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REGULAR ARTICLE

Analyzing Changes in Four Teachers' Knowledge and Practice of Inquiry-Based Mathematics Teaching

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Abstract Teachers' inquiry-based mathematics teaching (IBMT) competence is often acquired and developed through participation in professional development programs (PDPs), but researchers have not established an effective method of assessing teachers' performance in this area. Moreover, the definition of IBMT, and the relationship between teachers' mathematics inquiry knowledge and practice, both remain unclear. Therefore, this study first proposes a sophisticated definition of mathematics inquiry-based teaching, and goes on to use this definition as the basis of a systematic, integrated method for assessing teachers' knowledge and practice of such teaching. Changes in four case teachers' knowledge and practice of IBMT during their participation in a 1-year PDP in Taiwan are reported and analyzed. The data sources included classroom observations as well as interviews and teachers' concept maps. Data were analyzed using an analytic framework of concept-map structure; essential constructs of knowledge regarding mathematics inquiry; and an analytic framework of inquiry-based teaching practice. The research revealed (1) that all four case teachers made positive progress in both their knowledge of mathematics inquiry and their related teaching practices; (2) that the two sets of gains described in point (1) above did not appear to have any causal relationship; and (3) that although all four teachers' knowledge and practices were found to have changed, radical changes in their teaching practices were still rare in the short term.

Keywords Inquiry-based mathematics teaching · Knowledge about mathematics inquiry · Teaching practices

Introduction

Mathematics education reformers have strongly emphasized the importance of student-centered approaches to teaching and learning, in which students are encouraged to actively construct their own knowledge via inquiry processes such as problem-solving, reasoning and proof, communication, representation, and connection (e.g., National Council of Teachers of Mathematics [NCTM] 1989, 2000). The ultimate goal of this reform movement is to develop students' mathematical proficiency, including not only conceptual understanding but also positive attitudes and beliefs about learning mathematics (Kilpatrick et al. 2001). Inquiry-based mathematics teaching (IBMT) can be considered an essential tool for achieving these goals (Artigue and Blomhøj 2013; Chapman 2011; Jaworski 1994; Siegel et al. 1998).

IBMT is generally thought of as an effective means of facilitating students' development of mathematics knowledge and mathematical thinking (e.g., Chapman and Heater 2010). It is characterized by the teacher's creation of an environment in which students extend their mathematics proficiency through actively conjecturing about, justifying and reconciling mathematical ideas (e.g., Artigue and Blomhøj 2013), collecting and analyzing data, reasoning and making conclusions, and communicating the results to their peers and the teacher (Jaworski 1994; NCTM, 1991; Siegel et al. 1998).

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Despite long-standing theoretical support (e.g., Dewey 1938), however, school teachers tend to refrain from using the inquiry approach (Saad and BouJaoude 2012). Though the reasons behind this lack of uptake may be various and complicated, teachers' lack of knowledge of mathematics inquiry and how to apply it in the classroom seem to be two of the most crucial issues (Wee et al. 2007). As such, it has been widely argued that professional development programs (PDPs) should include components that help teachers increase their understanding of mathematics inquiry (Chapman and Heater 2010; Stonewater 2005; Wilkins 2008). The issue of how to assess teachers' mathematics inquiry knowledge and practices therefore emerges as a critical one: not least so that feedback can be provided to the designers of PDPs.

Although research on science education has yielded several potential methods for assessing teachers' mathematics inquiry knowledge and practices, few studies of mathematics education have utilized them, apparently due to a lack of consensus regarding whether an assessment method derived from science education would be transferable. In other words, the question of how best to assess teachers' mathematical inquiry knowledge and practices has yet to be answered.

In addition, although teachers' mathematics inquiry knowledge is crucial to the implementation of inquiry-based teaching, researchers do not yet have a sufficient understanding of how teachers develop their inquiry knowledge in PDPs, or how such knowledge may influence their implementation of IBMT. Moreover, researchers have not arrived at any consensus about whether different academic backgrounds or personal experiences may influence teachers' development of inquiry knowledge. Investigation of the above questions will considerably broaden our understanding of the nature of teachers' development of inquiry knowledge and IBMT, and thus benefit students' learning.

Purpose of this Study

Accordingly, the purpose of this study was to analyze changes in teachers' mathematics inquiry knowledge and IBMT performance at the beginning and the end of their participation in a PDP lasting 1 year.

Literature Review

The Essential Constructs of Knowledge about Mathematics Inquiry

The precise nature of teachers' knowledge about mathematics inquiry (i.e., mathematics inquiry knowledge) remains ambiguous to mathematics educators (Artigue and Blomhøj 2013; Chapman 2011; Chapman and Heater 2010). However, it may be possible to clarify it via an examination of meaningful parallels from science education. In the latter field, researchers have used various terms to define knowledge about inquiry, including *understanding*, *conception*, *beliefs*, and *practical knowledge* (Anderson 2002; Asay and Orgill 2010). Each of these terms reflects a different research focus: with *conception* tending to imply an emphasis on teachers' knowledge of the nature of inquiry and how to implement it in the classroom (Morrison 2012), whereas the discourse of *practical knowledge* implies that knowledge, beliefs, values, and practical contexts cannot be separated from each other (Anderson 2002; Chapman 2011). In this study, we use the term *knowledge* to emphasize *knowledge structure*, which allows us to divide teachers' knowledge about inquiry into several distinct components.

Generally, knowledge about inquiry can be seen as synonymous or coterminous with understanding of it, and includes two subcomponents: (1) understanding of what inquiry is, and (2) knowing how to teach via inquiry (Morrison 2012; Wee et al. 2007). Research on science education usually cites the standards set by the American National Research Council ([NRC] 2000) as the framework for teacher knowledge of scientific inquiry. These standards recommend that students: (1) "create their own scientifically oriented questions"; (2) "give priority to evidence in responding to questions"; (3) "formulate explanations from evidence"; (4) "connect explanations to scientific knowledge"; and (5) "communicate and justify explanations" (p. 27). However, it should be noted that these five essentials focus exclusively on students' *inquiry process*, and that other dimensions—notably, the role of the teacher—are absent.

Extending the NRC's model, the PRIMAS project (http://www.primas-project.eu) proposed essential integrates of teacher knowledge in inquiry-based teaching (Fig. 1). As such, the PRIMAS model contained more dimensions than the NRC's, including "type of questions," "teacher guidance," "classroom culture," and "valued outcomes" (in addition to the student-inquiry process). However, while the PRIMAS model's developers asserted that it could be applied in mathematics as well as in science, it more clearly emphasized scientific inquiry, including the five-E cycle, which is generally not used in mathematics.

Difference Between Mathematics Inquiry and Scientific Inquiry

Inquiry in mathematics education has been conceived of as broader than inquiry in science education (Artigue and

Blomhøj 2013). There are at least two major differences between these two types of inquiry: the first relating to how questions arise, and the second to the inquiry process per se. With regard to the first, science education relies heavily on daily-life experience or natural phenomena, whereas mathematics education may rely on mathematics itself. For example, the statement "The sum of two odd numbers is always even" could lead to the questions, "Will this still be true for the sum of three odd numbers?" "How about the product of two odd numbers?" and so forth. Second, the inquiry process in science education can be broken down as *predicting*, *designing experiments*, *collecting data*, *interpreting data,* and *drawing conclusions* (Anderson 2002; Artigue and Blomhøj 2013). This contrasts fairly sharply with the inquiry process in mathematics education, which emphasizes *problem-solving*, *metacognition*, *modeling and mathematizing*, *reasoning*, *arguing and proving*, *connecting*, *representing*, *communicating*, and so on, collectively making up a distinctive *mathematics thinking process* (Artigue and Blomhøj 2013; Chapman 2011).

