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Rigidity Results on Lagrangian and Symplectic Translating Solitons

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Abstract In this short note, we prove that an almost calibrated Lagrangian translating soliton must be a plane if it has weighted integrable mean curvature vector or weighted quadratic area growth. Similar results are also true for symplectic translating solitons.

Keywords Rigidity · Translating soliton · Almost calibrated Lagrangian · Symplectic

Mathematics Subject Classification 53C44 · 53C21

1 Introduction

In recent years, translating solitons to the mean curvature flow have attracted much attention. It is well known that [3,6,7,12] translating solitons play an important role in classifying Type-II singularity of mean curvature flow.

Recall that a surface Σ^n in $\mathbb{R}^{n+k}$ is called a translating soliton (or translator) of the mean curvature flow, if it satisfies

$$\mathbf{T}^{\perp} = \mathbf{H}, \tag{1.1}$$

where $\mathbf{H}$ is the mean curvature vector of Σ in $\mathbb{R}^{n+k}$. Let $\mathbf{V}$ be the tangent part of $\mathbf{T}$. Then we have

$$\mathbf{T} = \mathbf{V} + \mathbf{H}. \tag{1.2}$$

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It is well known that a translating soliton can be viewed as a critical point of the functional

$$L(\Sigma) = \int_\Sigma e^{\langle \mathbf{T}, \mathbf{x} \rangle} \mathrm{d}\mu, \tag{1.3}$$

where $\mathbf{x}$ is the position vector in $\mathbb{R}^{n+k}$, and $\mathrm{d}\mu$ is the volume form on Σ induced from the Euclidean space $\mathbb{R}^{n+k}$. For our convenience, we denote $\mathrm{d}\tilde{\mu} = e^{\langle \mathbf{T}, \mathbf{x} \rangle}\mathrm{d}\mu$.

Recently, Xin [13] systematically studied translating solitons with arbitrary dimension and codimension, and proved some rigidity results. In particular, he proved that any n-dimensional complete translating soliton with $\int_\Sigma |\mathbf{A}|^n d\mu$ small enough and $|\mathbf{A}| \in L^n(d\tilde{\mu})$ must be a flat plane.

Since it is proved by Chen and Li [1,2] and Wang [11] that there is no finite time Type I singularities for symplectic mean curvature flow and almost calibrated Lagrangian mean curvature flow, translating solitons to such flows have its own interests. There are already several rigidity results on symplectic and Lagrangian translating solitons under various assumptions. See, for example, [4,5,8,9] and [10], etc.

In this short note, we continue to study symplectic and almost calibrated Lagrangian translating solitons. It is known that a complete translating soliton cannot be compact. Our first result states that

Main Theorem 1 *Suppose Σ^2 is a complete almost calibrated Lagrangian translating soliton in $\boldsymbol{C}^2$ with $\cos\theta \geq \delta > 0$ and mean curvature vector $\boldsymbol{H} \in L^1(d\tilde{\mu})$. Then Σ must be a plane.*

A similar argument gives us the following result for symplectic translating solitons:

Main Theorem 2 *Suppose Σ^2 is a complete symplectic translating soliton in $\boldsymbol{C}^2$ with $\cos\alpha \geq \delta > 0$ and second fundamental form $\boldsymbol{A} \in L^1(d\tilde{\mu})$. Then Σ must be a plane.*

In order to state the next result, we first give the definition of weighted quadratic area growth:

Definition 1.1 We say a surface Σ^2 in $\mathbb{R}^4$ has weighted quadratic area growth, if there is a constant $D_0 > 0$, such that

$$\tilde{\mu}(\Sigma \cap B(r)) := \int_{\Sigma \cap B(r)} e^{\langle \mathbf{T}, \mathbf{x} \rangle} \mathrm{d}\mu \leq D_0 r^2, \tag{1.4}$$

for any $r \geq 1$ holds, where $B(r)$ is the ball of radius r in $\mathbb{R}^4$.

Then we have

Main Theorem 3 *Suppose Σ^2 is a complete almost calibrated Lagrangian translating soliton in $\boldsymbol{C}^2$ with $\cos\theta \geq \delta > 0$ and weighted quadratic area growth. Then Σ must be a plane.*

Similarly, we have

Main Theorem 4 *Suppose Σ^2 is a complete symplectic translating soliton in $\boldsymbol{C}^2$ with $\cos\alpha \geq \delta > 0$ and weighted quadratic area growth. Then Σ must be a plane.*

As a corollary, we get that

Corollary 1.1 *Any translating soliton with weighted quadratic area growth cannot arise as blow up limit of symplectic mean curvature flow or almost calibrated Lagrangian mean curvature flow.*

By the monotonicity formula to the mean curvature flow, we know that the blow up limit of mean curvature flow must have quadratic area growth. Namely, we always have

$$\mu(\Sigma \cap B(r)) \leq D_1 r^2,$$

for each $r \geq 1$. It is not clear whether the blow up limit has weighted quadratic area growth.

