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Isospectral Operators

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Abstract For a large class of integral operators or second-order differential operators, their isospectral (or cospectral) operators are constructed explicitly in terms of h-transform (duality). This provides us a simple way to extend the known knowledge on the spectrum (or the estimation of the principal eigenvalue) from a smaller class of operators to a much larger one. In particular, an open problem about the positivity of the principal eigenvalue for birth–death processes is solved in the paper.

Keywords Isospectral · Harmonic function · Integral operator · Differential operator

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1 Introduction

Let us consider the elliptic operators

$$L = \sum_{i,j} a_{ij}(x)\partial_{ij}^2 + \sum_i b_i(x)\partial_i + c(x),$$
$$\widetilde{L} = \sum_{i,j} \tilde{a}_{ij}(x)\partial_{ij}^2 + \sum_i \tilde{b}_i(x)\partial_i,$$

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on $L^2(\mu)$ and $L^2(\tilde{\mu})$ (real), respectively, where $\tilde{\mu} = h^2\mu$ for a given measure μ and some $h \neq 0$. Their main difference is that $c(x) \not\equiv 0$. We are interested in when the operators L and $\widetilde{L}$ are L^2-isospectral in the following sense:

$$(Lf, f)_\mu = \big(\widetilde{L}\tilde{f}, \tilde{f}\big)_{\tilde{\mu}}, \quad \text{for every } \tilde{f} := f/h,\ f \in \mathscr{D}(L).$$

Here is one of our typical results in the note (cf. Theorems 3.1 and 3.6 in Sect. 3).

Theorem 1.1 (1) *Given L on $L^2(\mu)$ having domain $\mathscr{D}(L)$, let $h \neq 0$, μ-a.e. be L-harmonic*: $Lh = 0$, *μ-a.e., then, L is L^2-isospectral to $\widetilde{L}$*:

$$\widetilde{L} = L_0 + 2h^{-1}\langle a\nabla h, \nabla\rangle, \quad \mathscr{D}\big(\widetilde{L}\big) = \{f : fh \in \mathscr{D}(L)\},$$

where $L_0 = L - c$.

(2) *Given $\widetilde{L}$ on $L^2(\tilde{\mu})$ having domain $\mathscr{D}\big(\widetilde{L}\big)$, then for each $h \neq 0$, μ-a.e., $\widetilde{L}$ is L^2-isospectral to L*:

$$L = \widetilde{L} - \frac{2}{h}\langle\tilde{a}\nabla h, \nabla\rangle + \left[\frac{2}{h^2}\langle\tilde{a}\nabla h, \nabla h\rangle - \frac{1}{h}\widetilde{L}h\right],$$
$$\mathscr{D}(L) = \big\{f : f/h \in \mathscr{D}\big(\widetilde{L}\big)\big\},$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product.

As a typical application of Theorem 1.1, we obtain the next result. To state it, we need to explain the meaning of eigenvalue in different sense. We say that λ is an eigenvalue of L in the ordinary sense if $Lg = \lambda g$ for some $g \neq 0$. It is called a L^2-eigenvalue if additionally, $g \in L^2(\mu)$.

Corollary 1.2 *For each $h \in \mathscr{C}^2(\mathbb{R})$, $h \neq 0$, a.e., the operator*

$$L^h = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(x + \frac{h'}{h}\right)\frac{\mathrm{d}}{\mathrm{d}x} + \left[\left(\frac{h'}{h}\right)^2 + x\frac{h'}{h} - \frac{h''}{2h}\right]$$

has L^2-eigenvalues $\lambda_n\big(L^h\big) = -n$ with eigenfunctions

$$g_n(x) = (-1)^n h(x) e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(e^{-x^2}\big), \quad n \geqslant 0,$$

respectively. A particular class of L^h is the following:

$$L^b = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - b(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\left[b(x)^2 - b'(x) - x^2 + 1\right], \quad b \in \mathscr{C}^1(\mathbb{R}).$$

Proof Noting that the Ornstein–Uhlenbeck operator

$$\widetilde{L} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathscr{D}\big(\widetilde{L}\big) \supset \mathscr{C}_0^\infty(\mathbb{R})$$

has ordinary eigenvalues $\lambda_n(\widetilde{L}) = -n$ with eigenfunctions

$$g_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right), \quad n \geqslant 0,$$

respectively (cf. [3, Example 5.1]). Clearly, the polynomial function $g_n \in L^2(\tilde{\mu})$ for every $n \geqslant 0$, where $\tilde{\mu}(dx) = \exp(-x^2)dx$. Hence, the eigenvalues are all L^2-ones. Now, the first assertion follows from part (2) of Theorem 1.1. The last assertion then follows by setting $h = \exp \psi$ with $\psi' = b - x$:

$$\left(\frac{h'}{h}\right)^2 + x\frac{h'}{h} - \frac{h''}{2h} = \psi'^2 + x\psi' - \frac{1}{2}\left(\psi'' + \psi'^2\right) = \psi'\left(x + \frac{1}{2}\psi'\right) - \frac{1}{2}\psi''.$$

□

Corollary 1.2 says that a large class of operators are all isospectral to the rather simple Ornstein–Uhlenbeck operator. This indicates the value of the study on isospectral operators. It should be pointed out that the technique is still valuable even if you know only some estimates of the principal eigenvalue of $\widetilde{L}$ but have no knowledge on the other part of the spectrum of $\widetilde{L}$, since our knowledge on the principal eigenvalue of L is still rather limited.

Actually, Theorem 1.1 comes from a very simple observation. For completeness, here we write its complex version, even though we will use only its real version later on.

