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The Logarithmic Sobolev and Sobolev Inequalities Along the Ricci Flow

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Abstract Based on Perelman's entropy monotonicity, uniform logarithmic Sobolev inequalities along the Ricci flow are derived. Then uniform Sobolev inequalities along the Ricci flow are derived via harmonic analysis of the integral transform of the relevant heat operator. These inequalities are fundamental analytic properties of the Ricci flow. They are also extended to the volume-normalized Ricci flow and the Kähler–Ricci flow.

Keywords Sobolev inequality · Logarithmic Sobolev inequality · Ricci flow · Heat operator

Mathematics Subject Classification 53C44 · 35K55

1 Introduction

Consider a compact manifold M of dimension $n \geq 3$. Let $g = g(t)$ be a smooth solution of the Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric} \tag{1.1}$$

on $M \times [0, T)$ for some (finite or infinite) $T > 0$ with a given initial metric $g(0) = g_0$.

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Theorem 1.1 *For each $\sigma > 0$ and each $t \in [0, T)$, there holds*

$$\int_M u^2 \ln u^2 \mathrm{dvol} \le \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \mathrm{dvol} - \frac{n}{2} \ln \sigma + A_1 \left(t + \frac{\sigma}{4} \right) + A_2 \tag{1.2}$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 \mathrm{dvol} = 1$, where

$$A_1 = \frac{4}{\tilde{C}_S(M, g_0)^2 \mathrm{vol}_{g_0}(M)^{\frac{2}{n}}} - \min R_{g_0},$$
$$A_2 = n \ln \tilde{C}_S(M, g_0) + \frac{n}{2}(\ln n - 1),$$

and all geometric quantities are associated with the metric $g(t)$ (e.g., the volume form dvol *and the scalar curvature R), except the scalar curvature R_{g_0}, the modified Sobolev constant $\tilde{C}_S(M, g_0)$ (see Sect. 2 for its definition) and the volume* $\mathrm{vol}_{g_0}(M)$ *which are those of the initial metric g_0.*

Consequently, there holds for each $t \in [0, T)$

$$\int_M u^2 \ln u^2 \mathrm{dvol} \le \frac{n}{2} \ln \left[\alpha_I \left(\int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \mathrm{dvol} + \frac{A_1}{4} \right) \right] \tag{1.3}$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 \mathrm{dvol} = 1$, where

$$\alpha_I = \frac{2e}{n} e^{\frac{2(A_1 t + A_2)}{n}}. \tag{1.4}$$

Indeed, a more general result holds true, in which the logarithmic Sobolev inequality along $g(t)$ is derived from a logarithmic Sobolev inequality for g_0, see Theorem 4.2. The exact factor $\frac{n}{2}$ in the term $-\frac{n}{2} \ln \sigma$ in the logarithmic Sobolev inequality (1.2) (also in (1.5) and (1.8) below) is crucial for the purpose of Theorems 1.5 and 1.6. Note that an upper bound for the Sobolev constant $C_S(M, g_0)$ and the modified Sobolev constant $\tilde{C}_S(M, g_0)$ can be obtained in terms of a lower bound for the diameter rescaled Ricci curvature and a positive lower bound for the diameter rescaled volume, see Sect. 2. In particular, a lower bound for the Ricci curvature, a positive lower bound for the volume, and an upper bound for the diameter lead to an upper bound for the Sobolev constant and the modified Sobolev constant.

The logarithmic Sobolev inequality in Theorem 1.1 is uniform for all time which lies below a given bound, but deteriorates as time becomes large. This is not a deficiency in the result, however. In general, it is impossible to obtain a uniform logarithmic Sobolev inequality along the Ricci flow which is independent of an upper bound for time. Indeed, by [5], there are smooth solutions of the Ricci flow on torus bundles over the circle which exist for all time, have bounded curvature, and collapse as $t \to \infty$. In view of the proofs of Theorems 1.5 and 1.7, a uniform logarithmic Sobolev inequality fails to hold along these solutions.

To obtain a uniform logarithmic Sobolev inequality, we employ a natural geometric condition. Let $\lambda_0 = \lambda_0(g_0)$ denote the first eigenvalue of the operator $-\Delta + \frac{R}{4}$ for the initial metric g_0.

Theorem 1.2 *Assume that the first eigenvalue $\lambda_0 = \lambda_0(g_0)$ of the operator $-\Delta + \frac{R}{4}$ for the initial metric g_0 is positive. Let $\delta_0 = \delta_0(g_0)$ be the number defined in (3.12). Let $t \in [0, T)$ and $\sigma > 0$ satisfy $t + \sigma \geq \frac{n}{8} C_S(M, g_0)^2 \delta_0$. Then there holds*

$$\int_M u^2 \ln u^2 \text{dvol} \leq \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \text{dvol} - \frac{n}{2} \ln \sigma + \frac{n}{2} \ln n + n \ln C_S(M, g_0) + \sigma_0(g_0) \tag{1.5}$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$, where all geometric quantities are associated with the metric $g(t)$ (e.g., the volume form dvol *and the scalar curvature R), except the Sobolev constant $C_S(M, g_0)$ and the number $\sigma_0(g_0)$ (defined in (3.13)) which are those of the initial metric g_0.*

Consequently, there holds for each $t \in [0, T)$

$$\int_M u^2 \ln u^2 \text{dvol} \leq \frac{n}{2} \ln \left[\alpha_{II} \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \text{dvol} \right] \tag{1.6}$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$, where

$$\alpha_{II} = 2e C_S(M, g_0)^2 e^{\frac{2}{n} \sigma_0(g_0)}. \tag{1.7}$$

Combining Theorems 1.1 and 1.2, we obtain a uniform logarithmic Sobolev inequality along the Ricci flow.

Theorem 1.3 *Assume that $\lambda_0(g_0) > 0$. For each $t \in [0, T)$ and each $\sigma > 0$, there holds*

$$\int_M u^2 \ln u^2 \text{dvol} \leq \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \text{dvol} - \frac{n}{2} \ln \sigma + C \tag{1.8}$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$, where C depends only on the dimension n, a positive lower bound for $\text{vol}_{g_0}(M)$, a nonpositive lower bound for R_{g_0}, an upper bound for $C_S(M, g_0)$, and a positive lower bound for $\lambda_0(g_0)$.

Consequently, there holds for each $t \in [0, T)$

$$\int_M u^2 \ln u^2 \text{dvol} \leq \frac{n}{2} \ln \left[\alpha_{III} \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \text{dvol} \right] \tag{1.9}$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$, where

$$\alpha_{III} = \frac{2e}{n} e^{\frac{2}{n} C}. \tag{1.10}$$

The class of Riemannian manifolds (M, g_0) with $\lambda_0(g_0) > 0$ or, more generally, $\lambda_0(g_0) \geq 0$ is a very large one and particularly significant from a geometric point of view. (For example, the condition $\lambda_0(g_0) > 0$ holds true when the scalar curvature of g_0 is nonnegative and somewhere positive.) On the other hand, we would like to remark that if $\lambda_0 > 0$, then there holds $T < \frac{n}{2\lambda_0}$ by the proof of Proposition 1.2 in [6]. Hence, Theorem 1.3 can also be derived as a corollary of Theorem 1.1 alone. Next we note a special consequence of Theorem 1.3.

Corollary 1.4 *Assume that $\lambda_0(g_0) > 0$. Then we have at any time $t \in [0, T)$*

$$\mathrm{vol}_{g(t)}(M) \geq e^{-\frac{1}{4}-C} \tag{1.11}$$

when $\hat{R}(t) \leq 0$, and

$$\mathrm{vol}_{g(t)}(M) \geq e^{-\frac{1}{4}-C}\hat{R}(t)^{-\frac{n}{2}} \tag{1.12}$$

when $\hat{R}(t) > 0$. Here $\hat{R}$ denotes the average scalar curvature.

Similar volume bounds follow from Theorem 1.1 without the condition $\lambda_0(g_0) > 0$, but they also depend on a (finite) upper bound of T.

For a brief account of the logarithmic Sobolev inequalities on the euclidean space, we refer to Appendix 1, which serve as the background for the idea of the logarithmic Sobolev inequality. Both Theorems 1.1 and 1.2 are consequences of Perelman's entropy monotonicity [6]. We obtained these two results, Theorems 1.3 and 4.2 in 2004 (around the time of the author's differential geometry seminar talk "An introduction to the logarithmic Sobolev inequality" at UCSB in June 2004). They have also been prepared as part of the notes [13].

Next we apply the theory as presented in Chap. 2 of [2] to derive from Theorem 1.5 a Sobolev inequality along the Ricci flow without any restriction on time. (We came to notice [2] in the paper [19]. Note that the main result presented in [19] (and [20]) is incorrect, as pointed out in [14] (the archive version of the present paper), based on the example in [5]. Subsequently, a correction of this mistake was made in [21].) A particularly nice feature of the theory in Chap. 2 of [2] is that no additional geometric data (such as the volume) are involved in the passage from the logarithmic Sobolev inequality to the Sobolev inequality. Only the non-integral terms in the logarithmic Sobolev inequality and a nonpositive lower bound for the potential function Ψ (see Theorem 5.5) come into play. This leads to the form of the geometric dependence in the following theorem.

Theorem 1.5 *Assume $T < \infty$. There are positive constants A and B depending only on the dimension n, a nonpositive lower bound for R_{g_0}, a positive lower bound for $\mathrm{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, and an upper bound for T, such that for each $t \in [0, T)$ and all $u \in W^{1,2}(M)$ there holds*

$$\left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{dvol}\right)^{\frac{n-2}{n}} \leq A \int_M \left(|\nabla u|^2 + \frac{R}{4}u^2\right) \mathrm{dvol} + B \int_M u^2 \mathrm{dvol}, \tag{1.13}$$

where all geometric quantities except A and B are associated with $g(t)$.

Under the assumption $\lambda_0(g_0) > 0$, this theorem can be improved as follows.

Theorem 1.6 *Assume that* $\lambda_0(g_0) > 0$. *There is a positive constant A depending only on the dimension n, a nonpositive lower bound for* R_{g_0}, *a positive lower bound for* $\mathrm{vol}_{g_0}(M)$, *an upper bound for* $C_S(M, g_0)$, *and a positive lower bound for* $\lambda_0(g_0)$, *such that for each* $t \in [0, T)$ *and all* $u \in W^{1,2}(M)$ *there holds*

$$\left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{dvol}\right)^{\frac{n-2}{n}} \leq A \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2\right) \mathrm{dvol}, \tag{1.14}$$

where all geometric quantities except A are associated with $g(t)$.

We also obtain two results which extend Theorems 1.5 and 1.6 to the set-up of $W^{1,p}(M)$ for all $1 < p < n$, see Theorems 9.6 and 9.7 in Appendix 3. (Theorems 1.5 and 1.6 correspond to the case $p = 2$.) These two general results can be thought of as nonlocal versions of Sobolev inequality, because they involve nonlocal pseudo-differential operators. Further results on (conventional) Sobolev inequalities for $2 < p < n$ and $1 < p < 2$ will be presented in [17]. (Part of these results are derived from Theorems 1.5, 1.6, 9.6 and 9.7.) We would like to point out that the $p = 2$ case of the Sobolev inequality is the most important for analytic and geometric applications.

The theory in Chap. 2 of [2] is formulated in a general and abstract set-up of symmetric Markov semigroups. By Lemma 5.2, e^{-tH} is a symmetric Markov semigroup, where $H = -\Delta + \frac{R}{4}$ in the case $\lambda_0(g_0) > 0$ and $H = -\Delta + \frac{R}{4} - \frac{\min R^-}{4}$ in the general case. Hence, the general theory and results in Chap. 2 of [2] can be applied to our situation. However, to obtain the precise geometric dependence of the Sobolev inequalities in Theorems 1.5 and 1.6, one has to verify the exact geometric nature of the constants which would appear in the many steps of the involved (and tightly formulated) arguments in [2]. Our proofs of Theorems 1.5 and 1.6 would be unclear and non-transparent if we go through a multitude of checking processes. Instead, we adapt the theory in [2] to our geometric set-up and work it out in complete, self-contained details. Another reason for doing so is to obtain some useful extensions of the theory as presented in Sect. 5, Appendix 2 and Appendix 3 (in particular Theorems 9.5–9.7). On the other hand, we think that our presentation makes the theory easily accessible to the general audience of geometric analysis. In particular, our presentation demonstrates in detail how the theory of the Ricci flow interacts with the basic theory of harmonic analysis.