2 Proof of the Main Theorems

In [4], Han and Li computed the following identities on translating solitons, which will be used later:

Lemma 2.1 *On the translating soliton to the Lagrangian mean curvature flow, the Lagrangian angle satisfies the following equation*

$$-\Delta \cos\theta = |\boldsymbol{H}|^2 \cos\theta + \boldsymbol{V} \cdot \nabla \cos\theta. \tag{2.1}$$

Lemma 2.2 *On the translating soliton to the symplectic mean curvature flow, the Kähler angle satisfies the following equation*

$$-\Delta \cos\alpha = |\overline{\nabla} J|^2 \cos\alpha + \boldsymbol{V} \cdot \nabla \cos\alpha. \tag{2.2}$$

Lemma 2.3 *On the two-dimensional translating soliton in $\boldsymbol{C}^2$, at the points where $|\boldsymbol{V}| \neq 0$,*

$$|\boldsymbol{A}|^2 = |\boldsymbol{H}|^2 + 2\frac{|\nabla \boldsymbol{H}|^2}{|\boldsymbol{V}|^2} + \frac{\boldsymbol{V} \cdot \nabla |\boldsymbol{H}|^2}{|\boldsymbol{V}|^2}. \tag{2.3}$$

Now we can start to prove the Main Theorems.

Proof of the Main Theorem 1 We set $u = \frac{1}{\cos\theta}$, then by (2.1), we can easily see that

$$\Delta u + \langle \mathbf{V}, \nabla u \rangle = |\mathbf{H}|^2 u + 2u^{-1} |\nabla u|^2, \tag{2.4}$$

where Δ and ∇ are the Laplacian and gradient operator on Σ with respect to the induced metric, respectively. Multiplying both sides of (2.4) by $\phi^2 u e^{\langle \mathbf{T}, \mathbf{x} \rangle}$, where ϕ is a cutoff function, and integrating by parts, we get that

$$
\begin{aligned}
\int_\Sigma \phi^2(|\mathbf{H}|^2u^2+2|\nabla u|^2)e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu &= \int_\Sigma \phi^2 u(\Delta u+\langle \mathbf{V},\nabla u\rangle)e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu\\
&= \int_\Sigma \phi^2 u\, div_\Sigma(e^{\langle \mathbf{T},\mathbf{x}\rangle}\nabla u)\mathrm{d}\mu\\
&= -\int_\Sigma \langle \nabla(\phi^2 u),\nabla u\rangle e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu\\
&= -\int_\Sigma \phi^2|\nabla u|^2 e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu\\
&\quad -2\int_\Sigma \phi u\langle \nabla\phi,\nabla u\rangle e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu,
\end{aligned}
$$

which implies that

$$
\int_\Sigma \phi^2(|\mathbf{H}|^2u^2+3|\nabla u|^2)e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu \le 2\int_\Sigma \phi u|\nabla\phi||\nabla u|e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu. \tag{2.5}
$$

Since $\cos\theta \ge \delta > 0$, we see that $1 \le u \le \frac{1}{\delta}$. Furthermore, from the fact that on a Lagrangian submanifold, $\mathbf{H} = J\nabla\theta$, we have $|\mathbf{H}| = |\nabla\theta|$, which implies that

$$
|\nabla u| = \frac{\sin\theta}{\cos^2\theta}|\mathbf{H}| \le \frac{1}{\delta^2}|\mathbf{H}|.
$$

Therefore, by (2.5), we have

$$
\int_\Sigma \phi^2(|\mathbf{H}|^2u^2+3|\nabla u|^2)\mathrm{d}\tilde{\mu} \le \frac{2}{\delta^3}\int_\Sigma |\nabla\phi||\mathbf{H}|\mathrm{d}\tilde{\mu}. \tag{2.6}
$$

Now for any fixed $R > 0$, we take the cutoff function $\phi = \phi_R$ such that $\phi \in C_0^\infty(B(2R))$, $\phi \equiv 1$ on $B(R)$, and $|\nabla\phi| \le |D\phi| \le \frac{C_1}{R}$. Here, $B(R)$ is the ball of radius in $\mathbb{R}^4$, $D\phi$ is the gradient of ϕ with respect to the Euclidean metric in $\mathbb{R}^4$, and C_1 is an absolute constant. Taking $\phi = \phi_R$ in (2.6) yields

$$
\int_{\Sigma\cap B(R)} (|\mathbf{H}|^2u^2+3|\nabla u|^2)\mathrm{d}\tilde{\mu} \le \frac{2C_1}{\delta^3 R}\int_\Sigma |\mathbf{H}|\mathrm{d}\tilde{\mu}. \tag{2.7}
$$

