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REGULAR ARTICLE

# An Approach to Enactivist Perspective on Learning: Mathematics-Grounding Activities

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**Abstract** Based on an enactivist perspective on learning mathematics, we articulate three key processes of designing mathematics-grounding activities (MGAs) where students' mathematical thinking can be motivated and shaped with the interactions between their enactments and the evolving tasks in the activities. Then, evaluation criteria and design steps will be derived in terms of the key processes. The key processes of designing MGAs, the criteria for evaluating quality MGAs and the design steps also emerged from the reciprocal relationships between theories and practices in the context of the Just Do Math (JDM) program. The processes and steps of designing MGAs suggested in this article can benefit researchers and educators to develop original activities for advancing the learning of mathematics in line with the enactivist perspective. Additionally, the key processes can be further referred to for explanations of how metaphorical grounds of mathematics can emerge under systemic interactions between learners, tasks and social contexts, and how learners' motivation is integrated into the evolving tasks. Criteria could be applied for not only evaluating the potential of MGAs but also for identifying the weaknesses needed to be modified.

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## Introduction

Based on an enactivist perspective on learning, we articulate three key processes of designing mathematics-grounding activities (MGAs) where students' mathematical thinking can be motivated and shaped with the interactions between their enactments and the evolving tasks in the activities. Then, evaluation criteria and design steps will be derived in terms of the key processes. The key processes of designing MGAs, the criteria for evaluating quality MGAs and design steps also emerged from the reciprocal relationships between theories and practices in the context of the Just Do Math (JDM) program.

We firstly introduced the background of JDM and its core component of MGAs. Next, an enactivist perspective on learning mathematics was presented to provide a base for reflecting on the design of MGAs. Then, three key processes of designing MGAs were proposed and followed by the evaluation criteria for quality MGAs and the design steps, which were provided for educators and teachers to select, adapt or design quality MGAs to motivate and shape students' mathematical thinking and conceptions. The innovation and contribution of our approach are concluded by comparing with current approaches to learning mathematics.

## Background of JDM and MGAs

### Goals and Products of JDM

Students' negative affective performance in mathematics learning is always a global problem (Mullis et. al., 2012; OECD, 2014). Taiwan's Ministry of Education launched the JDM program in collaboration with Shi-Da Institute for Mathematics Education in 2014 to resolve this problem (Lin & Chang, 2019). The program aimed at engaging students in mathematics cognitively and affectively as well as enhancing mathematics teachers' professional development. The scope of the scaling up has encompassed more than 23,000 primary and lower secondary school mathematics teachers and more than 180,000 students (Wang, et al., 2021). Until mid-2018, there have been 175 MGAs designed and evaluated to accept. The mathematical content of these MGAs is connected to numbers, algebra, measurement, geometry and probability (Lin & Chang, 2019).

To effectively engage students in mathematics cognitively and affectively, MGAs are the core of the Just Do Math program. MGAs were designed under careful consideration of mathematics content, mediated tools, and student learning to help students establish the fundamental prerequisite ideas before learning a mathematics topic in regular classes. The theories referred to the design of MGAs included to provide students opportunities for meaningful learning (Ausubel, 1961), for schema construction (Skemp, 1989), for experiencing different abstract levels of concepts with enactive, iconic and symbolic representations (Bruner, 1964), and for game-based learning adapted from Dienes (1973) in the mode stage of the JDM program (Lin & Chang, 2019). However, these theories were reactively related to the elaboration of MGAs rather than proactively referred to facilitating and improving the design of MGAs.

### An Exemplary MGA

Before reflecting on the design of MGAs, a classical activity, Skemp's (1989) rectangular numbers game, is introduced and elaborated because it is acknowledged as the first MGA in the JDM program (Lin & Chang, 2019). The primary rule of the game is that one player gives the other a number of counters, and the other tries to make a rectangle using all of the counters. If the number of counters can or cannot be used to make a rectangle, the rectangular and non-rectangular shapes may represent two different categories of shapes in the beginning of the activity. Figure 1 shows an example of a rectangular shape. Players as learners can meaningfully act on the counters

**Fig. 1** A rectangular shape

with the goal of making as many rectangles as possible. When taking turns, the goal is to give a number of counters which cannot be used to make a rectangular shape. As players further classify the different shapes, the different categories of shapes could further be transformed as different categories of numbers. It is noted that players' active classification may not be completely correspondent with prime and composite numbers. For instance, students may find that no matter how many counters they have, they can make them into a line.

