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On Boundary Factors and Traces of Subgroups of Finite Groups

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Abstract A subgroup E of a finite group G is called hypercyclically embedded in G if every chief factor of G below E is cyclic. Let A be a subgroup of a group G. Then we call any chief factor H/A_G of G a G-boundary factor of A. For any G-boundary factor H/A_G of A, we call the subgroup $(A \cap H)/A_G$ of G/A_G a G-trace of A. On the basis of these notions, we give some new characterizations of hypercyclically embedded subgroups.

Keywords Finite group · Hypercyclically embedded subgroup · G-boundary factor · G-trace of subgroup · Meet-irreducible subgroup

Mathematics Subject Classification 20D10 · 20D15 · 20D30

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group, p is supposed to be a prime.

It is obvious that a maximal subgroup M of G cannot be written as a proper intersection of subgroups of G. Proper subgroups of G with this property are called primitive

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[1] or meet-irreducible [2] since, in fact, they are just the meet-irreducible elements in the sense of Birkhoff [3, p. 93] of the lattice $L(G)$ of all subgroups of G.

Recall that a normal subgroup of G is called hypercyclically embedded [4, p. 217] in G if either $E = 1$ or $E \neq 1$ and every chief factor of G below E is cyclic. With study and applications of hypercyclically embedded subgroup are connected a large number of researches. Some results related to such subgroups are discussed in the books [2,4,5]. Among recent papers in this line of researches, see for example [6–13].

Our goal here is to give new characterizations of hypercyclically embedded subgroups on the basis of the following new notion.

Definition 1.1 Let A be a proper subgroup of G. We call any chief factor H/A_G of G/H_G a G-boundary factor or simply boundary factor of A. For any G-boundary factor H/A_G of A, we call the subgroup $(A \cap H)/A_G$ of G/A_G a G-trace of A or simply a trace of A.

Example 1.2 A subgroup A of G is said to be a CAP-subgroup [14, p. 37] if A either covers or avoids each chief factor of G; a partial CAP-subgroup [15,16] or semi CAP-subgroup [17] if H either covers or avoids each factor of some chief series of G. It is clear that if A is a CAP-subgroup of G, then every G-trace of A is identity. If A is a proper partial CAP-subgroup, then some G-trace of A is identity. Indeed, let $1 = G_0 < G_1 < \cdots < G_t = G$ be a chief series of G such that H either covers or avoids each factor of this series. Then there is an index i such that $G_i \leq A$ and $G_{i+1} \nleq A$. Hence A does not cover the factor G_{i+1}/G_i, so A avoids it, that is, $A \cap G_{i+1} \leq G_i$. Thus the trace $(A \cap G_{i+1})/G_i$ is identity.

Example 1.3 Let M be a maximal subgroup of G and H/M_G and H_1/M_G be chief factors of G. Then $|H/M_G| = |H_1/M_G|$ by Baer's theorem [14, Ch.A, 15.2(3)], and the number $|H/M_G|$ is called the normal index of M (Deskins). In general, the chief trace $(H \cap M)/M_G$ may be non-trivial (for example, if G is a simple non-abelian group). But if G is soluble, then every chief factor of G is abelian and so any trace of M is trivial by [14, Ch.A,15.2(1)(2)]. Hence in this case, the normal index of M coincides with its index $|G : M|$ (Deskins).

Example 1.4 Let G be p-soluble (respectively, p-supersoluble). Let A be a proper subgroup of G and H/A_G be any G-boundary factor of A such that p divides $|H/A_G|$. Then H/A_G is an abelian p-group (respectively, H/A_G is a group of order p). Hence the trace $(H \cap A)/A_G$ is subnormal in G/A_G, and it is trivial in the case where G is p-supersoluble. Therefore, if G is soluble, then every trace of any proper subgroup of G is subnormal; if G is supersoluble, then every trace of any proper subgroup of G is trivial.

Example 1.5 Every subgroup A of order 2 of the alternative group A_4 of degree 4 is meet-irreducible and A has a unique boundary factor which coincides with the Sylow 2-subgroup of A_4.

Recall that a subgroup A of G is said to permute with a subgroup B if $AB = BA$. Now we can state our first result.

Theorem A *A non-identity normal subgroup E of G is hypercyclically embedded in G if and only if for every meet-irreducible subgroup X of E and some G-boundary factor H/X_G of X, where $H \leq E$, the trace $(H \cap X)/X_G$ permutes with some Sylow q-subgroup of G/X_G for every prime q dividing $|G/X_G|$.*

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. A non-empty class $\mathfrak{F}$ of groups is said to be a formation provided, for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be solubly saturated if $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}} \leq \Phi(S)$ for some normal soluble subgroup S of G.

It is not difficult to show that if $G/E \in \mathfrak{F}$, where $\mathfrak{F}$ is a solubly saturated formation, and E is hypercyclically embedded in G, then $G \in \mathfrak{F}$ (see [13, Lemma 2.16]). Hence, the following result follows directly from Theorem A.