Next we deduce from Theorems 1.5 and 1.6 κ-noncollapsing estimates for the Ricci flow which are measured relative to upper bounds of the scalar curvature to improve Perelman's κ-noncollapsing result [6]. The original κ-noncollapsing result of Perelman in [6] is formulated relative to bounds for $|Rm|$. Later, a κ-noncollapsing result for bounded time measured relative to upper bounds of the scalar curvature was obtained independently by Perelman and the present author (see [11]). The κ-noncollapsing estimates below improve these results in two ways. First, they provide explicit estimates with clear geometric dependence on the initial metric. The estimates below are formulated in terms of more familiar geometric quantities of the initial metric. (If we apply Theorem 4.2, then we obtain κ-noncollapsing estimates which only depend on the logarithmic Sobolev inequality of the initial metric.) Moreover, the estimates are uniform up to $t = 0$ (under a given upper bound for T). Secondly, the strategy of deriving the κ-noncollapsing estimate from the Sobolev inequality is particularly powerful and flexible, and has among others important applications to the Ricci flow with surgeries as constructed by Perelman in his work on the Poincaré conjecture and the geometrization conjecture [7]. It leads to a considerable clarification and simplification of an important step of the main arguments in [7], see Theorem 1.12 below.

Theorem 1.7 *Assume that $T < \infty$. Let $L > 0$ and $t \in [0, T)$. Consider the Riemannian manifold (M, g) with $g = g(t)$. Assume $R \le \frac{1}{r^2}$ on a geodesic ball $B(x, r)$ with $0 < r \le L$. Then there holds*

$$\mathrm{vol}(B(x, r)) \ge \left(\frac{1}{2^{n+3}A + 2BL^2} \right)^{\frac{n}{2}} r^n, \tag{1.15}$$

where A and B are from Theorem 1.5.

Theorem 1.8 *Assume that $\lambda_0(g_0) > 0$. Let $t \in [0, T)$. Consider the Riemannian manifold (M, g) with $g = g(t)$. Assume $R \le \frac{1}{r^2}$ on a geodesic ball $B(x, r)$ with $r > 0$. Then there holds*

$$\mathrm{vol}(B(x, r)) \ge \left(\frac{1}{2^{n+3}A} \right)^{\frac{n}{2}} r^n, \tag{1.16}$$

where A is from Theorem 1.6. In other words, the flow $g = g(t), t \in [0, T)$ is κ-noncollapsed relative to upper bounds of the scalar curvature on all scales.

As is well-known, a major application of κ-noncollapsing estimates is to obtain smooth blow-up limits of the Ricci flow at singularities, which is crucial for analysing the structures of singularities of the Ricci flow.

Now we discuss how Theorems 1.5–1.8 lead to uniform Sobolev inequalities and uniform κ-noncollapsing estimates independent of any upper bound of time for various modified Ricci flows. In particular, they hold both on finite and infinite time intervals.

Consider the modified Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric} + \lambda(g, t) g \tag{1.17}$$

with a smooth scalar function $\lambda(g, t)$ independent of $x \in M$. The volume-normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric} + \frac{2}{n} \hat{R} g \tag{1.18}$$

on a closed manifold, with $\hat{R}$ denoting the average scalar curvature, is an example of the modified Ricci flow. The λ-normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2\text{Ric} + \lambda g \tag{1.19}$$

for a constant λ is another example. (Of course, it reduces to the Ricci flow when $\lambda = 0$.) The normalized Kähler–Ricci flow is a special case of it.

Let $g = g(t)$ be a smooth solution of the modified Ricci flow (1.17) on $M \times [0, T)$ for some $T > 0$, which is allowed to be finite or infinite. Let $g_0 = g(0)$ denote the initial metric. First we have the following results.

Theorem 1.9 *Theorem 1.6 and 1.8 extend to the above $g = g(t)$.*

This result simply follows from scaling invariance of the estimates in Theorems 1.6 and 1.8. Here no additional condition is required. In the general case without the assumption $\lambda_0(g_0) > 0$, we need an additional condition. Set

$$T^* = \int_0^T e^{-\int_0^t \lambda(g(s),s)ds} \mathrm{d}t. \tag{1.20}$$

Theorem 1.10 *Assume that $T^* < \infty$.*

(1) *There are positive constants A and B depending only on the dimension n, a nonpositive lower bound for R_{g_0}, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, and an upper bound for T^*, such that for each $t \in [0, T)$ and all $u \in W^{1,2}(M)$ there holds*

$$\left(\int_M |u|^{\frac{2n}{n-2}} \text{dvol}\right)^{\frac{n-2}{n}} \leq A \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2\right) \text{dvol} + Be^{-\int_0^t \lambda(g(s),s)ds} \int_M u^2 \text{dvol}. \tag{1.21}$$

(2) *Let $L > 0$ and $t \in [0, T)$. Consider the Riemannian manifold (M, g) with $g = g(t)$. Assume $R \leq \frac{1}{r^2}$ on a geodesic ball $B(x, r)$ with $0 < r \leq L$. Then there holds*

$$\text{vol}(B(x,r)) \geq \left(\frac{1}{2^{n+3}A + 2Be^{-\int_0^t \lambda(g(s),s)ds} L^2}\right)^{\frac{n}{2}} r^n. \tag{1.22}$$

This theorem is a simple consequence of the scaling behavior of the estimates in Theorems 1.5 and 1.7. Combining Theorems 1.9 and 1.10 with Perelman's scalar curvature estimate [10], we obtain the following corollary.

Theorem 1.11 *Let $g = g(t)$ be a smooth solution of the normalized Kähler–Ricci flow*

$$\frac{\partial g}{\partial t} = -2\text{Ric} + 2\gamma g \tag{1.23}$$

on $M \times [0, \infty)$ with a positive first Chern class, where γ is the positive constant such that the Ricci class equals γ times the Kähler class. (We assume that M carries such a Kähler structure.) Then the Sobolev inequality (1.24) holds true with $\lambda(g(s), s) = 2\lambda$. Moreover, there is a positive constant L depending only on the initial metric $g_0 = g(0)$ and the dimension n such that the inequality (1.22) holds true for all $t \in [0, T)$ and $0 < r \leq L$.

If $\lambda_0(g_0) > 0$, then the Sobolev inequality (1.14) holds true for g. Moreover, there is a positive constant depending only on the initial metric g_0 and the dimension n such that the inequality (1.16) holds true for all $t \in [0, T)$ and $0 < r \leq L$. Consequently, blow-up limits of g at the time infinity satisfy (1.16) for all $r > 0$ and the Sobolev inequality

$$\left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{dvol}\right)^{\frac{n-2}{n}} \leq A \int_M |\nabla u|^2 \mathrm{dvol} \tag{1.24}$$

for all u. (In particular, they must be noncompact.)

Finally, we would like to mention that Theorems 1.5 and 1.6 hold true for the Ricci flow with surgeries of Perelman [7], with suitable modifications as stated below.

Theorem 1.12 *Let $n = 3$ and $g = g(t)$ be a Ricci flow with surgeries as constructed in [7] on its maximal time interval $[0, T_{max})$, with suitably chosen surgery parameters. Let $g_0 = g(0)$. Then there holds at each $t \in [0, T_{max})$*

$$\left(\int_M |u|^6 \mathrm{dvol}\right)^{\frac{1}{3}} \leq A(t) \int_M \left(|\nabla u|^2 + \frac{R}{4}u^2\right) \mathrm{dvol} + B(t) \int_M u^2 \mathrm{dvol} \tag{1.25}$$

for all $u \in W^{1,2}(M)$, where $A(t)$ and $B(t)$ are bounded from above in terms of a nonpositive lower bound for R_{g_0}, a positive lower bound for $\mathrm{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, and an upper bound for t.

If $\lambda_0(g_0) > 0$, then there holds at each $t \in [0, T_{max})$

$$\left(\int_M |u|^6 \mathrm{dvol}\right)^{\frac{1}{3}} \leq A(t) \int_M \left(|\nabla u|^2 + \frac{R}{4}u^2\right) \mathrm{dvol} \tag{1.26}$$

for all $u \in W^{1,2}(M)$, where $A(t)$ is bounded from above in terms of a nonpositive lower bound for R_{g_0}, a positive lower bound for $\mathrm{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, a positive lower bound for $\lambda_0(g_0)$, and an upper bound for $m(t)$.

κ-noncollapsing estimates follow as before, which lead to a considerable simplification of the arguments in [7] about preserving the κ-noncollapsing property after surgeries. Similar results hold true in higher dimensions whenever similar surgeries are performed. (The constants also depend on the dimension n.)

This result follows from Theorems 1.5 and 1.6, and a general result on Sobolev inequalities under surgeries. The details can be found in [18] and its sequel. In [7], the surgery parameters are chosen such that several key properties of the Ricci flow are preserved after surgery. One is the κ-noncollapsing property. Since the Sobolev inequalities (1.25) and (1.26) are derived without using the κ-noncollapsing property, the choice of the surgery parameters is also simplified. The κ-noncollapsing property follows as a consequence of (1.25) and (1.26).

The results in this paper (except Theorem 1.12) extend to the dimension $n = 2$, see [15].

This paper first appeared on the archive in 2007 as [14].

2 The Sobolev Inequality

Consider a compact Riemannian manifold (M, g) of dimension $n \geq 3$. Its Poincaré-Sobolev constant (for the exponent 2) is defined to be

$$C_{P,S}(M, g) = \sup\{\|u - u_M\|_{\frac{2n}{n-2}} : u \in C^1(M), \|\nabla u\|_2 = 1\}, \tag{2.1}$$

where $\|u\|_p$ denotes the L^p norm of u with respect to g, i.e., $\|u\|_p = (\int_M |u|^p d\mathrm{vol})^{1/p}$ ($d\mathrm{vol} = d\mathrm{vol}_g$). In other words, $C_{P,S}(M, g)$ is the smallest number such that the Poincare–Sobolev inequality

$$\|u - u_M\|_{\frac{2n}{n-2}} \leq C_{P,S}(M, g)\|\nabla u\|_2 \tag{2.2}$$

holds true for all $u \in C^1(M)$ (or all $u \in W^{1,2}(M)$). The Sobolev constant of (M, g) (for the exponent 2) is defined to be

$$C_S(M, g) = \sup\{\|u\|_{\frac{2n}{n-2}} - \frac{1}{\mathrm{vol}(M)^{\frac{1}{n}}}\|u\|_2 : u \in C^1(M), \|\nabla u\|_2 = 1. \tag{2.3}$$

In other words, $C_S(M, g)$ is the smallest number such that the inequality

$$\|u\|_{\frac{2n}{n-2}} \leq C_S(M, g)\|\nabla u\|_2 + \frac{1}{\mathrm{vol}(M)^{\frac{1}{n}}}\|u\|_2 \tag{2.4}$$

holds true for all $u \in W^{1,2}(M)$.

Definition We define the modified Sobolev constant $\tilde{C}_S(M, g)$ to be $\max\{C_S(M, g), 1\}$.

The Hölder inequality leads to the following basic fact.

Lemma 2.1 *There holds for all $u \in W^{1,2}(M)$*

$$\|u\|_{\frac{2n}{n-2}} \leq C_{P,S}(M, g)\|\nabla u\|_2 + \frac{1}{\mathrm{vol}(M)^{\frac{1}{n}}}\|u\|_2. \tag{2.5}$$

In other words, there holds $C_S(M, g) \leq C_{P,S}(M, g)$.

Another basic constant, the Neumann isoperimetric constant of (M, g), is defined to be

$$C_{N,I}(M, g) = \sup\left\{\frac{\mathrm{vol}(\Omega)^{\frac{n-1}{n}}}{A(\partial\Omega)} : \Omega \subset M \text{ is a } C^1 \text{ domain, } \mathrm{vol}(\Omega) \le \frac{1}{2}\mathrm{vol}(M)\right\}, \tag{2.6}$$

where $A(\partial\Omega)$ denotes the $n-1$-dimensional volume of $\partial\Omega$.