Lemma 1.3 *Let $(E, \mathscr{E}, \mu)$ be a measure space and let h be Lebesgue measurable: $E \to \mathbb{C}$, $h \neq 0$, μ-a.s. Then,*

(1) $\tilde{f} := \mathbb{1}_{[h \neq 0]} f/h$ *is an isometry from $L^2(E, \mu)$ to $L^2(E, \tilde{\mu})$ (complex), where $\tilde{\mu} = |h|^2\mu$.*

(2) *Let L be an operator on $L^2(E, \mu)$ with domain $\mathscr{D}(L)$. Define an operator $\widetilde{L}$ as follows*:

$$\widetilde{L}\tilde{f} = \mathbb{1}_{[h \neq 0]} \frac{1}{h} L\left(\tilde{f}h\right), \quad \mathscr{D}\left(\widetilde{L}\right) = \left\{\tilde{f} \in \mathscr{E} : \tilde{f}h \in \mathscr{D}(L)\right\}. \qquad (1)$$

Then, the operators $(L, \mathscr{D}(L))$ on $L^2(E, \mu)$ and $\left(\widetilde{L}, \mathscr{D}\left(\widetilde{L}\right)\right)$ on $L^2\left(E, \tilde{\mu}\right)$ are isospectral (say L and $\widetilde{L}$ are L^2-isospectral, for short) (in the following sense):

$$(Lf, f)_\mu = \left(\widetilde{L}\tilde{f}, \tilde{f}\right)_{\tilde{\mu}}, \quad f \in \mathscr{D}(L).$$

(3) *If additionally, $h \in \mathscr{D}(L)$, then $\widetilde{L}\mathbb{1} = 0$, $\tilde{\mu}$-a.e. iff h is L-harmonic*: *$Lh = 0$, μ-a.s.*

Proof Recall the inner product in a complex L^2-space:

$$(f, g)_\mu = \int_E f\bar{g}d\mu.$$

The first assertion is obvious:

$$\int_E |f|^2 \mathrm{d}\mu = \int_{E\,[h\neq 0]} |\tilde{f}|^2 |h|^2 \mathrm{d}\mu = \int_E |\tilde{f}|^2 \mathrm{d}\tilde{\mu}.$$

By definition, for $\tilde{f} \in \mathscr{D}(\widetilde{L})$, we have $\tilde{f}h \in \mathscr{D}(L) \subset L^2(E, \mu)$. Then, we have not only $\tilde{f} \in L^2(E, \tilde{\mu})$ but also $L(\tilde{f}h) \in L^2(E, \mu)$. This means that $\widetilde{L}\tilde{f} \in L^2(E, \tilde{\mu})$. Hence, as an operator on $L^2(E, \tilde{\mu})$, $\widetilde{L}$ is well defined. Furthermore, we have

$$(Lf, f)_\mu = \big(L(\tilde{f}h), \tilde{f}h\big)_\mu = \int_E \overline{\tilde{f}h}\, L(\tilde{f}h)\mathrm{d}\mu = \int_E \bar{\tilde{f}}(\bar{h}h)\frac{1}{h}L(\tilde{f}h)\mathrm{d}\mu = \big(\widetilde{L}\tilde{f}, \tilde{f}\big)_{\tilde{\mu}}.$$

We have thus proved the second assertion. Clearly, if $h \in \mathscr{D}(L)$, then $\mathbb{1}h = h \in L^2(E, \mu)$, and hence $\mathbb{1} \in L^2(E, \tilde{\mu})$ which implies that $\tilde{\mu}(E) < \infty$. Furthermore, $\mathbb{1} \in \mathscr{D}(\widetilde{L})$ by definition of $\mathscr{D}(\widetilde{L})$. Therefore, the last assertion follows by definition of $\widetilde{L}$. □

For non-symmetric operators, their spectrum can be complex. Hence, it is natural to use the complex L^2-theory. However, in this note, we use the real L^2-spaces only. Thus, the L^2-isospectral (real) here means the spectrum of their symmetrized operators. The last assertion of the lemma suggests us, as we will do often later, to choose h as an L-harmonic function in a weak (pointwise) sense (in other words, h is in a weak domain of L) without assuming $h \in \mathscr{D}(L)$. Then, $\widetilde{L}\mathbb{1} = 0$ is meaningful in the weak sense. In this way, we can construct the operator $\widetilde{L}$ explicitly, which is the main goal of this note. Furthermore, part (3) of the lemma has the following extension.

Remark 1.4 For fixed $B \in \mathscr{E}$, $\widetilde{L}\mathbb{1} = 0$, $\tilde{\mu}$-a.e. on B iff $Lh = 0$, μ-a.s. on B.

We will illustrate later an application of this assertion in the context of Markov chains. Clearly, the L-harmonic function is an eigenfunction corresponding to the eigenvalue $\lambda = 0$. However, $\lambda = 0$ is not necessary an eigenvalue in the L^2-sense unless $h \in L^2(E, \mu)$.

One may write $\widetilde{L} = h^{-1}L(h\,\bullet)$ (μ-a.e.) for short. Because of this, $\widetilde{L}$ is called a h-transform of L. Alternatively, define an operator H:

$$Hf = hf, \quad \mathscr{D}(H) = \{f \in L^2(E, \mu) : hf \in \mathscr{D}(L)\}.$$

Then, we indeed have $\widetilde{L} = H^{-1}LH$. In view of this, L and $\widetilde{L}$ are similar and so are L^2-isospectral. More generally (without assuming the invertibility of H),

$$H\widetilde{L} = LH.$$

Because of this, L and $\widetilde{L}$ are called dual with respect to H. Therefore, the h-transform is indeed a special duality. For a different dual, refer to [2, §5 and §10]. Note that in the later case, we were interested in the principal eigenvalue only, but the transform used there is still isospectral. The reason is that the isospectral transform is easier to handle even though it looks rather strong. We remark that when E has boundary ∂E, one may deduce a boundary condition for $\widetilde{L}$ from that of L, based on the transform $\tilde{f} = \mathbb{1}_{[h\neq 0]} f/h$.

