int_\Theta \{x_1^\varepsilon(t)\ell_x(t,\theta) + x_1^\varepsilon(t)E(\ell_{\widetilde{x}}(t,\theta))$$
$$-y_1^\varepsilon(t)\ell_y(t,\theta) - y_1^\varepsilon(t)E(\ell_{\widetilde{y}}(t,\theta)) - z_1^\varepsilon(t)\ell_z(t,\theta) - z_1^\varepsilon(t)E(\ell_{\widetilde{z}}(t,\theta))$$
$$- r_1^\varepsilon(t,\theta)\ell_r(t,\theta) - \ell_u(t,\theta)u(t)\}\,\mu(\mathrm{d}\theta)\mathrm{d}t + E\int_0^T H_u(t)u(t)\mathrm{d}t.$$

This completes the proof of (3.6). □

The second Lemma presents the estimates of the perturbed state process $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot), r^\varepsilon(\cdot,\cdot))$.

Lemma 3.3 *Under assumptions* (H1) *and* (H2), *the following estimations hold*

$$E\left(\sup_{0\le t\le T}\left|x_1^\varepsilon(t)\right|^2\right) \to 0,\ as\ \varepsilon\to 0,$$
$$E\left(\sup_{0\le t\le T}\left|y_1^\varepsilon(t)\right|^2\right) + E\int_0^T [\left|z_1^\varepsilon(s)\right|^2$$
$$+\int_\Theta \left|r_1^\varepsilon(s,\theta)\right|^2\mu(\mathrm{d}\theta)]ds \to 0,\ as\ \varepsilon\to 0, \tag{3.15}$$
$$\sup_{0\le t\le T}\left|E\left(x_1^\varepsilon(t)\right)\right|^2 \to 0,\ as\ \varepsilon\to 0,$$
$$\sup_{0\le t\le T}\left|E\left(y_1^\varepsilon(t)\right)\right|^2 + \int_t^T \left|E\left(z_1^\varepsilon(s)\right)\right|^2\mathrm{d}s$$
$$+\int_0^T\int_\Theta \left|E\left(r_1^\varepsilon(s,\theta)\right)\right|^2\mu(\mathrm{d}\theta)\mathrm{d}s \to 0,\ as\ \varepsilon\to 0,. \tag{3.16}$$
$$E\left(\sup_{0\le t\le T}\left|x^\varepsilon(t)-x^*(t)\right|^2\right) \to 0,\ as\ \varepsilon\to 0,$$
$$E\left(\sup_{0\le t\le T}\left|y^\varepsilon(t)-y^*(t)\right|^2\right) + E\left(\int_0^T \left|z^\varepsilon(t)-z^*(t)\right|^2\right)\mathrm{d}t$$
$$+E\int_0^T\int_\Theta \left|r^\varepsilon(t,\theta)-r^*(t,\theta)\right|^2\mu(\mathrm{d}\theta)\mathrm{d}t \to 0,\ as\ \varepsilon\to 0, \tag{3.17}$$

and

$$E\left(\sup_{0\le t\le T}\left|\frac{1}{\varepsilon}\left[x^\varepsilon(t)-x^*(t)\right]-x_1^\varepsilon(t)\right|^2\right) \to 0,\ as\ \varepsilon\to 0,$$
$$E\left(\sup_{0\le t\le T}\left|\frac{1}{\varepsilon}\left[y^\varepsilon(t)-y^*(t)\right]-y_1^\varepsilon(t)\right|^2\right) \to 0,\ as\ \varepsilon\to 0,$$
$$E\int_0^T\left|\frac{1}{\varepsilon}\left[z^\varepsilon(s)-z^*(s)\right]-z_1^\varepsilon(s)\right|^2\mathrm{d}s \to 0,\ as\ \varepsilon\to 0,$$

$$E\int_0^T\int_\Theta\left|\frac{1}{\varepsilon}\left[r^\varepsilon(s,\theta)-r^*(s,\theta)\right]-r_1^\varepsilon(s,\theta)\right|^2\mu(\mathrm{d}\theta)\mathrm{d}s\to 0,\ as\ \varepsilon\to 0. \tag{3.18}$$

Let us also point out that the above estimates (3.15)–(3.17) can be proved using similar arguments developed in ([21, Lemmas 4.2 and 4.3]) and ([24, Lemmas 2.1]). So we omit their proofs.

Proof **of** (3.18). We set

$$\begin{aligned}
\widehat{x}^\varepsilon(t)&=\frac{1}{\varepsilon}\left[x^\varepsilon(t)-x^*(t)\right]-x_1^\varepsilon(t),\\
\widehat{y}^\varepsilon(t)&=\frac{1}{\varepsilon}\left[y^\varepsilon(t)-y^*(t)\right]-y_1^\varepsilon(t),\\
\widehat{z}^\varepsilon(t)&=\frac{1}{\varepsilon}\left[z^\varepsilon(t)-z^*(t)\right]-z_1^\varepsilon(t),\\
\widehat{r}^\varepsilon(t,\theta)&=\frac{1}{\varepsilon}\left[r^\varepsilon(t,\theta)-r^*(t,\theta)\right]-r_1^\varepsilon(t,\theta),
\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}
f(t)&=f(t,x^*(t),E(x^*(t)),u^*(t)),\sigma(t)=\sigma(t,x^*(t),E(x^*(t)),u^*(t)),\\
c(t,\theta)&=c(t,x^*(t),E(x^*(t)),u^*(t),\theta)\\
g(t,\theta)&=g(x^*(t),E(x^*(t)),y^*(t),E(y^*(t)),z^*(t),E(z^*(t)),r^*(t,\theta),u^*(t)).
\end{aligned}$$