Lemma 2.2 *There holds for all* $u \in W^{1,2}(M)$

$$\|u - u_M\|_{\frac{2n}{n-2}} \le 2(1+\sqrt{2})\frac{n-1}{n-2}C_{N,I}(M, g)\|\nabla u\|_2. \tag{2.7}$$

In other words, there holds $C_{P,S}(M, g) \le 2(1+\sqrt{2})\frac{n-1}{n-2}C_{N,I}(M, g)$.

For the proof, see [12]. The following estimate of the Neumann isoperimetric constant follows from S. Gallot's estimate in [4]. We define the diameter rescaled Ricci curvature $\hat{Ric}(v, v)$ of a unit tangent vector v to be $\mathrm{diam}(M)^2\mathrm{Ric}(v, v)$, and set $\kappa_{\mathrm{Ric}} = \min_v\{\hat{Ric}(v, v)\}$. Then we set $\hat{\kappa}_{\hat{Ric}} = |\min\{\kappa_{\mathrm{Ric}}, -1\}|$. We also define the diameter rescaled volume $\hat{vol}(M)$ to be $\mathrm{vol}(M)\mathrm{diam}(M)^{-n}$.

Theorem 2.3 *There holds*

$$C_{N,I}(g, M) \le C(n, \hat{\kappa}_{\hat{Ric}})\hat{vol}(M)^{-\frac{1}{n}}, \tag{2.8}$$

where $C(n, \hat{\kappa}_{\hat{Ric}})$ *is a positive constant depending only on n and* $\hat{\kappa}_{\hat{Ric}}$.

Note that $\hat{\kappa}_{\hat{Ric}}$ can be replaced by a certain integral lower bound of the Ricci curvature, see [3].

3 The Logarithmic Sobolev Inequalities on a Riemannian Manifold

The various versions of the logarithmic Sobolev inequality on the Euclidean space as presented in Appendix 1 allow suitable extensions to Riemannian manifolds. We formulate a log gradient version and a straight version, cf. Appendix 1. As in the last section, let (M, g) be a compact Riemannian manifold of dimension n.

Theorem 3.1 *There holds*

$$\int_M u^2 \ln u^2 d\mathrm{vol} \le n \ln\left(C_S(M, g)\|\nabla u\|_2 + \frac{1}{\mathrm{vol}_g(M)^{\frac{1}{n}}}\right), \tag{3.1}$$

provided that $u \in W^{1,2}(M)$ *and* $\|u\|_2 = 1$.

Proof Set $q = \frac{2n}{n-2}$. Since ln is concave and $\int_M u^2 \mathrm{dvol} = 1$, we have by Jensen's inequality

$$\ln \int_M u^q \mathrm{dvol} = \ln \int_M u^2 \cdot u^{q-2} \mathrm{dvol} \geq \int_M u^2 \ln u^{q-2}. \tag{3.2}$$

It follows that

$$\begin{aligned} \int_M u^2 \ln u &\leq \frac{1}{q-2} \ln \int_M u^q \mathrm{dvol} \\ &= \frac{q}{q-2} \ln \|u\|_q \\ &\leq \frac{n}{2} \ln \left(C_S(M,g) \|\nabla u\|_2 + \frac{1}{\mathrm{vol}_g(M)^{\frac{1}{n}}} \|u\|_2 \right). \end{aligned} \tag{3.3}$$

□

Lemma 3.2 *There holds*

$$\ln(x+B) \leq \alpha x + \alpha B - 1 - \ln \alpha \tag{3.4}$$

for all $B \geq 0, \alpha > 0$ *and* $x > -B$.

Proof Consider the function $y = \ln(x+B) - \alpha x$ for $x > -B$. Since $y \to -\infty$ as $x \to -B$ or $x \to \infty$, it achieves its maximum somewhere. We have

$$y' = \frac{1}{x+B} - \alpha. \tag{3.5}$$

Hence, the maximum point is $x_0 = \frac{1}{\alpha} - B$. It follows that the maximum of y is $y(x_0) = \alpha B - 1 - \ln \alpha$. □

Theorem 3.3 *For each* $\alpha > 0$ *and all* $u \in W^{1,2}(M)$ *with* $\|u\|_2 = 1$, *there holds*

$$\int_M u^2 \ln u^2 \leq \frac{n\alpha C_S(M,g)^2}{2} \int_M |\nabla u|^2 - \frac{n}{2} \ln \alpha + \frac{n}{2}(\ln 2 + \alpha \mathrm{vol}_g(M)^{-\frac{2}{n}} - 1) \tag{3.6}$$

and

$$\begin{aligned} \int_M u^2 \ln u^2 \leq{}& \frac{n\alpha C_S(M,g)^2}{2} \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) - \frac{n}{2} \ln \alpha \\ &+ \frac{n\alpha}{2} \left(\mathrm{vol}_g(M)^{-\frac{2}{n}} - \frac{\min R^-}{4} C_S(M,g)^2 \right) + \frac{n}{2}(\ln 2 - 1). \end{aligned} \tag{3.7}$$

(*The notation of the volume is omitted.*)

Proof By (3.1), we have for $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$

$$\int_M u^2 \ln u^2 \le \frac{n}{2} \ln \left(C_S(M,g)\|\nabla u\|_2 + \frac{1}{\mathrm{vol}_g(M)^{\frac{1}{n}}} \right)^2$$
$$\le \frac{n}{2}\ln 2 + \frac{n}{2} \ln \left(C_S(M,g)^2 \int_M |\nabla u|^2 + \frac{1}{\mathrm{vol}_g(M)^{\frac{2}{n}}} \right). \quad (3.8)$$

Applying Lemma 3.2 with $x = C_S(M,g)^2 \int_M |\nabla u|^2$ and $B = 1$, we then arrive at (3.6). The inequality (3.7) follows from (3.6). □

Lemma 3.4 *Let $A > 0$, $B > 0$ and $\gamma > 0$ such that $A \ge \frac{1}{\gamma + B}$. Then we have*

$$\ln(x+B) \le Ax - \ln A + \ln(\gamma + B) - \ln \gamma - 1 \quad (3.9)$$

for all $x \ge \gamma$.

Proof First consider the function $y = \ln t - \gamma t$ for $t > 0$. Since $y \to -\infty$ as $t \to 0$ or $t \to \infty$, y achieves its maximum somewhere. We have $y' = \frac{1}{t} - \gamma$. Hence, the maximum is achieved at $\frac{1}{\gamma}$. It follows that the maximum is $y(\frac{1}{\gamma}) = -\ln\gamma - 1$. We infer

$$\ln A - \gamma A \le -\ln \gamma - 1. \quad (3.10)$$

Next we consider the function $y = \ln(x+B) - Ax + \ln A$ for $x \ge \gamma$. By (3.10), we have $y(\gamma) = \ln(\gamma + B) - A\gamma + \ln A \le \ln(\gamma + B) - \ln\gamma - 1$. On the other hand, we have $y' = \frac{1}{x+B} - A \le \frac{1}{\gamma+B} - A \le 0$. We arrive at (3.9). □

Theorem 3.5 *Assume that the first eigenvalue $\lambda_0 = \lambda_0(g)$ of the operator $-\Delta + \frac{R}{4}$ is positive. For each $A \ge \delta_0$ and all $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$, there holds*

$$\int_M u^2 \ln u^2 \le \frac{nAC_S^2}{2} \int_M \left(|\nabla u|^2 + \frac{R}{4}u^2 \right) - \frac{n}{2}\ln A + \frac{n}{2}\ln 2 + \sigma_0, \quad (3.11)$$

where

$$\delta_0 = \delta_0(g) = \left(\lambda_0 C_S^2 + \frac{1}{\mathrm{vol}_g(M)^{\frac{2}{n}}} - C_S^2 \frac{\min R^-}{4} \right)^{-1}, \quad (3.12)$$

$$\sigma_0 = \sigma_0(g) == \frac{n}{2} \left[\ln \left(\lambda_0 C_S^2 + \frac{1}{\mathrm{vol}_g(M)^{\frac{2}{n}}} - C_S^2 \frac{\min R^-}{4} \right) - \ln(\lambda_0 C_S^2) - 1 \right], \quad (3.13)$$

and $C_S = C_S(M,g)$.

Proof Arguing as in the proof of Theorem 3.3, we deduce for $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$

$$\begin{aligned}\int_M u^2 \ln u^2 &\le \frac{n}{2}\ln 2 + \frac{n}{2}\ln\left(C_S^2\int_M |\nabla u|^2 + \frac{1}{\mathrm{vol}_g(M)^{\frac{2}{n}}}\right)\\ &\le \frac{n}{2}\ln 2 + \frac{n}{2}\ln\left[C_S^2\int_M\left(|\nabla u|^2 + \frac{R}{4}u^2\right) + \frac{1}{\mathrm{vol}_g(M)^{\frac{2}{n}}} - C_S^2\frac{\min R^-}{4}\right]. \end{aligned} \tag{3.14}$$

Applying (3.9) with $\gamma = \lambda_0 C_S^2$, $B = \frac{1}{\mathrm{vol}_g(M)^{\frac{2}{n}}} - C_S^2\frac{\min R^-}{4}$ and $x = C_S^2\int_M(|\nabla u|^2 + \frac{R}{4}u^2)$, we then arrive at (3.11) for each $A \ge (\gamma + B)^{-1}$. □

4 The logarithmic Sobolev Inequality Along the Ricci Flow

Let M be a compact manifold of dimension n. Consider Perelman's entropy functional

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\mathrm{dvol}, \tag{4.1}$$

where τ is a positive number, g is a Riemannian metric on M, and $f \in C^\infty(M)$ satisfies

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\mathrm{dvol} = 1. \tag{4.2}$$

All geometric quantities in (4.1) and (4.2) are associated with g. To relate to the idea of logarithmic Sobolev inequalities, we make a change of variable

$$u = \frac{e^{-\frac{f}{2}}}{(4\pi\tau)^{\frac{n}{4}}}. \tag{4.3}$$

Then (4.2) leads to

$$\int_M u^2\mathrm{dvol} = 1 \tag{4.4}$$

and we have

$$\mathcal{W}(g, f, \tau) = \mathcal{W}^*(g, u, \tau) - \frac{n}{2}\ln\tau - \frac{n}{2}\ln(4\pi) - n, \tag{4.5}$$

where

$$\mathcal{W}^*(g, u, \tau) = \int_M \left[\tau(4|\nabla u|^2 + Ru^2) - u^2\ln u^2\right]\mathrm{dvol}. \tag{4.6}$$

We define $\mu^*(g, \tau)$ to be the infimum of $\mathcal{W}^*(g, u, \tau)$ over all u satisfying (4.4).

Next let $g = g(t)$ be a smooth solution of the Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric} \tag{4.7}$$

on $M \times [0, T)$ for some (finite or infinite) $T > 0$. Let $0 < t^* < T$ and $\sigma > 0$. We set $T^* = t^* + \sigma$ and $\tau = \tau(t) = T^* - t$ for $0 \le t \le t^*$. Consider a solution $f = f(t)$ of the equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \tag{4.8}$$

on $[0, t^*]$ with a given terminal value at $t = t^*$ (i.e., $\tau = \sigma$) satisfying (4.2) with $g = g(t^*)$. Then (4.2) holds true for $f = f(t), g = g(t)$, and all $t \in [0, t^*]$. Perelman's monotonicity formula says

$$\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} = 2\tau \int_M |Ric + \nabla^2 f - \frac{1}{2\tau} g|^2 \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} \mathrm{dvol} \ge 0, \tag{4.9}$$

where $\mathcal{W} = \mathcal{W}(g(t), f(t), \tau(t))$. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}^*(g, u, \tau) \ge \frac{n}{2}\frac{\mathrm{d}}{\mathrm{d}t} \ln \tau, \tag{4.10}$$

where $g = g(t), \tau = \tau(t)$, and

$$u = u(t) = \frac{e^{-f(t)/2}}{(4\pi\tau(t))^{\frac{n}{4}}}, \tag{4.11}$$

which satisfies the equation

$$\frac{\partial u}{\partial t} = -\Delta u + \frac{|\nabla u|^2}{u} + \frac{R}{2} u. \tag{4.12}$$

It follows that

$$\mu^*(g(t_1), \tau(t_1)) \le \mu^*(g(t_2, \tau(t_2)) + \frac{n}{2} \ln \frac{\tau_1}{\tau_2}, \tag{4.13}$$

for $t_1 < t_2$, where $\tau_1 = \tau(t_1)$ and $\tau_2 = \tau(t_2)$. Choosing $t_1 = 0$ and $t_2 = t^*$, we then arrive at

$$\mu^*(g(0), t^* + \sigma) \le \mu^*(g(t^*), \sigma) + \frac{n}{2} \ln \frac{t^* + \sigma}{\sigma}. \tag{4.14}$$

Since $0 < t^* < T$ is arbitrary, we can rewrite (4.14) as follows

$$\mu^*(g(t), \sigma) \ge \mu^*(g(0), t + \sigma) + \frac{n}{2} \ln \frac{\sigma}{t + \sigma} \tag{4.15}$$

for all $t \in [0, T)$ and $\sigma > 0$ (the case $t = 0$ is trivial).