Essential Constructs of Knowledge About Mathematics Inquiry

Taking as our starting point a detailed appreciation of these differences between science and mathematics education, we have revised the PRIMAS project's definition of inquiry to make it more suitable for mathematics education. The revised versions of the essential constructs are as follows:

1. Setting context (SC) refers to where inquiry questions emerge from, such as daily-life problems, natural phenomena, or mathematics itself (e.g., "Does multiplication always increase a number?").
2. Students' work (SW) refers to learners' mathematical thinking processes that take place during inquiry, including (but not limited to) elaborating questions, problem-solving, conjecturing, modeling and mathematizing, and reasoning.
3. Teacher guidance (TG) covers the teaching strategies used in IBMT, e.g., fostering students' generation of inquiry questions by asking, "What happens if…?"
4. Classroom environment (CE) refers to the building of an atmosphere in which students can share, justify, discuss, and challenge ideas during the mathematics inquiry process.
5. Theoretical understanding (TU) is teachers' understanding of mathematics inquiry from a theoretical/researchers' perspective, including the nature of mathematics, the benefits of mathematics inquiry (e.g., fostering inquiring minds, preparation for lifelong learning), and related theories such as constructivism that support the implementation of mathematics inquiry.

To some extent, our five constructs (SC, SW, TG, CE, and TU) are parallel to PRIMAS' (1) type of questions, (2) what students do, (3) teacher guidance, (4) classroom culture, and (5) valued outcomes, respectively.

Further Explanation of Theoretical Understanding (TU)

It is worth mentioning why we added "understanding related theories" in TU. Research has shown that teacher educators or expert teachers have better pedagogical knowledge than novice teachers (Rink et al. 1994), perhaps because the former are able to build more holistic understandings of an idea. Anderson (2002) found that teachers' understanding of inquiry teaching mainly stemmed from a more practical approach (e.g., reflecting on classroom events) rather than a theoretical or propositional one. Along the same lines, we speculate that if a practical teacher can develop a theoretical understanding of mathematics inquiry, he or she may have a better chance of developing a high level of mathematics inquiry knowledge.

Among various theories related to mathematics inquiry (e.g., problem-solving) that have been discussed in the literature, the most fundamental ones would appear to be constructivism and the nature of mathematics (Artigue and Blomhøj 2013). Abd-El-Khalick et al. (2004) have argued that "discussions of inquiry cannot, at least presently, be divorced from discussions of constructivism," which "necessarily bring along" issues pertaining to the nature of science and mathematics (p. 406). Constructivism holds that learning only occurs when learners are actively participating in learning tasks. Understanding constructivism is therefore equivalent to understanding inquiry, and the two share many educational objectives (Abd-El-Khalick et al. 2004). In addition, teachers' understanding of constructivism or inquiry is closely linked to the nature of mathematics: those who know that constructivism is a theoretical basis for mathematics inquiry may be more likely to emphasize the importance of students' prior knowledge or experience when doing IBMT (Artigue and Blomhøj 2013).

Mathematics Inquiry Knowledge and Inquiry-Based Mathematics Teaching

Researchers have acknowledged that teachers' pedagogical knowledge affects their teaching practices (Chapman and Heater 2010), but since IBMT is a relatively new idea, we do not yet have sufficient understanding of how mathematics inquiry knowledge may influence teachers' IBMT skills (Artigue and Blomhøj 2013). Nevertheless, research

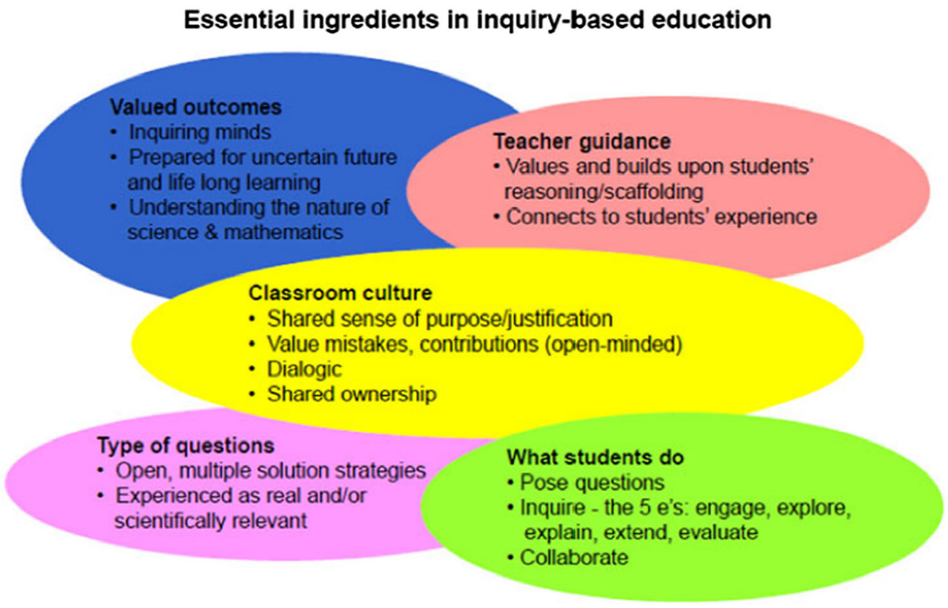


Fig. 1 PRIMAS's model of essential ingredients in science and mathematics education. Adapted from Artigue and Blomhøj (2013, p. 801)

on inquiry-based science teaching (IBST) may provide some relevant clues (Anderson 2002; Asay and Orgill 2010).

Researchers studying IBST have indicated that teachers' inquiry knowledge may impact how inquiry is implemented in the classroom (Crawford 2007; Lotter et al. 2007; Morrison 2012), and teachers' differing levels of inquiry knowledge may affect their implementation of IBST in various ways (Crawford 2000; Kang et al. 2008; Wallace and Kang 2004; Wee et al. 2007). Morrison (2012) has indicated that teachers' understanding of learners as having inquiring minds—i.e., curious, persistent, and maintaining a positive attitude—is crucial to the successful implementation of IBST. Kang et al.'s (2008) analysis of teachers' narratives about inquiry-based teaching found that they only focused on three of the NRC's five essential features of inquiry. This suggested that, for science teachers at least, inquiry knowledge was limited to a traditional view of inquiry, including collecting data and drawing conclusions, but not connecting explanations to scientific knowledge or communicating/justifying explanations. In other words, teachers did not appreciate either the use of inquiry to create or advance scientific knowledge, or the role of scientific argumentation in the inquiry process. Using concept maps drawn by four teachers before and after they implemented inquiry-based teaching, Wee et al. (2007) examined the relation between implanting such teaching and the development of inquiry knowledge. They found that actual use of inquiry in the classroom had only a small effect on the teachers' development of inquiry knowledge, with only two of the four exhibiting even a slight improvement; and that, among the NRC's essential planks of inquiry, "communicate and justify explanations" was the most challenging to develop: a finding similar to Kang et al.'s (2008) mentioned above.

From the above-cited IBST research, it is reasonable to conjecture that teachers' mathematics inquiry knowledge may be limited—or even quite narrow—and that the relationship between such knowledge and IBMT may be very complicated. While it is possible that increased mathematics inquiry knowledge may lead to better IBMT, the mere fact of their having implemented IBMT does not necessarily reflect teachers' development of such knowledge.

Inquiry-Based Mathematics Teaching and its Influence on Mathematics Teaching and Learning

IBMT has been recognized as one of the best ways to teach mathematics (Artigue and Blomhøj 2013; Stonewater 2005). In contrast to traditional chalk-and-talk teaching, IBMT is expected to include more open-ended and authentic problems; allow students to ask their own questions; value students' prior knowledge; help students use different ways of knowing; and encourage discussion of multiple viewpoints. In broad outline, then, IBMT is related to the concepts of learner-centered instruction, discovery-based learning, constructivist learning, and problem-solving or problem-based learning (Artigue and Blomhøj 2013; Chapman 2011).

Although some research has found that the inquiry-based classroom had no positive effects on student learning (e.g., Kremer and Schluter 2006), most studies have shown that IBMT can benefit learning (Anderson 2002). Bruder and Prescott's (2013) comprehensive review identified three major benefits of students' inquiry-based learning: (1) increased motivation, (2) a better understanding of mathematics, and (3) more positive attitudes or beliefs about mathematics and its relevance to life and society.