From Eq. (1.1) we have

$$\begin{aligned}
\mathrm{d}\widehat{x}^\varepsilon(t)&=\frac{1}{\varepsilon}\left[\mathrm{d}x^\varepsilon(t)-\mathrm{d}x^*(t)\right]-\mathrm{d}x_1^\varepsilon(t)\\
&=\frac{1}{\varepsilon}[f(t,x^*(t)+\varepsilon(\widehat{x}^\varepsilon(t)+x_1^\varepsilon(t)),E(x^*(t)+\varepsilon(\widehat{x}^\varepsilon(t)+x_1^\varepsilon(t)),u^\varepsilon(t))-f(t)]\mathrm{d}t\\
&\quad-\left[f_x(t)x_1^\varepsilon(t)+f_{\widetilde{x}}(t)E\left(x_1^\varepsilon(t)\right)+f_u(t)u(t)\right]\mathrm{d}t\\
&\quad+\frac{1}{\varepsilon}[\sigma(t,x^*(t)+\varepsilon(\widehat{x}^\varepsilon(t)+x_1^\varepsilon(t)),E(x^*(t)+\varepsilon(\widehat{x}^\varepsilon(t)+x_1^\varepsilon(t))),u^\varepsilon(t))\\
&\quad-\sigma(t)]dW(t)-\left[\sigma_x(t)x_1^\varepsilon(t)+\sigma_{\widetilde{x}}(t)E\left(x_1^\varepsilon(t)\right)+\sigma_u(t)u(t)\right]dW(t)\\
&\quad+\int_\Theta[c(t,x^*(t)+\varepsilon(\widehat{x}^\varepsilon(t)+x_1^\varepsilon(t)),E(x^*(t)+\varepsilon(\widehat{x}^\varepsilon(t)+x_1^\varepsilon(t))),u^\varepsilon(t),\theta)\\
&\quad-c(t,\theta)]N(\mathrm{d}\theta,\mathrm{d}t)-\int_\Theta[c_x(t,\theta)x_1^\varepsilon(t)+c_{\widetilde{x}}(t,\theta)E(x_1^\varepsilon(t))\\
&\quad+c_u(t,\theta)u(t)]N(\mathrm{d}\theta,\mathrm{d}t).
\end{aligned}\tag{3.20}$$

We denote

$$\begin{aligned}
x^{\lambda,\varepsilon}(t)&=x^*(t)+\lambda\varepsilon(\widehat{x}^\varepsilon(t)+x_1^\varepsilon(t)),\\
y^{\lambda,\varepsilon}(t)&=y^*(t)+\lambda\varepsilon(\widehat{y}^\varepsilon(t)+y_1^\varepsilon(t)),
\end{aligned}$$

$$
\begin{aligned}
z^{\lambda,\varepsilon}(t) &= z^*(t) + \lambda\varepsilon(\widehat{z}^{\varepsilon}(t) + z_1^{\varepsilon}(t)),\\
r^{\lambda,\varepsilon}(t,\theta) &= r^*(t,\theta) + \lambda\varepsilon(\widehat{r}^{\varepsilon}(t,\theta) + r_1^{\varepsilon}(t,\theta)),\\
u^{\lambda,\varepsilon}(t) &= u^*(t) + \lambda\varepsilon u(t).
\end{aligned} \tag{3.21}
$$

By Taylor's expansion with a simple computations, we show that

$$
\widehat{x}^{\varepsilon}(t) = \frac{1}{\varepsilon}\left[x^{\varepsilon}(t) - x^*(t)\right] - x_1^{\varepsilon}(t) = \widetilde{I}_1(\varepsilon) + \widetilde{I}_2(\varepsilon) + \widetilde{I}_3(\varepsilon), \tag{3.22}
$$

where

$$
\begin{aligned}
\widetilde{I}_1(\varepsilon) &= \int_0^t \int_0^1 f_x(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s))(\widehat{x}^{\varepsilon}(s) + x_1^{\varepsilon}(s))\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 f_{\widetilde{x}}(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s))E(\widehat{x}^{\varepsilon}(s) + x_1^{\varepsilon}(s))\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 \left[f_x(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s)) - f_x(s)\right] x_1^{\varepsilon}(s)\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 \left[f_{\widetilde{x}}(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s)) - f_{\widetilde{x}}(s)\right] E(x_1^{\varepsilon}(s))\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 \left[f_u(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s)) - f_u(s)\right] u(s)\mathrm{d}\lambda\mathrm{d}s,
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
\widetilde{I}_2(\varepsilon) &= \int_0^t \int_0^1 \sigma_x(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s))[\widehat{x}^{\varepsilon}(s) + x_1^{\varepsilon}(s)]\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 \sigma_{\widetilde{x}}(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s))E[\widehat{x}^{\varepsilon}(s) + x_1^{\varepsilon}(s)]\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 [\sigma_x(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s)) - \sigma_x(s)]x_1^{\varepsilon}(s)\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 [\sigma_{\widetilde{x}}(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s)) - \sigma_{\widetilde{x}}(s)]E(x_1^{\varepsilon}(s))\mathrm{d}\lambda\mathrm{d}s\\
&\quad + \int_0^t \int_0^1 \left[\sigma_u(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s)) - \sigma_u(s)\right] u(s)\mathrm{d}\lambda\mathrm{d}s,
\end{aligned} \tag{3.24}
$$