We will also need the following elementary lemma.

Lemma 4.1 *Let* $a > 0$ *and* b *be constants. Then the minimum of the function* $y = a\sigma - \frac{n}{2}\ln\sigma + b$ *for* $\sigma > 0$ *is* $\frac{n}{2}\ln(\alpha a)$, *where*

$$\alpha = \frac{2e}{n} e^{\frac{2b}{n}}. \tag{4.16}$$

Proof Since $y \to \infty$ as $t \to 0$ or $t \to \infty$, it achieves its minimum somewhere. We have $y' = a - \frac{n}{2\sigma}$, whence the minimum is achieved at $\sigma = \frac{n}{2a}$. Then the minimum equals $y(\frac{n}{2a})$, which leads to the desired conclusion. □

Proof of Theorem 1.1 We apply Theorem 3.3 with $g = g_0$ to estimate $\mu^*(g_0, t+\sigma)$. Consider $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$. We choose

$$\alpha = \frac{8(t+\sigma)}{n\tilde{C}_S(M, g_0)^2} \tag{4.17}$$

in (3.6) and deduce

$$\begin{aligned}
\int_M u^2 \ln u^2 &\le 4(t+\sigma)\int_M |\nabla u|^2 - \frac{n}{2}\ln\frac{8(t+\sigma)}{n\tilde{C}_S^2} \\
&\quad + \frac{n}{2}\cdot\frac{8(t+\sigma)}{n\tilde{C}_S^2 \mathrm{vol}_{g_0}(M)^{\frac{2}{n}}} + \frac{n}{2}(\ln 2 - 1) \\
&\le (t+\sigma)\int_M (4|\nabla u|^2 + Ru^2) + (t+\sigma)\left(\frac{4}{n\tilde{C}_S^2 \mathrm{vol}_{g_0}(M)^{\frac{2}{n}}} - \min_{t=0} R\right) \\
&\quad - \frac{n}{2}\ln(t+\sigma) + \frac{n}{2}(2\ln\tilde{C}_S + \ln n - 2\ln 2 - 1),
\end{aligned} \tag{4.18}$$

where $\tilde{C}_S = \tilde{C}_S(M, g_0)$. It follows that

$$\begin{aligned}
\mu^*(g(0), t+\sigma) &\ge \frac{n}{2}\ln(t+\sigma) - (t+\sigma)\left(\frac{4}{n\tilde{C}_S^2 \mathrm{vol}_{g_0}(M)^{\frac{2}{n}}} - \min_{t=0} R\right) \\
&\quad - \frac{n}{2}(2\ln\tilde{C}_S + \ln n - 2\ln 2 - 1).
\end{aligned} \tag{4.19}$$

Combining this with (4.15) leads to

$$\begin{aligned}
\mu^*(g(t), \sigma) &\ge \frac{n}{2}\ln\sigma - (t+\sigma)\left(\frac{4}{n\tilde{C}_S^2 \mathrm{vol}_{g}(M)^{\frac{2}{n}}} - \min_{t=0} R\right) \\
&\quad - \frac{n}{2}(2\ln\tilde{C}_S + \ln n - 2\ln 2 - 1),
\end{aligned} \tag{4.20}$$

or

$$\mu^*\left(g(t), \frac{\sigma}{4}\right) \geq \frac{n}{2}\ln\sigma - \left(t + \frac{\sigma}{4}\right)\left(\frac{4}{n\tilde{C}_S^2 \mathrm{vol}_g(M)^{\frac{2}{n}}} - \min_{t=0} R\right)$$
$$-\frac{n}{2}(2\ln\tilde{C}_S + \ln n - 1), \tag{4.21}$$

which is equivalent to (1.2).

To see (1.3), we apply Lemma 4.1 to (1.2) with $a = \int_M (|\nabla u|^2 + \frac{R}{4}u^2)\mathrm{dvol} + \frac{A_1}{4}$ and $b = A_1 t + A_2$. □

Proof of Theorem 1.2 This is similar to the proof of Theorem 1.1. We apply Theorem 3.5 with $g = g_0$ to estimate $\mu^*(g_0, t+\sigma)$. Assume $t + \sigma \geq \frac{n}{8}C_S(M, g_0)^2\delta_0(g_0)$. We set

$$A = \frac{8(t+\sigma)}{nC_S(M, g_0)^2}. \tag{4.22}$$

Then there holds $A \geq \delta_0(g_0)$. Using this A in (3.11), we deduce for $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$

$$\int_M u^2 \ln u^2 \leq 4(t+\sigma)\int_M \left(|\nabla u|^2 + \frac{R}{4}u^2\right) - \frac{n}{2}\ln(t+\sigma)$$
$$+\frac{n}{2}(2\ln C_S(M, g_0) + \ln n - 2\ln 2) + \sigma_0(g_0). \tag{4.23}$$

It follows that

$$\mu^*(g_0, t+\sigma) \geq \frac{n}{2}\ln(t+\sigma) - \frac{n}{2}(2\ln C_S(M, g_0) + \ln n - 2\ln 2) - \sigma_0(g_0). \tag{4.24}$$

Combining this with (4.15) yields

$$\mu^*(g(t), \sigma) \geq \frac{n}{2}\ln\sigma - \frac{n}{2}(2\ln C_S(M, g_0) + \ln n - 2\ln 2) - \sigma_0(g_0). \tag{4.25}$$

Replacing σ by $\frac{\sigma}{4}$, we then arrive at (1.5).

To see (1.6), we apply Lemma 4.1 to (1.5) with $a = \int_M (|\nabla u|^2 + \frac{R}{4}u^2)$ and $b = \frac{n}{2}\ln n + n\ln C_S(M, g_0) + \sigma_0(g_0)$. Note that by the maximum principle and the evolution equation of the scalar curvature associated with the Ricci flow, min R is nondecreasing, which implies that $a > 0$. □

Note that the proofs of Theorems 1.1 and 1.2 lead to the following general result. Indeed, Theorems 1.1 and 1.2 follows from it.

Theorem 4.2 *Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ for some (finite or infinite) $T > 0$. Let $h(\sigma)$ be a scalar function for $\sigma > 0$. Assume that the initial metric $g_0 = g(0)$ satisfies the logarithmic Sobolev inequality*

$$\int_M u^2 \ln u^2 \text{dvol} \leq \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \text{dvol} + h(\sigma) \tag{4.26}$$

for each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$. Then there holds at each $t \in [0, T)$

$$\int_M u^2 \ln u^2 \text{dvol} \leq \sigma \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \text{dvol} + h(4(t + \sigma)) \tag{4.27}$$

for each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$.

Proof of Theorem 1.3 Let $t \in [0, T)$ and $\sigma > 0$. If $\sigma < \frac{n}{8} C_S(M, g_0)^2 \delta_0(g_0)$, we apply Theorem 1.1. Otherwise, we apply Theorem 1.2. Then we arrive at (1.8). To see (1.9), we note that by [6] the eigenvalue $\lambda_0(g(t))$ is nondecreasing. Hence, $\lambda_0(g(t)) > 0$ for all t, which implies that $\int_M (|\nabla u|^2 + \frac{R}{4} u^2) > 0$ for all t. Hence, we can apply Lemma 4.1 to (1.8) with $a = \int_M (|\nabla u|^2 + \frac{R}{4} u^2)$ and $b = C$ to arrive at the desired inequality. □

Proof of Corollary 1.4 to Theorem 1.3 Choosing $u = \text{vol}_{g(t)}(M)^{-\frac{1}{2}}$ in (1.8), we infer

$$\ln \frac{1}{\text{vol}_{g(t)}(M)} \leq \frac{\sigma}{4} \hat{R}(t) - \frac{n}{2} \ln \sigma + C. \tag{4.28}$$

If $\hat{R}(t) \leq 0$, we choose $\sigma = 1$ to arrive at (1.11). If $\hat{R}(t) > 0$, we choose $\sigma = \hat{R}(t)^{-1}$ to arrive at (1.12). □

5 The Sobolev Inequality Along the Ricci Flow

We first present a general result which converts a logarithmic Sobolev inequality to a Sobolev inequality. It follows straightforwardly from more general results in [2]. Consider a compact Riemannian manifold (M, g) of dimension $n \geq 1$. Let $\Psi \in L^\infty(M)$, which we call a potential function. We set $H = -\Delta + \Psi$. Its associated quadratic form is

$$Q(u) = \int_M (|\nabla u|^2 + \Psi u^2) \text{dvol}, \tag{5.1}$$

where $u \in W^{1,2}(M)$. We also use Q to denote the corresponding bilinear form, i.e.,

$$Q(u, v) = \int_M (\nabla u \cdot \nabla v + \Psi u v) \text{dvol}. \tag{5.2}$$

Consider the operator e^{-tH} associated with H. It is characterized by the property that for $u_0 \in L^2(M)$, $u = e^{-tH}u_0$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = -Hu \tag{5.3}$$

for $t > 0$ and the initial condition $u(0) = u_0$. We have the spectral formula

$$e^{-tH}u = \sum e^{-\lambda_i t}\phi_i < u, \phi_i >_2, \tag{5.4}$$

for $u \in L^2(M)$, where $\{\phi_i\}$ is a complete set of L^2-orthonormal eigenfunctions of H and $\lambda_1 \le \lambda_2 \le \cdots$ are the corresponding eigenvalues. Since $\lambda_i \to \infty$, $e^{-tH} : L^2(M) \to L^2(M)$ is a bounded operator. On the other hand, there holds

$$e^{-tH}u = \int_M K(\cdot, y, t)u\,\mathrm{dvol}_y, \tag{5.5}$$

where $K(x, y, t)$ denotes the heat kernel of H.

Lemma 5.1 *The extension of e^{-tH} for $t > 0$ to $L^1(M)$ by the spectral formula (5.4) defines a bounded linear operator $e^{-tH} : L^1(M) \to W^{2,p}(M)$ for each $0 < p < \infty$.*

Proof By elliptic regularity, we have $\phi_i \in W^{2,p}(M)$ for each i and $0 < p < \infty$. The elliptic $W^{2,p}$ estimates and Sobolev embedding lead to $\|\phi_i\|_{2,p} \le c_p(|\lambda_i| + 1)^{m_n}$ for some $c_p > 0$ independent of i and a natural number m_n depending only on n. The Sobolev embedding then implies $\|\phi_i\|_\infty \le c(\lambda_i| + 1)^{m_n}$ for some $c > 0$ independent of i. Now we have for $u \in L^1(M)$

$$\sum_{i\ge1} e^{-\lambda_i t}| < u, \phi_i >_2 | \cdot \|\phi_i\|_{2,p} \le \left(\sum_{i\ge1} e^{-\lambda_i t}\|\phi_i\|_\infty\|\phi_i\|_{2,p}\right)\|u\|_1. \tag{5.6}$$

By the above estimates, the last series converges. The desired conclusion follows. □

Lemma 5.2 *Assume $\Psi \ge 0$. Then e^{-tH} for $t > 0$ is a contraction on $L^p(M)$ for each $1 \le p \le \infty$, i.e.,*

$$\|e^{-tH}u\|_p \le \|u\|_p \tag{5.7}$$

for all $u \in L^p(M)$. It is also a contraction on $W^{1,2}(M)$ with respect to the norm $Q(u)^{\frac{1}{2}}$ (if $Q > 0$, i.e., $\lambda_1 > 0$) or the norm $(Q(u) + \int_M u^2\mathrm{dvol})^{\frac{1}{2}}$ (if $\lambda_1 = 0$). Moreover, it is positivity preserving, i.e., $e^{-tH}u \ge 0$ if $u \ge 0$ and $u \in L^2(M)$.