Teachers' Professional Development in Inquiry-Based Mathematics Teaching

However, teachers' learning of inquiry-based teaching takes time (Bucholtz and Kaiser 2013), and generally must be supported by PDPs aimed at the development of better inquiry knowledge and practices (Kang et al. 2008; Maaß and Artigue 2013; Wee et al. 2007). In light of prior research (e.g., Jaworski 1994; Maaß and Artigue 2013; Secada and Williams 2005), we designed the model of PDP activities illustrated in Fig. 2. In this model, "Understand Inquiry" refers to helping teachers learn about what mathematics inquiry is, by means of reading and discussing the relevant literature. "Experience Inquiry" indicates teachers' practical experience of mathematics inquiry, which occurs via two channels: participation in mathematics inquiry tasks as learners, and observation of an exemplary teacher's inquiry teaching to a class of actual students. "Implement Inquiry" refers to the PDP participants designing inquiry-based tasks and assigning them in their own classrooms, as a form of action research (Van Driel et al. 2001) grounded in introspection, discussion, and modification. Finally, the center of the model is "Discussion and Reflection," which occur during the entire process.

Methods

This qualitative case study aimed at achieving an in-depth understanding of changes in teachers' knowledge about IBMT, and changes in their teaching practices, from a holistic perspective (Yin 2014).

Description of the Professional Development Program

The PDP incorporated into the present study, which has been funded by the National Science Council of Taiwan, is intended to enhance teachers' knowledge about mathematics inquiry and their competence in implementing IBMT.

A total of 19 participants enrolled in the PDP. They included eight mathematics teachers who were new members of the program; seven mathematics teachers who had been in the program for 1 year (the PDP was running for 1 year before the study started; these seven teachers were in their second year of participation) and possessed fundamental understanding and some practical experience regarding IBMT; and four program staff (educators or researchers), who included a mathematics educator, two doctoral students, and a consultant teacher who had more than 10 years' teaching experience in IBMT.

The program adapted *co-learning* (Jaworski 2001; Wanger 1997) as its core design idea. Under this approach, participants with different backgrounds (e.g., new teachers, experienced teachers, and educators) can contribute to each other's learning, with the members of each group being allowed to play dual roles as learners and instructors. For example, educators could help teachers understand the theoretical basis of mathematics inquiry, while teachers could provide feedback when implementing mathematics inquiry teaching in a practical classroom context. The program activities for teachers were designed based on the model described in Fig. 2, above.

The PDP was a 3-year project, with the present study reporting the results of the second year of the program. New participants were recruited each year, so that eventually, there were three distinct pools of participants, one with no experience in IBMT, one with 1 year's experience, and one with 2 years' experience. It was hoped that this diversity in the participants' IBMT experience would lead to more fruitful conversations in the group meetings, and that this in turn would result in better mutual help and support. The program had a group meeting every 2 weeks and a total of 12 meetings per year. In these meetings, the teachers would experience several cycles of the three domains of activities (see Fig. 2). For example, in the first cycle, the teachers experienced two activities, one belonging to the Understanding Inquiry dimension and the other to the Experiencing Inquiry dimension. In the second cycle, they experienced three activities, related to Understanding Inquiry, Experiencing Inquiry, and Implementing Inquiry, respectively. Over the course of 1 year's

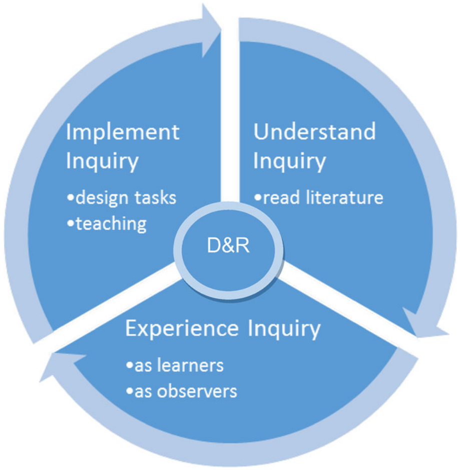


Fig. 2 Model of professional development program activities. D&R represents teacher discussion and reflection

Table 1 Backgrounds of the four research subjects

Name	Gender	Teaching experience (years)	Grade(s) taught	Undergraduate major
Yvonne	Female	8	7th–8th	Statistics
Jacky	Female	8	7th–8th	Applied Math
Wendy	Female	5	7th	International Business
Sara[a]	Female	10	7th–9th	Pure Math

[a] Sara graduated from a university specializing in teacher training, while the other three graduated from non-specialized universities

participation in the PDP, the teachers would experience four cycles.

Participants

This study focused on beginning learners of IBMT. From among the eight new teachers participating in the PDP, four female junior high school teachers were purposefully selected as the research subjects for this case study (Table 1). Three out of these four subjects had similar backgrounds, i.e., were graduates of general universities, had similar levels of teaching experience, and worked at the same school. The other participant graduated from a teacher-training university and had somewhat more teaching experience than the others. This allowed us to examine both a homogeneous sample (Wendy, Yvonne, and Jacky) and a heterogeneous one (Sara vs. the other three), which it was hoped would provide us with meaningful comparisons of their performance.

Selection of the Four Participants

A research meeting was convened to review the background data of all eight new participants. In this review, it was noted that three of the eight participants worked at the same school and had very similar backgrounds including gender, ages, academic background, teaching experience, and non-traditional pathways to becoming middle school mathematics teachers. Initially, they had all taught at vocational high schools and did not teach mathematics. Additionally, they took longer than usual to finally become middle school mathematics teachers, at a time when it was relatively easy to do so. It was also observed that they were contract teachers when they were serving in the vocational high schools. These factors stimulated research interest in comparing their growth in the PDP, given the above-mentioned similarities. Communication was thus initiated with them, leading to their inclusion as case teachers for the present study. After their agreement to participate, concerns arose among the researchers that perhaps these three case teachers had backgrounds that were too similar. Therefore, it was decided to include another teacher with a different background and who had become a teacher in Taiwan via the traditional route. This led to the final case teacher, Sara, being invited.

Data Collection

Concept Maps and Interviews

At the end of the third month and again at the end of the final (twelfth) month of our PDP, the four teachers were asked to draw concept maps of their knowledge about mathematics inquiry teaching and learning. Since at the time of our research they were all studying in the in-service mathematics education master's program at a university of education, each of them had already learned how to draw a concept map of a subject area. To better understand the gaps between the concept maps, the participants drew and what they really had in mind, the researchers interviewed them soon after the maps were drawn. Some example interview questions were "Why did you put back-and-forth arrows here?", and "What do you mean by 'creating cognitive conflict'?"

Classroom Observations

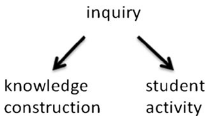
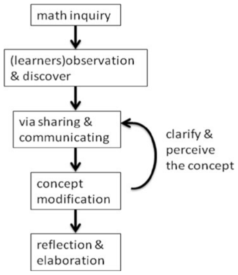
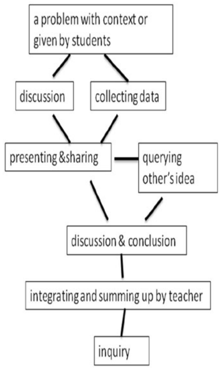
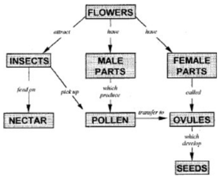
All four participants were asked to conduct their self-designed inquiry-based activities in their own classrooms at least once a month, excluding the summer and winter vacations. As such, each participant conducted eight of these classroom sessions, and all 32 sessions were observed and video-recorded by the researchers.