In order to win, the player has to make more rectangles than the other player. The task is evolved as players regulate their strategies by observing what happened in the game and transforming their strategies into an applicable conception for further enaction. In addition to considering the cognitive function of the rectangular numbers game, the motivational function of games can make players continuously and actively engage in the tasks. The classification of various lines and rectangles are assumed to be brought forth in players' mathematical world, which is actually determined by each player, according to the enactivist perspective (Simmt & Kieren, 2015). Both cognitive and motivational functions in Skemp's rectangular numbers game are regarded as a favourable context for students' learning of mathematics.

We acknowledge Skemp's rectangular numbers game as a quality MGA for the effective learning of mathematics in line with the enactivist perspective on learning, which will be described later, because this game provides the opportunity for the players to bring forth an intended mathematical world of the mathematical process (effective classification of different shapes) and mathematical object (categories of numbers) underlying the task (rectangular numbers game) through embodied thinking with the counters. Hence, we would base on an enactivist perspective on learning mathematics to reflect on MGAs and then propose three key processes underlying the design of MGAs.

## An Enactivist Perspective on Learning Mathematics

Enactivism defines cognition as “the enactment of a world and a mind on the basis of a history of a variety of actions that a being in a world performs” (Maturana & Varela, 1992, p. 9). This definition indicates that cognition is not the individual’s mind itself, but rather the individual’s action of bringing forth a world of significance where they interact with others in the environment (Maturana & Varela, 1992). The interactions that bring forth a world of significance have been recognized as one pivotal move in mathematics education research (Simmt & Kieren, 2015). Learning mathematics is not predetermined by either the individual or others, but rather by acting and interacting with each other such that the co-evolution (i.e. the structural coupling, Maturana & Varela, 1992) of the individual and others allows new actions to bring forth mathematical knowledge and practices. The mathematical knowledge and practices are occasioned by the interactions between the individual and others but determined by the individual’s structure.

Mathematics education researchers have paid attention to elaborating the implications of the enactivist perspective for pedagogical practice (e.g. Hutto et al., 2015; Kieren et al., 1995), and taken enactivism as research methodology to interpret the interactions between the individual and others, which bring forth a world of mathematical significance and generate inter-objectivity (Simmt & Kieren, 2015). These studies indicate that it is paramount to investigate how students or teachers bring forth a mathematical world that is not fixed and predetermined but rather is shaped and reshaped. Nonetheless, most studies focus on descriptions and explanations of student learning or teacher learning to teach. We would like to move one step forward to anticipate student learning through task design and evaluation.

An inevitable problem about task design is “the persistence of a gap between teacher intention and student activity” (Coles & Brown, 2016, p. 149). Coles and Brown, based on the enactivist perspective, have provided two mechanisms, including the making of distinctions and the having of an explicit image of mathematical thinking, to eliminate the gap by guiding teacher planning, teacher actions and student activity in the classroom. The making of distinctions through the use of similarity and difference can occasion transforming actions of teachers and students. For example, when students solve $1/4 \times 36 = 9$ by transforming it as $4 \times 9$, it is necessary to refer to their internal structure, an image of mathematical thinking (e.g. the unit portion of the unknown whole is the part), relevant to this task. Then, the task may be co-evolved as the whole is $4 \times 1$ if the part (i.e. 9) is changed to 1, or the whole is $2 \times 9$ if one unit portion (i.e. 1/4) is changed to 1/2. Such co-evolution implies that students may be aware of the common structure underlying the original and evolved tasks. Such awareness may occasion the individual’s image of the relationship between one unit portion, the whole and the part to construct the solution of $4 \times 9$. The occasioned transforming actions (the shift in attention, Watson & Mason, 2007) reciprocally extend the individual’s image of mathematical thinking to other tasks, such as $3/4 \times 12 = 9$.