Having figured out the dual operators, in the study of their spectrum for Markov processes, it is more convenient in practice to use their extension to the Dirichlet forms, especially for the operator $\big(\widetilde{L}, \mathscr{D}\big(\widetilde{L}\big)\big)$. Generally speaking, Lemma 1.3 says that for a given Dirichlet form $(D, \mathscr{D}(D))$ on $L^2(\mu)$, its dual form $\big(\widetilde{D}, \mathscr{D}\big(\widetilde{D}\big)\big)$ on $L^2(\tilde{\mu})$ is given by

$$\widetilde{D}\big(\tilde{f}\big) = D\big(\tilde{f}h, \tilde{f}h\big), \quad \mathscr{D}\big(\widetilde{D}\big) = \big\{\tilde{f} \in \mathscr{E} : \tilde{f}h \in \mathscr{D}(D)\big\}.$$

Certainly, one may go to the inverse way, defining $(D, \mathscr{D}(D))$ in terms of $\big(\widetilde{D}, \mathscr{D}\big(\widetilde{D}\big)\big)$. In particular, for the O.-U. operator used in the proof of Corollary 1.2, corresponding to $\big(\widetilde{L}, \mathscr{D}\big(\widetilde{L}\big)\big)$, the Dirichlet form $\big(\widetilde{D}(f), \mathscr{D}\big(\widetilde{D}\big)\big)$ is

$$\begin{aligned}
\widetilde{D}(f) &= \int_{\mathbb{R}} f'^2 \mathrm{e}^{-x^2} \mathrm{d}x, \\
\mathscr{D}\big(\widetilde{D}\big) &= \big\{f \in L^2(\tilde{\mu}) : \widetilde{D}(f) < \infty\big\} = \left\{f : \int_{\mathbb{R}} \big[f^2 + f'^2\big] \mathrm{e}^{-x^2} \mathrm{d}x < \infty\right\}.
\end{aligned}$$

In the case that the potential term c^h (the last term) in L^h is non-positive, then L^h corresponds to the operator of a diffusion having killing rate $-c^h$, to which we certainly have a Dirichlet form $\big(D^h, \mathscr{D}\big(D^h\big)\big)$ on $L^2(\mu^h)$:

$$\begin{aligned}
D^h(f) &= \int_{\mathbb{R}} \big[f'^2(x) - c^h(x) f^2(x)\big] \mathrm{e}^{-x^2} \frac{\mathrm{d}x}{h(x)^2}, \\
\mathscr{D}\big(D^h\big) &= \left\{f : \int_{\mathbb{R}} \big[f^2 + (f'h - fh')^2\big] \mathrm{e}^{-x^2} \mathrm{d}x < \infty\right\}, \\
c^h(x) &= \left[\left(\frac{h'}{h}\right)^2 + x\frac{h'}{h} - \frac{h''}{2h}\right](x), \quad \mu^h(\mathrm{d}x) = \mathrm{e}^{-x^2} \frac{\mathrm{d}x}{h(x)^2}.
\end{aligned}$$

Here, $\mathscr{D}\big(D^h\big)$ is deduced from $\mathscr{D}\big(\widetilde{D}\big)$, based on Lemma 1.3. For general $c^h(x) \in \mathbb{R}$, this symmetric form may not be a Dirichlet one even though it does have non-negative spectrum in view of our isospectral property. Actually, Lemma 1.3 is meaningful in a very general setup rather than Markov processes.

The h-transform, or the Doob's h-transform is a well-known topic in probability/potential theory. Here, we mention only two related papers [9,10] where the tool is used to study the principal eigenvalue. In [9], the following model

$$L = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x} a \frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}\left(\frac{b^2}{a} + b'\right),$$

$$\widetilde{L} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x} + b\frac{\mathrm{d}}{\mathrm{d}x},$$
$$h(x) = \exp\left[\int_0^x \frac{b}{a}(y)\mathrm{d}y\right]$$

is carefully handled and applied to multi-dimensional diffusion operators. In [10], a class of symmetric Markov processes having killings are studied, and some upper and lower estimates for the first eigenvalue are presented.

The remainder of this note is organized as follows. In the next two sections, we apply Lemma 1.3, respectively, to two special classes of operators: either integral operators for Markov pure jump processes or the operators for diffusions.

2 Integral Operators

Theorem 2.1 *Let $(q(x), q(x,\mathrm{d}y))$ be a totally stable and conservative q-pair on $(E, \mathscr{E}, \mu)$ (cf.* [1, Definition 1.9]). *For a given function $c \in \mathscr{E}$ with $c \leqslant q$, define an operator Ω*

$$\Omega f(x) = \int_E q(x,\mathrm{d}y)\big[f(y) - f(x)\big] + c(x)f(x), \quad x \in E$$

with domain $\mathscr{D}(\Omega) \subset L^2(E,\mu)$. Next, let h $(> 0, \mu$-a.e.) be Ω-harmonic (if exists): $\Omega h = 0$, μ-a.e. on E. Define a new totally stable and conservative q-pair $\big(\tilde{q}(x), \tilde{q}(x, cy)\big)$ as follows:

$$\tilde{q}(x, A) = \mathbb{1}_{[h(x)\neq 0]}\frac{1}{h(x)}\int_A q(x,\mathrm{d}y)h(y), \quad A \in \mathscr{E},$$
$$\tilde{q}(x) = \tilde{q}(x, E), \quad \mu\text{-a.e. } x \in E.$$

Set

$$\widetilde{\Omega} f(x) = \int_E \tilde{q}(x,\mathrm{d}y)\big[f(y) - f(x)\big], \quad \mu\text{-a.e. } x \in E,$$
$$\mathscr{D}\big(\widetilde{\Omega}\big) = \big\{\tilde{f} \in \mathscr{E} : \tilde{f}h \in \mathscr{D}(\Omega)\big\}.$$

Then, Ω and $\widetilde{\Omega}$ are L^2-isospectral.