Corollary 1.6 *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups and E a non-identity normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that for every meet-irreducible subgroup X of E and some G-boundary factor H/X_G of X, where $H \leq E$, the trace $(H \cap X)/X_G$ permutes with some Sylow q-subgroup of G/X_G for every prime q dividing $|G/X_G|$. Then $G \in \mathfrak{F}$.*

Corollary 1.7 *G is supersoluble if and only if for every meet-irreducible subgroup X of G and some G-boundary factor H/X_G of X, the trace $(H \cap X)/X_G$ permutes with some Sylow q-subgroup of G/X_G for every prime q dividing $|G/X_G|$.*

It is clear that if $|G : A| = p^n$, then for every prime $q \in \pi(G)$ there is a Sylow q-subgroup Q of G such that A permutes with Q. Thus from Corollary 1.7 we get

Corollary 1.8 (See Johnson [1]). *If the index $|G : X|$ is a prime power for every meet-irreducible subgroup X of G, then G is supersoluble.*

The following example shows that in general the inverse of Corollary 1.8 is not true.

Example 1.9 Let p and q be primes such that q divides $p-1$. Let $G = (\langle a\rangle \rtimes \langle b\rangle) \times \langle c\rangle$, where $|a| = |c| = p$, $|b| = q$ and $A = \langle a\rangle \rtimes \langle b\rangle$ is a non-abelian group of order pq. Then G is a supersoluble group, so any G-trace of the subgroup $E = \langle ac\rangle$ is trivial. Moreover, $|G : E| = pq$ and E does not permute with any Sylow q-subgroup of G. Hence E is a meet-irreducible subgroup of G.

The product of all normal quasinilpotent subgroups of G is denoted by $F^*(G)$ and it is called the generalized Fitting subgroup of G.

Theorem B *A non-identity normal subgroup E of G is hypercyclically embedded in G if and only if for every two meet-irreducible subgroups X and Y of E with $X_G = Y_G$, there are G-boundary factors H/X_G and H_1/X_G of X and Y, respectively, where $H \leq E$ and $H_1 \leq E$, such that the traces $(X \cap H)/X_G$ and $(Y \cap H_1)/X_G$ are conjugated by an element of the generalized Fitting subgroup $F^*(G/X_G)$.*

Corollary 1.10 *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups and E a non-identity normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that* *for every two meet-irreducible subgroups X and Y of E with $X_G = Y_G$, there are G-boundary factors H/X_G and H_1/X_G of X and Y, respectively, where $H \leq E$ and $H_1 \leq E$, such that the traces $(X \cap H)/X_G$ and $(Y \cap H_1)/X_G$ are conjugated by an element of the generalized Fitting subgroup $F^*(G/X_G)$. Then $G \in \mathfrak{F}$.*

It is well known [18, II, 3.2] that in a soluble group G any two maximal subgroups M and L with $M_G = L_G$ are conjugated. It is not difficult to show that the inverse statement is also true: *If any two maximal subgroups M and L with $M_G = L_G$ are conjugated, then G is soluble* (see the proof of Theorem B*). These observations are the motivation for our next result.

Corollary 1.11 *G is supersoluble if and only if for every two meet-irreducible subgroups X and Y of G with $X_G = Y_G$, there are G-boundary factors H/X_G and H_1/X_G of X and Y, respectively, such that the traces $(X \cap H)/X_G$ and $(Y \cap H_1)/X_G$ are conjugated by an element of the generalized Fitting subgroup $F^*(G/X_G)$.*

It is clear that if A is a meet-irreducible subgroup of G, then there is a unique subgroup $A \leq A_0$ of G such that A is a maximal subgroup of A_0. We say that A_0 is the covering subgroup for A, $|A_0 : A|$ is the *small index* of A and denote it by $|G : A|_0$.

Theorem C *A non-identity normal subgroup E of G is hypercyclically embedded in G if and only if for every meet-irreducible subgroup X of E and some of its G-boundary factor H/X_G, where $H \leq E$, we have $|E : X|_0 = |H/X_G|$.*

Corollary 1.12 *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups and E a non-identity normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that for every meet-irreducible subgroup X of E and some of its G-boundary factor H/X_G of X, where $H \leq E$, we have $|E : X|_0 = |H/X_G|$. Then $G \in \mathfrak{F}$.*

The following result is well known (see Deskins [19]): *A group G is soluble if and only if the normal index $|G : M|_n$ of any maximal subgroup M of G coincides with the index $|G : M|$.* Our next result is an analogue of this result for supersoluble groups.

Corollary 1.13 *G is supersoluble if and only if for every meet-irreducible subgroup X of G and some of its G-boundary factor H/X_G, we have $|E : X|_0 = |H/X_G|$.*

2 Proofs of Theorems A, B and C

The following results are useful in our proof.

Lemma 2.1 (See [2, Ch. 4, 5.1])). *Let $R \leq A \leq G$, where R is normal in G.*

(1) If $A \leq B \leq G$, where A is a meet-irreducible subgroup of B, then there is a meet-irreducible subgroup X of G such that $A = B \cap X$.
(2) A is a meet-irreducible subgroup of G if and only if A/R is a meet-irreducible subgroup of G/R.