After analyzing the complex process of learning algebra in the classroom, Coles and Brown (2016) developed the following principles of task design: “(1) considering at least two contrasting examples (where possible, images) and collecting responses; (2) asking students to comment on what is the same or different about contrasting examples and/or to pose questions; (3) introducing language and notation arising from student distinctions; (4) starting with a closed activity (which may involve teaching a new skill); (5) having a challenge prepared in case no questions are forthcoming; (6) opportunities for the teacher to teach further new skills and for students to practise skills in different contexts; (7) opportunities for students to spot patterns, make conjectures and work on proving them” (p. 157). Principle (1), (2) and (3) aim at closing the gap between teacher intention and student activity and thus relate more to the mathematical world, brought forth by the interaction between students and tasks through their making of distinctions. Principle (4), (5), (6) and (7) are specific to teaching in the particular way and thus relate more to the task itself and teacher instruction as otherness to make an intended mathematical world co-evolved more likely. However, these principles were developed mainly based on teaching with contrasting examples. It is worth proposing alternative approaches to the design of learning activities based on the enactivist perspective.

Tasks with certain characteristics will have the influence on learners’ enactments which simultaneously change the tasks (Lozano, 2017), so that tasks and learners are thus always co-evolving to bring forth a mathematical world. To allow students more likely to effectively bring forth an intended mathematical world with tasks co-evolved (Varela et al., 1991), how can we design tasks which can motivate learners’ active engagement and facilitate their effective, but not necessarily correct, enactments of mathematical thinking through interacting with other learners as well as the co-evolving tasks? To answer the questions, we reflect on MGAs with the three focuses: the essence of an intended mathematics, an approach to it’s palpable enactment, and motivation inter-related with it. The focuses are different from the art-of-the-state of research on the reciprocal interactions between tasks, students’ learning and classroom teaching. Herein, the factors considered to design tasks include motivational and cognitive processes, and processes of designing quality MGAs are proposed in terms of the above three focuses to facilitate the original design of tasks based on an enactivist perspective on learning mathematics.

## Key Processes of Designing MGAs: Metaphorizing, Scaffolding, Gamifying

From the enactivist perspective, mathematical processes and objects are enacted, not predetermined and brought forth from learners' embodied actions and interactions with the evolving tasks as well as others (Varela, 1999). Nonetheless, designers have to analyse what and how the intended mathematical process and objects could be enacted through the interactions when designing the tasks. Thus, we posed three questions with respect to the above three focuses for reflection on the design of MGAs. That is, (1) What could the essence of an intended mathematics be for enactment? (2) How can an intended mathematical world be enacted and brought forth? and (3) How can learners be motivated to enjoy the enactment of an intended mathematical world? Three key processes of designing quality MGAs, including metaphorizing, scaffolding and gamifying, are, respectively, derived from responses to the three questions with reflections on the design of MGAs.

### Metaphorizing

Without loss of nature, mathematics is about essences which make mathematics abstract and are characterized as embodied metaphors and semiotic objectification (Lakoff & Nunez, 2013; von Glasersfeld, 1990; Radford, 2000). In a mathematical world brought forth through the interactions between the individual and others (tasks, peers or teachers), mathematical abstraction can be counted as the evolving enactments of cross-domain mappings between a frame of a source domain (individuals' world) and a frame of a target domain (others' world) by carrying the embodied thinking and metaphorical inferences (an interaction world), which are composed of a *metaphorical ground* for metaphorizing (Lakoff, 2009). Like in Radford's study (2000), students developed metaphorical figures as linguistic devices for objectifying the 12th figure of the pattern in a quiz and as substance for a symbolic formula of a general term. Thus, mathematical abstraction can be counted as a social activity where a mathematical world is brought forth through the learners' embodied thinking and metaphorical inferences from manipulating or imagining the material or digital tools.

