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A Study of Albedo Effects on Libration Points in the Elliptic Restricted Three-Body Problem

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Abstract
This paper deals with the libration points in the elliptic restricted three-body problem. The combined effects of the primary bodies' orbital eccentricity and albedo on the existence and stability of libration points is analyzed. It is found that there exist five libration points, three collinear and two non-collinear. The non-collinear libration points are stable for a critical value of mass parameter μ_c, the collinear libration points are unstable for all values of μ.

Keywords Elliptic restricted three-body problem · Radiation pressure · Albedo effect · Libration points · Linear stability

Introduction

Albedo effect is one of the most interesting non-gravitational force having significant effects on the motion of an infinitesimal mass. Albedo is the fraction of solar energy reflected diffusely from the planet back into space [13]; the measure of the reflectivity of the planet's surface. Therefore, the Albedo can be defined as the fraction of incident solar radiation returned to the space from surface of the planet [31],

$$Albedo = \frac{radiation \quad reflected \quad back \quad to \quad space}{incident \quad radiation}.$$

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Albedo is a dimensionless quantity measured on a scale from 0 to 1. A body or surface that has zero albedo indicates the body is a 'black-body', which absorbs all the incident radiations. An albedo value of unity represents a 'white-body', which is a perfect reflector that reflects all incident radiations completely and uniformly in all directions. A high-albedo surface has lower temperature because it reflects the majority of the radiation that hits it. On the other hand, a low-albedo surface has higher temperature as it absorbs more of the incoming radiation. For instance, fresh snow has a high albedo of 0.95 as it reflects 95% of incoming radiation, while water reflects about 10% of the incoming radiation, resulting in a low albedo of 0.1. On average, the albedo of Earth is 0.3 as 30% of incident solar radiation is reflected by the entire Earth. Generally, dark surfaces have a low albedo and light surfaces have a high albedo.

In previous studies, authors did not consider the effect of reflected radiations upon the spacecraft. As this effect is much less than the direct radiation effect known as the photogravitational effect, it was generally neglected by authors in recent decades. If this effect is neglected, it means the primaries are considered as black-bodies, which is a contradiction to the fact that there is no planet in our solar system whose albedo is zero. No planet in our solar system is a black-body. The planets with their respective average albedos are as follows:

Planet	*Mercury*	*Venus*	*Earth*	*Mars*	*Jupiter*	*Saturn*	*Uranus*	*Neptune*
Albedo	0.12	0.75	0.30	0.16	0.34	0.34	0.30	0.29

In the light of all above facts, we have decided to develop a new model for the elliptic restricted three-body problem in which one primary is a source of radiation and the other a non-black-body. Albedo is studied by Anselmo et al. [3]; Nuss [28]; McInnes [24]; Bhanderi and Bak [5]; Pontus [29]; MacDonald and McInnes [21], etc. Idrisi [14, 15] considered the albedo effect in the circular restricted three-body problem considering the primaries as point masses, and found that the albedo not only effects the location of libration points but also the stability. Idrisi and Shahbaz Ullah (2017) studied the same problem with one more parameter, i.e. the oblateness of the smaller primary. Again, Idrisi [14, 15] studied the albedo effect on libration points in the circular restricted three-body problem taking into account the triaxiality of the smaller primary.

In this paper we are interested in investigating the albedo effect on the libration points in the elliptic restricted three-body problem (ER3BP) when the primaries are point masses but not black-bodies. The elliptic restricted three-body problem is a generalization of the circular restricted three-body problem in which two bodies with finite masses called primaries move around their center of mass in elliptic orbits under the influence of their mutual gravitational attraction, and an infinitesimal mass moving in the plane formed by the primaries is attracted and influenced by their motion without influencing them. There exist three collinear and two non-collinear libration points. The collinear libration points L_1, L_2 and L_3 are unstable for all values of mass parameter μ ($0 \le \mu \le ½$) while the non-collinear libration points $L_{4,5}$ are stable for a critical value of mass parameter $\mu < \mu_c = 0.03852$ …., Szehebely (1967). The ER3BP has been described in detailed by Danby [8], Bennett [4], Szebehely [36] and Markeev [22]. The influence of the eccentricity of the orbits of the primaries with or without radiation pressure on the existence and stability of the equilibrium points was studied by Gyorgyey [12], Kumar and Choudhry [17], Markellos et al. [23], Zimovshchikov and Tkhai [38], Ammar [2], Kumar and Ishwar [18], Narayan and Kumar [25], Singh and Umar [35], Narayan and Usha [26], Abd-El-Salam [1], Narayan et al. [27].

Some of the notable researches in photogravitational restricted three-body problem (PRTBP) are carried by Radzievskii [30]; Chernikov [7]; Schuerman D.W [32]; Simmons et. al. [34]; Kunitsyn and Tureshbaev [19]; Lukyanov [20]; Sharma [33]; Xuetang et.al. [37]; Grun et.al. [11]; Douskos [9]; Ershkov [10]; Katour et.al. [16] etc. This paper is divided into six sections. In section 2, the equations of motion are derived. The mean-motion of the primaries is obtained in section 3. The existence of non-collinear and collinear libration points is shown in section 4. In section 5, the stability of non-collinear and collinear libration points is discussed. Section 6 concludes the document.