and

$$
\begin{aligned}
&\widetilde{I}_3(\varepsilon)\\
&= \int_0^t \int_{\Theta} \int_0^1 c_x(s, x^{\lambda,\varepsilon}(s_-), E(x^{\lambda,\varepsilon}(s_-)), u^{\lambda,\varepsilon}(s), \theta)[\widehat{x}^{\varepsilon}(s) + x_1^{\varepsilon}(s)]\mathrm{d}\lambda N(\mathrm{d}\theta, \mathrm{d}s)\\
&+ \int_0^t \int_{\Theta} \int_0^1 c_{\widetilde{x}}(s, x^{\lambda,\varepsilon}(s_-), E(x^{\lambda,\varepsilon}(s_-)), u^{\lambda,\varepsilon}(s), \theta)E[\widehat{x}^{\varepsilon}(s) + x_1^{\varepsilon}(s)]\mathrm{d}\lambda N(\mathrm{d}\theta, \mathrm{d}s).\\
&+ \int_0^t \int_{\Theta} \int_0^1 \left[c_x(s, x^{\lambda,\varepsilon}(s_-), E(x^{\lambda,\varepsilon}(s_-)), u^{\lambda,\varepsilon}(s), \theta) - c_x(s,\theta)\right] x_1^{\varepsilon}(s)\mathrm{d}\lambda N(\mathrm{d}\theta, \mathrm{d}s)
\end{aligned}
$$

$$+\int_0^t \int_\Theta \int_0^1 [c_{\widetilde{x}}(s, x^{\lambda,\varepsilon}(s_-), E(x^{\lambda,\varepsilon}(s_-)), u^{\lambda,\varepsilon}(s), \theta) - c_{\widetilde{x}}(s,\theta)]E(x_1^\varepsilon(s))\mathrm{d}\lambda N(\mathrm{d}\theta, \mathrm{d}s)$$
$$+\int_0^t \int_\Theta \int_0^1 [c_u(s, x^{\lambda,\varepsilon}(s), E(x^{\lambda,\varepsilon}(s)), u^{\lambda,\varepsilon}(s), \theta) - c_u(s,\theta)]u(s)\mathrm{d}\lambda N(\mathrm{d}\theta, \mathrm{d}s), \quad (3.25)$$

we proceed as in Anderson and Djehiche [18, pp. 7–8], we get

$$E\left(\sup_{0\le t\le T} \left|\widetilde{I}_1(\varepsilon)\right|^2\right) \to 0, \quad as \quad \varepsilon \to 0,$$
$$E\left(\sup_{0\le t\le T} \left|\widetilde{I}_2(\varepsilon)\right|^2\right) \to 0, \quad as \quad \varepsilon \to 0. \quad (3.26)$$

Applying similar estimations for the third term with the help of *Proposition 3.2* (in Appendix Bouchard and Elie [27]), we have

$$E\left(\sup_{0\le t\le T} \left|\widetilde{I}_3(\varepsilon)\right|^2\right) \to 0, \quad as, \quad \varepsilon \to 0. \quad (3.27)$$

From (3.26) and (3.27) we obtain

$$E\left(\sup_{0\le t\le T} \left|\frac{1}{\varepsilon}\left[x^\varepsilon(t) - x^*(t)\right] - x_1^\varepsilon(t)\right|^2\right) \to 0, \quad as \quad \varepsilon \to 0. \quad (3.28)$$

We proceed to estimate the last terms in (3.18). First, from (3.19) and since $\widehat{y}^\varepsilon(t) = \frac{1}{\varepsilon}\left[y^\varepsilon(t) - y^*(t)\right] - y_1^\varepsilon(t)$, we get

$$\mathrm{d}\widehat{y}^\varepsilon(t) = -\frac{1}{\varepsilon}\int_\Theta [g(t, x^*(t) + \varepsilon(\widehat{x}^\varepsilon(t) + x_1^\varepsilon(t)), E(x^*(t) + \varepsilon(\widehat{x}^\varepsilon(t) + x_1^\varepsilon(t))),$$
$$y^*(t) + \varepsilon(\widehat{y}^\varepsilon(t) + y_1^\varepsilon(t)), E(y^*(t) + \varepsilon(\widehat{y}^\varepsilon(t) + y_1^\varepsilon(t))), z^*(t) + \varepsilon(\widehat{z}^\varepsilon(t) + z_1^\varepsilon(t)),$$
$$E(z^*(t)+\varepsilon(\widehat{z}^\varepsilon(t)+z_1^\varepsilon(t))), r^*(t,\theta)+\varepsilon(\widehat{r}^\varepsilon(t,\theta)+r_1^\varepsilon(t,\theta)), u^\varepsilon(t))-g(t,\theta)]\mu(\mathrm{d}\theta)\mathrm{d}t$$
$$-\int_\Theta [g_x(t,\theta)x_1^\varepsilon(t) + g_{\widetilde{x}}(t,\theta)E(x_1^\varepsilon(t)) + g_y(t,\theta)y_1^\varepsilon(t) + g_{\widetilde{y}}(t,\theta)E(y_1^\varepsilon(t))$$
$$+g_z + (t,\theta)z_1^\varepsilon(t) + g_{\widetilde{z}}(t,\theta)E(z_1^\varepsilon(t)) + g_r(t,\theta)r_1^\varepsilon(t,\theta) + g_u(t,\theta)u(t)]\mu(\mathrm{d}\theta)\mathrm{d}t$$
$$+\widehat{z}^\varepsilon(t)\mathrm{d}W(t) + \int_\Theta \widehat{r}^\varepsilon(t,\theta)N(\mathrm{d}\theta, \mathrm{d}t),$$

and

$$\widehat{y}^\varepsilon(T) = \frac{1}{\varepsilon}[h(x^\varepsilon(T), E(x^\varepsilon(T))) - h(x(T), E(x(T)))]$$
$$+[h_x(x(T), E(x(T))) + h_{\widetilde{x}}(x(T), E(x(T)))]x_1^\varepsilon(T).$$