Lemma 2.2 (See [20]). *Let H, K and N be pairwise permutable subgroups of G and H is a Hall subgroup of G. Then $N \cap HK = (N \cap H)(N \cap K)$.*

Recall that the largest normal subgroup E of G such that every chief factor of G below E is cyclic is called the $\mathfrak{U}$-hypercentre of G and is denoted by $Z_{\mathfrak{U}}(G)$ (see [14, p. 389]).

Theorem A is a corollary of the following more general result since $\bigcap_{p_i\in\pi(E)} O_{p_i'}(E) = 1$.

Theorem A* *Let E be a non-identity normal subgroup of G. Then $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$ if and only if for every meet-irreducible subgroup X of E such that p divides the order of some G-boundary factor H/X_G of X, where $H \leq E$, the trace $(H\cap X)/X_G$ permutes with some Sylow q-subgroup of G/X_G for all primes q dividing $|G/X_G|$.*

Proof Sufficiency. Assume that this is false and let G be a counterexample with $|G| + |E|$ minimal. Let R be a minimal normal subgroup of G contained in E. Then

(1) *The hypothesis holds for $(G/R, E/R)$. Hence $(E/R)/O_{p'}(E/R) \leq Z_{\mathfrak{U}}((G/R)/O_{p'}(E/R))$.*
Let X/R be any meet-irreducible subgroup of G/R such that p divides the order of some (G/R)-boundary factor $(H/R)/(X/R)_{G/R}$ of X/R, where $H/R \leq E/R$. Let q be any prime dividing $|(G/R)/(X/R)_{G/R}| = |(G/R)/(X_G/R)| = |G/X_G|$. Then, by hypothesis, there is a Sylow q-subgroup Q/X_G of G/X_G such that

$$\begin{aligned} Q(H\cap X)/X_G &= (Q/X_G)((H\cap X)/X_G) \\ &= ((H\cap X)/X_G)(Q/X_G) = (H\cap X)Q/X_G, \end{aligned}$$

so

$$(Q(H\cap X)/R)/(X_G/R) = ((H\cap X)Q/R)/(X_G/R).$$

Hence

$$\begin{aligned} &((Q/R)/(X/R)_{G/R})(((H/R)\cap(X/R))/(X/R)_{G/R}) \\ &\quad = (((H/R)\cap(X/R))/(X/R)_{G/R})((Q/R)/(X/R)_{G/R}), \end{aligned}$$

where $(Q/R)/(X/R)_{G/R} = (Q/R)/(X_G/R)$ is a Sylow q-subgroup of $(G/R)/(X/R)_{G/R}$. Therefore the hypothesis holds for $(G/R, E/R)$. The choice of G implies that $(E/R)/O_{p'}(E/R) \leq Z_{\mathfrak{U}}((G/R)/O_{p'}(E/R))$.

(2) *$O_{p'}(E) = 1$. Hence p divides $|R|$.*
Assume that $O_{p'}(E) \neq 1$. Without loss of generality, we can assume that $R \leq O_{p'}(E)$. Then $O_{p'}(E/R) = O_{p'}(E)/R$. Claim (1) implies that $(E/R)/O_{p'}(E/R) \leq Z_{\mathfrak{U}}((G/R)/O_{p'}(E/R))$. On the other hand, from the G-isomorphism $E/O_{p'}(E) \simeq (E/R)/(O_{p'}(E)/R)$ it follows that every chief factor of G between E and $O_{p'}(E)$ is cyclic. Hence $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$, contrary to the choice of G. Therefore $O_{p'}(E) = 1$ and so p divides $|R|$.

(3) *If R is the only minimal normal subgroup of G contained in E and X is a meet-irreducible subgroup of E, then $X \cap R$ permutes with some Sylow q-subgroup Q of G for every prime q dividing $|G|$.*
If $R \leq X$, it is evident. Now assume that $R \nleq X$. Then $X_G = 1$. Since R is the unique minimal normal subgroup of G contained in E, $R/1$ is the unique G-boundary of X. Therefore, by hypothesis, there is a Sylow q-subgroup Q of G such that $(X \cap R)Q = Q(X \cap R)$.

(4) *$C_E(R) \neq 1$. Hence E is p-soluble by (1) and R is a p-group.*
If G has two different minimal normal subgroups contained in E, then it follows from Claim (1). We may, therefore, assume that R is the unique minimal normal subgroup of G contained in E. Now, in view of Claims (1) and (2), it is enough to show that R is abelian. Suppose that this is false. Then by the Feit–Thompson theorem, 2 divides $|R|$ and R is not a 2-group. Let $R_0 = P_1 \cap R$ be a Sylow 2-subgroup of R, where P_1 is some Sylow 2-subgroup of G, and V a maximal subgroup of R_0. Then $N_R(V)/V$ is 2-nilpotent by [18, IV,2.8], so $N_R(V)$ has a subgroup W such that $|N_R(V) : W| = 2$ and $V \leq W$. Then W is a meet-irreducible subgroup of $N_R(V)$. Hence by Lemma 2.1, there is a meet-irreducible subgroup X_0 of R such that $W = X_0 \cap N_R(V)$. Clearly, 2 divides $|R : X_0|$ and 4 does not divide $|R : X_0|$. Again applying Lemma 2.1, we obtain that $X_0 = X \cap R$ for some meet-irreducible subgroup X of G.
Claim (3) implies that G has a Sylow 2-subgroup P such that $X_0P = PX_0$. Let $R_2 = P \cap R$ be a Sylow 2-subgroup of R. Then $X_0P \cap R = X_0(P \cap R) = X_0R_2$ is a subgroup of R, and $X_0 < X_0R_2$ since 2 divides $|R : X_0|$.
Suppose that some prime $q \neq 2$ divides $|R : X_0|$. Since R is the unique minimal normal subgroup of G contained in E, Claim (3) implies that there is a Sylow q-subgroup Q of G such that $X_0Q = QX_0$. Then $X_0Q \cap R = X_0(Q \cap R)$ is a subgroup of R and $X_0 < X_0R_q$, where $R_q = Q \cap R$ is a Sylow q-subgroup of R. It is also clear that $X_0Q \cap X_0R_2 = X_0$. Hence X_0 is not a meet-irreducible subgroup of R. This contradiction shows that $|R : X_0| = 2$, so X_0 is normal in R. It follows that $1 \neq O^2(R) \neq R$. Since $O^2(R)$ is characteristic in R, it is normal in G, which contradicts the minimality of R. Hence R is an abelian p-group.