Equations of Motion

Let m_1 and m_2 be the masses of the primaries such that m_1 is a source of radiation and $m_1 > m_2$. The primaries move in elliptic orbits around their center of mass O. An infinitesimal mass $m_3 << 1$, moves in the plane of motion of m_1 and m_2. The vectors from m_1, m_2 and O to m_3 are $\boldsymbol{r_1}$, $\boldsymbol{r_2}$ and $\boldsymbol{r}$, respectively. $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are the gravitational forces acting on m_3 due to m_1 and m_2, respectively. $\boldsymbol{F_p}$ is the force due to solar radiation pressure by m_1 on m_3 and $\boldsymbol{F_A}$ is the Albedo force due to solar radiation reflected by m_2 on m_3. Let the line joining m_1 and m_2 be taken as the X –axis and O, their center of mass, as origin. Let the line passing through O and perpendicular to OX and lying in the plane of motion of m_1 and m_2 be the Y –axis (Fig. 1). Let us consider a synodic system of co-ordinates $Oxyz$ initially coincident with the inertial system $OXYZ$, rotating with angular velocity $\dot{f}$ about the Z –axis (the z –axis is coincide with the Z –axis). We wish to find the equations of motion of m_3 using the terminology of Szebehely [36] in the synodic co-ordinate system with dimensionless variables such that the distance between the primaries is unity, the unit of time t is such that the gravitational constant $G = 1$ and the sum of the masses of the primaries is unity ($m_1 + m_2 = 1$).

The forces acting on m_3 due to m_1 and m_2 are $F_1\ (1 - F_p / F_1) = F_1\ (1 - \alpha)$ and $F_2\ (1 - F_A / F_2) = F_2\ (1 - \beta)$ respectively, where $\alpha = F_p / F_1 << 1$ and $\beta = F_A / F_2 << 1$. Also, α and β can be expressed as:

$$\alpha = \frac{\ell_1}{2\pi G m_1 c\sigma}; \beta = \frac{\ell_2}{2\pi G m_2 c\sigma};$$

where ℓ_1 is the luminosity of the larger primary m_1, $\ell_2$is the luminosity of smaller primary m_2, G is the gravitational constant, c is the speed of light and σ is mass per unit area of the infinitesimal mass m_3.

Now,

$$\frac{\beta}{\alpha} = \frac{m_1 \ell_2}{m_2 \ell_1} \Rightarrow \beta = \alpha\left(\frac{1-\mu}{\mu}\right)k. \tag{1}$$

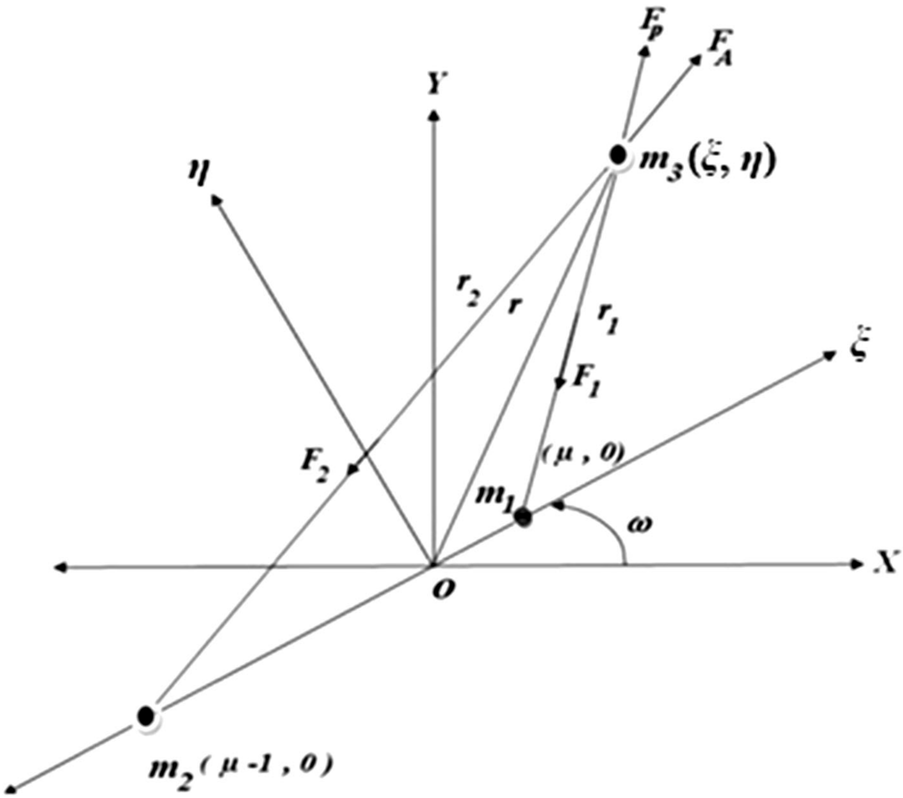

Fig. 1 The configuration of the ER3BP under albedo effect

Let $k = \frac{\ell_2}{\ell_1} =$ constant, $0 < \alpha < 1$, $0 < \beta < \alpha$ and $0 < k < 1$.