Proof The maximum principle implies that e^{-tH} is a contraction on $L^\infty(M)$ for $t > 0$. For $t > 0$ and $u \in L^1(M)$, we set $\phi = sgn(e^{-tH}u)$, i.e., $\phi = 1$ where $e^{-tH}u \ge 0$

and $\phi = -1$ where $e^{-tH}u < 0$. There holds

$$\|e^{-tH}u\|_1 = \int_M \phi e^{-tH} u \text{dvol} = \int_M u e^{-tH} \phi \text{dvol} \\ \leq \|e^{-tH}\phi\|_\infty \|u\|_1 \leq \|\phi\|_\infty \|u\|_1 = \|u\|_1. \tag{5.8}$$

Hence, e^{-tH} is a contraction on $L^1(M)$. By the Riesz–Thorin interpolation theorem (see Appendix 3), e^{-tH} is a contraction on $L^p(M)$ for each $1 < p < \infty$.

The contraction property of e^{-tH} on $W^{1,2}(M)$ follows from the spectral formula (5.4) because $\lambda_1 \geq 0$. (The contraction property of e^{-tH} on $L^2(M)$ also follows from (5.4).) Finally, the positivity preserving property of e^{-tH} is a consequence of the maximum principle. □

Theorem 5.3 *Let* $0 < \sigma^* \leq \infty$. *Assume that for each* $0 < \sigma < \sigma^*$ *the logarithmic Sobolev inequality*

$$\int_M u^2 \ln u^2 \text{dvol} \leq \sigma Q(u) + \beta(\sigma) \tag{5.9}$$

holds true for all $u \in W^{1,2}(M)$ *with* $\|u\|_2 = 1$, *where* β *is a nonincreasing continuous function. Assume that*

$$\tau(t) = \frac{1}{2t} \int_0^t \beta(\sigma) d\sigma \tag{5.10}$$

is finite for all $0 < t < \sigma^*$. *Then there holds*

$$\|e^{-tH}u\|_\infty \leq e^{\tau(t) - \frac{3t}{4} \inf \Psi^-} \|u\|_2 \tag{5.11}$$

for each $0 < t < \frac{1}{4}\sigma^*$ *and all* $u \in L^2(M)$. *There also holds*

$$\|e^{-tH}u\|_\infty \leq e^{2\tau(\frac{t}{2}) - \frac{3t}{4} \inf \Psi^-} \|u\|_1 \tag{5.12}$$

for each $0 < t < \frac{1}{4}\sigma^*$ *and all* $u \in L^1(M)$.

The proof of this theorem is presented in Appendix 2. Note that (5.12) is equivalent to an upper bound for the heat kernel. The nonincreasing condition on β can easily be removed (the function $\tau(t)$ needs to be slightly modified).

Theorem 5.4 (1) *Assume* $\Psi \geq 0$. *Let* $\mu > 2$ *and* $c > 0$. *Assume that the inequality*

$$\|e^{-tH}u\|_\infty \leq c t^{-\frac{\mu}{4}} \|u\|_2 \tag{5.13}$$

holds true for each $t > 0$ *and all* $u \in L^2(M)$. *Then the Sobolev inequality*

$$\|u\|^2_{\frac{2\mu}{\mu-2}} \leq C(\mu, c) Q(u) \tag{5.14}$$

holds true for all $u \in W^{1,2}(M)$, *where the positive constant* $C(\mu, c)$ *can be bounded from above in terms of upper bounds for* c, μ *and* $\frac{1}{\mu-2}$.

(2) *Let* $\mu > 2$ *and* $c > 0$. *Assume that the inequality*

$$\|e^{-tH}u\|_\infty \le c_1 t^{-\frac{\mu}{4}} \|u\|_2 \tag{5.15}$$

holds true for each $0 < t < 1$ *and all* $u \in L^2(M)$. *Then the Sobolev inequality*

$$\|u\|^2_{\frac{2\mu}{\mu-2}} \le C(\mu, c)(Q(u) + (1 - \inf \Psi^-)\|u\|_2^2) \tag{5.16}$$

holds true for all $u \in W^{1,2}(M)$, *where* $C(\mu, c)$ *has the same property as the above* $C(\mu, c)$.

The proof of this theorem is presented in Appendix 3. Combining Theorems 5.3 and 5.4, we arrive at the following result.

Theorem 5.5 *Let* $0 < \sigma^* < \infty$. *Assume that for each* $0 < \sigma < \sigma^*$ *the logarithmic Sobolev inequality*

$$\int_M u^2 \ln u^2 \mathrm{dvol} \le \sigma Q(u) - \frac{\mu}{2} \ln \sigma + C \tag{5.17}$$

holds true for all $u \in W^{1,2}(M)$ *with* $\|u\|_2 = 1$, *where* μ *and* c *are constants such that* $\mu > 2$. *Then we have the Sobolev inequality*

$$\|u\|^2_{\frac{2\mu}{\mu-2}} \le \left(\frac{\sigma^*}{4}\right)^{1-\frac{n}{\mu}} C(\bar{C}, \mu) \left(Q(u) + \frac{4 - \sigma^* \min \Psi^-}{\sigma^*}\|u\|_2^2\right) \tag{5.18}$$

for all $u \in W^{1,2}(M)$, *where* $C(\bar{C}, \mu)$ *is from Theorem* 5.4 *and* $\bar{C}$ *is defined in* (5.22) *below.*

Proof For $\lambda > 0$, we consider the metric $\bar{g} = \lambda^{-2}g$ and the potential function $\bar{\Psi} = \lambda^2\Psi$. Let $\bar{H} = -\Delta_{\bar{g}} + \bar{\Psi}$ and $\bar{Q}$ be the associated quadratic form. It follows from (5.17) that

$$\int_M u^2 \ln u^2 \mathrm{dvol}_{\bar{g}} \le \sigma \bar{Q}(u) - \frac{\mu}{2} \ln \sigma + (n - \mu) \ln \lambda + C \tag{5.19}$$

for $0 < \sigma < \lambda^{-2}\sigma^*$ and $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$. Choosing $\lambda = \frac{1}{2}\sqrt{\sigma^*}$, we obtain

$$\int_M u^2 \ln u^2 \mathrm{dvol}_{\bar{g}} \le \sigma \bar{Q}(u) - \frac{\mu}{2} \ln \sigma + \frac{n - \mu}{2}(\ln \sigma^* - 2\ln 2) + C \tag{5.20}$$

for each $0 < \sigma < 4$. By Theorem 5.3, we have for each $0 < t < 1$ and $u \in L^2(M)$

$$\|e^{-tH}u\|_\infty \le \bar{C} t^{-\frac{\mu}{4}} \|u\|_{2,\bar{g}}, \tag{5.21}$$

where

$$\bar{C} = 2^{\frac{\mu-n}{2}} (\sigma^*)^{\frac{n-\mu}{4}} e^{\frac{\mu}{4} - \frac{3\sigma^*}{16} \min \Psi^- + \frac{1}{2} C}. \tag{5.22}$$

Applying Theorem 5.4 and converting back to g, we then arrive at (5.18). □

Proof of Theorem 1.6 Applying Theorems 1.3 and 5.5 with $\Psi = \frac{R}{4}, \mu = n$, and $\sigma^* = 4$, we deduce

$$\|u\|^2_{\frac{2n}{n-2}} \leq c \left(\int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \mathrm{dvol} + \left(1 - \frac{\min_t R^-}{4} \right) \int_M u^2 \mathrm{dvol} \right), \tag{5.23}$$

where $c = c(C, -\min_t R^-)$. By the maximum principle, we have $\min_t R^- \geq \min_{t=0} R^-$. Hence, we arrive at

$$\|u\|^2_{\frac{2n}{n-2}} \leq c \left(\int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \mathrm{dvol} + \left(1 - \frac{\min_0 R^-}{4} \right) \int_M u^2 \mathrm{dvol} \right) \tag{5.24}$$

with $c = c(C, -\min_0 R^-)$. Since λ_0 is nondecreasing along the Ricci flow [6], we obtain

$$\|u\|^2_{\frac{2n}{n-2}} \leq c \left(1 + \frac{1}{\lambda_0(g_0)} \left(1 - \frac{\min_0 R^-}{4} \right) \right) \left(\int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \mathrm{dvol} \right) \tag{5.25}$$

which leads to (1.14). □

Proof of Theorem 1.5 This is similar to the above proof. □

6 The κ-Noncollapsing Estimate

It is obvious that Theorems 1.7 and 1.8 follow from Theorem 1.5, Theorem 1.6, and the following result.

Theorem 6.1 *Consider the Riemannian manifold (M, g) for a given metric g, such that for some $A > 0$ and $B > 0$ the Sobolev inequality*

$$\left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{dvol} \right)^{\frac{n-2}{n}} \leq A \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2 \right) \mathrm{dvol} + B \int_M u^2 \mathrm{dvol} \tag{6.1}$$

holds true for all $u \in W^{1,2}(M)$. Let $L > 0$. Assume $R \leq \frac{1}{r^2}$ on a geodesic ball $B(x, r)$ with $0 < r \leq L$. Then there holds

$$\mathrm{vol}(B(x, r)) \geq \left(\frac{1}{2^{n+3} A + 2BL^2} \right)^{\frac{n}{2}} r^n. \tag{6.2}$$

Proof Let $L > 0$. Assume that $R \leq \frac{1}{r^2}$ on a closed geodesic ball $B(x_0, r)$ with $0 < r \leq L$, but the estimate (6.2) does not hold, i.e.,

$$\mathrm{vol}(B(x_0, r)) < \delta r^n, \tag{6.3}$$

where

$$\delta = \left(\frac{1}{2^{n+3}A + 2BL^2}\right)^{\frac{n}{2}}. \tag{6.4}$$

We derive a contradiction. Set $\bar{g} = \frac{1}{r^2} g$. Then we have for $\bar{g}$

$$\mathrm{vol}(B(x_0, 1)) < \delta \tag{6.5}$$

and $R \leq 1$ on $B(x_0, 1)$. Moreover, (6.1) leads to the following Sobolev inequality for $\bar{g}$

$$\left(\int_M |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq A \int_M \left(|\nabla u|^2 + \frac{R}{4} u^2\right) + BL^2 \int_M u^2, \tag{6.6}$$

where the notation of the volume form is omitted. For $u \in C^\infty(M)$ with support contained in $B(x_0, 1)$, we then have

$$\left(\int_{B(x_0,1)} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq A \int_{B(x_0,1)} \left(|\nabla u|^2 + \frac{1}{4} u^2\right) + BL^2 \int_{B(x_0,1)} u^2. \tag{6.7}$$

By Hölder's inequality and (6.5), we have

$$\int_{B(x_0,1)} u^2 \leq \delta^{\frac{2}{n}} \left(\int_{B(x_0,1)} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}. \tag{6.8}$$

Hence, we deduce

$$\begin{aligned} \left(\int_{B(x_0,1)} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq A \int_{B(x_0,1)} |\nabla u|^2 + \left(\frac{A}{4} + BL^2\right) \delta^{\frac{2}{n}} \left(\int_{B(x_0,1)} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq A \int_{B(x_0,1)} |\nabla u|^2 + \frac{1}{2} \left(\int_{B(x_0,1)} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}. \end{aligned} \tag{6.9}$$

It follows that

$$\left(\int_{B(x_0,1)} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq 2A \int_{B(x_0,1)} |\nabla u|^2. \tag{6.10}$$

Next consider an arbitrary domain $\Omega \subset B(x_0, 1)$. For $u \in C^\infty(\Omega)$ with support contained in Ω, we deduce from (6.10) via Hölder's inequality

$$\int_{B(x_0,1)} |u|^2 \leq 2A\mathrm{vol}(\Omega)^{\frac{2}{n}} \int_\Omega |\nabla u|^2. \tag{6.11}$$

Hence, we arrive at the following Faber-Krahn inequality:

$$\lambda_1(\Omega)\mathrm{vol}(\Omega)^{\frac{2}{n}} \geq \frac{1}{2A}, \tag{6.12}$$

where $\lambda_1(\Omega)$ denotes the first Dirichlet eigenvalue of $-\Delta$ on Ω. By the proof of Proposition 2.4 in [1], we then infer

$$\mathrm{vol}(B(x,\rho)) \geq \left(\frac{1}{2^{n+3}A}\right)^{\frac{n}{2}} \rho^n \tag{6.13}$$

for all $B(x,\rho) \subset B(x_0, 1)$. Consequently, we have

$$\mathrm{vol}(B(x_0,1)) \geq \left(\frac{1}{2^{n+3}A}\right)^{\frac{n}{2}}, \tag{6.14}$$

contradicting (6.5).