To take Skemp's (1989) rectangular numbers game as an example, when players classify the different shapes (mathematical processes) and then map the shapes (source domain) into numbers (target domain), the source domain, target domain as well as the potential interactive process serve as the metaphorical ground of prime and composite numbers (an intended mathematical object), drawn in Fig. 2. The potential interactive process includes players' enactments of making a line or rectangle, transforming shapes into numbers and classifying numbers which can be metaphorically represented as geometric shapes and embodied in the action and re-action of making a rectangle. In order to make more rectangles than the other player, one would regulate their strategy with embodied thinking and develop an applicable conception with metaphorical inference (i.e. conceptual metaphor) in the evolving activity. That is, the rectangular numbers game provides players the opportunity to metaphorically represent numbers as geometric shapes and embody numbers as the enactments of making a rectangle, and thus builds a metaphorical ground for bring forth a conceptual metaphor of prime and composite numbers, i.e. an intended mathematical world.

A conceptual metaphor has been recognized as "the cognitive mechanism by which the abstract is comprehended in terms of the concrete" (Núñez, 2000, p. 6). For instance, numbers are points on a line, or an equation is a balance (Lakoff & Nunez, 2013). Nonetheless, we would like to create more metaphorical grounds to occasion conceptual metaphors of the target domain through learners' embodied experiences linked with active and reflective thinking. The source domain is realized as a task context with consideration of the subject's historical contexts in social contexts. In a task context, a *source domain* is embodied as regulative actions where substantial ideas in a *target domain* are co-evolved through carrying out the embodied thinking and metaphorical inferences by leaners and their interactions in social context. MGAs can be accounted as providing the opportunities to develop conceptual metaphors of the target domains involving intended mathematical worlds emerging with learners' embodied thinking and metaphorical inferences from their changing enactments in source domains.

### Scaffolding

When reflecting how MGAs could more likely bring forth conceptual metaphors of an intended mathematical world through learners' enactments, the need for scaffolding arises. In view of the fact that a metaphorical ground could be embedded in MGAs for shaping learners' conceptual metaphors and *making sense of* an intended mathematical concept, the scaffolding aims at supporting the learners' cognitive structuring and metacognitive reflection, which are required to make experiences, actions and thoughts

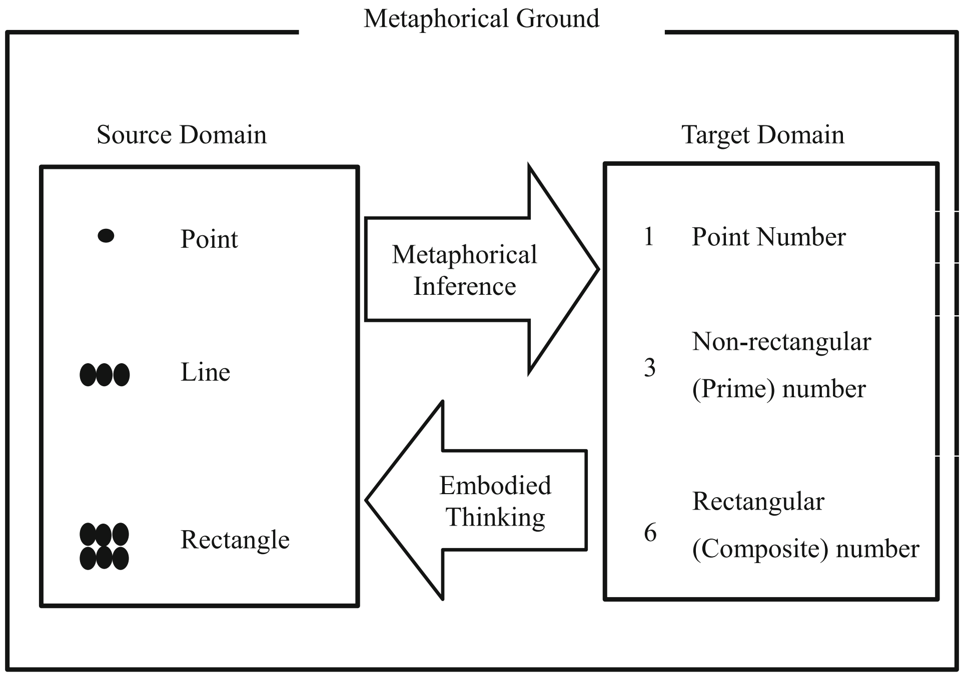


**Fig. 2** Metaphorical ground of prime and composite numbers

coherent (Menary, 2008). Cognitive structuring on embodied perceptions is not only facilitated by the source domain, but could be also scaffolded by physical and mental manipulation (e.g. making a rectangle) which can benefit to the co-evolution of learners' conceptions and the tasks.