The equations of motion of infinitesimal mass $m_3 << m_1$, m_2 in terms of pulsating coordinates (ξ, η) are given by

$$\left.\begin{aligned} \xi'' - 2\eta' &= \Omega^*_\xi, \\ \eta'' + 2\xi' &= \Omega^*_\eta \end{aligned}\right\} \tag{2}$$

where

$$\Omega^* = \frac{1}{\sqrt{1-e^2}}\left[\frac{\xi^2+\eta^2}{2} + \frac{\Omega}{n^2}\right],$$

$$\Omega = \frac{(1-\mu)(1-\alpha)}{r_1} + \frac{\mu(1-\beta)}{r_2},$$

n mean-motion of the primaries,

e common eccentricity of elliptic orbit described by the primaries $(0 < e < 1)$,

$$r_1^2 = (\xi - \mu)^2 + \eta^2, \tag{3}$$

$$r_2^2 = (\xi + 1-\mu)^2 + \eta^2,$$
$$0 < \mu = \frac{m_2}{m_1 + m_2} < \frac{1}{2} \Rightarrow m_1 = 1-\mu; m_2 = \mu. \tag{4}$$

Mean-Motion of the Primaries

In the elliptic case, the distance between the primaries is $r = \frac{a(1-e^2)}{1+e\cos f}$, and the mean distance between the primaries is given by $\frac{1}{2\pi}\int_0^{2\pi} r\,df = \frac{a(1-e^2)}{\sqrt{1+e^2}}$, a is semi-major axis of the elliptic orbit of one primary around the other.

Since, the orbits of the primaries with respect to the center of mass have semi-major axes $a_1 = am_2$ and $a_2 = am_1$, and the same eccentricity [36], their equations of motion are given by $\frac{n^2 am_1(1-e^2)}{\sqrt{1+e^2}} = \frac{Gm_1m_2}{r^2}$ and $\frac{n^2 am_2(1-e^2)}{\sqrt{1+e^2}} = \frac{Gm_2m_1}{r^2}$.

Adding these equations yields $n^2 = \frac{\sqrt{1+e^2}}{a(1-e^2)}$, since, $m_1 + m_2 = 1$ and we choose the unit of time such that the gravitational constant $G = 1$. Considering only terms of e^2 and neglect their product, we have

$$n^2 = 1 + \frac{3}{2}e^2, (a = 1). \tag{5}$$

Libration Points

The libration points are the solutions of the Eqs. $\Omega_\xi^* = 0$ and $\Omega_\eta^* = 0$, i.e.,

$$\frac{1}{\sqrt{1-e^2}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi-\mu)(1-\alpha)}{r_1^3} + \frac{\mu(\xi + 1-\mu)(1-\beta)}{r_2^3}\right\}\right] = 0, \tag{6}$$

$$\text{and}\ \frac{\eta}{\sqrt{1-e^2}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)(1-\alpha)}{r_1^3} + \frac{\mu(1-\beta)}{r_2^3}\right\}\right] = 0. \tag{7}$$

In general, the libration points are the intersection of the contours Ω_ξ^* and Ω_η^*. As shown in the Fig. 2 (for $\mu = 0.1$, $\alpha = 0.01$, $k = 0.001$ and $e = 0.01$) there exist five libration points, three collinear (L_1, L_2, L_3) and two non-collinear (L_4, L_5).

Non-collinear Libration Points

The non-collinear libration points are the solution of the Eqs. $\Omega_\xi^* = 0$ and $\Omega_\eta^* = 0$, $\eta \neq 0$, i.e.,