(5) *If R is the only minimal normal subgroup of G contained in E, then $|R| = p$.*
In view of Claim (4), R is a p-group. Let V be a maximal subgroup of R such that V is normal in a Sylow p-subgroup G_p of G. Let $q \neq p$ be any other prime dividing $|G|$. By Lemma 2.1, there is a meet-irreducible subgroup X of G such that $V = X \cap R$. Claim (3) implies that for some Sylow q-subgroup Q of G we have $VQ = QV$. It is also clear that $V = VQ \cap R$ is normal in VQ, so q does not divide $|G : N_G(V)|$. Hence V is normal in G and therefore $|R| = p$.

(6) *If $R \leq \Phi(G)$, then $O_{p'}(E/R) = 1$ and so $E/R \leq Z_{\mathfrak{U}}(G/R)$.*
Claim (1) implies that $(E/R)/O_{p'}(E/R) \leq Z_{\mathfrak{U}}((G/R)/O_{p'}(E/R))$. Let $V/R = O_{p'}(E/R)$. Clearly, V is p-soluble, so by the Frattini argument we have $G = VN_G(S) = RSN_G(S) = N_G(S)$, where S is a Hall p'-subgroup of V. But this contradicts Claim (2). Hence $O_{p'}(E/R) = 1$, and so $E/R \leq Z_{\mathfrak{U}}(G/R)$.

(7) *If for some minimal normal subgroup L of G we have $L \leq E$ and $L \neq R$, then R and L are cyclic p-groups.*

First note that p divides $|L|$ by Claim (2). On the other hand, by Claim (1), $(E/R)/O_{p'}(E/R) \leq Z_{\mathfrak{U}}((G/R)/O_{p'}(E/R))$, so from the G-isomorphism $L \simeq LR/R$ we see that L is either a p'-group or a cyclic group. Hence L must be a cyclic p-group. Similarly, R is cyclic p-groups.

(8) $\Phi(G) \cap O_p(E) = 1$.

Assume that $\Phi(G) \cap O_p(E) \neq 1$. Then E has a minimal normal subgroup $R \leq \Phi(G)$. Hence $E/R \leq Z_{\mathfrak{U}}(G/R)$ by Claim (6). But by Claims (5) and (7), R is cyclic. It follows that $E \leq Z_{\mathfrak{U}}(G)$. This contradiction shows that we have (8).

Final contradiction for the sufficiency.

Since $\Phi(O_p(E)) \leq \Phi(G)$, Claim (8) implies that $\Phi(O_p(E)) = 1$, so $O_p(E)$ is abelian. Therefore, again by Claim (8), for every minimal normal subgroup N of G contained in $O_p(E)$, there is a maximal subgroup M of G such that $O_p(E) = N \times (O_p(E) \cap M)$, where $O_p(E) \cap M$ is normal in G. Hence $O_p(E) = N_1 \times \cdots \times N_t$ for some minimal normal subgroups $N_1, \ldots, N_t$ of G. Let $C_i = C_G(N_i)$. Then $C = C_G(O_p(E)) = C_1 \cap \cdots \cap C_t$. Since E is p-soluble by Claim (4) and $O_{p'}(E) = 1$ by Claim (2), $O_p(E) = C \cap E$ by [18, VI,6.5]. Note that $|N_i| = p$ for all $i = 1, \ldots t$. Indeed, if $t > 1$, then it follows from Claim (7). Suppose that $t = 1$. Then Claims (2), (4) and (7) imply that $N_1 = R$ is a unique minimal normal subgroup of G contained in E, so $|N_1| = p$ by Claim (5). Hence G/C_i is cyclic for all $i = 1, \ldots, t$. This implies that G/C is an abelian group. Therefore from the G-isomorphism $CE/C \simeq E/C \cap E = E/O_p(E)$, we see that every chief factor of G between E and $O_p(E)$ is cyclic. Thus $E \leq Z_{\mathfrak{U}}(G)$ by the Jordan-Hölder theorem [14, Ch.A,3.2]. The final contradiction completes the proof of the sufficiency.