There are studies suggesting that physical models are not always necessarily effective because students may have difficulties perceiving physical models as representations of mathematical concepts and translating physical models into mathematical symbols for further problem solving (Uttal et al., 1997). From our analysis of Skemp's rectangular numbers game based on the enactivist perspective, learners confront alternative difficulties associated with metaphorizing, such as whether a hollow square belongs to the rectangular numbers or whether a bigger number is a rectangular number with $9 \times 9$ multiplication facts actively applied by leaners, because physical models are enacted by and evolved with learners to bring forth conceptual metaphors of the intended mathematical concepts, rather than needing to be represented as and translated into mathematical concepts. Representations, such as zero-point rectangular numbers or point-line numbers or square-rectangular numbers (i.e. square numbers), are co-evolved with the learners' cognitive structuring and metacognitive reflection on their interactions with others and tasks.

The scaffolding is gradually faded out as physical manipulation is evolved as mental manipulation (e.g. imagining a number of counters which can only make a line) and then reflectively abstracted as a conceptual metaphor (e.g. a number which cannot be divided by 'other' numbers) of the intended mathematical concept (e.g. a prime number). That is, the scaffolding of physical manipulation aims at facilitating metaphorical inferences, and assist visual as well as verbal representations of conceptual metaphors when it is gradually faded out. Because the representations are co-evolved with and thus made sense of by learners, the interconnections between physical models and the co-evolved representations may support further understanding and application of the formal representations of the mathematical concepts.

The metacognitive scaffolding is assumed to appear when the tasks can trigger the learners' awareness of the *insufficiency* of the emergent conceptions (different features of numbers) and the need to re-structure them (e.g. classification of different features of numbers), which may shift learners' attention and then advance the structures of attention from gazing, making distinctions, recognizing relationships, perceiving properties to reasoning on the basis of identified properties (Mason & Johnston-Wilder, 2004).

Accordingly, we assume that the attention to self- and others' enactments and *shifts* in structures of attention can become a metacognitive scaffolding for *self-reflection*. That is, through the interactions within and between the leaners and the tasks in social the context, an evolving system is provided for learners to create and maintain their own domain of meaningfulness under the scaffolding of physical as well as mental manipulation and the opportunity to be aware of insufficiency as well as to self-reflect on structures of attention which could constitute their vital significance and brings forth a mathematical world, i.e. a process of sense-making (Thompson, 2007). The complex process of learning is occasioned when they engage the activities "as reflections of a structure without losing sight of the directness of our own experience" (Varela et al., 1991, p.12).

### Gamifying

In addition to the essence of an intended mathematics and scaffolding for bringing forth it, the motivation power of games is considered to uplift learners' emotional engagement and enjoyment in a co-evolving activity. Thus, gamifying task contexts is applied for sustaining learners' motivation to resolve the insufficiency of the emergent conceptions. Particularly, an enactivist perspective on motivation emphasises the *intrinsic connections* between organism and environment through the organism's enaction (Shargel & Prinz, 2018). Such as in Skemp's rectangular numbers game, the atmosphere of competition and challenge in generating critical numbers produce a sense of emotion inter-related with the grasp of a non-rectangular number (i.e. prime numbers) named as "blow-up" numbers by learners.

The gamified task contexts are also employed to optimize *systemic interactions*. According to flow theory, 'educational games should stretch a player's mind to its limits in his effort to overcome worthwhile challenges' (Kiili et al., 2012, p. 89). Flow theory supposes that to accomplish something difficult and worthwhile may not be necessarily pleasant but can still be enjoyable (Csikszentmihalyi, 1991) because the experiential engagement with gamified tasks can trigger selecting, processing, integrating and discussing with each other, and then is intrinsically interesting and fun (Kiili et al., 2014). Such process benefits to optimize systemic interactions between internal and external ecology. For instance, Skemp's rectangular numbers game has been modified to develop the concept of factors, and such modification brought forth systemic progression of tasks, learners and all other participants in the JDM program.