Applying Taylor's expansion, we get

$$
\begin{aligned}
-\mathrm{d}\widehat{y}^{\varepsilon}(t) = & \int_{\Theta}\int_0^1 g_x(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)),\\
& r^{\lambda,\varepsilon}(t,\theta), u^{\lambda,\varepsilon}(t)) \times (\widehat{x}^{\varepsilon}(t) + x_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 g_{\widetilde{x}}(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) \times E(\widehat{x}^{\varepsilon}(t) + x_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 [g_x(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) - g_x(t,\theta)]x_1^{\varepsilon}(t)\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 [g_{\widetilde{x}}(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) - g_{\widetilde{x}}(t,\theta)]E(x_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 [g_u(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) - g_u(t,\theta)]u(t)\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 g_y(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) \times (\widehat{y}^{\varepsilon}(t) + y_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 g_{\widetilde{y}}(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) \times E(\widehat{y}^{\varepsilon}(t) + y_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 [g_y(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) - g_y(t,\theta)]y_1^{\varepsilon}(t)\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 [g_{\widetilde{y}}(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) - g_{\widetilde{y}}(t,\theta)] \times E(y_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 g_z(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) \times (\widehat{z}^{\varepsilon}(t) + z_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 g_{\widetilde{z}}(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) \times E(\widehat{z}^{\varepsilon}(t) + z_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t\\
& + \int_{\Theta}\int_0^1 [g_z(t, x^{\lambda,\varepsilon}(t), E(x^{\lambda,\varepsilon}(t)), y^{\lambda,\varepsilon}(t), E(y^{\lambda,\varepsilon}(t)), z^{\lambda,\varepsilon}(t), E(z^{\lambda,\varepsilon}(t)), r^{\lambda,\varepsilon}(t,\theta),\\
& u^{\lambda,\varepsilon}(t)) - g_z(t,\theta)] \times z_1^{\varepsilon}(t)\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t
\end{aligned}
$$

$$+\int_{\Theta}\int_{0}^{1}[g_{\widetilde{z}}(t,x^{\lambda,\varepsilon}(t),E(x^{\lambda,\varepsilon}(t)),y^{\lambda,\varepsilon}(t),E(y^{\lambda,\varepsilon}(t)),z^{\lambda,\varepsilon}(t),E(z^{\lambda,\varepsilon}(t)),r^{\lambda,\varepsilon}(t,\theta),$$
$$u^{\lambda,\varepsilon}(t))-g_{\widetilde{z}}(t,\theta)]E(z_1^{\varepsilon}(t))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t$$
$$+\int_{\Theta}\int_{0}^{1}g_r(t,x^{\lambda,\varepsilon}(t),E(x^{\lambda,\varepsilon}(t)),y^{\lambda,\varepsilon}(t),E(y^{\lambda,\varepsilon}(t)),z^{\lambda,\varepsilon}(t),E(z^{\lambda,\varepsilon}(t)),r^{\lambda,\varepsilon}(t,\theta),$$
$$u^{\lambda,\varepsilon}(t))\times(\widehat{r}^{\varepsilon}(t,\theta)+r_1^{\varepsilon}(t,\theta))\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t$$
$$+\int_{\Theta}\int_{0}^{1}[g_r(t,x^{\lambda,\varepsilon}(t),E(x^{\lambda,\varepsilon}(t)),y^{\lambda,\varepsilon}(t),E(y^{\lambda,\varepsilon}(t)),z^{\lambda,\varepsilon}(t),E(z^{\lambda,\varepsilon}(t)),r^{\lambda,\varepsilon}(t,\theta),$$
$$u^{\lambda,\varepsilon}(t))-g_r(t,\theta)]r_1^{\varepsilon}(t,\theta)\mathrm{d}\lambda\mu(\mathrm{d}\theta)\mathrm{d}t$$
$$-\widehat{z}^{\varepsilon}(t)\mathrm{d}W(t)-\int_{\Theta}\widehat{r}^{\varepsilon}(t,\theta)N(\mathrm{d}\theta,\mathrm{d}t),$$

finally, using similar arguments developed in [24, pp. 222–224], the desired result follows. This completes the proof of (3.18). □.

Lemma 3.4 *Let assumptions* (H1) *and* (H2) *hold. The following variational inequality holds*

$$E\int_0^T\int_{\Theta}[\ell_x(t,\theta)x_1^{\varepsilon}(t)+\ell_{\widetilde{x}}(t,\theta)E(x_1^{\varepsilon}(t))+\ell_y(t,\theta)y_1^{\varepsilon}(t)+\ell_{\widetilde{y}}(t,\theta)E(y_1^{\varepsilon}(t))$$
$$+\ell_z(t,\theta)z_1^{\varepsilon}(t)+\ell_{\widetilde{z}}(t,\theta)E(z_1^{\varepsilon}(t))+\ell_r(t,\theta)r_1^{\varepsilon}(t,\theta)+\ell_u(t,\theta)u(t)]\mu(\mathrm{d}\theta)\mathrm{d}t$$
$$+E[\phi_x(T)x_1^{\varepsilon}(T)+\phi_{\widetilde{x}}(T)E(x_1^{\varepsilon}(T))]+E[\varphi_y(0)y_1^{\varepsilon}(0)+\varphi_{\widetilde{y}}(0)E\left(y_1^{\varepsilon}(0)\right)]\geq o(\varepsilon).$$