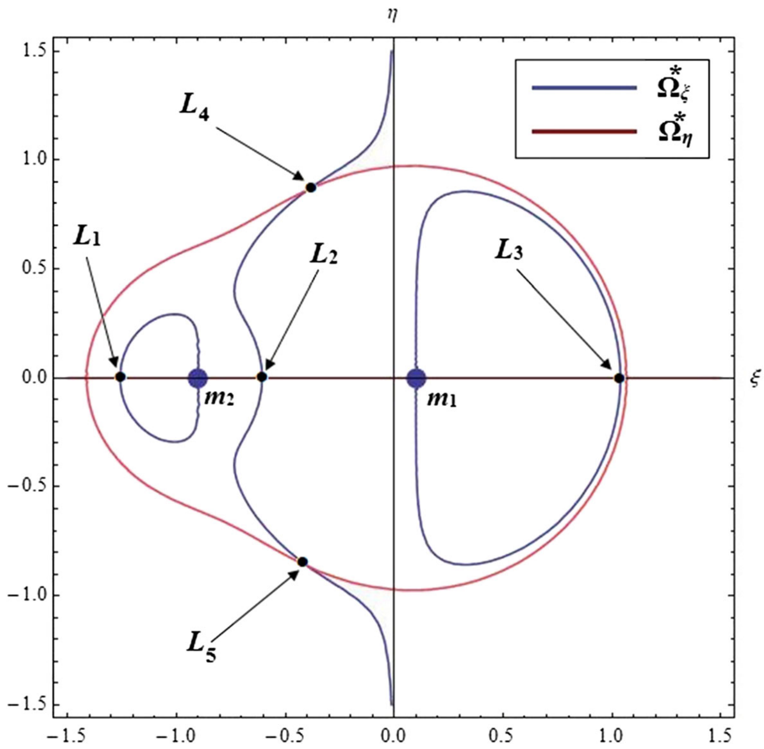

Fig. 2 Libration points; $\mu = 0.1$, $k = 0.001$, $\alpha = 0.01$, $e = 0.01$

$$\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi-\mu)(1-\alpha)}{r_1^3} + \frac{\mu(\xi+1-\mu)(1-\beta)}{r_2^3}\right\} = 0, \tag{8}$$

$$1 - \frac{1}{n^2}\left\{\frac{(1-\mu)(1-\alpha)}{r_1^3} + \frac{\mu(1-\beta)}{r_2^3}\right\} = 0. \tag{9}$$

The solution of Eqs. (8) and (9) is $r_1^3 = \frac{1-\alpha}{n^2}$ and $r_2^3 = \frac{1-\beta}{n^2}$.

Substitute these values of r_1 and r_2 into Eqs. (3) and (4) and solving for ξ and η, we have

$$\xi = \mu - \frac{1}{2} + \frac{(1-e^2)}{3}(\alpha-\beta),$$
$$\eta = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}e^2 - \frac{2}{9}(1-e^2)(\alpha+\beta)\right].$$

Using the relation (1), $\beta = \alpha\,(1-\mu)\,k\,/\,\mu$, we have

$$
\left.
\begin{aligned}
\xi &= \mu - \frac{1}{2} + \frac{1}{3}\alpha\left(1-e^2\right)\left[1 - \frac{(1-\mu)}{\mu}k\right], \\
\eta &= \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}e^2 - \frac{2}{9}\alpha\left(1-e^2\right)\left(1 + \frac{(1-\mu)}{\mu}k\right)\right].
\end{aligned}
\right\} \tag{10}
$$

Thus, there exist two non-collinear libration points $L_{4,5}$ forming a scalene triangle with the primaries as $r_1 \neq r_2$ and affected by Albedo and eccentricity of elliptic orbits of the primaries. From Eq. (8), the following results can be verified:

- For $e = 0$, the results are conform with Idrisi [14, 15].
- For $e = 0$ and $k = 0$, the results agree with Bhatnagar and Chawla [6].
- For $e = 0$ and $\alpha = 0$, the results agree with Szebehely [36].

In the Figs. 3 and 4, it is shown that the coordinates of L_4 (ξ, η) vary with respect to e and α. From Fig. 3, it is observed that as e increases ξ changes little while η is displaced downward; the point L_4 moves toward the ξ-axis (Fig. 5). Hence, the shape of the triangle formed by L_4 with the primaries reduces. From Fig. 4, it has been observed that as α increases, ξ rapidly move towards the center of mass while the value of η decreases uniformly. Thus, the triangle formed by L_4 with the primaries reduces, and L_4 moves toward the η-axis (Fig. 6). The location of libration points $L_{4,5}$ for fixed values of k (0.001) and μ(0.1), and different values of e and α are given in Table 1.

Collinear Libration Points

The collinear libration points are the solutions of Eqs. $\Omega^*_\xi = 0$ and $\Omega^*_\eta = 0$, $\eta = 0$, i.e.,

$$
f(\xi) = n^2\xi - \frac{(1-\mu)(\xi-\mu)(1-\alpha)}{r_1^3} - \frac{\mu(\xi+1-\mu)(1-\beta)}{r_2^3} = 0, \tag{11}
$$

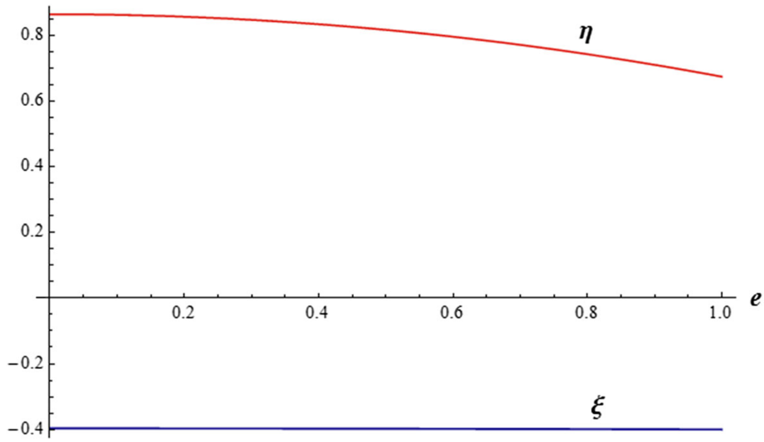

Fig. 3 e versus ξ and η; $\alpha = 0.01$, $k = 0.001$, $\mu = 0.1$.