For the convenience of the reader, we reproduce here the arguments in the proof of Proposition 2.4 in [1]. Consider $B(x,\rho) \subset B(x_0, 1)$. Set $u(y) = \rho - d(x, y)$. Then we obtain

$$\lambda_1(B(x,\rho)) \equiv \lambda_1(int\ B(x,\rho)) \leq \frac{\mathrm{vol}(B(x,r))}{\int_{B(x,\rho/2)} u^2} \leq \frac{4\mathrm{vol}(B(x,\rho))}{\rho^2\mathrm{vol}(B(x,\rho/2))}. \tag{6.15}$$

By (6.12), we then infer

$$\mathrm{vol}(B(x,\rho)) \geq \left(\frac{\rho^2}{2A}\right)^{\frac{n}{n+2}} 4^{-\frac{n}{n+2}} vol\left(B\left(x,\frac{\rho}{2}\right)\right)^{\frac{n}{n+2}}. \tag{6.16}$$

Iterating (6.16), we obtain

$$\mathrm{vol}(B(x,\rho)) \geq \left(\frac{\rho^2}{2A}\right)^{\sum_{l=1}^m \left(\frac{n}{n+2}\right)^l} 4^{-\sum_{l=1}^m l\left(\frac{n}{n+2}\right)^l} \mathrm{vol}\left(B\left(x,\frac{\rho}{2^m}\right)\right)^{\left(\frac{n}{n+2}\right)^m} \tag{6.17}$$

for all natural numbers $m \geq 1$. Letting $m \to \infty$, we finally arrive at

$$\begin{aligned}\operatorname{vol}(B(x,\rho)) &\geq \left(\frac{\rho^2}{2A}\right)^{\sum_{l=1}^{\infty}\left(\frac{n}{n+2}\right)^l} 4^{-\sum_{l=1}^{\infty} l\left(\frac{n}{n+2}\right)^l} \\ &= \left(\frac{\rho^2}{2A}\right)^{\frac{n}{2}} 4^{-\frac{n(n+2)}{4}} = \left(\frac{1}{2^{n+3}A}\right)^{\frac{n}{2}} \rho^n. \end{aligned} \tag{6.18}$$

□

Appendix 1: The Logarithmic Sobolev Inequalities on the Euclidean Space

In this appendix, we review several versions of the logarithmic Sobolev inequality on the euclidean space for the purpose of presenting the background of the logarithmic Sobolev inequalities. These versions are equivalent to each other.

The Gaussian version

This is the original version of L. Gross.

Theorem 7.1 *Let* $u \in W^{1,2}_{loc}(\mathbf{R}^n)$ *satisfy* $\int_{\mathbf{R}^n} u^2 d\mu = 1$, *where*

$$d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \mathrm{d}x. \tag{7.1}$$

Then

$$\int_{\mathbf{R}^n} u^2 \ln u^2 \mathrm{d}\mu \leq 2\int_{\mathbf{R}^n} |\nabla u|^2 \mathrm{d}\mu. \tag{7.2}$$

The straight (Euclidean volume element) version

Theorem 7.2 *There holds*

$$\int u^2 \ln u^2 \mathrm{d}x \leq 2\int |\nabla u|^2 \mathrm{d}x, \tag{7.3}$$

provided that $u \in W^{1,2}(\mathbf{R}^n)$ *and* $\int u^2 \mathrm{d}x = (2\pi)^{n/2}e^n$. *Equivalently, for* $\beta > 0$,

$$\int u^2 \ln u^2 \mathrm{d}x \leq 2\int |\nabla u|^2 \mathrm{d}x + \beta \ln \beta - \frac{n}{2}\beta \ln(2\pi e^2), \tag{7.4}$$

provided that $u \in W^{1,2}(\mathbf{R}^n)$ *and* $\int u^2 = \beta$.

The log gradient version

It appears to be stronger than the other versions because of the logarithm in front of the Dirichlet integral of u.

Theorem 7.3 *There holds*

$$\int u^2 \ln u^2 \mathrm{d}x \le \frac{n}{2} \ln \left[\frac{2}{\pi n e} \int |\nabla u|^2 \mathrm{d}x \right], \tag{7.5}$$

provided that $u \in W^{1,2}(\mathbf{R}^n)$ *and* $\int u^2 \mathrm{d}x = 1$.

The entropy version (as formulated in [6])

This version is intimately related to Perelman's entropy functional $\mathcal{W}$. Indeed, it can be viewed as the motivation for $\mathcal{W}$.

Theorem 7.4 *There holds*

$$\int \left(\frac{1}{2} |\nabla f|^2 + f - n \right) e^{-f} \mathrm{d}x \ge 0, \tag{7.6}$$

provided that $f \in W^{1,2}_{loc}(\mathbf{R}^n)$ *and* $\int e^{-f} \mathrm{d}x = (2\pi)^{n/2}$.

Appendix 2: The Estimate for e^{tH}

In this appendix, we present the proof of Theorem 5.3. The global case $\sigma^* = \infty$ of this theorem follows from Corollary 2.2.8 in [2]. On the other hand, the proof of this corollary in [2] can easily be extended to cover the local case $\sigma^* < \infty$, as is done below. The global case is customarily phrased in terms of "ultracontractivity," i.e., the logarithmic Sobolev inequality implies the ultracontractivity of e^{-tH}, see e.g., [2]. Note that the global case suffices for the main purpose of this paper. The local case should be useful for further applications.

Proof of Theorem 5.3 **Part 1** We first assume $\Psi \ge 0$, i.e., $\min \Psi^- = 0$. It follows from (5.9)

$$\int_M u^2 \ln u^2 \le \sigma Q(u) + \beta(\sigma) \|u\|_2^2 + \|u\|_2^2 \ln \|u\|_2^2 \tag{8.1}$$

for all $u \in W^{1,2}(M)$. Here the notation of the volume form is omitted. Replacing u by $|u|^{p/2}$ for $p > 2$ and $u \in W^{1,2}(M) \cap L^\infty(M)$, we deduce

$$p \int_M |u|^p \ln |u| \le \sigma Q(|u|^{\frac{p}{2}}) + \beta(\sigma) \|u\|_p^p + p \|u\|_p^p \ln \|u\|_p^2. \tag{8.2}$$

Since

$$Q(|u|^{\frac{p}{2}}) = \frac{p^2}{4(p-1)} Q(|u|, |u|^{p-1}), \tag{8.3}$$

we arrive at

$$\int_M |u|^p \ln |u| \le \frac{\sigma p}{4(p-1)} Q(|u|, |u|^{p-1}) + \frac{\beta(\sigma)}{p} \|u\|_p^p + \|u\|_p^p \ln \|u\|_p. \tag{8.4}$$

By the nonincreasing property of β, we then infer, replacing σ by $\frac{4(p-1)}{p}\sigma$

$$\int_M |u|^p \ln |u| \le \sigma Q(|u|, |u|^{p-1}) + \frac{\beta(\sigma)}{p} \|u\|_p^p + \|u\|_p^p \ln \|u\|_p \tag{8.5}$$

for $\sigma \in (0, \frac{p}{4(p-1)}\sigma^*]$.

Part 2 We continue with the assumption $\Psi \ge 0$. Consider $0 < t \le \frac{1}{4}\sigma^*$. Let $\sigma(p)$ be a nonnegative continuous function for $p \ge 2$ such that $\sigma(p) \in (0, \frac{p}{4(p-1)}\sigma^*]$ for $p > 2$, which will be chosen later. Then we have

$$\int_M |u|^p \ln |u| \le \sigma(p) Q(|u|, |u|^{p-1}) + \Gamma(p) \|u\|_p^p + \|u\|_p^p \ln \|u\|_p \tag{8.6}$$

for each $p > 2$ and all $u \in W^{1,2}(M) \cap L^\infty(M)$, where $\Gamma(p) = \frac{\beta(\sigma(p))}{p}$. Define the function $p(s)$ for $0 \le s < t$ by

$$\frac{\mathrm{d}p}{\mathrm{d}s} = \frac{p}{\sigma(p)}, \; p(0) = 2. \tag{8.7}$$

Assume that

$$p(s) \to \infty \tag{8.8}$$

as $s \to t$. We also define the function $N(s)$ for $0 \le s < t$ by

$$\frac{\mathrm{d}N}{\mathrm{d}s} = \frac{\Gamma(p(s))}{\sigma(s)}, \; N(0) = 0 \tag{8.9}$$

and set

$$N^* = \lim_{s \to t} \equiv \int_2^\infty \frac{\Gamma(p)}{p} \mathrm{d}p. \tag{8.10}$$

For $u \in W^{1,2}(M) \cap L^\infty(M)$ with $u \ge 0$, we set $u_s = e^{-sH}u$ for $0 < s < t$. By the contraction properties of e^{-sH}, we have $u_s \in W^{1,2}(M) \cap L^\infty(M)$ for all s. If

$\Psi \in C^\infty(M)$, we have for a fixed $q > 2$

$$\frac{\mathrm{d}}{\mathrm{d}s}\|u_s\|_q^q = q\int_M \frac{\partial u_s}{\partial s} \cdot u_s^{q-1} = -q\int_M Hu_s \cdot u_s^{q-1}. \tag{8.11}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}s}\|u_s\|_q^q = -qQ(u_s, u_s^{q-1}). \tag{8.12}$$

In the general case $\Psi \in L^\infty(M)$, this formula follows from the spectral formula for e^{-sH}. Using this formula, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\ln(e^{-N(s)}\|u_s\|_{p(s)}) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(-N(s) + \frac{1}{p(s)}\ln\|u_s\|_{p(s)}^{p(s)}\right)\\ &= \frac{\Gamma}{\sigma} - \frac{1}{p^2}\frac{p}{\sigma}\ln\|u_s\|_p^p + \frac{1}{p}\|u_s\|_p^{-p}\left(-pQ(u_s, u_s^{p-1}) + \frac{p}{\sigma}\int_M u_s^p \ln u_s\right)\\ &= \frac{1}{\sigma}\|u_s\|_p^{-p}\left(\int_M u_s^p \ln u_s - \sigma Q(u_s, u_s^{p-1}) - \Gamma\|u_s\|_p^p - \|u_s\|_p^p \ln\|u_s\|_p\right).\end{aligned} \tag{8.13}$$

By (8.6), this is nonpositive. Hence, $e^{-N(s)}\|u_s\|_{p(s)}$ is nonincreasing, which leads to

$$\|e^{-sH}u\|_{p(s)} \le e^{N(s)}\|f\|_2 \tag{8.14}$$

for all $0 \le s < t$. By the contraction properties, we have $\|e^{-tH}u\|_{p(s)} \le \|e^{-sH}u\|_{p(s)}$, whence

$$\|e^{-tH}u\|_{p(s)} \le e^{N(s)}\|f\|_2 \tag{8.15}$$

for all $0 \le s < t$. It follows that

$$\|e^{-tH}u\|_\infty \le e^{N^*}\|u\|_2. \tag{8.16}$$

This estimate extends to $u \in L^2(M)$ with $u \ge 0$ by an approximation. For a general $u \in L^2(M)$, we use the pointwise inequality $|e^{-tH}u| \le e^{-tH}|f|$ (a consequence of the positivity preserving property) to deduce

$$\|e^{-tH}u\|_\infty \le \|e^{-tH}|u|\|_\infty \le e^{N^*}\|u\|_2. \tag{8.17}$$

Now we choose

$$\sigma(p) = \frac{2t}{p} \tag{8.18}$$

for $p \geq 2$. Then $p(s) = \frac{2t}{t-s}$. One readily sees that $\sigma(p) \in (0, \frac{p}{4(p-1)}\sigma^*]$ for $p > 2$ and $p(s) \to \infty$ as $s \to t$. We have for this choice

$$N^* = \frac{1}{2t}\int_0^t \beta(\sigma)\mathrm{d}\sigma. \tag{8.19}$$

Hence, we arrive at

$$\|e^{-tH}u\|_\infty \leq e^{\tau(t)}\|u\|_2 \tag{8.20}$$

for all $u \in L^2(M)$ and $0 < t \leq \frac{1}{4}\sigma^*$.