Necessity. Let X be any meet-irreducible subgroup of E such that p divides the order of some G-boundary factor H/X_G of X, where $H \leq E$. Since $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$, E is p-supersoluble, so $|H/X_G| = p$. Hence $(H \cap X)/X_G = 1$. The theorem is proved. □

A group G is called semisimple if either $G = 1$ or G is the direct product of some simple non-abelian groups.

Theorem B is a corollary of the following our result.

Theorem B* *Let E be a non-identity normal subgroup of G. Then $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$ if and only if for every two meet-irreducible subgroups X and Y of E with $X_G = Y_G$ such that p divides the order of some G-boundary factors H/X_G and H_1/X_G of X and Y respectively, where $H \leq E$ and $H_1 \leq E$, the traces $(X \cap H)X_G/X_G$ and $(Y \cap H_1)X_G/X_G$ are conjugated by an element of the generalized Fitting subgroup $F^*(G/X_G)$.*

Proof *Sufficiency.* Assume that this is false and let G be a counterexample with $|G| + |E|$ minimal. Let R be a minimal normal subgroup of G contained in E. We first show that the hypothesis holds for $(G/R, E/R)$.

Let X/R and Y/R be meet-irreducible subgroups of E/R with $(X/R)_{G/R} = (Y/R)_{G/R}$ such that p divides the orders of some (G/R)-boundary factors $(H/R)/(X/R)_{G/R}$ and $(H_1/R)/(X/R)_{G/R}$ of X/R and Y/R respectively, where $H/R \leq E/R$ and $H_1/R \leq E/R$. Then X and Y are meet-irreducible subgroups of E with $X_G = Y_G$

such that p divides the orders of the G-boundary factors H/X_G and H_1/X_G of X and Y respectively, where $H \leq E$ and $H_1 \leq E$. By hypothesis, for some element $xX_G \in F^*(G/X_G)$ we have

$$((X \cap H)/X_G)^{xX_G} = (Y \cap H_1)/X_G.$$

Let $f : G/X_G \to (G/R)/(X_G/R)$ be the natural isomorphism. Then

$$f(xX_G) = xR(X_G/R) \in F^*((G/R)/(X_G/R))$$

and

$$\begin{aligned} f(((X \cap H)/X_G))^{xX_G}) &= (((X \cap H)/R)/(X_G/R))^{xR(X_G/R)} \\ &= ((X/R) \cap (H/R))/(X/R)_{G/R})^{xR(X_G/R)} \\ &= f((Y \cap H_1)/X_G) = ((Y/R) \cap (H_1/R))/(X/R)_{G/R}. \end{aligned}$$

Therefore the hypothesis holds for $(G/R, E/R)$.

Then $O_{p'}(E) = 1$ and so p divides $|R|$ (see Claim (2) in the proof of Theorem A^*). Moreover, if for some minimal normal subgroup L of G we have $L \leq E$ and $L \neq R$, then R and L are cyclic p-groups (see Claim (6) in the proof of Theorem A^*).

Now assume that R is the only minimal normal subgroup of G contained in E. We now show that $|R| = p$. First we claim that R is a p-group. Assume that this is false. Then R is a non-abelian group, so the Frattini argument implies that, for any prime q dividing $|R|$, there is a maximal subgroup L of E such that $R \nleq L$ and a Sylow q-subgroup R_q of R is contained in $R \cap L$. On the other hand, if M is a maximal subgroup of E such that $R \nleq M$, then $R/1 = R/M_G$ is the only G-boundary factor of M. Hence for any two maximal subgroups M and L not containing R, we have $|M \cap R| = |L \cap R|$ by hypothesis. It follows that $R \cap M = R$, a contradiction. Thus R is a p-group.

Next we show that $|R| = p$. Indeed, assume that $|R| > p$. Then it has two different maximal subgroups V and W. Hence V and W are meet-irreducible subgroups of R. By Lemma 2.1, there are meet-irreducible subgroups X and Y of E such that $V = X \cap R$ and $W = Y \cap R$. Since R is the only minimal normal subgroup of G contained in E, $R/1 = R/M_G$ is the only G-boundary factor of V and W satisfying $R \leq E$. Therefore by hypothesis there is an element $x \in F^*(G)$ such that $W = V^x$. If $F^*(G) \cap E = 1$, then $F^*(G) \leq C_G(E)$ and so $W = V^x = V$, a contradiction. Hence $F^*(G) \cap E \neq 1$. But then $R \leq F^*(G) \cap E$. By [21, X, 13.6], $F^*(G)/Z_\infty(F^*(G))$ is a semisimple group. Since R is a p-group, it follows that $R \leq Z_\infty(F^*(G))$. Thus $F^*(G)/C_{F^*(G)}(R)$ is a p-group by [22, Ch.5, 3.2]. On the other hand, $Z_\infty(F^*(G)) = F(G)$ by [21, X, 13.7]. Hence $Z_\infty(F^*(G)) \leq C_G(R)$ by [2, AppendixC, 3.2]. But then we have $C_{F^*(G)}(R) = F^*(G)$ and so again we get that $W = V$. This contradiction shows that $|R| = p$.