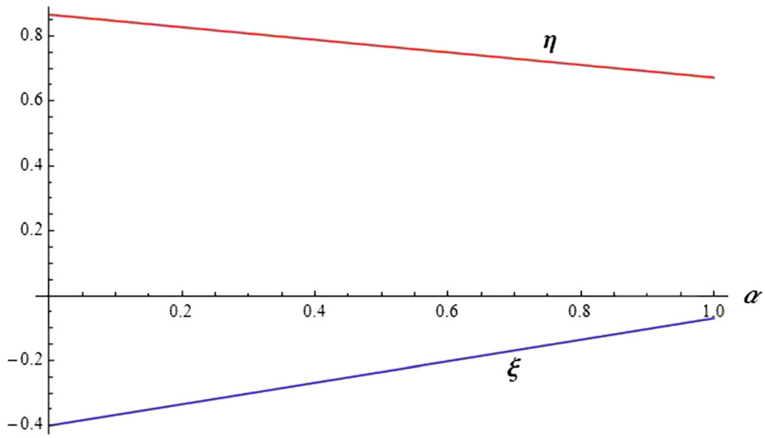

Fig. 4 α versus ξ and η; $e = 0.01$, $k = 0.001$, $\mu = 0.1$.

where $r_i = |\ \xi - \xi_i\ |$, $i = 1, 2$. The solution is a fifth degree equation in ξ.

On plotting the function $f(\xi)$ for $\mu = 0.1$, $\alpha = 0.01$, $k = 0.001$ and $e = 0.01$, three collinear libration points, L_i (ξ_i, 0), $i = 1, 2, 3$ are observed in the intervals $(-\infty, \mu - 1)$, $(\mu - 1, \mu)$ and (μ, ∞), respectively. For other values of μ, α, k and e, the result is same as there exist only three collinear libration points L_i (ξ_i, 0), $i = 1, 2, 3$ in the intervals $(-\infty, \mu - 1)$, $(\mu - 1, \mu)$ and (μ, ∞), respectively. Thus, there exist only three collinear libration points (Fig. 7).

Eq. (11) determines the location of the collinear libration points $L_1(\xi_1, 0)$, L_2 $(\xi_2, 0)$ and L_3 $(\xi_3, 0)$ to lie in the intervals $(-\infty, \mu - 1)$, $(\mu - 1, \mu)$ and (μ, ∞), respectively, where