In gamified task contexts, the structural coupling is more likely regulated by reflective and collective abstraction of the intra- and inter-learners accompanying the systemic interactions between learners, tasks and social contexts because the regulation of the structural coupling relies on learners' historical contexts under task (intra-learners) and social (inter-learners) contexts in a quality MGA.

## Criteria for Evaluating MGAs

In the JDM program, MGAs were reflected to design for creating an effective environment including learners, tasks and social contexts where learners' needs for and attention to structures co-evolving with tasks can be sustained. An empirical study has shown that the MGAs can enhance students' interest, confidence and engagement in bringing forth fundamental mathematical conceptions in gamified activities before learning mathematics in classes (Lin et al., 2018). Based on the experience of the JDM program, three criteria for evaluating the quality of these MGAs were identified in terms of the above three processes: metaphorizing, scaffolding, and gamifying.

When we are concerned about what constitutes a good metaphorical ground for an intended mathematical world, two conditions: *sufficient* and *necessary* are considered. To elaborate our sufficient and necessary conditions, we refer to the first two features of good generic examples—representative and developmental (Mason & Pimm, 1984)—being easy to learn and enjoyable to learn (Cheng, 2000). Representative, considered as a sufficient condition of the source domain, means that the source domain has to contain the essential substance and a potential form of a target domain. Developmental, considered as a necessary condition of the target domain, means that the metaphorical ground for the target mathematical domain, such as rectangular numbers, should be beneficial to developing a new concept, such as prime numbers. The other two features, easy and enjoyable, are, respectively, correspondent with the other two processes of designing MGAs: scaffolding and gamifying.

As for the scaffolding criterion, we also consider good generic examples, which can substantiate students' mathematics sense-making (Mason & Pimm, 1984). Similarly, MGAs should provide learners with the opportunity to generate examples under the scaffolding of physical manipulation and of self-awareness of the insufficiency of emergent conceptions. Particularly, learners can make structural coupling by generating, comparing as well as testing different examples, and become aware of the need to restructure their insufficient conceptions through the shift in structures of attention.

In addition to the generation of generic examples, the second criterion of scaffolding is drawn on optimizing the third feature of good generic examples: being easy to learn. It is suggested that the underlying structures (the source and target domains) of task contexts be connected with learners' historical contexts, such as their pre-knowledge and experience of rectangles under social contexts, because the intrinsic connections make it easier for learners to make sense of conceptual metaphors, emerging from interactions of complex systems (Fenwick, 2000). This process of embodying an abstract concept through MGAs could be counted as a form of generic abstraction where conceptual metaphors, instead of formal concepts, are gradually made sense of and evolved (cf. Harell & Tall, 1991).

The fourth feature of good generic examples, being enjoyable to learn, is considered with relation to the third process of gamifying. When evaluating the quality of gamifying, we focus on the two criteria for keeping players in the flow state: *challenge* and *playability*. The appropriate level of challenge requires a balance between the learners' confidence and the task difficulty (internal ecology) for developing conceptual metaphors of an intended mathematical world in social contexts (external ecology). Playability resides not only in gamified task contexts but also in learners' historical and social contexts where the regulative reflection on the insufficiency of the emergent conceptions can be kept motivated.

In sum, being *representative* and *developmental* are the criteria for evaluating the quality of metaphorizing. For evaluating the quality of scaffolding, we are concerned with the opportunities to *generate generic examples* as well as the *intrinsic connections* of learners with tasks and social contexts, which make the generation of generic examples easier for students. The evaluation criteria for the quality of gamifying are *challenge* and *playability* under the consideration of the balance within and between internal and external ecology.

## Designing Steps and One Exemplary Application

From the three processes of designing MGAs: metaphorizing, scaffolding and gamifying, three missions of designing MGAs emerged: (1) to transform a mathematical idea into *a metaphorical ground*, (2) to materialize the metaphorical ground as *the scaffolding* of physical manipulation as well as self-awareness of the insufficiency of the emergent conceptions, and (3) to *gamify a task context* for sustaining learners' interest and perseverance in bringing forth a conceptual metaphor to succeed in the game.