The above shows that every minimal normal subgroup of G contained in E is a cyclic p-group. Now, the final part of the proof of the sufficiency can be proved similarly as the final part of the proof of the sufficiency in Theorem A^*.

Necessity. Let X and Y be any two meet-irreducible subgroups with $X_G = Y_G$ such that p divides the order of some G-boundary factors H/X_G and H_1/X_G of X and Y, respectively, where $H \leq E$ and $H_1 \leq E$. Then, since $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$, $|H/X_G| = p = |H_1/X_G|$ and so the traces $(X \cap H)/X_G$ and $(Y \cap H_1)/X_G$ are the identities.

The theorem is proved. □

Lemma 2.3 *Let R be a normal subgroup and A a meet-irreducible subgroup of G.*

(1) *If $R \leq A$, then $|G : A|_0 = |G/R : A/R|_0$.*
(2) *If $R \nleq A$, then $A_0 = A(A_0 \cap R)$ and $|G : A|_0 = |(A_0 \cap R) : (A \cap R)|$.*

Proof Both assertions are evident. Indeed, if, for example, $R \nleq A$, then $A < AR$ and so $A_0 \leq AR$. Hence $A_0 = A(A_0 \cap R)$. It follows that $|G : A|_0 = |A_0 : A| = |(A_0 \cap R) : (A \cap A_0 \cap R)| = |(A_0 \cap R) : (A \cap R)|$. □

Theorem C is a corollary of the following result.

Theorem C* *Let E be a non-identity normal subgroup of G. Then $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E)$ if and only if for every meet-irreducible subgroup X of E such that p divides the order of some G-boundary factor H/X_G of X, where $H \leq E$, we have $|E : X|_0 = |H/X_G|$.*

Proof *Sufficiency.* Assume that this is false and let G be a counterexample with $|G| + |E|$ minimal. Let R be a minimal normal subgroup of G contained in E.

We first show that the hypothesis holds for $(G/R, E/R)$. Indeed, let X/R be a meet-irreducible subgroup of E/R such that p divides the order of a (G/R)-boundary factor $(H/R)/(X/R)_{G/R}$ of X/R satisfying $H/R \leq E/R$. Then X is a meet-irreducible subgroup of E and H/X_G is a G-boundary factor of X such that $H \leq E$ and p divides $|H/X_G|$. Hence $|E : X|_0 = |H/X_G|$ by hypothesis, so

$$|E/R : X/R|_0 = |E : X|_0 = |(H/R)/(X_G/R)| = |(H/R)/(X/R)_{G/R}|.$$

This shows that the hypothesis holds for $(G/R, E/R)$. It follows that $O_{p'}(E) = 1$ and so p divides $|R|$ (see Claim (2) in the proof of Theorem A^*).

If for some minimal normal subgroup L of G we have $L \leq E$ and $L \neq R$, then R and L are cyclic p-groups (see Claim (7) in the proof of Theorem A^*).

Now assume that R is the only minimal normal subgroup of G contained in E. If R is a non-abelian group, then for some Sylow p-subgroup R_p of R we have $N_E(R_p) \neq E$. Let M be a maximal subgroup of E such that $N_E(R_p) \leq M$. The Frattini argument implies that $R \nleq M$. Then, in view of the G-isomorphism $RM_G/M_G \simeq R$, we have that RM_G/M_G is a G-boundary factor of M such that p divides $|RM_G/M_G|$ and $RM_G \leq E$. The hypothesis implies that $|RM_G/M_G| = |E : M|_0 = |E : M|$. Hence p divides $|E : M|$. But for a Sylow p-subgroup E_p of E containing R_p we have $E_p \cap R = R_p$, so $E_p \leq N_E(R_p) \leq M$. Hence p does not divide $|E : M|$. This contradiction show that R is a p-group.

Now we show that $|R| = p$. Assume that this is false. Let V be a maximal subgroup of R. Then $V \neq 1$ and, by Lemma 2.1, for some meet-irreducible subgroup X of E we have $V = X \cap R$. It is clear that $R \nleq X$, so $X_G = 1$ since R is the only minimal normal subgroup of G contained in E. Hence $R/1 = R/M_G$ is the only G-boundary factor of X satisfying $R \leq E$. Thus $|E : X|_0 = |R|$ by hypothesis. But $|E : X|_0 = |R/V| = p$, so $V = 1$. This contradiction shows that $|R| = p$.

The above shows that every minimal normal subgroup of G contained in E is a cyclic p-group. Now, the final part of the proof of the sufficiency can be proved similarly as the final proof of the sufficiency in Theorem A^*.

Necessity. Let X be any meet-irreducible subgroup of E such that p divides the order of some G-boundary factor H/X_G of X where $H \leq E$. Since $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$, E is p-supersoluble. This implies that $|H/X_G| = p$. Hence by Lemma 2.3(2), $1 \neq |E : X|_0 = |H \cap X_0 : X \cap H| = |H \cap X_0 : X_G|$. Consequently $|E : X|_0 = |H/X_G|$. The theorem is thus proved. □

3 Some Other Applications

In this section, we discuss some other applications of the notions of boundary factor and trace of a subgroup.

1. Recall that a series $M < T < G$, where T is a maximal subgroup of G and M is a maximal subgroup of T, is said to be a maximal chain of G of length 2.