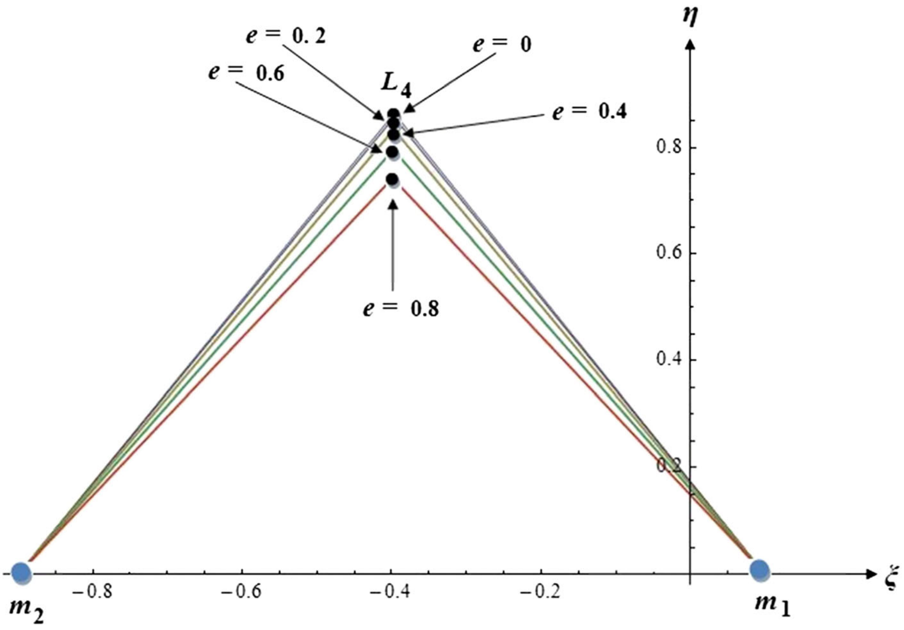

Fig. 5 L_4 versus e; $\alpha = 0.01$, $k = 0.001$, $\mu = 0.1$

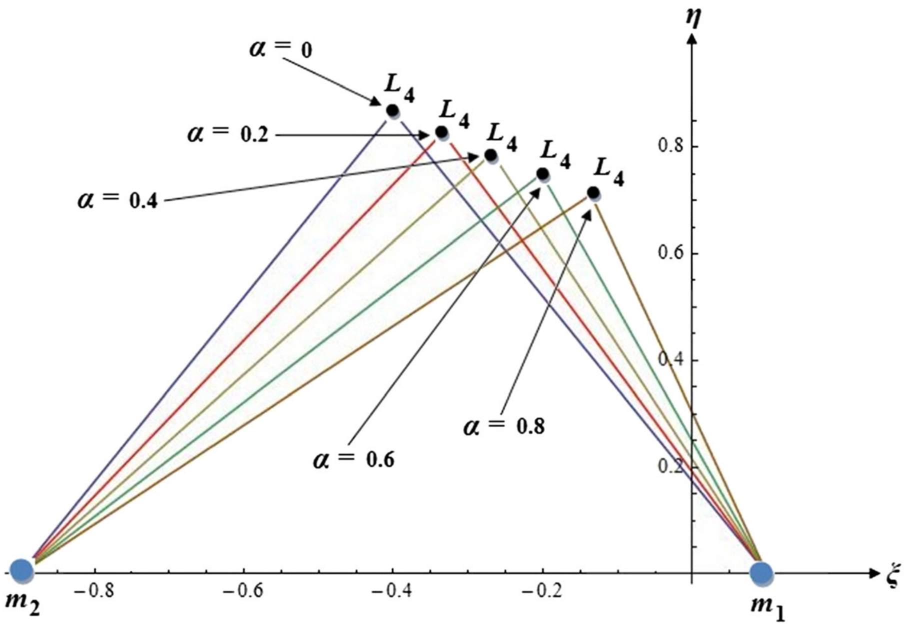

Fig. 6 L_4 versus α; $e = 0.01$, $k = 0.001$, $\mu = 0.1$

$$\left.\begin{aligned}\xi_1 &= \mu - 1 - \varepsilon_1,\\ \xi_2 &= \mu - 1 + \varepsilon_2,\\ \xi_3 &= \mu + \varepsilon_3.\end{aligned}\right\} \tag{12}$$

Since the libration point L_1 lies in the interval $(-\infty,\ \mu - 1)$, i.e., left of the smaller primary, we have $r_1 = \mu - \xi_1$ and $r_2 = \mu - 1 - \xi_1$, which when substituted in Eq. (11), give

Table 1 Non-collinear libration points $L_{4,5}$ $(\xi, \pm\eta)$

e	$\alpha = 0.01$, $k = 0.001$, $\mu = 0.1$		α	$e = 0.01$, $k = 0.001$, $\mu = 0.1$	
	ξ	$\pm\eta$		ξ	$\pm\eta$
0.0	−0.39669666	0.864083582	0.0	−0.40000000	0.866006158
0.1	−0.39672971	0.862178499	0.1	−0.36696997	0.846589886
0.2	−0.39682881	0.856463251	0.2	−0.33393994	0.827173614
0.3	−0.39699396	0.846937838	0.3	−0.30090991	0.807757342
0.4	−0.39722521	0.833602259	0.4	−0.26787988	0.788341069
0.5	−0.39752251	0.816456515	0.5	−0.23484985	0.768924797
0.6	−0.39788586	0.795500605	0.6	−0.20181982	0.749508525
0.7	−0.39831531	0.770734531	0.7	−0.16878979	0.730092253
0.8	−0.39881081	0.742158290	0.8	−0.13575976	0.710675981
0.9	−0.39937236	0.709771885	0.9	−0.10272973	0.691259708

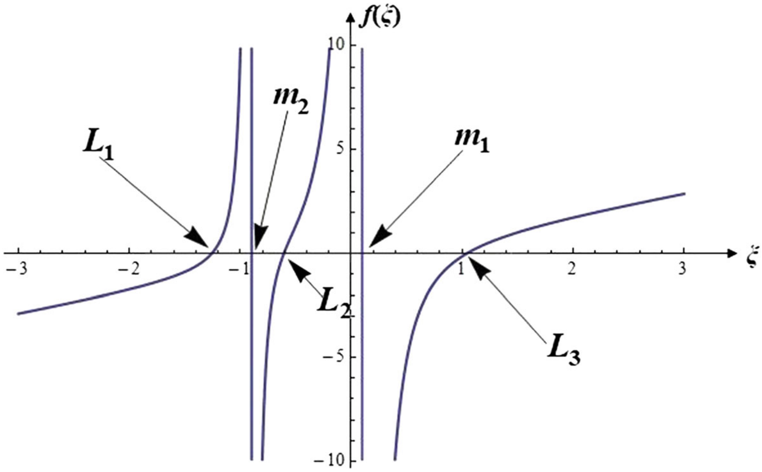

Fig. 7 Collinear libration points L_i (ξ_i, 0), i = 1, 2, 3

$$n^2\xi_1+\frac{(1-\mu)(\xi_1-\mu)(1-\alpha)}{r_1^3}+\frac{\mu(\xi_1+1-\mu)(1-\beta)}{r_2^3}=0 \tag{13}$$