Proof From (3.2) we have

$$J\left(u^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot)\right)$$
$$=E\Big\{\int_0^T\int_{\Theta}[\ell(t,x^{\varepsilon}(t),E(x^{\varepsilon}(t)),y^{\varepsilon}(t),E(y^{\varepsilon}(t)),z^{\varepsilon}(t),E(z^{\varepsilon}(t)),r^{\varepsilon}(t,\theta),u^{\varepsilon}(t))$$
$$-\ell(t,x^{*}(t),E(x^{*}(t)),y^{*}(t),E(y^{*}(t)),z^{*}(t),E(z^{*}(t)),r^{*}(t,\theta),u^{*}(t)]\mu(\mathrm{d}\theta)\,\mathrm{d}t$$
$$+\left[\phi\left(x^{\varepsilon}(T),E(x^{\varepsilon}(T))\right)-\phi\left(x^{*}(T),E(x^{*}(T))\right)\right]$$
$$+\left[\varphi\left(x^{\varepsilon}(0),E\left(x^{\varepsilon}(0)\right)\right)-\varphi\left(y^{*}(0),E\left(y^{*}(0)\right)\right)\right]\Big\}\geq 0. \tag{3.29}$$

By applying Taylor's expansion and Lemma 3.3, we have

$$\frac{1}{\varepsilon}E[\phi(x^{\varepsilon}(T),\widetilde{x}^{\varepsilon}(T))-\phi(x^{*}(T),\widetilde{x}^{*}(T))]$$
$$=\frac{1}{\varepsilon}E\left\{\int_0^1\phi_x(x^{*}(T)+\lambda(x^{\varepsilon}(T)-x^{*}(T)),\widetilde{x}^{*}(T)\right.$$
$$+\lambda(\widetilde{x}^{\varepsilon}(T)-\widetilde{x}^{*}(T)))\mathrm{d}\lambda(x^{\varepsilon}(T)-x^{*}(T))$$
$$+\int_0^1\phi_{\widetilde{x}}(x^{*}(T)+\lambda(x^{\varepsilon}(T)-x^{*}(T)),\widetilde{x}^{*}(T)$$
$$\left.+\lambda(\widetilde{x}^{\varepsilon}(T)-\widetilde{x}^{*}(T)))\mathrm{d}\lambda(\widetilde{x}^{\varepsilon}(T)-\widetilde{x}^{*}(T))\right\}+o(\varepsilon).$$

From estimate (3.18), we get

$$\begin{aligned}
&\frac{1}{\varepsilon}E[\phi(x^{\varepsilon}(T),\widetilde{x}^{\varepsilon}(T))-\phi(x^{*}(T),\widetilde{x}^{*}(T))] \\
&\quad\to E[\phi_{x}(x^{*}(T),E(x^{*}(T)))x_{1}^{\varepsilon}(T)+\phi_{\widetilde{x}}(x^{*}(T),E(x^{*}(T)))E(x_{1}^{\varepsilon}(T))] \\
&\quad= E\left[\phi_{x}(T)x_{1}^{\varepsilon}(T)+\phi_{\widetilde{x}}(T)E(x_{1}^{\varepsilon}(T))\right], \quad as\ \varepsilon\to 0.
\end{aligned} \tag{3.30}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}E[\varphi(y^{\varepsilon}(0),\widetilde{y}^{\varepsilon}(0))-\varphi(y^{*}(0),\widetilde{y}^{*}(0))] \\
&\quad\to E[\varphi_{y}(y^{*}(0),\widetilde{y}^{*}(0))y_{1}^{\varepsilon}(0)+\varphi_{\widetilde{y}}(y^{*}(0),\widetilde{y}^{*}(0))E\left(y_{1}^{\varepsilon}(0)\right)] \\
&\quad= E\left[\varphi_{y}(0)y_{1}^{\varepsilon}(0)+\varphi_{\widetilde{y}}(0)E\left(y_{1}^{\varepsilon}(0)\right)\right], \quad as\ \varepsilon\to 0,
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon}E\int_{0}^{T}\int_{\Theta}[\ell(t,x^{\varepsilon}(t),E(x^{\varepsilon}(t)),y^{\varepsilon}(t),E(y^{\varepsilon}(t)),z^{\varepsilon}(t),E(z^{\varepsilon}(t)),r^{\varepsilon}(t,\theta),u^{\varepsilon}(t)) \\
&\quad-\ell(t,x^{*}(t),E(x^{*}(t)),y^{*}(t),E(y^{*}(t)),z^{*}(t),E(z^{*}(t)),r^{*}(t,\theta),u^{*}(t))]\mu(\mathrm{d}\theta)\,\mathrm{d}t \\
&\quad\to E\int_{0}^{T}\int_{\Theta}[\ell_{x}(t,\theta)x_{1}^{\varepsilon}(t)+\ell_{\widetilde{x}}(t,\theta)E(x_{1}^{\varepsilon}(t))+\ell_{y}(t,\theta)y_{1}^{\varepsilon}(t)+\ell_{\widetilde{y}}(t,\theta)E(y_{1}^{\varepsilon}(t)) \\
&\quad+\ell_{z}(t,\theta)z_{1}^{\varepsilon}(t)+\ell_{\widetilde{z}}(t,\theta)E(z_{1}^{\varepsilon}(t))+\ell_{r}(t,\theta)r_{1}^{\varepsilon}(t,\theta)+\ell_{u}(t,\theta)u(t)]\mu(\mathrm{d}\theta)\mathrm{d}t, \\
&\quad as\ \ \varepsilon\to 0.
\end{aligned} \tag{3.32}$$

The desired result follows by combining (3.29)–(3.32). This completes the proof of Lemma 3.4. □

Proof of Theorem 3.1 The desired result follows immediately by combining (3.6) in Lemmas 3.3 and 3.4. □