By our assumption, $\int_\Sigma |\mathbf{H}|\mathrm{d}\tilde{\mu} < \infty$. Letting $R \to \infty$ in (2.7), we finally obtain that $\mathbf{H} \equiv 0$ on Σ. By (2.3), we see that $\mathbf{A} \equiv 0$ on Σ. Thus it must be a flat plane. □

As a corollary, we have

Corollary 2.1 *Any translating soliton with $\mathbf{H} \in L^1(d\tilde{\mu})$ cannot arise as blow up limit of almost calibrated Lagrangian mean curvature flow.*

Proof of the Main Theorem 2 We set $u = \frac{1}{\cos\alpha}$, then arguing in the same way as in the proof of Main Theorem 1, replacing (2.1) by (2.2), we obtain

$$
\int_\Sigma \phi^2(|\overline{\nabla} J|^2u^2+3|\nabla u|^2)\mathrm{d}\tilde{\mu} \le \frac{2}{\delta^3}\int_\Sigma |\nabla\phi||\nabla\alpha|\mathrm{d}\tilde{\mu}. \tag{2.8}
$$

On the other hand, notice that on a symplectic surface, we have $|\nabla\alpha| \leq |\overline{\nabla} J|$ (see [4]). Also, Chen-Li proved that [1]

$$2|\mathbf{A}|^2 \geq |\overline{\nabla} J|^2 \geq \frac{1}{2}|\mathbf{H}|^2. \tag{2.9}$$

Therefore, we see that

$$|\nabla\alpha| \leq \sqrt{2}|\mathbf{A}|.$$

Then we have

$$\int_\Sigma \phi^2(|\overline{\nabla} J|^2 u^2 + 3|\nabla u|^2)\mathrm{d}\tilde{\mu} \leq \frac{2\sqrt{2}}{\delta^3}\int_\Sigma |\nabla\phi||\mathbf{A}|\mathrm{d}\tilde{\mu}.$$

The remaining part of the proof is the same as in the proof of the Main Theorem 1, combined with (2.9). □

Corollary 2.2 *Any translating soliton with* $\mathbf{A} \in L^1(d\tilde{\mu})$ *cannot arise as blow up limit of symplectic mean curvature flow.*

The proof of The Main Theorem 3 will depend on the choice of logarithm cutoff function.

Proof of the Main Theorem 3 We set $u = \frac{1}{\cos\theta}$. Then arguing in the same way as in the proof of the Main Theorem 1, we get (2.5). By triangle inequality, we obtain that

$$\int_\Sigma \phi^2(|\mathbf{H}|^2 u^2 + 3|\nabla u|^2)e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu \leq 3\int_\Sigma \phi^2|\nabla u|^2 e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu + \frac{1}{3}\int_\Sigma u^2|\nabla\phi|^2 e^{\langle \mathbf{T},\mathbf{x}\rangle}\mathrm{d}\mu,$$

which implies that

$$\int_\Sigma \phi^2|\mathbf{H}|^2u^2\mathrm{d}\tilde{\mu} \leq \frac{1}{3}\int_\Sigma u^2|\nabla\phi|^2\mathrm{d}\tilde{\mu} \leq \frac{1}{3\delta^2}\int_\Sigma |\nabla\phi|^2\mathrm{d}\tilde{\mu}. \tag{2.10}$$

Now we choose cutoff function as follows:

$$\phi = \begin{cases} 1 & r^2 \leq R, \\ 2 - 2\frac{\log r}{\log R} & R < r^2 \leq R^2, \\ 0 & r^2 > R^2. \end{cases}$$

From (2.10) and our assumption (1.4), we have

$$
\begin{aligned}
\int_{\Sigma\cap B(0,\sqrt{R})} |\mathbf{H}|^2 u^2 \mathrm{d}\tilde{\mu} &\leq \frac{1}{3\delta^2}\int_{\Sigma} |\nabla\phi|^2 \mathrm{d}\tilde{\mu} \\
&\leq \frac{4}{3\delta^2(\log R)^2} \sum_{\frac{\log R}{2}\leq l\leq \log R} \int_{\Sigma\cap(B(0,e^l)\setminus B(0,e^{l-1}))} r^{-2}\mathrm{d}\tilde{\mu} \\
&\leq \frac{4}{3\delta^2(\log R)^2} \sum_{\frac{\log R}{2}\leq l\leq \log R} e^{-2(l-1)} D_0 e^{2l} \\
&\leq \frac{4D_0e^2}{3\delta^2 \log R}.
\end{aligned}
$$

Letting $R\to\infty$, we get that $|\mathbf{H}|^2u^2\equiv 0$ on Σ. Since $u\geq 1$, we get that $\mathbf{H}\equiv 0$ on Σ. As before, by (2.3), we obtain that $\mathbf{A}\equiv 0$ and thus Σ is a plane. □

The proof of the Main Theorem 4 is similar and we omit the details here.

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