Similarly, for L_2 (ξ 2, 0) and L_3 (ξ 3, 0), Eq. (11) becomes.

$$n^2\xi_2+\frac{(1-\mu)(\xi_2-\mu)(1-\alpha)}{r_1^3}-\frac{\mu(\xi_2+1-\mu)(1-\beta)}{r_2^3}=0 \tag{14}$$

Table 2 Collinear Libration Points L_i (ξ_i, 0), i = 1, 2, 3

e	α = 0.01, k = 0.001, μ = 0.1			α	e = 0.01, k = 0.001, μ = 0.1		
	ξ_1	ξ_2	ξ_3		ξ_1	ξ_2	ξ_3
0.0	−1.25888	−0.60777	1.03841	0.0	−1.25967	−0.60902	1.04156
0.1	−1.25579	−0.60712	1.03357	0.1	−1.25157	−0.59561	1.00855
0.2	−1.24695	−0.60517	1.01959	0.2	−1.24368	−0.58018	0.97301
0.3	−1.23351	−0.60191	0.99798	0.3	−1.23602	−0.56223	0.93439
0.4	−1.21696	−0.59734	0.93995	0.4	−1.22859	−0.54101	0.89193
0.5	−1.19878	−0.59145	0.90741	0.5	−1.22138	−0.51536	0.84448
0.6	−1.18018	−0.58427	0.87447	0.6	−1.21441	−0.48355	0.79026
0.7	−1.16201	−0.57584	0.84215	0.7	−1.20764	−0.44251	0.72614
0.8	−1.14478	−0.56623	0.81104	0.8	−1.20111	−0.38604	0.64576
0.9	−1.12879	−0.55555	0.78151	0.9	−1.19481	−0.29725	0.53155

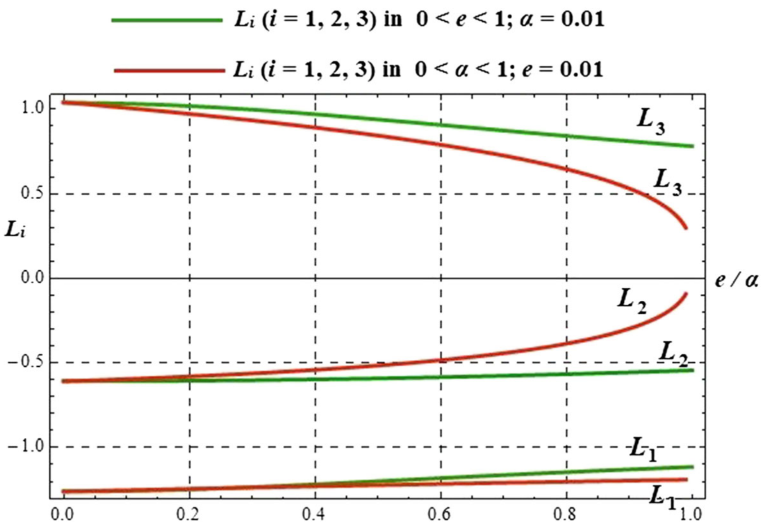

Fig. 8 Collinear libration points for $k = 0.001$ and $\mu = 0.1$

$$n^2\xi_3 - \frac{(1-\mu)(\xi_3-\mu)(1-\alpha)}{r_1^3} - \frac{\mu(\xi_3+1-\mu)(1-\beta)}{r_2^3} = 0 \tag{15}$$

The solutions of Eqs. (13), (14) and (15) are given in Table 2 for on specific values of the parameters appearing in the first line.

In Fig. 8, the green curves are the collinear libration point values of ξ_i ($i = 1, 2, 3$) in the interval $0 < e < 1$; $\alpha = 0.01$ and red curves show the collinear libration point values of ξ_i ($i = 1, 2, 3$) in $0 < \alpha < 1$; $e = 0.01$; for $k = 0.001$; $\mu = 0.1$. It is observed that the first collinear libration point L_1 always lies to the left of the primary m_2, the second libration point L_2, lies between the center of mass of the primaries O and m_1, and the third libration point L_3 lies to the right of the primary m_1. It is also observed that the libration point L_1 moves toward m_2, L_2 moves toward center of mass and L_3 moves toward m_1 in $0 < e < 1$ and $0 < \alpha < 1$.