4 Application: Mean-Variance Portfolio Selection Problem Mixed with a Recursive Utility Functional, Time-Inconsistent Solution

The mean-variance portfolio selection theory, which was first proposed in Markowitz [31] is a milestone in mathematical finance and has laid down the foundation of modern finance theory. Using sufficient maximum principle, the authors in [30] gave the expression for the optimal portfolio selection in a jump diffusion market with time consistent solutions. The near-optimal consumption-investment problem has been discussed in Hafayed, Abbas and Veverka [28]. The continuous time mean-variance portfolio selection problem has been studied in Zhou and Li [32]. The mean-variance portfolio selection problem where the state driven by SDE (without jump terms) has been studied in [18]. Optimal dividend, harvesting rate, and optimal portfolio for systems governed by jump diffusion processes have been investigated in [10]. Mean-variance portfolio selection problem mixed with a recursive utility functional has been studied by Shi and Wu [24], under the condition that

$$E(x^{\pi}(T)) = c,$$

where c is a given real positive number.

In this section, we will apply our mean-field stochastic maximum principle of optimality to study a mean-variance portfolio selection problem mixed with a recursive utility functional time-inconsistent solutions in a financial market and we will derive the explicit expression for the optimal portfolio selection strategy. This optimal control is represented by a state feedback form involving both $x(\cdot)$ and $E(x(\cdot))$.

Suppose that we are given a mathematical market consisting of two investment possibilities:

1. *Risk-free security* (*Bond price*). The first asset is a risk-free security whose price $P_0(t)$ evolves according to the ordinary differential equation

$$\begin{cases} \mathrm{d}P_0(t) = \rho(t)P_0(t)\,\mathrm{d}t, \quad t \in [0, T], \\ P_0(0) > 0, \end{cases} \tag{4.1}$$

where $\rho(\cdot) : [0, T] \to \Re_+$ is a locally bounded and continuous deterministic function.

2. *Risk-security* (*Stock price*). A risk-security (e.g., a stock), where the price $P_1(t)$ at time t is given by

$$\begin{cases} \mathrm{d}P_1(t) = P_1(t_-)\left[\varsigma(t)\mathrm{d}t + G(t)\mathrm{d}W(t) + \int_\Theta \xi(t,\theta)\,N(\mathrm{d}\theta, \mathrm{d}t)\right], \\ P_1(0) > 0, \quad t \in [0, T]. \end{cases} \tag{4.2}$$

Assumptions. In order to ensure that $P_1(t) > 0$ for all $t \in [0, T]$, we assume

1. The functions $\varsigma(\cdot) : [0, T] \to \Re$, $G(\cdot) : [0, T] \to \Re$ are bounded deterministic such that

$$\varsigma(t), G(t) \neq 0, \quad \varsigma(t) > \rho(t), \forall t \in [0, T],$$

2. $\xi(t, \theta) > -1$ for $\mu-$almost all $\theta \in \Theta$ and all $t \in [0, T]$,
3. $\int_\Theta \xi^2(t, \theta)\,\mu(d\theta)$ is bounded.

Portfolio strategy, *the price dynamic with recursive utility process.* A portfolio is a $\mathcal{F}_t-$predictable process $e(t) = (e_1(t), e_2(t))$ giving the number of units of the risk-free and the risky security held at time t. Let $\pi(t) = e_1(t)\,P_0(t)$ denote the amount invested in the risky security. We call the control process $\pi(\cdot)$ a portfolio strategy.

Let $x^{\pi}(0) = \zeta > 0$ be an initial wealth. By combining (4.1) and (4.2), we introduce the wealth process $x^{\pi}(\cdot)$ and the recursive utility process $y^{\pi}(\cdot)$ corresponding to $\pi(\cdot) \in \mathcal{U}([0, T])$ as solution of the following FBSDEJs

$$\begin{cases} \mathrm{d}x^{\pi}(t) = [\rho(t)x^{\pi}(t) + (\varsigma(t) - \rho(t))\pi(t)]\,\mathrm{d}t \\ \qquad\qquad +G(t)\pi(t)\mathrm{d}W(t) + \int_\Theta \xi(t,\theta)\,\pi(t)N(\mathrm{d}\theta, \mathrm{d}t), \\ \qquad\qquad -\mathrm{d}y^{\pi}(t) = [\rho(t)x^{\pi}(t) + (\varsigma(t) - \rho(t))\pi(t) - \alpha y^{\pi}(t)]\,\mathrm{d}t \\ \qquad\qquad -z^{\pi}(t)dW(t) - \int_\Theta r^{\pi}(t,\theta)\,N(\mathrm{d}\theta, \mathrm{d}t), \\ x^{\pi}(0) = \zeta, \quad y^{\pi}(T) = x^{\pi}(T). \end{cases} \tag{4.3}$$

Mean-variance portfolio selection problem mixed with a recursive utility functional: In this section, the objective is to apply our maximum principle to study the mean-variance portfolio selection problem mixed with a recursive utility functional maximization.