Proof Noting that h $(> 0$, μ-a.e.) is Ω-harmonic by assumption, we have

$$\big[q(x) - c(x)\big]h(x) = \int_E q(x,\mathrm{d}y)h(y) \geqslant 0.$$

Hence, h is $q(x,\cdot)$-integrable for a.e.-$x \in E$, and moreover $q \geqslant c$. Therefore, the new q-pair $\big(\tilde q(x), \tilde q(x,\mathrm{d}y)\big)$ is totally stable. It is clearly conservative. By definition of $\widetilde\Omega$, we have on the set $[h>0]$,

$$\begin{aligned}
\widetilde\Omega(f)(x) &= \int_E \tilde q(x,\mathrm{d}y)\big[f(y)-f(x)\big] \\
&= \frac{1}{h(x)}\int_E q(x,\mathrm{d}y)\Big\{\big[(fh)(y)-(fh)(x)\big]+f(x)\big[h(x)-h(y)\big]\Big\}\Big] \\
&= \frac{1}{h(x)}\bigg[\int_E q(x,\mathrm{d}y)\big[(fh)(y)-(fh)(x)\big]-f(x)\int_E q(x,\mathrm{d}y)\big[h(y)-h(x)\big] \\
&= \frac{1}{h(x)}\big[\Omega(fh)(x)-c(fh)(x)-f(x)[\Omega h(x)-(ch)(x)]\big] \\
&= \frac{1}{h(x)}\big[\Omega(fh)(x)-f(x)\Omega h(x)\big].
\end{aligned}$$

Now, by harmonic property of h, the right-hand side is equal to

$$\frac{1}{h(x)}\Omega(fh)(x) \quad \text{on } [h>0].$$

The assertion then follows from Lemma 1.3. □

We mention that the positive condition of h used in the theorem is to keep $\big(\tilde q(x), \tilde q(x,\mathrm{d}y)\big)$ to be a q-pair. This is certainly not necessary in a general context: considering general integral kernel instead of the non-negative one.

The inverse of the last theorem goes as follows.

Theorem 2.2 *Given a totally stable and conservative q-pair $\big(\tilde q(x), \tilde q(x,\mathrm{d}y)\big)$ and a positive $\mathscr{E}$-measurable function h such that h^{-1} is $\tilde q(x,\cdot)$-integrable for each $x \in E$, the operator $\big(\widetilde\Omega, \mathscr{D}(\widetilde\Omega)\big)$ on $L^2(E,\tilde\mu)$ corresponding to the q-pair $\big(\tilde q(x), \tilde q(x,\mathrm{d}y)\big)$ is L^2-isospectral to the following operator Ω on $L^2(E,\mu)$ $(\mu := h^{-2}\tilde\mu)$:*

$$\Omega f(x) = \int_E q(x,\mathrm{d}y)[f(y)-f(x)] + c(x)f(x),$$

$$\mathscr{D}(\Omega) = \big\{f \in \mathscr{E} : f/h \in \mathscr{D}\big(\widetilde\Omega\big)\big\} \subset L^2(E,\mu),$$

where

$$\begin{aligned}
q(x,\mathrm{d}y) &= h(x)\frac{\tilde q(x,\mathrm{d}y)}{h(y)}, \\
c(x) &= \int_E \tilde q(x,\mathrm{d}y)\bigg[\frac{h(x)}{h(y)}-1\bigg], \quad x \in E.
\end{aligned}$$

Proof It is simply a use of the duality $\Omega = H\widetilde{\Omega}H^{-1}$, noting the property that $\Omega h = 0$ is now automatic since $\widetilde{\Omega}1 = 0$. The remainder of the proof is mainly a careful computation. □

It is the place to discuss the existence of a positive Ω-harmonic function. Let $c(x) < q(x),\ x \in E$. Choose and fix a reference point $\theta \in E$. By [1, Theorem 2.2], there exists uniquely the minimal solution $(h^*(x) : x \in E)$ with $h^*(\theta) = 1$ to the following non-negative equation:

$$h(x) = \int_{E\setminus\{\theta\}} \frac{q(x, \mathrm{d}y)}{q(x) - c(x)} h(y) + \frac{q(x, \{\theta\})}{q(x) - c(x)}, \quad x \neq \theta. \tag{2}$$

Moreover, the solution can be obtained in the following way: let

$$h^{(1)}(x) = \frac{q(x, \{\theta\})}{q(x) - c(x)}, \quad x \neq \theta,$$
$$h^{(n+1)}(x) = \int_{E\setminus\{\theta\}} \frac{q(x, \mathrm{d}y)}{q(x) - c(x)} h^{(n)}(y) + \frac{q(x, \{\theta\})}{q(x) - c(x)}, \quad x \neq \theta,\ n \geqslant 1.$$

Then, for each $x \neq \theta$, $h^{(n)}(x) \uparrow h^*(x) \in [0, \infty]$ as $n \to \infty$.