To accomplish the first mission, two steps are suggested: (1) recognizing what the target domain of an intended mathematical concept is, and (2) recognizing a source domain to embody the target domain which could be inferred as conceptual metaphors of the source domain. To accomplish the second mission, two steps are suggested: (3) providing tools which can be manipulated to generate examples, and (4) providing goals which can trigger learners' awareness of the insufficiency of their emergent conceptions and enact the development of conceptual metaphors. To accomplish the third mission, two steps are suggested: (5) generating multiple task processes and outcomes, and (6) generating relevant feedback as tasks evolve.

Taking factorization of quadratic expressions in one variable as the intended mathematical concept, the first step is to recognize the target domain of this concept including quadratic, linear and constant terms (e.g. $x^2$, 5x, 6), quadratic expressions (the addition or subtraction of these terms), the multiplication of linear expressions [e.g. $(x + 2)(x + 3)$] and their structural relationship. The second step is to recognize the source domain including the sides and area of one rectangle, the sides and area of one rectangle composed of several rectangles, as well as the relationship of the sides and area of the composed rectangle and the other rectangles. The source domain directs us to use algebra tiles (Leitze & Kitt, 2000) as physical models which can embody metaphorical concepts of the target domain, such as different representations of the area of one rectangle (visual, geometric as well as symbolic) and the transformation between geometric and symbolic representations from which structural relationships are co-evolved instead of mapped in the MGA, which can be viewed as conceptual understanding (Caglayan, 2013). This direction has followed the third and fourth steps to provide manipulable tools and the goals to trigger the insufficiency of free arrangements of the tiles.

Differing from previous studies of using algebra tiles for understanding factorization of quadratic polynomials, algebra tiles are used to make sense of quadratic expressions through manipulating geometric representations and attending to the underlying structural relationships where learners are asked to generate and compare various examples of rectangles and express the structural relationships. Such process is assumed to bring forth metaphorical concepts of the target domain when shifting the attention to the structural relationships between geometric and symbolic representations of the sides and area of rectangles to the structural relationships within the coefficients of quadratic polynomials and factored terms.

Accordingly, tasks are designed to construct one rectangle using up a set of rectangles and to record the area of the set of rectangles as well as the two adjacent sides of the composed rectangle. In order to trigger learners' awareness of insufficiency of emergent conceptions (e.g. the composed rectangle and the set of rectangles used up have the same area), two contexts, with and without using physical models, to accomplish the tasks are provided. In the context without using physical models, learners can encounter the need to grasp an effective and efficient approach to compose a set of rectangles because trial-and-error processes are unfavourable without using physical models, and then reflect on the process of using physical models to regulate the attention.

Several sets of rectangles are written on blank cards, for example, one rectangle with x length and x width, 4 rectangles with x length and 1 width, and 3 rectangles with 1 length and 1 width (unit squares). In order to generate multiple task processes and outcomes, which is the fifth step, we can vary the sets of rectangles by varying the length and width parameters, such as one rectangle with 2x length and x width, and the number of rectangles with x length and 1 width as well as unit squares. It can be assumed that one would get a higher score when performing a task without using physical models. The direct feedback is the shape of a rectangle. Indirect feedback is generated through observing and reflecting on others' strategies to regulate one's own. The sixth step is accomplished by both feedback types provide information relevant to the goals. The board game could end when all cards are used up or when one player scores a certain number of points (Deng, 2014).

Applying the criteria for evaluating the above MGA of rectangular tiles, algebra tiles are *representative* of algebraic terms and polynomials. Various numbers of rectangles of different categories make emergent conceptions *developmental* and the generation of various examples possible. *Generic examples* reside in generic quadratic polynomials, such as $x^2 + 4x + 3$ or $x^2 + 6x + 5$, which is both not too simple (e.g. $x^2 + 2x + 1$) and yet simple enough to evolve emergent conceptions. The physical models make the tasks easier for learners to accomplish because of being manipulative and transformative, resulted from the *intrinsic connections* and self-regulation. Uncertainty about task processes brings *challenge* and curiosity. In addition to a *playable* activity, the relevant feedback sustains learners' interest and perseverance.