The cost functional, to be minimized, is given by

$$J(\pi(\cdot)) = \frac{\gamma}{2}\mathrm{Var}(x^\pi(T)) - E(x^\pi(T)) - y^\pi(0). \tag{4.4}$$

By a simple computation, we can show that

$$J(\pi(\cdot)) = E\left[\frac{\gamma}{2}x^\pi(T)^2 - x^\pi(T)\right] - \frac{\gamma}{2}\left[E(x^\pi(T))\right]^2 - y^\pi(0), \tag{4.5}$$

where the wealth process $x^\pi(\cdot)$ and the recursive utility process $y^\pi(\cdot)$ corresponding to $\pi(\cdot) \in \mathcal{U}([0,T])$ are given by FBSDEJ-(4.3). We note that the cost functional (4.5) becomes a time-inconsistent control problem. Let $\mathcal{A}$ be a compact convex subset of $\Re$. We denote $\mathcal{U}([0,T])$ the set of admissible $\mathcal{F}_t-$predictable portfolio strategies $\pi(\cdot)$ valued in $\mathcal{A}$. The optimal solution is denoted by $(x^*(\cdot), \pi^*(\cdot))$. The Hamiltonian functional (2.2) gets the form

$$\begin{aligned}
&H(t, x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, r, \pi, \Psi, Q, K, R)\\
&\quad = [\rho(t)x(t) + (\varsigma(t) - \rho(t))\pi(t)](\Psi(t) + K(t))\\
&\quad + G(t)\pi(t)Q(t) - \alpha K(t)y(t) + \int_\Theta \xi(t,\theta)\pi(t)R(t,\theta)\mu(\mathrm{d}\theta).
\end{aligned}$$

According to the maximum condition ((3.1), Theorem 3.1), and since $\pi^*(\cdot)$ is optimal we immediately get

$$\begin{aligned}
&(\varsigma(t) - \rho(t))\left(\Psi^*(t) + K^*(t)\right) + G(t)Q^*(t)\\
&\quad + \int_\Theta \xi(t,\theta) R^*(t,\theta)\mu(\mathrm{d}\theta) = 0.
\end{aligned} \tag{4.6}$$

The adjoint equation (2.1) being

$$\begin{cases}
\mathrm{d}\Psi^*(t) = -\rho(t)(K^*(t) + \Psi^*(t))\,\mathrm{d}t + Q^*(t)\mathrm{d}W(t)\\
\qquad\qquad + \int_\Theta R^*(t,\theta)N(\mathrm{d}\theta, \mathrm{d}t).\\
\Psi^*(T) = \gamma(x^*(T) + E(x^*(T))) - 1 - K^*(T),\\
\mathrm{d}K^*(t) = -\alpha K^*(t)\mathrm{d}t, \quad K^*(0) = 1, \quad t \in [0,T].
\end{cases} \tag{4.7}$$

In order to solve the above equation (4.7) and to find the expression of optimal portfolio strategy $\pi^*(\cdot)$, we conjecture a process $\Psi^*(t)$ of the form:

$$\Psi^*(t) = A_1(t)x^*(t) + A_2(t)E\left(x^*(t)\right) + A_3(t), \tag{4.8}$$

where $A_1(\cdot)$, $A_2(\cdot)$, and $A_3(\cdot)$ are deterministic differentiable functions. (see Shi and Wu [24], Shi [9], Framstad, Øksendal and Sulem [30], Li [19], Yong [20], for other models of conjecture). From last equation in (4.7), which is a simple ordinary differential equation (ODE in short), we get immediately

$$K^*(t) = \exp(-\alpha t). \tag{4.9}$$

Noting that from (4.3), we get

$$\mathrm{d}(E(x^*(t)) = \left\{\rho(t)E(x^*(t)) + (\varsigma(t) - \rho(t))E(\pi^*(t))\right\} \mathrm{d}t.$$

Applying Itô's formula to (4.8) (see Lemma 6.1, Appendix) in virtue of SDE-(4.3), we get

$$\begin{aligned}
\mathrm{d}\Psi^*(t) &= A_1(t)\left\{\left[\rho(t)x^*(t) + (\varsigma(t) - \rho(t))\pi^*(t)\right]\mathrm{d}t\right. \\
&\quad \left. + G(t)\pi^*(t)dW(t) + \int_\Theta \xi\left(t_-, \theta\right)\pi^*(t)N\left(\mathrm{d}\theta, \mathrm{d}t\right)\right\} \\
&\quad + x^*(t)A_1'(t)\mathrm{d}t + A_2(t)\left[\rho(t)E(x^*(t)) + (\varsigma(t) - \rho(t))E(\pi^*(t))\right]\mathrm{d}t \\
&\quad + E\left(x^*(t)\right)A_2'(t)\mathrm{d}t + A_3'(t)\mathrm{d}t,
\end{aligned}$$

which implies that

$$\begin{cases}
\mathrm{d}\Psi^*(t) = \left\{A_1(t)\left[\rho(t)x^*(t) + (\varsigma(t) - \rho(t))\,\pi^*(t)\right] + x^*(t)A_1'(t)\right. \\
\qquad + A_2(t)\left[\rho(t)E(x^*(t)) + (\varsigma(t) - \rho(t))E(\pi^*(t))\right] \\
\qquad \left. + A_2'(t)E\left(x^*(t)\right) + A_3'(t)\right\}\mathrm{d}t + A_1(t)G(t)\pi^*(t)\mathrm{d}W(t) \\
\qquad + \int_\Theta A_1(t)\xi\left(t_-, \theta\right)\pi^*(t)N\left(\mathrm{d}\theta, \mathrm{d}t\right), \\
\Psi^*(T) = A_1(T)x^*(T) + A_2(T)E\left(x^*(T)\right) + A_3(T),
\end{cases} \tag{4.10}$$

where $A_1'(t)$, $A_2'(t)$ and $A_3'(t)$ denote the derivatives with respect to t.
Next, comparing (4.10) with (4.7), we get

$$\begin{aligned}
-\rho(t)\left(K^*(t) + \Psi^*(t)\right) &= A_1(t)\left[\rho(t)x^*(t) + (\varsigma(t) - \rho(t))\pi^*(t)\right] + x^*(t)A_1'(t) \\
&\quad + A_2(t)\left[\rho(t)E(x^*(t)) + (\varsigma(t) - \rho(t))E(\pi^*(t))\right] \\
&\quad + A_2'(t)E\left(x^*(t)\right) + A_3'(t), \qquad (4.11) \\
Q^*(t) &= A_1(t)G(t)\pi^*(t), \qquad (4.12) \\
R^*(t,\theta) &= A_1(t)\xi\left(t,\theta\right)\pi^*(t). \qquad (4.13)
\end{aligned}$$