Proposition 2.3 *Let $c(x) < q(x)$ for every $x \in E$ and assume that $q(x, \{\theta\}) > 0$ for some $x \neq \theta$. Then, the equation $\Omega h = 0$ has a non-trivial (finite) solution iff the minimal solution $(h^*(x) : x \in E)$ to (2) is finite. Equivalently, there is a finite f satisfying the inequality*

$$f(x) \geqslant \int_{E\setminus\{\theta\}} \frac{q(x, \mathrm{d}y)}{q(x) - c(x)} f(y) + \frac{q(x, \{\theta\})}{q(x) - c(x)}, \quad x \neq \theta.$$

Then, we actually have $f(x) \geqslant h^*(x)$ for every $x \in E$.

Proof For a given finite non-trivial Ω-harmonic function h, choosing $h(\theta) = 1$, one may write down immediately Eq. (2).

Conversely, a finite solution h^* to (2) is clearly a Ω-harmonic function. From the construction given above, it is also clear that $h^*(x) > 0$ once $q(x, \{\theta\}) > 0$. The last assertion of the proposition is essentially a comparison theorem [1, Theorem 2.6]. □

It is clear from the proof above, to obtain a positive harmonic h, some irreducible condition is necessary. Noting that it is often practical to find an explicit comparison function f, and $h^{(n)}$ for each n is already explicit, we have explicit estimates of h^* which may not be easy to obtain explicitly.

Before moving further, we discuss an alternative way to describe the Ω-harmonic function. Suppose that $\sup_x c(x) < \infty$. Then by a shift if necessary, we may and will assume for a moment that $\sup_x c(x) \leqslant 0$. Define

$$z^{(0)}(x) = 1, \qquad\qquad x \in E,$$
$$z^{(n+1)}(x) = \int_E \frac{q(x, \mathrm{d}y)}{q(x) - c(x)} z^{(n)}(y), \quad x \in E,\ n \geqslant 1.$$

Then, $z^{(n)}(x) \downarrow \bar{z}(x)$ as $n \to \infty$ for each $x \in E$. This is an analog of the maximal exit solution in the study of q-processes, cf. [1, Lemma 2.39]. The proof for the conclusion is easy, simply use the property

$$\frac{q(x, E)}{q(x) - c(x)} \leqslant 1, \quad x \in E.$$

Remark 2.4 Let $\sup_x c(x) \leqslant 0$. Then, a bounded Ω-harmonic function is non-zero iff so is the maximal solution $\bar{z}$ constructed above.

To apply the previous results, Theorem 2.1 for instance, to finite state spaces, say $E = \{0, 1, \ldots, N\}$ for some $N \geqslant 3$, one meets a problem about the existence of positive Ω-harmonic h. For which, there $N + 1$ homogeneous equations with $N + 1$ variables $h_0, h_1, \ldots, h_N$. Because of the homogeneous property in h, one may assume that $h_0 = 1$ once a non-trivial solution h exists with $h_0 \neq 0$ for instance. Thus, we have only N free variables in $N + 1$ equations. Then, a finite non-trivial solution often does not exist (or equivalently, the minimal solution given in Proposition 2.3 may be infinite). To overcome this difficulty, one has to decrease the number of equations. This is the reason we will adopt a local harmonic condition below. Then, one needs non-trivial $\tilde{c}_i$ in the corresponding operator $\widetilde{\Omega}$.

Theorem 2.5 *Let $E = \{0, 1, \ldots, N\}$ for some $N \geqslant 3$ and let $Q = (q_{ij})$ be a conservative Q-matrix on E. For given $(c_i : i = 0, 1, \ldots, N)$ with $c_i \leqslant q_i := -q_{ii}$ for $i = 0, 1, \ldots, N - 1$, set $\Omega = Q + diag(c_i)$. Next, let $h > 0$ be Ω-harmonic on $\{0, 1, \ldots, N - 1\}$, i.e.,*

$$\Omega h = 0 \quad on\ \{0, 1, \ldots, N - 1\}.$$

Define $\tilde{q}_{ij}$ $(i, j \in E)$ as in Theorem 2.1:

$$\tilde{q}_{ij} = h_i^{-1} q_{ij} h_j, \quad i, j \in E.$$

Next, define $\tilde{c}_i = 0$ on $\{0, 1, \ldots, N - 1\}$ and

$$\tilde{c}_N = c_N + \sum_{j \leqslant N} q_{Nj} \left(\frac{h_j}{h_N} - 1 \right).$$

Denote by $\widetilde{\Omega}$ the operator corresponding to the matrix $\left(\tilde{q}_{ij}\right) + diag(\tilde{c}_i)$. Then, Ω and $\widetilde{\Omega}$ are L^2-isospectral.

Proof Following the proof of Theorem 2.1, restricted to $\{0, 1, \ldots, N-1\}$, we see that

$$\widetilde{\Omega}\tilde{f}(i) = \frac{1}{h_i}\Omega\big(\tilde{f}h\big)(i) \quad \text{on } \{0, 1, \ldots, N-1\}.$$

We now show that this equality also holds for $i = N$.

$$\begin{aligned}
\widetilde{\Omega} f(N) &= \sum_{j\leqslant N} \tilde{q}_{Nj}(f_j - f_N) + \tilde{c}_N f_N \\
&= \frac{1}{h_N}\sum_{j\leqslant N} q_{Nj}\big[(fh)_j - (fh)_N\big] - \frac{f_N}{h_N}\sum_{j\leqslant N} q_{Nj}(h_j - h_N) + \tilde{c}_N f_N \\
&= \frac{1}{h_N} Q(fh)(N) - \frac{1}{h_N} c_N h_N f_N - \frac{f_N}{h_N}\sum_{j\leqslant N} q_{Nj}(h_j - h_N) + \tilde{c}_N f_N \\
&= \frac{1}{h_N}\Omega(fh)(N).
\end{aligned}$$

From Remark 1.4, it follows that $c_i = 0$ on $\{0, 1, \ldots, N-1\}$. The required assertion now follows from Lemma 1.3. □