Part 3 For a general Ψ, we consider $\bar{\Psi} = \Psi - \min \Psi^-$ and denote the corresponding operator and quadratic form by $\bar{H}$ and $\bar{Q}$, respectively. We have by (5.9)

$$\int_M u^2 \ln u^2 \mathrm{dvol} \leq \sigma \bar{Q}(u) + \bar{\beta}(\sigma) \tag{8.21}$$

for all $u \in L^2(M)$ with $\|u\|_2 = 1$, where $\bar{\beta}(\sigma) = \beta(\sigma) + \sigma \min \Psi^-$. We apply (8.20) to deduce for $0 < t \leq \frac{1}{4}\sigma^*$ and $u \in L^2(M)$

$$\|e^{-t\bar{H}}u\|_\infty \leq e^{\frac{1}{2t}\int_0^t \bar{\beta}(\sigma)\mathrm{d}\sigma}\|u\|_2 = e^{\tau(t)+\frac{t}{4}\min\Psi^-}\|u\|_2. \tag{8.22}$$

The desired estimate (5.11) follows.

The estimate (5.12) follows from (5.11) in terms of duality, namely we have for $u, v \in L^2(M)$

$$\int_M v e^{-tH}u = \int_M u e^{-tH}v \leq \|e^{-tH}v\|_\infty \|u\|_1 \leq e^{\tau(t)-\frac{3t}{4}\min\Psi^-}\|v\|_2\|u\|_1. \tag{8.23}$$

It follows that

$$\|e^{-tH}u\|_2 \leq e^{\tau(t)-\frac{3t}{4}\min\Psi^-}\|u\|_1 \tag{8.24}$$

and then

$$\|e^{-tH}u\|_\infty \leq e^{\tau(\frac{t}{2})-\frac{3t}{8}\min\Psi^-}\|e^{-\frac{t}{2}H}u\|_2 \leq e^{2\tau(\frac{t}{2})-\frac{3t}{4}\min\Psi^-}\|u\|_1. \tag{8.25}$$

By Lemma 5.1, we arrive at (5.12) for all $u \in L^1(M)$. The estimate (5.12) also follows from the arguments in Part 2 by choosing $\sigma(p) = \frac{t}{p}$ and $p(s) = \frac{t}{t-s}$. □

Appendix 3: From the Estimate for e^{-tH} to the Sobolev Inequality

In this appendix, we present the proof of Theorem 5.4. We also present a more general result Theorem 9.5, and its implication for the Ricci flow. Consider a compact Riemannian manifold (M, g) of dimension $n \geq 1$ and $\Psi \in L^\infty(M)$ as in the set-up for

Theorem 5.4. If $Q \geq 0$, then we define the spectral square root $H^{\frac{1}{2}}$ of the operator $H = -\Delta + \Psi$ as follows. For $u = \sum_{i \geq 1} a_i \phi_i \in L^2(M)$, we set

$$H^{\frac{1}{2}} u = \sum_{i \geq 1} \lambda_i^{\frac{1}{2}} a_i \phi_i, \tag{9.1}$$

whenever the series converges in $L^2(M)$.

Lemma 9.1 *Assume $Q \geq 0$. Then $H^{\frac{1}{2}}$ is a bounded operator from $W^{1,2}(M)$ to $L^2(M)$. Indeed, there holds for all $u \in W^{1,2}(M)$*

$$\|H^{\frac{1}{2}} u\|_2^2 = Q(u). \tag{9.2}$$

Proof For $u = \sum_{1 \geq 1} a_i \phi_i \in C^2(M)$, there holds $Q(u) = < Hu, u >_2 = \sum_{i \geq 1} \lambda_i a_i^2$. By approximation, we derive $Q(u) = \sum_{i \geq 1} \lambda_i a_i^2$ for all $u \in W^{1,2}(M)$. Now we have for $N \geq 1$

$$\left\| \sum_{1 \leq i \leq N} \lambda_i^{\frac{1}{2}} a_i \phi_i \right\|_2^2 = \sum_{1 \leq i \leq N} \lambda_i a_i^2. \tag{9.3}$$

Taking the limit as $N \to \infty$, we infer $\|H^{\frac{1}{2}} u\|_2^2 = Q(u)$. □

If $Q > 0$, i.e., the first eigenvalue of H is positive, then the inverse $H^{-\frac{1}{2}}$: $L^2(M) \to W^{1,2}(M)$ of $H^{\frac{1}{2}}$ exists. We have $H^{-\frac{1}{2}} u = \sum_{i \geq 1} \lambda_i^{-\frac{1}{2}} a_i \phi_i$ for $u = \sum_{i \geq 1} a_i \phi_i \in L^2(M)$. More generally, we define $H^{-\frac{1}{2}}$ in the case $Q \geq 0$ by $H^{-\frac{1}{2}} u = \sum_{\lambda_i > 0} \lambda_i^{-\frac{1}{2}} a_i \phi_i$ for $u = \sum_{i \geq 1} a_i \phi_i \in L^2(M)$.

Lemma 9.2 *Assume $Q \geq 0$. We set $\phi_1^* = \phi_1$ if $\lambda_1 = 0$ and $\phi_1^* = 0$ if $\lambda_1 > 0$. There holds*

$$H^{-\frac{1}{2}} u = \Gamma(\frac{1}{2})^{-1} \int_0^\infty t^{-\frac{1}{2}} e^{-tH} u \mathrm{d}t \tag{9.4}$$

for all $u \in L^2(M)$ with $u \perp \phi_1^*$. Moreover, if $u \in L^2(M)$ with $u \perp \phi_1^*$ satisfies $\|e^{-tH} u\|_\infty \leq \phi(t)$ on an open interval $(a, b) \subset (0, \infty)$ for a nonnegative continuous function ϕ, then there holds

$$\left\| \int_a^b t^{-\frac{1}{2}} e^{-tH} u dt \right\|_\infty \leq \int_a^b t^{-\frac{1}{2}} \phi(t) \mathrm{d}t. \tag{9.5}$$

Proof For $u \in L^2(M)$ with $u \perp \phi_1^*$, we write $u = \sum_{\lambda_i > 0} a_i \phi_i$, where the series converges in $L^2(M)$. We have $e^{-tH}u = \sum_{\lambda_i>0} e^{-\lambda_i t} a_i \phi_i$. We have

$$\sum_{\lambda_i > 0} \int_0^\infty t^{-\frac{1}{2}} e^{-\lambda_i t} a_i \phi_i \mathrm{d}t = \Gamma\left(\frac{1}{2}\right) \sum_{\lambda_i>0} \lambda_i^{-\frac{1}{2}} a_i \phi_i = \Gamma\left(\frac{1}{2}\right) H^{-\frac{1}{2}} u. \tag{9.6}$$

Hence, the formula (9.4) follows. Next we note that convergence in L^2 implies almost everywhere convergence. Moreover, if u_k converges to u almost everywhere, then $\|u\|_\infty \leq \liminf \|u_k\|_\infty$. These two facts lead to (9.5). □

Next we recall, for the sake of clarity and precise estimates, the Marcinkiewicz interpolation theorem [8] and the Riesz–Thorin interpolation theorem [8], which we formulate in the special case of the measure space (M, μ), where μ denotes the Lebesgue measure associated with the volume element dvol of g.

Theorem 9.3 (Marcinkiewicz interpolation theorem) *Let L be an additive operator from $L^\infty(M)$ to the space of measurable functions on M. Let $1 \leq p_0 \leq q_0 \leq \infty$ and $1 \leq p_1 \leq q_1 \leq \infty$ with $q_0 \neq q_1$. Assume that L is of weak type (p_0, q_0) with constant K_0 and of weak type (p_1, q_1) with constant K_1, i.e.,*

$$\mu(\{|L(u)| > \alpha\}) \leq \left(K_0 \frac{\|u\|_{p_0}}{\alpha}\right)^{q_0} \tag{9.7}$$

and

$$\mu(\{|L(u)| > \alpha\}) \leq \left(K_1 \frac{\|u\|_{p_1}}{\alpha}\right)^{q_1} \tag{9.8}$$

for all $u \in L^\infty(M)$. Then L is of type (p_t, q_t) on $L^\infty(M)$ with constant K_t for each $0 < t < 1$, i.e.,

$$\|L(u)\|_{q_t} \leq K_t \|u\|_{p_t} \tag{9.9}$$

for all $u \in L^\infty(M)$ and $\alpha > 0$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, \tag{9.10}$$

$$K_t \leq K K_0^{1-t} K_1^t, \tag{9.11}$$

and $K = K(p_0, q_0, p_1, q_1, t)$ is bounded for $0 < \epsilon \leq t \leq 1 - \epsilon$ with each given $\epsilon > 0$, but tends to infinity as $t \to 0$ or $t \to 1$.

It follows that for each $0 < t < 1$, L extends uniquely to an additive operator $L : L^{p_t}(M) \to L^{q_t}(M)$ with the bound (9.10).

This follows from [8, Theorem 5.2]. The space of simple functions is used in [8, Theorem 5.2] instead of $L^\infty(M)$. Moreover, L is only assumed to be sublinear. Note that Theorem 9.3 holds both in the set-up of real-valued functions and the set-up of complex-valued functions.

Theorem 9.4 (Riesz–Thorin interpolation theorem) *Let L be a linear operator from $L^\infty_C(M)$, i.e., the complex-valued $L^\infty(M)$, to the space of complex-valued measurable functions on M. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$. Assume that L is of type (p_0, q_0) on $L^\infty_C(M)$ with constant K_0, and of type (p_1, q_1) on $L^\infty_C(M)$ with constant K_1. Then L is of type (p_t, q_t) on $L^\infty_C(M)$ with constant K_t for each $0 \le t \le 1$, where p_t and q_t are given by (9.10) and*

$$K_t \le K_0^{1-t} K_1^t. \tag{9.12}$$

Consequently, for each $0 \le t \le 1$, L extends uniquely to a linear operator $L : L^{p_t}_C(M) \to L^{q_t}_C(M)$ with the bound

$$\|L(u)\|_{q_t} \le K_t \|u\|_{p_t} \tag{9.13}$$

for all $u \in L^{p_t}_C(M)$, where $L^p_C(M)$ denotes the complex-valued $L^p(M)$.

If we replace the complex-valued spaces by real-valued spaces, then the same holds except that (9.12) is replaced by

$$K_t \le 2K_0^{1-t} K_1^t. \tag{9.14}$$

The bound (9.12) still holds in the set-up of real-valued functions, provided that $p_0 \le q_0$, $p_1 \le q_1$, or T is a positive operator.

Now we are ready to prove Theorem 5.4. We present a more general result which implies Theorem 5.4.

Theorem 9.5 (1) *Let $\mu > 1$. Assume that $\Psi \ge 0$ and for some $c > 0$ the inequality*

$$\|e^{-tH} u\|_\infty \le c t^{-\frac{\mu}{4}} \|u\|_2 \tag{9.15}$$

holds true for each $t > 0$ and all $u \in L^2(M)$. Let $1 < p < \mu$. Then there holds

$$\|H^{-\frac{1}{2}} u\|_{\frac{\mu p}{\mu - p}} \le C(c, \mu, p) \|u\|_p \tag{9.16}$$

for all $u \in L^p(M)$, where the positive constant $C(\mu, c, p)$ can be bounded from above in terms of upper bounds for c, μ, $\frac{1}{\mu - p}$ and $\frac{1}{p-1}$. Consequently, there holds

$$\|u\|_{\frac{\mu p}{\mu - p}} \le C(c, \mu, p) \|H^{\frac{1}{2}} u\|_p \tag{9.17}$$

for all $u \in W^{1,p}(M)$.