Data Analysis

Concept Maps and Interviews

Concept maps refer to one's understanding of something in relation to cognitive structures as proposed in psychology research (Frederiksen 1984). Such maps represent both what a person knows and how the information is structured and organized (Rink et al. 1994). Researchers agree that using concept maps is an advantageous tool because in concept maps, there are no predetermined structures and no limitations on the number of concepts, resulting in a better

Table 2 Classification of concept map types. Adapted from Wee et al. (2007, p. 74).

Map characteristic	Spoke	Chain I	Chain II	Net
Conceptual structure	Radial structures where related aspects of a topic are linked directly to the core concept but not to each other	A linear sequence of understanding in which each concept is linked to those above and below it	A somewhat more complex or refined variant of Chain I	A highly integrated network demonstrating deep understanding of the topic
Structural links	Simple associations with little integration of concepts	No complex connections between concepts	Slightly more complex connections with a degree of feedback in the inquiry process	Complex connections occur at different conceptual levels
Examples[a]				

Note Adapted from "Teaching and learning about inquiry: Insights and challenges in professional development" by Wee et al. (2007). Copyright 2007 by Association for Science Teacher Education

[a] The example of the NET structure was adopted from Kinchin et al. (2000, p. 47) which has complex connections among concepts

Table 3 Five essential constructs of knowledge about mathematics inquiry

	Definition	Examples
SC	Refers to origin of emerging inquiry questions, such as daily-life problems, natural phenomena, or mathematics itself	Does multiplication always increase a number?
SW	Refers to learners' mathematical thinking processes that take place during inquiry	Elaborating questions, problem-solving, conjecturing, modeling and mathematizing, reasoning
TG	The teaching strategies used in IBMT	Fostering students' generation of inquiry questions by asking "What happens if…?"
CE	The building of an atmosphere in which students can share, justify, discuss, and challenge ideas during the mathematics inquiry process	Students' challenges to a fellow student who is presenting a math problem
TU	Teachers' understanding of mathematics inquiry from a theoretical (researchers') perspective, including the nature of mathematics, the benefits of mathematics inquiry and related theories	Constructivism as a theoretical basis for mathematics inquiry teaching; fostering inquiring minds; preparation for lifelong learning

TU Theoretical understanding, *SC* setting context, *SW* Students' work, *TG* teacher guidance, *CE* classroom environment

reflection of the thinking process. Additionally, research has successfully used concept maps to investigate the relationship between teachers' knowledge and practice. In particular, studies have found that teachers who have more coherent and organized knowledge structures perform better when teaching than those who do not. Therefore, concept maps have traditionally been a common technique for evaluating teachers' knowledge structure (Rink et al. 1994).

There are two general approaches to analyzing concept-map structure, incorporating either a quantitative or a qualitative approach (Wee et al. 2007). In the former, researchers consider "an aggregate score of factors including the number of valid links presented; the degree of cross-linkage indicated; the amount of branching; and the hierarchical structure" (Kinchin et al. 2000, pp. 45–46). The quantitative approach has been critiqued in numerous studies (Kinchin et al. 2000). Researchers have indicated that the emphasis on "valid links" is incompatible with constructivist psychology as it fails to recognize the significance of people's perspectives. Moreover, the consistency of the scoring scheme has been identified as being somewhat problematic to control.

In the qualitative approach, in contrast, invalid links are deemed as important as valid ones, because the former may reveal much about the thinking process and may also help people to create valid links in the future. Guided by this insight, Kinchin et al.'s (2000) analysis framework included all links that were drawn by the respondents. Their classification highlighted three main patterns: *Spoke, Chain*, and *Net*. Wee et al. (2007) then revised the "*Chain*" structure as *Chain I* and *Chain II* in order to better analyze maps. More explanation of these classifications is offered below.

As previously mentioned, there are two important elements in concept maps: (1) how the knowledge is organized and (2) how many concepts are included. Accordingly, the concept maps in this study were analyzed and coded according to (a) concept-map structure, and (b) essential constructs of mathematics inquiry.

(a) Concept-map structure. We adopted a primarily qualitative approach to map structure. Each of Wee et al.'s (2007) four concept-map structures reflects a different level of conceptual understanding, complexity, degree of relatedness between concepts (including feedback levels), and ability to accommodate new materials—with Spoke being the lowest level, and Net the highest (Table 2).
(b) Essential constructs of mathematics inquiry.

The five essential constructs of mathematics inquiry identified in an earlier section are shown in Table 3. When analyzing the teachers' concept maps, the coder assigned one point respectively for the presence of each of our five essential constructs of mathematics inquiry knowledge. For example, the left side of Fig. 5 indicated that the respective relationships between inquiry and knowledge construction (coded as TU), inquiry and student activity (coded as SW), and finally, the map were awarded two points. Hence the number of points that might be awarded to a given map ranged from 0 to 5 (based on the presence or absence of an essential construct). That is, the points reflected the total number of the essential constructs appearing in concept maps.

Concept Maps Coding Process

Each participant's concept map was first analyzed by two of the authors, who would give codes for it individually. Subsequently, a researchers' meeting would be held in which all three authors met together to discuss the differences between the first two authors' coding results. The final codes were then deliberated until the entire research team had reached an agreement. For example, in Sara's posttest map (the right-hand side of Fig. 4), one of the authors thought it should be coded as Chain II because it contained a feedback loop between "via sharing and communicating" and "concept modification"; however, another author argued for the Chain I classification for the map because it appeared to be almost a linear process. In the meeting, all three authors shared their opinions and conferred with each other, finally deciding to code Sara's posttest map as Chain I and likewise formulating a more clear definition of Chain I and Chain II, whereby a linear map structure with only one feedback in the inquiry process would still be coded as Chain I.

Interviews Coding Process

All the audio-recorded interviews were transcribed verbatim and were used as supplementary resources to further explain what the teachers had drawn in their concept maps. In the analysis process, one of the authors read all the interview transcripts and made notes about his first impressions. Then the author read the transcripts very carefully again and started to label relevant sentences or sections in the transcripts. These labels related to how the interviewees explained their concept maps. For example, the case teacher Sara wrote "student activity" in her pretest concept map, thus leading the interviewer to request her to explain her idea in the interview. She responded that "this means inquiry is learners constructing their own knowledge through various hands-on activities." Then, this sentence in the transcript was labeled as "explain student activity." Finally, these coding results were used to support us toward a better understanding of teachers' concept maps.

Classroom Observation

The 32 classroom sessions were divided into two stages, corresponding to the first and the last 6 months of the PDP. By means of an analytic framework for IBMT practice adapted from Wood et al. (2006) and set forth in Table 4, the participants' teaching sessions were classified into three different levels, according to the type of interaction that took up the highest percentage of classroom time. For example, if in a particular session, the most time was devoted to "exposition," that session was classified as "level 0." If most interaction time consisted of "challenge," the session was classified as "level 2."

Further Explanation of the Inquiry Level in Classroom Observations

The present section provides further explanation about the inquiry levels set forth in Table 4. Wood et al. (2006) based their framework on psychology and sociology, for

Table 4 Analytic framework for inquiry-based teaching practice

Dominant interaction	Level	Explanation
Exposition	0	Teacher-centered expository teaching: the teacher illustrates the main mathematical concepts in the curriculum, and demonstrates the problem-solving process for non-routine or open-ended problems. Students merely watch and listen to the teacher's exposition to learn mathematics. In this kind of classroom teaching, the teacher is the one who talks and asks questions, and the students' responses are simply "Yes" or "No,"
Discussion	1	Discussion and focusing attention: the teacher guides the students to the key point in the process of problem-solving by summarizing the results of group discussion and posing probing questions in order to help students solve the problems successfully
Explanation	1	Student-centered explanation: students present their answers and explain how they arrived at them. The focus here should be arousing students' diverse strategies for solving problems
Elaboration	1	Teacher-centered elaboration: the teacher integrates, extends, and adds more information to students' explanations
Clarification	2	Clarifying students' understanding: the teacher encourages students to clarify the presenter's explanation, approach an idea by means of asking questions, or simply point out what they do not understand
Challenge	2	Challenge leading to better understanding: the teacher encourages students to challenge others' solutions, strategies, and thinking through debate and argument, in order to develop a more rounded understanding

the core purpose of analyzing social features and the quality of students' thinking in reform-oriented classrooms. They identified two major types of reform-oriented classrooms, the first being termed a *strategy reporting classroom.* The main pattern in this classroom was "on children's presentation of different strategies for the problems solved" (p. 224). Students in such a classroom might be asked to explain how they solved the problem by the teacher, but not by the other students.