A typical application of Theorem 2.1 to the single-birth processes is presented in [12]. In this case, the Ω-harmonic function has a very simple expression (cf. [5, Theorem 1.1]). In particular, for the killing case, the function is not only positive but also non-decreasing. It is interesting to note that for single-birth processes, the function h-dual is again the same type, but the measure μ-dual

$$\bar{q}_{ij} = \frac{\mu_j q_{ji}}{\mu_i}, \quad i, j \in E$$

maps the single birth type to the single death type. Next, for birth–death processes with birth and death rates b_i and a_i, respectively, and with killing rates $-c_i \geqslant 0$, we have

$$\tilde{a}_i = a_i \frac{h_{i-1}}{h_i} \,(\leqslant a_i),\ i \geqslant 1, h_0 = 1, \quad \tilde{b}_i = b_i \frac{h_{i+1}}{h_i} \,(\geqslant b_i),\ i \geqslant 0.$$

Then,

$$\tilde{\mu}_i = \frac{\tilde{b}_0 \ldots \tilde{b}_{i-1}}{\tilde{a}_1 \ldots \tilde{a}_i} = \frac{b_0 \ldots b_{i-1}}{a_1 \ldots a_i} h_i^2 = h_i^2 \mu_i, \quad \hat{\tilde{\nu}}_i = \frac{1}{\tilde{\mu}_i \tilde{b}_i} = \frac{1}{h_i h_{i+1}} \hat{\nu}_i, \quad i \geqslant 0.$$

For finite state space, we have

$$\tilde{c}_N = c_N + a_N\left(\frac{h_{N-1}}{h_N} - 1\right).$$

Clearly, $\tilde{c}_N \leqslant 0$ since so does c_N. However, the story is still meaningful for general $c_i \in \mathbb{R}$ satisfying $c_i \leqslant a_i + b_i$ for all $i \geqslant 0$.

To conclude this section, we answer an open question for birth–death processes with state space $\{0, 1, 2, \ldots\}$. For this, we need some notation. Given birth rates $b_i > 0\,(i \geqslant 0)$, death rates $a_i > 0\,(i \geqslant 1)$, and killing rates $-c_i \geqslant 0\,(i \geqslant 0)$, define

$$\tilde{q}_n^{(k)} = \begin{cases} -c_n, & 0 \leqslant k \leqslant n-2 \\ a_n - c_n, & k = n-1, \end{cases}$$

$$\widetilde{F}_i^{(i)} = 1, \quad \widetilde{F}_n^{(i)} = \frac{1}{b_n} \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \quad n > i \geqslant 0,$$

$$h_n = 1 - \sum_{0 \leqslant k \leqslant n-1} \sum_{0 \leqslant j \leqslant k} \widetilde{F}_k^{(j)} \frac{c_j}{b_j}, \quad n \geqslant 0.$$

Next, define the principal eigenvalue λ_0 as follows:

$$\lambda_0 = \inf \left\{ \sum_{k \geqslant 0} \left[b_k (f_{k+1} - f_k)^2 - c_k f_k^2 \right] : \sum_{k \geqslant 0} \mu_k f_k^2 = 1,\; f \text{ has finite support} \right\}.$$

Here is a solution to the Open Problem 9.13 in [2].

Theorem 2.6 *For birth–death processes as above, we have $\tilde{\delta} \leqslant \lambda_0^{-1} \leqslant 4\tilde{\delta}$, where*

$$\tilde{\delta} = \sup_{n \geqslant 0} \sum_{j=0}^{n} \tilde{\mu}_j \sum_{k \geqslant n} \hat{\tilde{\nu}}_k = \sup_{n \geqslant 0} \sum_{j=0}^{n} \mu_j h_j^2 \sum_{k \geqslant n} \frac{1}{h_k h_{k+1} \mu_k b_k}.$$

In particular, $\lambda_0 > 0$ iff $\tilde{\delta} < \infty$.

Proof The harmonic function h we need for applying Theorem 2.1 is given by [5, Theorem 1.1]. Then, the result follows by applying [2, Theorem 3.1] to the process with rates $(\tilde{b}_i, \tilde{a}_i)$ and using $\tilde{\mu}_i$ and $\hat{\tilde{\nu}}_k$ just computed above. □

3 Differential Operators

We now turn to study the second-order differential operators.

Theorem 3.1 *Consider the elliptic operator*

$$L = \sum_{i,j} a_{ij}(x) \partial_{ij}^2 + \sum_i b_i(x) \partial_i + c(x)$$

with a domain $\mathscr{D}(L)$, and let $h \neq 0$ a.e. (with respect to Lebesgue measure) be L-harmonic. Here,

$$\partial_i = \mathrm{d}/\mathrm{d}x_i, \quad \partial_{ij}^2 = \partial_i \partial_j.$$

Define

$$\widetilde{L} = \sum_{i,j} \tilde{a}_{ij}(x)\partial_{ij}^2 + \sum_i \tilde{b}_i(x)\partial_i,$$

with domain $\mathscr{D}(\widetilde{L})$ *defined in Lemma* 1.3, *where*

$$\tilde{a}_{ij}(x) = a_{ij}(x), \quad \tilde{b}_i(x) = b_i(x) + \frac{2}{h(x)} \sum_j a_{ij}(x)\partial_j h(x)$$

for all i, j, *and a.e.-*x. *Then,* L *and* $\widetilde{L}$ *are* L^2*-isospectral.*