A large number of results are based on a stronger condition for subgroups "cover or avoid chief factors". For example, it is known that if either every maximal subgroup of G is a (partial) CAP-subgroup of G or every second maximal subgroup of G is a (partial) CAP-subgroup of G (see [17,23]), then G is soluble. Note that the subgroup A in Example 1.5 is 2-maximal and it is not a partial CAP subgroup of A_4.

Nevertheless, we can prove the following theorem which contains the above-mentioned results in [17,23].

Theorem 3.1 *G is soluble if and only if every maximal chain of G of length 2 contains a proper subgroup M of G such that some G-trace of M is subnormal.*

In order to prove Theorem 3.1, the following well-known result (see [24, (6.6.3)]) is used.

Lemma 3.2 *Let $G = R \rtimes M$. If M is a soluble maximal subgroup of G, then R is abelian.*

Proof of Theorem 3.1. In view of Example 1.4, it is enough to prove that if every maximal chain of G of length 2 contains a proper subgroup M of G such that some G-trace of M is subnormal, then G is soluble. Assume that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G, q the largest prime dividing $|R|$ and R_q a Sylow q-subgroup of R. Then:

(1) *G/R is soluble. Hence R is the unique minimal normal subgroup of G, $R \nsubseteq \Phi(G)$, $C_G(R) = 1$ and so $q > 3$.*
Let $M/R < T/R < G/R$ be any maximal chain of G/R of length 2. Then $M < T < G$ is a maximal chain of G of length 2, so for one of the subgroups M

or T (we denote it by L) some G-trace $(H_1 \cap L)/L_G$ of L is subnormal in G/L_G by hypothesis. But then the (G/R)-trace

$$((H_1/R) \cap (L/R))/(L/R)_{G/R} = ((H_1 \cap L)/R)/(L_G/R)$$

of L/R is subnormal in $(G/R)/(L/R)_{G/R}$. This shows that the hypothesis holds for G/R. The choice of G implies that G/R is soluble, and it follows that R is the unique minimal normal subgroup of G, $R \nsubseteq \Phi(G)$, $C_G(R) = 1$. It is clear that $2 \in \pi(R)$ and $q > 3$ by the well-known Feit–Thompson theorem and Burnside $p^a q^b$-theorem.

(2) *For some maximal subgroup M of G and some Sylow q-subgroup G_q of G, we have $R_q \le G_q \le N_G(R_q) \le M$ and $M_G = 1$.* (This follows from Claim (1) and Frattini argument).

(3) *$R \neq G$, so $D = M \cap R \neq M$.*
Suppose that $R = G$ is a simple non-abelian group. Then $G/1$ is the only G-boundary factor of any proper subgroup of G. By Claim (1), G is not 2-nilpotent and so G has a 2-closed Schmidt subgroup $H = H_2 \rtimes H_t$ by [18, IV, 5.4]. It is clear that $|H_2| \neq 2$ and $H \neq G$. Hence G has a maximal chain $T < L < G$, where $T \neq 1$. Since p divides $|G|$, at least one of the subgroups T or L is a proper non-identity subnormal subgroup of G by hypothesis, so G is not a simple non-abelian group. This contradiction shows that $R \neq G$. Since $M_G = 1$, $MR = G$ and so $D = M \cap R \neq M$.

(4) *D is a normal non-nilpotent subgroup of M, so $D \nleq \Phi(M)$.*
Clearly, R_q is a Sylow subgroup of D. Suppose that D is nilpotent. Then R_q is a characteristic subgroup of D, so $M \le N_G(R_q)$ since D is normal in M. Hence, in view of Claim (1), we have $M = N_G(R_q)$ and so $N_R(R_q) = D$ is nilpotent. Then $N_R(R_q)/C_R(R_q)$ is a q-group and hence $O^q(R) \neq R$ by [21, X, 8.13] since $q > 3$ by Claim(1). But in view of Claim (1) again, every composition factor of R is non-abelian, a contradiction. Thus we have (4).

(5) *M has a maximal subgroup T such that $M = DT$ and $D \cap T \neq 1$.*
In view of Claim (4), there is a maximal subgroup T of M such that $M = DT$. Assume that $D \cap T = 1$. Then D is a minimal normal subgroup of M. Note also that

$$G/R \simeq MR/R \simeq M/M \cap R = M/D \simeq T$$

is soluble by Claim (1). Then D is a abelian group by Lemma 3.2, which contradicts Claim (4). Hence $D \cap T \neq 1$.

(6) *At least one of the subgroups $T \cap R$ or $D = M \cap R$ is subnormal in G.*
By hypothesis, a maximal chain $T < M < G$ contains a proper subgroup L of G such that some G-trace of L is subnormal. In view of Claim (1), $R/1$ is the unique G-boundary factor of L. Therefore, $L \cap R$ is subnormal in G by [14, Ch.A, 14.2]
Final contradiction.
By [14, A, 14.3], R normalizes every subnormal subgroup of G. Hence, in view of Claim (6), either $R \le N_G(T \cap R)$ or $R \le N_G(D)$. Then since $(T \cap R)^G = (T \cap R)^{RM} \le (T \cap R)^M \le D^M \le M_G$ and $1 \neq D \cap T \le T \cap R$, we obtain

that $M_G \neq 1$. But this contradicts Claim (2). The final contradiction completes the proof. □

Question 3.3 *Is it true that G is p-soluble if and only if every maximal chain of G of length 2 contains a proper subgroup M of G such that either some G-trace of M is subnormal or every G-boundary factor of M is a p'-group?*

2. It is clear that every trace of any maximal subgroup of a soluble group is abelian. This fact is a motivation for our next observations, which can be proved similarly as Theorem 3.1.