(2) *Let* $\mu > 1$. *Assume that for some* $c > 0$ *the inequality*

$$\|e^{-tH}u\|_\infty \le ct^{-\frac{\mu}{4}}\|u\|_2 \tag{9.18}$$

holds true for each $0 < t < 1$ *and all* $u \in L^2(M)$. *Set* $H_0 = H - \inf \Psi^- + 1$. *Let* $1 < p < \mu$. *Then there holds*

$$\|H_0^{-\frac{1}{2}}u\|_{\frac{\mu p}{\mu - p}} \le C(\mu, c, p)\|u\|_p \tag{9.19}$$

for all $u \in L^p(M)$, *where the positive constant* $C(\mu, c, p)$ *has the same property as the* $C(\mu, c, p)$ *above. Consequently, there holds*

$$\|u\|_{\frac{\mu p}{\mu - p}} \le C(\mu, c, p)\|H_0^{\frac{1}{2}}u\|_p \tag{9.20}$$

for all $u \in W^{1,p}(M)$.

Proof (1) For simplicity, we work in the set-up of real-valued functions. The case $p = 2$ follows from [2, Theorem 2.4.2]. The proof of that theorem in [2] extends in a standard way to the general case of (9.16), so we follow it here. By the proof of Theorem 5.3 in Appendix 2, we have with $\tau(t) = \ln c_1 - \frac{\mu}{4}\ln t$

$$\|e^{-tH}u\|_\infty \le e^{2\tau(\frac{t}{2})}\|u\|_1 = \frac{2^{\frac{\mu}{2}}c^2}{t^{\frac{\mu}{2}}}\|u\|_1 \tag{9.21}$$

for all $u \in L^1(M)$ and all $t > 0$. On the other hand, we have by Lemma 5.2 $\|e^{-tH}u\|_\infty \le \|u\|_\infty$ for all $u \in L^\infty(M)$. By Theorem 9.4, we then have

$$\|e^{-tH}u\|_\infty \le \left(\frac{2^{\frac{\mu}{2}}c^2}{t^{\frac{\mu}{2}}}\right)^{\frac{1}{p}}\|u\|_p \tag{9.22}$$

for each $1 \le p \le \infty$ and all $u \in L^p(M)$.

Next we consider $1 \le p < \mu$ and set

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{\mu}, \quad \text{i.e.,} \quad q = \frac{\mu p}{\mu - p}. \tag{9.23}$$

Observe that (9.15) implies that the first eigenvalue λ_1 of H is positive. Otherwise, since $Q \ge 0$, λ_1 would be zero. Then $e^{-tH}\phi_1 = \phi_1$ for all $t > 0$. This contradicts (9.15). Thus (9.4) is valid for all $u \in L^2(M)$. We show that $H^{-\frac{1}{2}} : L^\infty(M) \to L^2(M)$ is of weak type (p, q). For a given $T \in (0, \infty)$, we write $H^{-\frac{1}{2}}(u) = G_{0,T}(u) + G_{T,\infty}(u)$, where

$$G_{a,b}(u) = \Gamma\left(\frac{1}{2}\right)^{-1}\int_a^b t^{-\frac{1}{2}}e^{-tH}u\,\mathrm{d}t. \tag{9.24}$$

We have by Lemma 9.2 and (9.22)

$$\|G_{T,\infty}(u)\|_\infty \le \Gamma\left(\frac{1}{2}\right)^{-1}\int_T^\infty t^{-\frac{1}{2}}\left(\frac{2^{\frac{\mu}{2}}c_1^2}{t^{\frac{\mu}{2}}}\right)^{\frac{1}{p}}\|u\|_p\mathrm{d}t$$
$$= c_1(\mu, p)T^{\frac{1}{2}-\frac{\mu}{2p}}\|u\|_p \tag{9.25}$$

for all $u \in L^p(M)$, where

$$c_1(\mu, p) = \frac{2p}{\mu - p}\Gamma\left(\frac{1}{2}\right)^{-1}\left(2^{\frac{\mu}{2}}c^2\right)^{\frac{1}{p}}. \tag{9.26}$$

On the other hand, we have by Lemma 5.2

$$\|G_{0,T}(u)\|_p \le \Gamma\left(\frac{1}{2}\right)^{-1}\int_0^T t^{-\frac{1}{2}}\|u\|_p\mathrm{d}t = 2\Gamma\left(\frac{1}{2}\right)^{-1}T^{\frac{1}{2}}\|u\|_p \tag{9.27}$$

for all $u \in L^p(M)$. Given $u \in L^\infty(M)$ and $\alpha > 0$, we define T by

$$\frac{\alpha}{2} = c_1(\mu, p)\|u\|_p T^{\frac{1}{2}-\frac{\mu}{p}}. \tag{9.28}$$

Then $\|G_{T,\infty}(u)\|_\infty \le \frac{\alpha}{2}$, and hence

$$\mu(\{|H^{-\frac{1}{2}}(u)| > \alpha\}) \le \mu\left(\left\{|G_{0,T}(u)| > \frac{\alpha}{2}\right\}\right) \le \left(\frac{\alpha}{2}\right)^{-p}\|G_{0,T}(u)\|_p^p$$
$$\le \left(\frac{\alpha}{2}\right)^{-p}\left(2\Gamma\left(\frac{1}{2}\right)^{-1}T^{\frac{1}{2}}\right)^p\|u\|_p^p$$
$$= c_2(\mu, p)\left(\frac{\|u\|_p}{\alpha}\right)^q, \tag{9.29}$$

where

$$c_2(\mu, p) = 2^{\frac{p(2\mu-p)}{\mu-p}}c_1(\mu, p)^{\frac{p^2}{\mu-p}}\Gamma\left(\frac{1}{2}\right)^{-p}. \tag{9.30}$$

It follows that $H^{-\frac{1}{2}}$ is of weak type (p, q) with constant $c_2(\mu, p)^{1/q}$.

Given $1 < p < \mu$, we set $\gamma = \max\{\frac{p}{p-1}, 2, \frac{2\mu-p}{\mu-p}\} + 1$, $p_0 = \frac{\gamma-1}{\gamma}p$ and $p_1 = \frac{\gamma-1}{\gamma-2}p$. Then $1 < p_0 < p_1 < \mu$ and

$$\frac{1}{p_0} + \frac{1}{p_1} = \frac{2}{p}, \quad \frac{1}{q_0} + \frac{1}{q_1} = \frac{2}{q}, \tag{9.31}$$

where $1/q_0 = 1/p_0 - 1/\mu$ and $1/q_1 = 1/p_1 - 1/\mu$, and q is the same as before, i.e., $1/q = 1/p - 1/\mu$. Applying (9.29) and Theorem 9.3 with $t = \frac{1}{2}$, we then arrive at (9.16) with

$$C(\mu, c, p) = K\left(p_0, q_0, p_1, q_1, \frac{1}{2}\right) c_2(\mu, p_0)^{\frac{1}{2q_0}} c_2(\mu, p_1)^{\frac{1}{2q_1}}. \tag{9.32}$$

The property of $C(\mu, c, p)$ is easy to see from this formula.

By [9], the operator $H^{\frac{1}{2}}$ is a pseudo-differential operator of order 1. Since M is compact, it follows that $H^{\frac{1}{2}}$ is a bounded operator from $W^{1,p}(M)$ into $L^p(M)$ for all $1 < p < \infty$. (The special case $p = 2$ is contained in Lemma 9.1). For $2 \le p < \mu$ (assuming $\mu > 2$), we have $W^{1,p}(M) \subset W^{1,2}(M)$, and hence $H^{-\frac{1}{2}} H^{\frac{1}{2}} u = u$ by Lemma 9.1. Replacing u in (9.16) by $H^{\frac{1}{2}} u$ for $u \in W^{1,p}(M)$, we then arrive at (9.17). For $1 < p < \min\{2, \mu\}$, we can argue this way to arrive at (9.16) for $u \in C^\infty(M)$. By the boundedness of $H^{\frac{1}{2}} : W^{1,p}(M) \to L^p(M)$, we then arrive at (9.16) for all $u \in W^{1,p}(M)$ via approximation.

By [9], the operator $H^{-\frac{1}{2}}$ is a pseudo-differential operator of order -1. It follows that $H^{-\frac{1}{2}}$ is a bounded map from $L^p(M)$ into $W^{1,p}(M)$ for all $1 < p < \infty$. It also follows that $H^{-\frac{1}{2}} : L^p(M) \to W^{1,p}(M)$ is the inverse of $H^{\frac{1}{2}} : W^{1,p}(M) \to L^p(M)$. Moreover, by approximation, the inequality (9.17) also implies the inequality (9.16).
(2) For $0 < t < 1$, we have for $u \in L^2(M)$

$$\|e^{-tH_0} u\|_\infty = e^{-t(1-\inf \Psi^-)} \|e^{-tH} u\|_\infty \le c t^{-\frac{\mu}{4}} \|u\|_2. \tag{9.33}$$

For $t \ge 1$, we write $t = \frac{m}{2} + t_0$ for a natural number m such that $1/2 \le t_0 < 1$. Then we have for $u \in L^2(M)$

$$\begin{aligned} \|e^{-tH_0} u\|_\infty &= e^{-t(1-\inf \Psi^-)} \|e^{-\frac{m}{2} t H} e^{-t_0 H} u\|_\infty \le e^{-t} \|e^{-t_0 H} u\|_\infty \\ &\le c e^{-t} t_0^{-\frac{\mu}{4}} \|u\|_2 \le c 2^{\frac{\mu}{4}} e^{-t} \|u\|_2 \\ &\le c 2^{\frac{\mu}{4}} e^{-\frac{\mu}{4}} \left(\frac{\mu}{4}\right)^{\frac{\mu}{4}} t^{-\frac{\mu}{4}} \|u\|_2. \end{aligned} \tag{9.34}$$

Hence, we can apply the result in (1) to arrive at the desired inequalities (9.19) and (9.20). (Note that by the above arguments they are equivalent to each other.) □

Proof of Theorem 5.4 1) Let $u \in W^{1,2}(M)$. Applying (9.17) with $p = 2$, we arrive at

$$\|u\|_{\frac{2\mu}{\mu-2}} \le C(\mu, c, 2) \|H^{\frac{1}{2}} u\|_2. \tag{9.35}$$

Combining this with (9.2), we then obtain the desired inequality. 2) This is similar to 1). Note that the quadratic form of $H - \inf \Psi^- + 1$ is $Q(u) + (1 - \inf \Psi^-)\|u\|_2^2$. □

Combining Theorems 1.1, 1.3, 5.3 and 9.5, we obtain the following two results for the Ricci flow, which extend Theorems 1.5 and 1.6. Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ as before.

Theorem 9.6 *Assume that $R_{g_0} \geq 0$ and $\lambda_0(g_0) > 0$. Let $1 < p < n$. There is a positive constant C depending only on the dimension n, a positive lower bound for $\lambda_0(g_0)$, a positive lower bound for $\mathrm{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, an upper bound for $\frac{1}{p-1}$, and an upper bound for $\frac{1}{n-p}$, such that for each $t \in [0, T)$ and all $u \in W^{1,p}(M)$ there holds*

$$\|u\|_{\frac{np}{n-p}} \leq C \| \left(-\Delta + \frac{R}{4}\right)^{\frac{1}{2}} u\|_p. \tag{9.36}$$

Theorem 9.7 *Assume $T < \infty$ and $1 < p < n$. There is a positive constant C depending only on the dimension n, a nonpositive lower bound for R_{g_0}, a positive lower bound for $\mathrm{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, an upper bound for T, an upper bound for $\frac{1}{p-1}$, and an upper bound for $\frac{1}{n-p}$, such that for each $t \in [0, T)$ and all $u \in W^{1,p}(M)$ there holds*

$$\|u\|_{\frac{np}{n-p}} \leq C \|H_0^{\frac{1}{2}} u\|_p, \tag{9.37}$$

where

$$H_0 = -\Delta + \frac{R}{4} - \frac{\min R_{g_0}^-}{4} + 1. \tag{9.38}$$

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