Proof Noting that by the symmetry of the matrix (a_{ij}), we have

$$\begin{aligned} L(fh) &= \sum_{i,j} a_{ij}\partial_{ij}^2(fh) + \sum_i b_i\partial_i(fh) + cfh \\ &= \sum_{i,j} a_{ij}\big[\big(\partial_{ij}^2 f\big)h + 2\partial_i f\partial_j h + f\big(\partial_{ij}^2 h\big)\big] \\ &\quad + \sum_i b_i\big[\big(\partial_i f\big)h + f\partial_i h\big] + f(ch) \\ &= hLf + fLh - cfh + 2\sum_{i,j} a_{ij}\partial_j h\partial_i f \quad \text{a.e.} \end{aligned}$$

Because h is L-harmonic, we obtain

$$\frac{1}{h}L(fh) = (Lf - cf) + \frac{2}{h}\sum_i\bigg(\sum_j a_{ij}\partial_j h\bigg)\partial_i f, \quad \text{a.e.}$$

From which, one reads out the coefficients $\tilde{a}_{ij}(x)$ and $\tilde{b}_i(x)$ of $\widetilde{L}$. □

For short, if we set $L_0 = L - c$, then we have

$$\begin{aligned} \widetilde{L} &= L_0 + \frac{2}{h}\langle a\nabla h, \nabla\rangle \\ &= L_0 + 2\langle a\nabla \log h, \nabla\rangle \quad \text{if } h > 0. \end{aligned}$$

Remark 3.2 In one-dimensional case, denoting by $(a(x), b(x), \textit{and } c(x))$ the coefficients of L, we can represent L as

$$L = \frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}}{\mathrm{d}\hat{\nu}} + c(x),$$

where

$$\mathrm{d}\mu(x) = \frac{\mathrm{e}^{C(x)}}{a(x)}\mathrm{d}x, \quad \mathrm{d}\hat{\nu}(x) = \mathrm{e}^{-C(x)}\mathrm{d}x, \quad C(x) = \int_\theta^x \frac{b}{a}(z)\mathrm{d}z,$$

and θ is a reference point. Then, the (dual) operator $\widetilde{L}$ can be written as

$$\widetilde{L} = \frac{\mathrm{d}}{\mathrm{d}\tilde{\mu}}\frac{\mathrm{d}}{\mathrm{d}\hat{\tilde{\nu}}} = \frac{\mathrm{d}}{\mathrm{d}(h^2\mu)}\frac{\mathrm{d}}{\mathrm{d}\big(h^{-2}\hat{\nu}\big)}.$$

Here are simple examples of L-harmonic functions.

Example 3.3 Let $E = \mathbb{R}$ or $(0, \infty)$.

(1) The function $h(x) = x$ is L-harmonic (a.e.) on E for

$$L = \gamma(x)\big(\partial_{xx}^2 + V(x)\partial_x - V(x)/x\big),$$

where the functions V and γ are arbitrary.

(2) The function $h(x) = x^2$ is L-harmonic (a.e.) on E for

$$L = \gamma(x)\big(x\partial_{xx}^2 + \partial_x - 4/x\big),$$

where the function γ is again arbitrary.

In dimension one, the existence and uniqueness of L-harmonic function, as well as an approximating (constructing) procedure, can be found from [11, Theorems 1.2.1 and 2.2.1]. To see the positivity of h in general dimensions, suppose that L is self-adjoint and $\sup_x c(x) \leqslant 0$. Then, the spectrum of $-L$ should be non-negative. If the principal eigenvalue λ_0 of L (i.e., the minimal eigenvalue of $-L$) is zero, then the L-harmonic function is just a non-trivial eigenfunction corresponding to the eigenvalue $\lambda_0 = 0$ and hence should be non-negative. The function h should be positive inside the domain based on the maximum principal. Next, if $\lambda_0 > 0$, then replacing L by a shift $L + \lambda_0$, its principal eigenvalue becomes zero, we can continue the study as above, and finally shifting back to the original operator.

In higher-dimensional case, the harmonic function may not be unique. We remark that the positive solution of L-harmonic functions for Schrödinger operator $L = \mathrm{dd}z + c(x)$ was examined in [7] in detail, and for elliptic operators in [8] with probabilistic representation.

Example 3.4 ([7, (1.2)]) The L-harmonic function h for $L = \mathrm{dd}z - 1$ can be represented as

$$h(x) = \int_{S^{n-1}} \mathrm{e}^{x\cdot\omega}\mathrm{d}\mu(\omega),$$

where μ is a non-negative measure on the unique sphere S^{n-1}.

The next example is a particular case of Corollary 1.2. Its duality relation was mentioned in [6, §6. Example of O.U.-process and harmonic oscillator], without mentioning the L-harmonic property of h.

Example 3.5 On $\mathbb{R}$, the function $h(x) = \exp[-x^2/2]$ is L-harmonic:

$$L = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1 - x^2\right).$$

Its dual is the O.U.-operator:

$$\widetilde{L} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x}.$$

Furthermore, L has L^2-eigenvalues $\lambda_n = n\ (n \geqslant 0)$ with eigenfunctions

$$g_n(x) = (-1)^n \mathrm{e}^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\mathrm{e}^{-x^2}\big), \quad n \geqslant 0,$$

respectively.

We have just seen an example of the application of known results having $\tilde{c}(x) = 0$ to the one having $c(x) \neq 0$. This indicates a general result as follows.