The second type of reform-oriented classroom was named an *inquiry/argument classroom*. The focus of this type of classroom was that students explained the thinking behind their solutions in order to make sense of them to others. In addition, students receive feedback in the form of challenges or disagreement from other students or teachers, and must give justifications for their ideas.

Table 4 reflects Wood et al.'s (2006) delineation of two types of classroom. The major interactions that occur in the *strategy reporting classroom* are "Discussion," "Explanation," and "Elaboration." This type of classroom was therefore defined as *Level I* of IBMT since students have a lower quality of mathematics thinking in this type of classroom as compared to students in the *inquiry/argument classroom.* "Clarification" and "Challenge" are the major interaction types that characterize the inquiry/argument classroom, which is further classified as *Level II* of IBMT based on the higher quality of students' mathematics thinking. Finally, if a classroom remains a traditional teacher-centered classroom, then it is *Level 0* since it would not be considered a reform-oriented classroom.

Classroom Observation Coding Process

In the first stage of the coding process, all three research team members worked together on a total of four classroom observation video recordings (one for each teacher, for a total of four video recordings). The co-authors discussed the recordings and identified the time intervals that could be classified as specific types of interactions according to Table 4. Non-teaching time intervals (e.g., the teacher telling a joke) were excluded from coding by the team members. If there was a disagreement during the coding process, all three members would share their opinions and recheck the video repeatedly until a final decision was made.

Triangulation

As described in the above section, the design of this study satisfies the requirement for triangulation between data and researchers (Patton 2002). Three distinct types of data (concept maps, interviews, and classroom observations) were collected and analyzed by the three authors, who repeatedly discussed them until an agreement was reached. The concept maps, for example, were coded individually by two authors, and the differences were subsequently discussed and reconciled. At the same time, the transcriptions of the interviews about the teachers' concept maps were taken into account to assist in further analyzing the maps.

Findings

Overall Analysis of the Four Case Teachers' Knowledge and Practice of Inquiry-Based Mathematics Teaching

Table 5 presents the results of spectrum analysis of the four case teachers' knowledge about mathematics inquiry and

Table 5 Spectrum analysis of case teachers' knowledge of mathematics inquiry and their practice of inquiry-based teaching

	Knowledge Level			
	Spoke	Chain I	Chain II	Net
Pre-test	Wendy(1)Sara(2)Yvonne(2)------	Jacky(4) ----------	----------	----------
Post-test	----------	------Wendy(2)Sara(3)----------	Jacky(4)Yvonne(4)----------	----------
	Practice Level			
	Exposition-oriented ←----------→ Inquiry-oriented			
	Level 0	Level 1		Level 2
Stage 1	Jacky-Wendy-Yvonne-Sara----------	----------	----------	----------
Stage 2	----------	Jacky-Wendy----------	----------	Yvonne-Sara----

Note. The number in parentheses refers to the points received for including essential constructs of knowledge about mathematics inquiry in concept maps. The pretest was conducted at the end of the third month; the posttest, at the end of the year. Stage 1 refers to the first 6 months of the PDP; Stage 2, to the remaining 6 months

their practice of IBMT. It indicates that these four teachers all had, to some degree, increased their knowledge about inquiry and changed their IBMT practices following their participation in the PDP.

Of the four case teachers, Yvonne changed the most in her understanding of mathematics inquiry: moving from Spoke to Chain II, and from 2 to 4 points for the essential constructs (Table 5). Jacky appeared to possess better knowledge about mathematics inquiry than the other three teachers in the pretest (ranked as Chain I, with 4 points for constructs); however, she only moved up one level (i.e., to Chain II) in the post-test, and her construct points remained unchanged. Wendy and Sara had the lowest levels of knowledge about mathematics inquiry in the pretest—both ranked as Spoke, with 1 point for constructs—and both moved up to Chain I in the posttest, with 2 points and 3 points, respectively, for constructs.

Regarding IBMT practices, the posttest showed that all four case teachers had modified their teaching practice in a more inquiry-oriented direction, with Yvonne and Sara showing more improvement than Jacky and Wendy (Table 5). Specifically, Yvonne's and Sara's practice moved from level 1 to level 2, while Jacky's and Wendy's moved from level 0 to level 1.

It is worth noting that although Wendy's and Yvonne's inquiry knowledge was consistent with their IBMT performance (Table 5), the other two teachers' understanding of mathematics inquiry does not appear to have been directly reflected in their teaching. Sara's inquiry knowledge only improved by one level, which would seem somewhat inconsistent with the dramatic improvement in her IBMT (from Level 0 to Level 2). Jacky, on the other hand, exhibited relatively high levels of knowledge in both the pre- and posttests, yet her IBMT did not improve as much as her inquiry knowledge over the course of the study.

Analysis of Case Teachers' Knowledge of Inquiry-Based Mathematics Teaching

Figures 3, 4, 5, and 6 represent the pre- and posttest concept maps created by Wendy, Sara, Yvonne, and Jacky, respectively.

Wendy's, Sara's, and Yvonne's pretest concept maps are somewhat similar to one another, both conceptually and in terms of map structure. It would appear that the major concept underlying all of their pretest maps was that constructivism serves as a basis theory for inquiry. For example, Wendy's pretest concept map directly presents inquiry as construction. As she explained, "I think my idea of 'inquiry' is very simple. 'Inquiry' is the same as 'construction,' and then equals to something I don't know [laugh]. I only know it's student-centered; the teacher plays a role as a guide." Although she had mentioned these terms in the interview, neither the idea of student-centeredness nor the teacher's role was included in her map. Sara's pretest concept map was slightly more sophisticated than Wendy's: though also starting from the term "construction," it had more content attached to it, such as "understanding how mathematicians construct math concepts," "complete understanding," and so on. In addition to inquiry as knowledge construction, Yvonne's pretest concept map included "student activity." She explained that "inquiry is learners constructing their own knowledge through some 'hands-on activities'," which addressed a key aspect of how learning by inquiry is meant to proceed.

Fig. 3 Wendy's pretest (*top left*) and posttest (*top right*) concept maps, with the original Chinese-language versions shown below

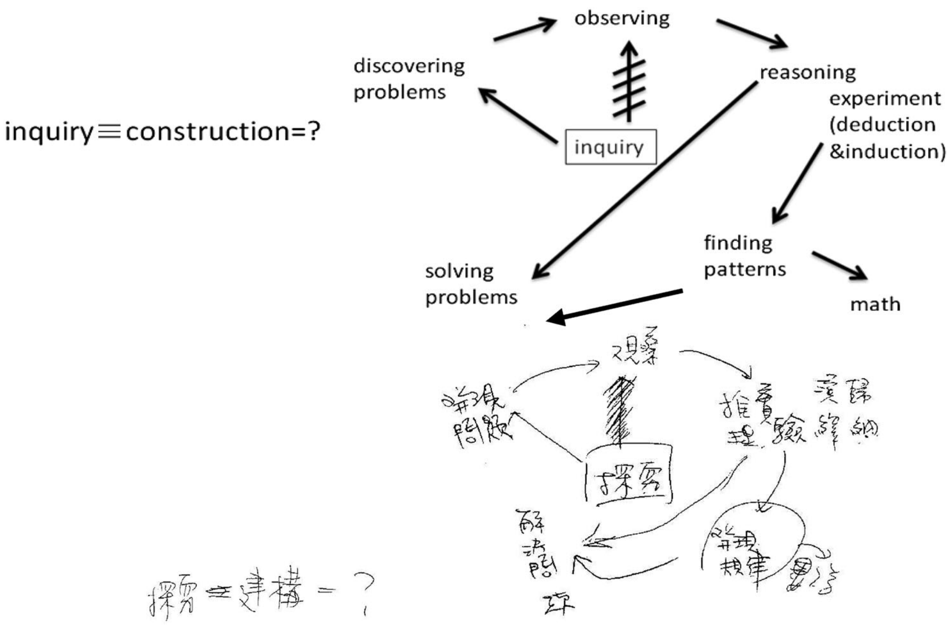


Fig. 4 Translations of Sara's pretest (*left*) and posttest (*right*) concept maps