By looking at the terminal condition of $\Psi^*(t)$, in (4.10), it is reasonable to get

$$A_1(T) = \gamma, \quad A_2(T) = -\gamma, \quad A_3(T) = -1 - K^*(T). \tag{4.14}$$

Combining (4.11) and (4.8), we deduce that $A_1(\cdot)$, $A_2(\cdot)$, and $A_3(\cdot)$ satisfying the following ODEs:

$$\begin{cases} A_1'(t) = -2\rho(t)A_1(t), \quad A_1(T) = \gamma, \\ A_2'(t) = -2\rho(t)A_2(t), \quad A_2(T) = -\gamma, \\ A_3'(t) + \rho(t)A_3(t) = \rho(t)\exp\{-\alpha t\}, \ A_3(T) = -\exp\{-\alpha T\} - 1. \end{cases} \tag{4.15}$$

By solving the first two ordinary differential equations in (4.15), we obtain

$$A_1(t) = -A_2(t) = \gamma \exp\left\{2\int_t^T \rho(s)\mathrm{d}s\right\}. \tag{4.16}$$

Using integrating factor method for the third equation in (4.15), we get

$$A_3(t) = -\chi(t)^{-1}\left[\exp(-\alpha T) + 1 + \int_t^T \chi(s)\rho(s)\exp\{-\alpha s\}\,\mathrm{d}s\right], \tag{4.17}$$

where the integrating factor is $\chi(t) = \exp\left\{\int_t^T \rho(s)\mathrm{d}s\right\}, \quad \chi(T) = 1.$
Combining (4.6), (4.9), (4.12) and (4.13) and denoting

$$\Gamma(t) = A_1(t)\left(G^2(t) + \int_\Theta \xi^2(t,\theta)\,\mu(\mathrm{d}\theta)\right), \tag{4.18}$$

we get

$$\pi^*(t) = \Gamma(t)^{-1}(\rho(t) - \varsigma(t))\left[A_1(t)\left(x^*(t) - E(x^*(t))\right) + A_3(t) - \exp(-\alpha t)\right], \tag{4.19}$$

and

$$E(\pi^*(t)) = \Gamma(t)^{-1}(\rho(t) - \varsigma(t))\left[A_3(t) - \exp\{-\alpha t\}\right]. \tag{4.20}$$

Finally, we give the explicit optimal portfolio selection strategy in the state feedback form involving both $x^*(\cdot)$ and $E(x^*(\cdot))$.

Theorem 4.1 *The optimal portfolio strategy $\pi^*(t)$ of our mean-variance portfolio selection* problems (4.3)–(4.5) *is given in feedback form by*

$$\begin{aligned} \pi^*(t, x^*(t), E(x^*(t)) = \Gamma(t)^{-1}(\rho(t) \\ -\varsigma(t))\left[A_1(t)\left(x^*(t) - E(x^*(t))\right) + A_3(t) - \exp\{-\alpha t\}\right], \end{aligned}$$

and

$$E(\pi^*(t, x^*(t), E(x^*(t))) = \Gamma(t)^{-1}(\rho(t) - \varsigma(t))\left[A_3(t) - \exp\{-\alpha t\}\right],$$

where $A_1(t)$, $A_3(t)$, and $\Gamma(t)$ are given by (4.16), (4.17) *and* (4.18) *respectively.*

5 Conclusions

In this paper, we have discussed the necessary conditions for optimal stochastic control of mean-field forward–backward stochastic differential equations with Poisson jumps (FBSDEJs). Time-inconsistent mean-variance portfolio selection mixed with recursive utility functional optimization problem has been studied to illustrate our theoretical results. We would like to indicate that the general maximum principle for fully coupled mean-field FBSDEJs is not addressed, and we will work for this interesting issue in the future research.

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6 Appendix

The following result gives special case of the Itô formula for mean-field jump diffusions.

Lemma 6.1 (*Integration by parts formula for mean-field jump diffusions*). *Suppose that the processes $x_1(t)$ and $x_2(t)$ are given by for $i = 1, 2, t \in [0, T]$*

$$\begin{cases} dx_i(t) = f\,(t, x_i(t), E(x_i(t)), u(t))\,dt + \sigma\,(t, x_i(t), E(x_i(t)), u(t))\,dW(t) \\ \qquad\qquad + \int_{\Theta} g\,(t, x_i(t_-), E(x_i(t_-)), u(t), \theta)\,N\,(d\theta, dt)\,, \\ x_i(0) = 0. \end{cases}$$

Then we get

$$\begin{aligned} E\,(x_1(T)x_2(T)) &= E\left[\int_0^T x_1(t)dx_2(t) + \int_0^T x_2(t)dx_1(t)\right] \\ &+E\int_0^T \sigma\,(t, x_1(t), E(x_1(t)), u(t))\,\sigma\,(t, x_2(t), E(x_2(t)), u(t))\,dt \\ &+E\int_0^T\int_{\Theta} g\,(t, x_1(t), E(x_1(t)), u(t), \theta)\,g\,(t, x_2(t), E(x_2(t)), u(t), \theta)\,\mu(d\theta)dt. \end{aligned}$$

Applying a similar method as in [30, Lemma 2.1], *for the proof of the above Lemma.*

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