Theorem 3.6 *Given an elliptic operator*

$$\widetilde{L} = \sum_{i,j} \tilde{a}_{ij}(x)\partial_{ij}^2 + \sum_i \tilde{b}_i(x)\partial_i, \quad \mathscr{D}\big(\tilde{L}\big) \subset L^2\big(\tilde{\mu}\big),$$

for each $h \in \mathscr{C}^2$, $h \neq 0$ a.e., $\widetilde{L}$ is L^2-isospectral to L:

$$L = \sum_{i,j} a_{ij}(x)\partial_{ij}^2 + \sum_i b_i(x)\partial_i + c(x), \quad \mathscr{D}(L) = \big\{f \in \mathscr{E} : f/h \in \mathscr{D}\big(\widetilde{L}\big)\big\},$$

where

$$\begin{aligned} a_{ij}(x) &= \tilde{a}_{ij}(x),\\ b_i(x) &= \tilde{b}_i(x) - \frac{2}{h(x)}\sum_j \tilde{a}_{ij}(x)\partial_j h(x) \ \textit{ on } [h \neq 0],\\ c(x) &= \frac{2}{h(x)^2}\sum_{i,j}\tilde{a}_{ij}(x)\partial_i h(x)\partial_j h(x) - \frac{1}{h(x)}\widetilde{L}h(x) \quad \textit{on } [h \neq 0]. \end{aligned}$$

Briefly,

$$\begin{aligned} L &= \widetilde{L} - \frac{2}{h}\langle \tilde{a}\nabla h, \nabla\rangle + \left[\frac{2}{h^2}\langle \tilde{a}\nabla h, \nabla h\rangle - \frac{1}{h}\widetilde{L}h\right]\\ &= \widetilde{L} - 2\langle \tilde{a}\nabla \log h, \nabla\rangle + \Big\{2\langle \tilde{a}\nabla\log h, \nabla\log h\rangle - h^{-1}\langle \tilde{a}\nabla, \nabla h\rangle\\ &\quad + \langle \tilde{b}, \nabla\log h\rangle\Big\} \quad \textit{if } h > 0. \end{aligned}$$

Proof In parallel to the pure jump case, this is simply a use of the duality $L = H\widetilde{L}H^{-1}$, noting the property that $Lh = 0$ is now automatic since $\widetilde{L}1 = 0$. The remainder of the proof is mainly a careful computation. Actually,

$$\widetilde{L}\left(\frac{f}{h}\right) = \frac{1}{h}\widetilde{L}f + f\widetilde{L}\left(\frac{1}{h}\right) + 2\left\langle \tilde{a}\nabla\left(\frac{1}{h}\right), \nabla f\right\rangle.$$

Hence,

$$h\widetilde{L}\left(\frac{f}{h}\right) = \widetilde{L}f + 2h\left\langle \tilde{a}\nabla\left(\frac{1}{h}\right), \nabla f\right\rangle + fh\widetilde{L}\left(\frac{1}{h}\right).$$

From this, it is ready to write down the coefficients of L. □

Corollary 3.7 *For given $\widetilde{L}$ and $h = \exp\psi$, the dual operator L takes the following form*:

$$L = \widetilde{L} - 2\langle\tilde{a}\nabla\psi, \nabla\rangle + \left\{\langle\tilde{a}\nabla\psi, \nabla\psi\rangle - \widetilde{L}\psi\right\}.$$

We remark that Corollary 3.7 provides us an alternative way to construct the isospectral operator in dimension one. Suppose that we are given an operator

$$\overline{L} = \bar{a}(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \bar{b}(x)\frac{\mathrm{d}}{\mathrm{d}x} + \bar{c}(x).$$

We want to construct $\widetilde{L}$ in terms of the operator L given in Corollary 3.7. First, instead of solving the second-order harmonic equation $\overline{L}h = 0$, we need to solve the first-order Riccati equation for ϕ:

$$\bar{a}\phi' + \bar{a}\phi^2 + \bar{b}\phi + \bar{c} = 0$$

to which there is a standard iterative procedure in ODE. Next, let ψ satisfy $\psi' = \phi$ and define $\tilde{b} = 2\bar{a}\phi + \bar{b}$. Then, we have $L = \overline{L}$. With this $\tilde{b}$ and $\tilde{a} := \bar{a}$, we obtain the operator $\widetilde{L}$ as required.

As an application of the last theorem, one can obtain a lot of examples from [3,4]. We remark that each $\widetilde{L}$ corresponds to a large class of L since h is quite arbitrary.

The natural higher-dimensional extension of Example 3.5 is as follows.

Example 3.8 The dual of $L = \frac{1}{2}\sum_i\left(\partial_{ii}^2 + 1 - x_i^2\right)$ is $\widetilde{L} = \frac{1}{2}\sum_i\left(\partial_{ii}^2 - 2x_i\partial_i\right)$. The function h takes the form $h(x) = \exp[-|x|^2/2]$ rather than $\sum_i \exp\left[-x_i^2/2\right]$. The operator L has eigenvalue n ($n \geqslant 0$) with multiplicity $\#\{(k_1, k_2, \ldots, k_d) : k_1 + k_2 + \cdots + k_d = n\}$, here # means the cardinality of the set following.

Proof For the higher-dimensional O.U.-operator $\widetilde{L}$, we have eigenvalues $\{\sum_{i=1}^d k_i : k_i = 0, 1, \ldots\}$. Corresponding to each $\sum_{i=1}^d k_i$, the eigenfunction is $g(x) := \prod_{i=1}^d g_{k_i}^{(i)}(x_i)$ (where each $g_n^{(i)}$ is the function g_n given in the proof of Corollary 1.2):

$$\widetilde{L}g(x) = -\sum_{i=1}^{d} k_i g_{k_i}^{(i)}(x_i) \prod_{j \neq i} g_{k_j}^{(j)}(x_j) = -\left(\sum_{i=1}^{d} k_i\right) g(x).$$

Therefore, $\widetilde{L}$ has eigenvalue n $(n \geqslant 0)$ with multiplicity $\#\{(k_1, k_2, \ldots, k_d) : k_1 + k_2 + \cdots + k_d = n\}$. From here, it is easy to write down the eigenvalues of L and their corresponding eigenfunctions. □

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