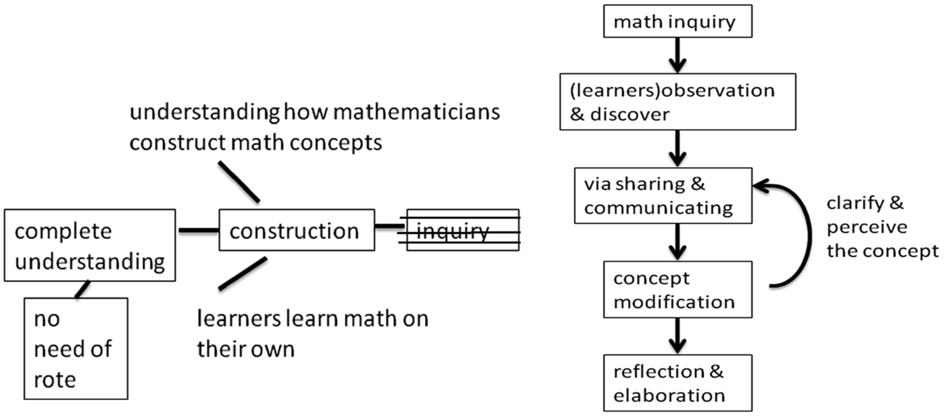
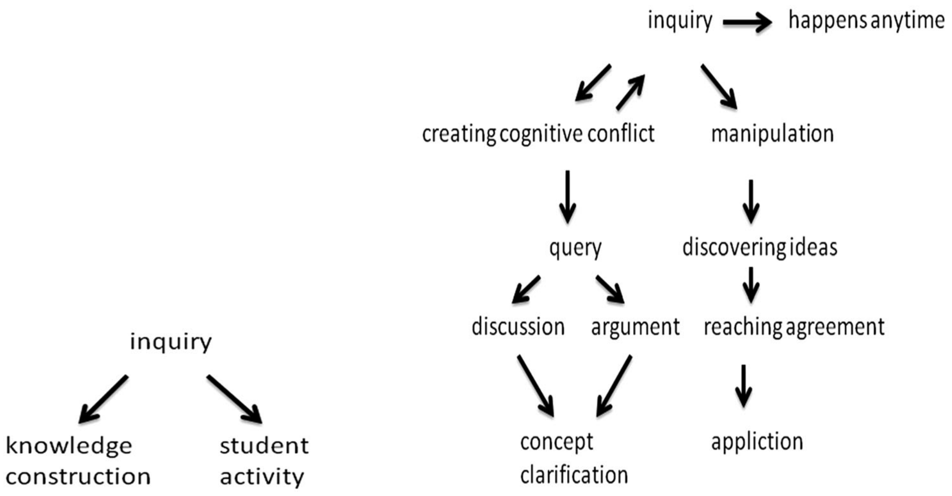


Fig. 5 Translations of Yvonne's pretest (*left*) and posttest (*right*) concept maps



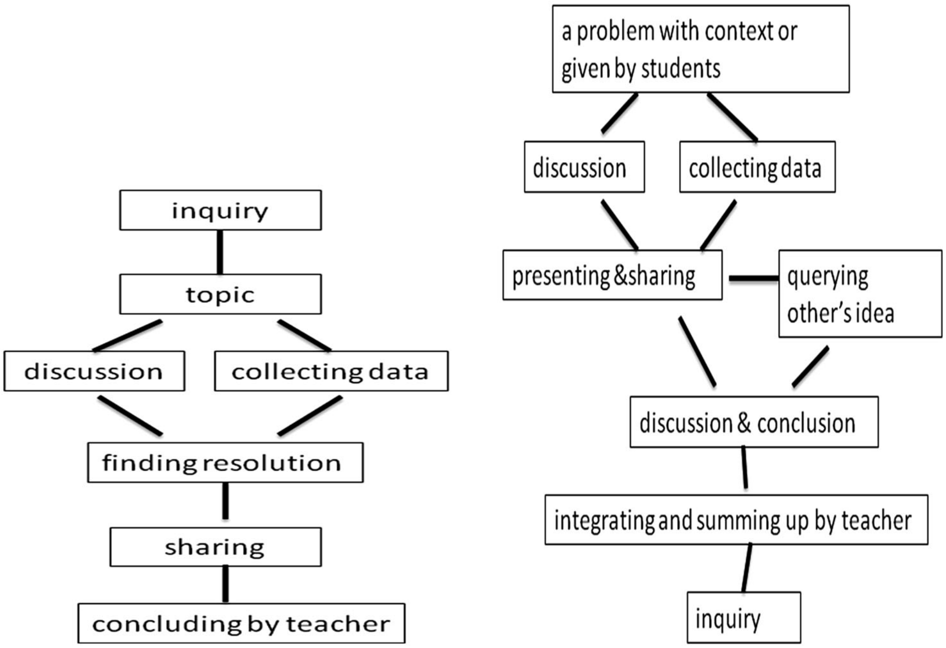


Fig. 6 Translations of Jacky's pretest (*left*) and posttest (*right*) concept maps

Jacky's pretest concept map was distinct from the other three. Despite the fact that it did *not* contain the term "construction," the process of inquiry that it set forth was the most sophisticated of the four pretest maps.

All four teachers' posttest concept maps exhibited advancement in comparison to their pretest maps. Each participant was able to establish a more complex structure that included more concepts. The major common feature of these posttest maps was that they were elaborated versions of the pretest maps, and that all of the elaborations included the incorporation of inquiry teaching knowledge. For example, "discussion," "sharing," "querying," or "observing (observation)" could be found in all posttest maps.

Of the posttest concept maps, Yvonne's was the most sophisticated, emphasizing two key elements of IBMT: "creating cognitive conflict" and "manipulation." As she put it, "inquiry emerges from students' cognitive conflict. Therefore, a teacher should try to create conflict for students, like giving an ill-posed problem …. Sometimes, this conflict can also come from students' alternative or unexpected ideas or questions." In her second concept-map interview, she also stressed the importance of hands-on activity, in particular, allowing students to develop abstract mathematical concepts by manipulating concrete objects. When asked to explain the phrase "happens anytime" that was used on her map, she stated that students' inquiry "does not necessarily happen during well-designed teaching activity; sometimes it just happens when a student asks a good question," i.e., a question that is considered worth pursuing by the whole class.

Jacky's posttest concept map, though quite similar to her pretest map, was more detailed in terms of the explanations it provided: for example, adding how an inquiry problem was generated ("a problem with context or given by students"), as well as how "querying other's idea" could interact with "presenting and sharing" and "discussion and conclusion." In the second interview she stated that any problem presented to the students as part of IBMT "had better be a realistic problem," as this realism "could encourage students to do inquiry spontaneously." Likewise, "it is good for students to create their own inquiry problems. They can find solutions, discuss, and share with each other … sometimes they made an incorrect reasoning during the [inquiry] process but then, other students could ask questions to challenge [their reasoning]."

Sara's posttest concept map referred directly to the IBMT process, strongly emphasizing the elements of clarification and reflection. She said that, in comparison to her first map, "my thinking about inquiry did not change too much … but I think now I have a greater appreciation of the process of clarification and reflection …. In the beginning, I thought inquiry was just letting students learn by themselves without any intervention, but now I know students need some help. They should have opportunities to elaborate their initial thinking."