Thus, the MGA of rectangular tiles should be qualified to provide an opportunity to make sense of and develop metaphorical concepts of factorizing quadratic expressions in line with the enactivist perspective on learning. Learning through engaging in the MGA of rectangular tiles enables learners to experience algebra as the expression of relationships and to overcome the difficulties in manipulations and applications of algebraic symbols because the structural relationships between physical models and algebraic symbols are co-evolved rather than residing in interpreting abstract algebraic symbols with concrete physical models.

## Conclusion

Based on an enactivist perspective on learning mathematics, MGAs are reflected to develop a systemic ecology where mathematical practices are enacted, rather than relying on either learners or tasks, based on a metaphorical ground analyzed by designers (metaphorizing), provided with the scaffolding of physical manipulation and self-awareness of insufficient conceptions (scaffolding) and through motivating learners to interact and interconnect with evolving gamified tasks (gamifying). This means that learners, tasks and others in MGAs are three interactive and interconnected systems, and are structurally coupled to develop metaphorical concepts. Tasks act as triggers for learners to change and transform as much as learners act as triggers for tasks to change and transform. Our approach to structural coupling is to make it not only occasioned and co-emergent at the moment of systemic interactions between learners, tasks and social contexts, but also scaffolded by manipulating physical models to generate examples and to be aware of the insufficiency of their emergent conceptions which can facilitate the need for abstraction and make metaphorical inference in MGAs.

Enactivism does not account too much for how the structured coupling of learners and tasks can be encouraged. The three processes, six criteria and six steps considered for designing and evaluating MGAs pave the way to putting the enactivist theory into practice. Compared to Skemp's (1989) notion of knowledge construction, a combination of building and testing, we further pay attention to how building and testing can occur more likely based on the enactivist perspective on learning. From Freudenthal's (1991) view, reality is based on common sense experience, and the boundary between horizontal and vertical mathematization relies on the learners' background. Metaphorical grounds of MGAs extend the meaning of reality to a source domain in contexts, and then provide opportunities for learners to horizontally and vertically mathematize with the scaffolding of physical manipulation, self-awareness of the insufficiency of emergent conceptions and shifts in structures of attention for self-reflection.

Proulx (2013) has applied the enactivist theory for analysing students' emergent strategies when solving mental mathematics problems, following Threlfall's (2002) ideas about strategy development in mental calculations. Proulx's illustrations shifted the focus of investigation of problem solving from the nature of strategies used by students to the nature of mathematical activities involving students and tasks (Davis, 1995). Our application of the enactivist perspective moves one step forward to originally design mathematical activities (e.g. MGAs) which can generate the learning opportunities for learners to bring forth intended mathematical concepts through systemic interactions between learners, tasks and social contexts.

Our key processes concur with the embodied design by considering two epistemic modes of the immediate "doing" and mediated "thinking" to emphasize physical experience and guided signification (i.e. the conveying of meaning) where learners are engaged in immersive action and then structured reflection (Abrahamson & Lindgren, 2014). In MGAs, such reflection is facilitated by self-awareness of the insufficiency of emergent conceptions and sustained through the interactions within and between the leaners and the tasks. Thus, MGAs resemble embodied design, with the aim of shaping and reshaping learners' physical action schemes with the evolving tasks, and shifting their structures of attention to bring forth metaphorical concepts of the target domain.

The processes and steps of designing MGAs suggested in this article can benefit researchers and educators to develop original activities for advancing the learning of mathematics in line with the enactivist perspective. Additionally, the key processes can be further referred to for explanations of how metaphorical grounds of mathematics can emerge under systemic interactions between learners, tasks and social contexts, and how learners' motivation is integrated into the evolving tasks. Criteria could be applied for not only evaluating the potential of MGAs but also for identifying the weaknesses needed to be modified. To sum up, our approach contributes to the application of the enactivist perspective to the articulation of the key processes of metaphorizing, scaffolding and gamifying for the design and evaluation of MGAs, which make task design more accessible and students' learning predictable. Further investigation is to develop effective instructional principles from analyzing MGAs from teachers' perspective in practical contexts.

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