Theorem 3.4 *G is soluble if and only if every maximal subgroup of G has a nilpotent trace.*

Corollary 3.5 (O. Yu. Schmidt [14, Ch.A, 10.7]). *If every maximal subgroup of G is nilpotent, then G is soluble.*

Nevertheless, we do not know the answer to the following

Question 3.6 *Suppose that every maximal subgroup of G has a supersoluble trace. Does it true then that G is soluble?*

3. It is well known that a p-soluble group is p-supersoluble if and only if for every its maximal subgroup M, we have that $|G : M|$ is either p or a p'-number [18, VI,9.2,9.3]. Note that if G is p-soluble and p divides the order of some boundary factor H/M_G of a maximal subgroup M of G, then H/M_G is abelian p-group and so $|H/M_G| = |G : M|$. This elementary observation is a motivation for the following generalization of the Theorems 9.2 and 9.3 in [18].

Theorem 3.7 *G is p-supersoluble if and only if for every maximal subgroup M of G such that p divides the order of some boundary factor H/M_G of M, we have $|H/M_G| = |G : M|$.*

Proof See the proof of Theorem C^*. □

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References

1. Johnson, D.L.: A note on supersoluble groups. Canadian J. Math. **23**, 562–564 (1971)
2. Weinstein, M. (ed.): Between Nilpotent and Solvable. Polygonal Publishing House, Passaic (1982)
3. Birkhoff, G.: Lattices Theory, 3rd edn. AMS Coll. Publ., Providence (1984)
4. Schmidt, R.: Subgroup Lattices of Groups. Walter de Gruyter, Berlin (1994)
5. Ballester-Bolinches, A., Esteban-Romero, R., Asaad, M.: Products of Finite Groups. Walter de Gruyter, Berlin-New York (2010)
6. Shemetkov, L.A., Skiba, A.N.: On the $\mathfrak{X}\Phi$-hypercentre of finite groups. J. Algebra **322**, 2106–2117 (2009)

7. Asaad, M.: Finite groups with certain subgroups of Sylow subgroups complemented. J. Algebra **323**(7), 1958–1965 (2010)
8. Skiba, A.N.: On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups. J. Group Theory **13**, 841–850 (2010)
9. Li, B.: On Π-property and Π-normality of subgroups of finite groups. J. Algebra **334**, 321–337 (2011)
10. Skiba, A.N.: A characterrization of hypercyclically embedded subgroups of finite groups. J. Pure Appl. Algebra **215**, 257–261 (2011)
11. Guo, W., Skiba, A.N.: On $\mathfrak{F}\Phi^*$-hypercentral subgroups of finite groups. J. Algebra **372**, 275–292 (2012)
12. Yi, X., Alexander, N.: Skiba, some new characterizations of PST-groups. J. Algebra **399**, 39–54 (2014)
13. Chen, X., Guo, W., Skiba, A.N.: Some conditions under which a finite group belongs a Baer local formation. Comm. Algebra **42**, 4188–4205 (2014)
14. Doerk, K., Hawkes, T.: Finite Soluble Groups. Walter de Gruyter, Berlin-New York (1992)
15. Skiba, A.N.: Finite groups with given systems of generalized permutable subgroups. Proc. F.Scorina Gomel State Univ. **36**(3), 12–31 (2006)
16. Ballester-Bolinches, A., Ezquerro, L.M., Skiba, A.N.: Local embeddings of some families of subgroups of finite groups. Acta Math. Sin. **25**, 869–882 (2009)
17. Guo, X., Wang, J., Shum, K.P.: On semi-cover-avoiding maximal subgroups and solvability of finite groups. Comm. Algebra **34**, 3235–3244 (2006)
18. Huppert, B.: Endliche Gruppen I. Springer-Verlag, Berlin, Heidelberg, New York (1967)
19. Deskins, W.E.: On maximal subgroups. Proc. Symp. Pure Math., Am. Math. Soc **1**, 100–104 (1959)
20. Knyagina, B.N., Monakhov, V.S.: On π'-properties of finite groups having a Hall π-subgroup. Sib. Math. J. **522**, 309–398 (2011)
21. Huppert, B., Blackburn, N.: Finite Groups III. Springer-Verlag, Berlin-New York (1982)
22. Gorenstein, D.: Finite Groups. Harper & Row Publishers, New York, Evanston, London (1968)
23. Guo, X., Shum, K.P.: Cover-avoidance properties and the structure of finite groups. J. Pure Appl. Algebra **181**, 297–308 (2003)
24. Kurzweil, H., Stellmacher, B.: The Theory of Finite Groups: An Introduction. Springer-Verlag, New York-Berlin-Heidelberg (2004)