Wendy's posttest concept map was slightly different from the other three teachers', in that it focused more on how a person constructs his or her own knowledge through the mathematics inquiry process, in contrast to the other teachers' maps which all involved some inquiry-based teaching processes ("discussion," "sharing," "presenting," and so on). Wendy explained: "The main purpose of inquiry should be solving daily-life problems. Lots of things happen around you in your daily life. If you can see

Table 6 Analysis of essential constructs of knowledge of inquiry-based mathematics teaching

	Wendy		Sara		Yvonne		Jacky	
	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
TU	1	0	1	0	1	0	0	0
SC	0	1	0	0	0	1	1	1
SW	0	1	1	1	1	1	1	1
TG	0	0	0	1	0	1	1	1
CE	0	0	0	1	0	1	1	1
Total	1	2	2	3	2	4	4	4

TU Theoretical understanding, *SC* setting context, *SW* students' work, *TG* teacher guidance, *CE* classroom environment

Table 7 Analysis of the four case teachers' teaching practice

	Wendy		Sara		Yvonne		Jacky	
	Stage 1	Stage 2	Stage 1	Stage 2	Stage 1	Stage 2	Stage 1	Stage 2
Time (min)[a]	384	456	459	456	480	462	420	263
Exposition (%)	49	4	51	8	78	14	75	17
Discussion (%)	38	48	38	23	15	32	23	50
Explanation (%)	9	34	8	14	7	4	1	25
Elaboration (%)	2	3	3	9	0	5	1	4
Clarification (%)	2	5	0	21	0	26	0	2
Challenge (%)	0	6	0	25	0	19	0	2
Inquiry level[b]	0	1	0	2	0	2	0	1

Stage 1 = the first 6 months of the PDP; Stage 2 = the remaining 6 months

[a] Total classroom teaching time excluded in-class time that was not spent teaching, e.g., telling a joke

[b] The teachers' mathematics inquiry teaching was categorized into different levels based on the dominant classroom interactions

them with interest and curiosity, you'll find many interesting problems and you definitely can do some experiments to find results."

Table 6 sets forth the results of our analysis of the five essential constructs of inquiry-based mathematics knowledge that were contained in our participants' pre- and posttest concept maps. As previously mentioned, each teacher's maps showed some degree of increased sophistication between the pre- and posttest. For example, Wendy's pretest map only had one essential construct, TU, but in the posttest, her maps included two, SC and SW. This result seemed to indicate that the teachers' initial understanding of mathematics inquiry was limited, generally to the realm of theory; but that over the course of the year-long PDP, their understanding incorporated more practical knowledge.

Analysis of Case Teachers' Practice of Inquiry-Based Mathematics Teaching

Table 7 presents our analysis of the four case teachers' teaching practices.

Wendy

In stage 1, the dominant types of interaction in Wendy's classroom were exposition (49 %) and discussion (38 %). It is obvious that even after Wendy had started trying inquiry-based instruction, her teaching was still teacher-centered and expository. In her former inquiry-based teaching, she had only arranged more time for students' discussion. The other expected classroom interactions of IBMT—e.g., clarification and challenge—seldom appeared in her teaching sessions that we observed. When scrutinizing her teaching practice, we noted that Wendy always let students discuss a given problem, but then quickly took the reins of control back: giving the students direct exposition and seldom providing them with opportunities to present their own ideas, or posing probing questions that might have helped them clarify their understanding. Hence, the students did not have any significant opportunities to question or challenge each other's ideas, and Wendy's stage 1 IBMT practice was classified as level 0.

In stage 2, Wendy made a number of changes in her teaching. The main interaction types in this stage were discussion (48 %) and explanation (34 %), complemented by clarification (5 %), challenge (6 %), and elaboration (3 %). Her use of exposition, meanwhile, had substantially decreased: from 49 to 4 %. In the in-depth analysis, Wendy began to stress “discussion” and “explanation,” in which students discussed problems in groups and then explained their solutions to the whole class. Wendy also had commenced questioning students’ ideas, to prompt them to make deeper interpretations of their answers. She spent a considerable amount of time on setting the contexts of problems; helping students engage in inquiry tasks; confirming whether they understood the presenter’s idea; and guiding them to challenge or query each other’s ideas. Since Wendy’s teaching practice in stage 2 was mainly focused on “discussion” and “explanation” (82 % in total), along with the highly inquiry-oriented “clarification” and “challenge,” it was classified as level 1.

Sara

Sara made more significant progress than Wendy did, i.e., from level 0 to level 2 (Table 7). In stage 1, Sara’s teaching was still teacher-centered, with her classroom interactions being exposition (51 %), discussion (38 %), explanation (8 %), and elaboration (3 %). She did not merely follow the content of the textbook, however, but reorganized it into activities (albeit still presented through exposition). She interacted with students and let them discuss some problems, and sometimes provided them with opportunities to explain their ideas.

The major differences in Sara’s stage 2 teaching were the emergence of challenge (25 %) and clarification (21 %), alongside significant decreases in exposition and discussion: from 51 to 8 %, and 38 to 23 %, respectively. Explanation increased from 8 to 14 %. The in-depth analysis of Sara’s teaching revealed that she always brought up a main activity, then let students discuss it in groups and pose some questions if necessary/relevant. The students had to present their findings and respond to one another’s queries and challenges. Finally, Sara would help students integrate different ideas into a conclusion regarding the activity. Based on all of this evidence, Sara was classified as level 2 in stage 2.

Yvonne

In stage 1, Yvonne used 78 % of class time in exposition, and allowed students to discuss and explain their ideas to others for the rest of the time (Table 7). Yvonne’s teaching was classified as level 0 at this stage. In stage 2, the major types of interaction in Yvonne’s classroom were discussion (32 %), clarification (26 %), and challenge (19 %). Even though exposition still took up 14 % of class time, the level-2 interaction types we observed (clarification and challenge, 45 % in total) were the most prominent. Therefore, Yvonne was classified as level 2 in stage 2.

In stage 1, Yvonne would present the problem on the board or via student worksheets, then explain it, guide students to solve it in groups, and invite one or two students to present their solutions on the board. Finally, she would present a conclusion to the whole class. In stage 2, she reduced the time allocated to exposition, instead giving students more time with their peers to clarify and challenge their own and others’ ideas. In other words, she had come to value students’ construction of their own knowledge.

Jacky

In stage 1 (Table 7), the major types of interaction in Jacky’s class were exposition (75 %) and discussion (23 %). In stage 2, she changed the focus to discussion (50 %) and explanation (25 %), supplemented by exposition (17 %), clarification (4 %), elaboration (2 %), and challenge (2 %). Thus, her teaching practice was classified as level 0 in the first stage and level 1 in the second.

Further analysis of Jacky’s teaching revealed that, in stage 1, she usually introduced the concept directly. She would sometimes allow students to discuss it in groups and present their solutions on the board, but would then step into draw conclusions on the students’ behalf. In other words, Jacky’s teaching in stage 1 was clearly teacher-centered. In stage 2, she made a few changes in her teaching: inviting students to join the discussion while she was lecturing, and providing them with more time for discussing and expressing their ideas. However, she continued to intervene, to re-explain what her students had just presented. Consequently, discussion, explanation, and exposition were her three major types of interaction in stage 2.

Discussion

Teachers with Similar Backgrounds Do Not Necessarily Possess Similar Development Patterns of Mathematics Inquiry Knowledge and Inquiry-Based Mathematics Teaching

This study intentionally selected three participants with similar backgrounds—Wendy, Yvonne, and Jacky—and one, Sara, with a different background from the other three. However, Wendy, Yvonne, and Jacky did not exhibit similar development processes in terms of mathematics inquiry knowledge and IBMT. Wendy’s and Yvonne’s IBMT development trajectories were somewhat consistent with their development of mathematics inquiry knowledge, but Jacky's development of mathematics inquiry knowledge was noticeably better than was reflected in her IBMT performance. This implies that teachers with similar backgrounds may not necessarily have similar development patterns in mathematics inquiry knowledge and IBMT. Certainly, it confirms that teachers' professional development is a complex process, and as such, many other factors may have influenced the participant teachers' growth (Chapman 2011; Maaß and Doorman 2013; Saad and BouJaoude 2012).

Sara, whose background was different from the other three participants, developed the best IBMT among all four participants, though she did not develop the best mathematics inquiry knowledge. We will further discuss her performance in the following section.

Knowledge About Mathematics Inquiry Is Not Necessarily Reflected in Teaching Practice

Our participant Jacky exhibited better knowledge of mathematics inquiry than the others in the pretest, and ranked second of the four in the posttest; however, her teaching practice did not match this performance. Indeed, our results taken as a whole indicate no obvious correlation between a person's knowledge of mathematics inquiry and her corresponding teaching practice. This is in sharp contrast to prior research findings that knowledge about mathematics inquiry is a crucial factor influencing teachers' inquiry practice (e.g., Seung et al. 2013; Wallace and Kang 2004; Wee 2007). As such, knowledge may be a necessary but not a sufficient condition for the successful practice of inquiry-based teaching. Other factors, such as attitudes or beliefs about mathematics, may have to be taken into account if we are to obtain a more fully rounded picture of teachers' IBMT abilities (Rushton et al. 2011; Saad and BouJaoude 2012).

Of our four case teachers, Sara was the most successful at inquiry teaching, but her knowledge about mathematics inquiry was only ranked third in the pretest. It is worth seeking possible explanations for this gap. From the point of view of declarative knowledge and procedural knowledge (e.g., Anderson 1980), the knowledge that is captured by the concept maps in this study is much like the former; but what might represent teachers' procedural knowledge of inquiry-based teaching? As Chapman (2011) has pointed out, teachers' mathematical knowledge for teaching (MKT) may be one kind of procedural knowledge, since MKT is inherently practice-based (Ball et al. 2008; Hill et al. 2008a, b). When we re-examined the classroom observation data in light of this insight, Sara seemed to show better MKT than the other three teachers. This may also suggest that declarative knowledge (i.e., knowledge of what IBMT *is*) cannot directly become procedural knowledge (i.e., knowledge of *how to do* IBMT, such as MKT), a finding that would echo those of earlier studies (e.g., Paradis 1994). As such, our results imply that teacher educators should not only help teachers learn what inquiry-based teaching is, but provide meaningful activities to help them connect both these types of knowledge. Our framework set forth in Fig. 2—which integrates understanding, experience and practice—seems to be a useful approach to achieving this goal, insofar as all of our case teachers improved their IBMT to some extent. However, in future analyses of teachers' knowledge about inquiry teaching, procedural knowledge (e.g., MKT) should be taken into account.

Teachers' Initial Knowledge About Mathematics Inquiry Is Limited, and Implementing Inquiry-Based Mathematics Teaching Can Help Teachers Increase This Knowledge

Three of the four teachers' knowledge was at the lowest level at the beginning of this study. This finding seemed consistent with previous research in science education that showed teachers' knowledge about inquiry was limited (Kang et al. 2008; Seung et al. 2013; Wallace and 2004; Wee et al. 2007). In addition, all four of our participants improved their knowledge about mathematics inquiry after having experienced IBMT implementation. This finding is consistent with the majority of relevant teacher education research, which indicates teachers learn from their practice (Bell et al. 2010; Borasi et al. 1999; Goldsmith et al. 2013). However, our finding was somewhat different from that of a previous study on scientific inquiry, that implementing inquiry had little or no effect on the improvement of inquiry knowledge (Wee et al. 2007). This discrepancy may imply the importance of PDP design: if a program cannot provide enough support, the participating teachers might still fail to learn inquiry teaching.

Teachers Tend to Understand Mathematics Inquiry from a Theoretical Perspective

Typically, research shows that most teachers understand mathematics inquiry in a practical sense (Anderson 2002; Chapman 2011; et al. 2008; Morrison 2012), i.e., how to implement IBMT in the classroom. However, the teachers in the present study all tended to understand mathematics inquiry from a theoretical perspective. We believe this difference may stem from how they learned IBMT in the beginning.

Methodologically, it should be noted that the participants in the present study began their professional development activity by reading literature and books, in contrast to those in Anderson's study, some of whom began by viewing classroom teaching videos. Future study to compare outcomes across these contrasting theoretical and practical methodologies for two groups of teachers could yield interesting and productive findings.

Radical Change in Teaching Practice Is Unlikely to Occur in the Short Term

Our results further indicate that, across stages 1 and 2, the quality of the four participants' IBMT was Sara > Yvonne > Wendy > Jacky; in other words, those who performed better in the early part of the PDP always performed better as it went on. This would seem to suggest that radical change in the quality of an individual's IBMT practice is not likely to happen over the course of a 1-year program, and supports prior studies' findings that changes in mathematics teachers' practice tend to occur slowly (e.g., Buchholtz and Kaiser 2013).

A Systematic and Integrated Method for Analyzing Mathematics Teachers' Inquiry-Based Teaching and Practice

One of the valuable contributions of this study is that it provides a systematic and integrated method for assessing teachers' mathematical inquiry knowledge and practice, based on a novel articulation of the character of mathematical inquiry (e.g., Table 3). Few such attempts have previously been made (et al. 2008; Saad and BouJaoude 2012). Additionally, using the tools the present study has provided, researchers will be able to conduct parallel research involving a larger number of participants.

Conclusion

In order to understand mathematics teachers' performance in PDPs, an instrument capable of measuring their mathematics inquiry knowledge and the quality of their IBMT is needed. In this study, we made important steps toward this goal. Our analysis frameworks in mathematics inquiry knowledge and IBMT can help us understand, at least in part, some important aspects of the former and its relation to the latter. The findings of the present study also represent an important contribution to the body of research on teachers' learning of, and professional program design for, IBMT.

The present findings support the assertion that PDPs can successfully assist teachers to develop mathematical inquiry knowledge and IBMT. Although understanding *how* the program affected the four case teachers' mathematical inquiry knowledge and practice was not the main focus of the present study, briefly touching on such aspects in the present discussion appears worthwhile.

The present research found that reading literature, observing experienced teachers' teaching (participating as an observer), and engaging in inquiry activities (experience as an inquiry learner) were the three most useful activities for helping teachers, since our four participant teachers frequently mentioned in the group meetings how these activities facilitated their growth. In addition, the teachers said that group meetings not only provided resources for them to implement IBMT but also promoted the process for their sharing, discussing, and reflecting on their teaching.

Suggestions for Future Studies

This study's analysis frameworks for concept maps and IBMT provide a certain degree of ability to measure teachers' mathematics inquiry knowledge and practice. These frameworks may also be useful to those engaging in further larger-scale research. However, this would require a large-scale parallel study geared toward obtaining additional and substantial evidence of teachers' mathematics inquiry knowledge and IBMT. Our selection of teachers with similar and different backgrounds was also found to be productive, and could usefully be extended to such a larger-scale study. While in the present research, there seemed to be no obvious correlational patterns regarding the participants' similar or contrasting backgrounds, the possible emergence of important patterns could be more readily captured if the numbers of participants were increased.

With the aim of helping teachers to understand inquiry, this study initially adopted a more theoretical approach (proceeding from reading literature about inquiry), as opposed to a more practical one (e.g., starting from classroom observations). In order to investigate these two differing initial approaches in the PDP design, we suggest that further studies can try to compare outcomes across PDPs with these two different approaches.

Regarding knowledge of IBMT, further study should pay due attention to both declarative knowledge and procedural knowledge, and to how these types of knowledge might influence teachers' development of IBMT. Other factors, such as attitudes and beliefs, may also influence teachers' development of IBMT, and future researchers should also consider taking these into account when studying teachers' IBMT.

The use of other types of instruments in future studies is also recommended, for instance, teaching scenarios or videos (e.g., Kersting et al. 2009) to assess teachers' procedural knowledge, in particular MKT. Likewise, the relationship between procedural knowledge and IBMT is also deserving of further research exploration.

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