Stability of Libration Points

The variational equations are obtained by substituting $\xi = \xi_0 + \varepsilon$ and $\eta = \eta_0 + \delta$ in the equations of motion (2), where (ξ_o, η_o) are the coordinates of a particular libration point and $\varepsilon, \delta << 1$, i.e.,

$$\left.\begin{aligned} \varepsilon'' - 2\delta' &= \varepsilon\,\Omega^{*0}_{\xi\xi} + \delta\,\Omega^{*0}_{\xi\eta}, \\ \delta'' + 2\varepsilon' &= \varepsilon\,\Omega^{*0}_{\xi\eta} + \delta\,\Omega^{*0}_{\eta\eta}. \end{aligned}\right\} \tag{16}$$

Here we have taken only linear terms in ε and δ. The subscript in Ω^* indicates the second partial derivative of Ω^* and superscript 0 indicates that the derivative is to be evaluated at the libration point (ξ_0, η_0). The characteristic equation corresponding to Eq. (16) is

$$\lambda^4 + \left(4-\Omega_{\xi\xi}^{*0}-\Omega_{\eta\eta}^{*0}\right)\lambda^2 + \Omega_{\xi\xi}^{*0}\,\Omega_{\eta\eta}^{*0}-\left(\Omega_{\xi\eta}^{*0}\right)^2 = 0. \tag{17}$$

Letting $\lambda^2 = \Pi$, Eq. (17) becomes

$$\Pi^2 + q_1\,\Pi + q_2 = 0, \tag{18}$$

which is a quadratic equation in Π, and its roots are given by

$$\Pi_{1,2} = \frac{1}{2}\left(-q_1 \pm \sqrt{D}\right) \tag{19}$$

where $q_1 = 4-\Omega_{\xi\xi}^{*0}-\Omega_{\eta\eta}^{*0};\ q_2 = \Omega_{\xi\xi}^{*0}\,\Omega_{\eta\eta}^{*0}-\left(\Omega_{\xi\eta}^{*0}\right)^2;\ D = q_1^2-4q_2.$

Therefore, the roots of characteristic Eq. (17) are given by, $\lambda_{1,2} = \pm\sqrt{\Pi_1}$ and $\lambda_{3,4} = \pm\sqrt{\Pi_2}$.

The libration point (ξ_0, η_0) is said to be stable if $\Delta_1 < 0$ and $\Delta_2 < 0$ or $D \geq 0$.

Stability of Non-collinear Libration Points

At the non-collinear libration point $L_4(\xi_0,\ \eta_0)$,

$$\Omega_{\xi\xi}^{*0} = \frac{3}{4}\left[1-\frac{2}{3}(1-3\mu)\alpha+\frac{2}{3}(2-3\mu)\beta\right]-\frac{3}{8}\left[1-\frac{2}{3}(5-11\mu)\alpha+\frac{2}{3}(6-11\mu)\beta\right]e^2,$$

$$\Omega_{\xi\eta}^{*0} = \frac{3\sqrt{3}}{2}\left[\mu-\frac{1}{2}+\frac{1}{9}(1+\mu)\alpha-\frac{1}{9}(2-\mu)\beta\right]-\frac{13}{4\sqrt{3}}\left[\mu-\frac{1}{2}+\frac{1}{39}(27-5\mu)\alpha-\frac{1}{39}(22+5\mu)\beta\right]e^2,$$

$$\Omega_{\eta\eta}^{*0} = \frac{9}{4}+\frac{1}{2}(1-3\mu)\alpha-\frac{1}{2}(2-3\mu)\beta-\frac{1}{4}\left[\frac{17}{2}-\frac{1}{9}(11+51\mu)\alpha-\frac{1}{9}(62-51\mu)\beta\right]e^2.$$

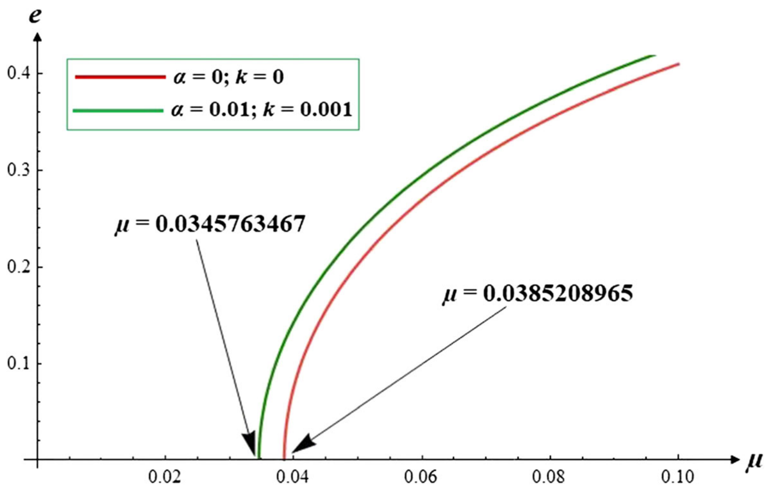

Fig. 9 μ versus e

The motion near the Libration point (ξ_0, η_0) is said to be bounded if $D \geq 0$ i.e.

$$1-27\mu+27\mu^2+6(1-\mu)k\alpha-3\frac{k(1-m)\alpha}{m}-6m\alpha+\left(5+39\mu-39\mu^2-\frac{28}{9}\alpha-12k(1-\mu)\alpha+\frac{31}{18}\frac{k(1-\mu)\alpha}{\mu}+2\mu\alpha\right)e^2\geq 0 \tag{20}$$

As shown in the Fig. 9, for $\alpha=0$, $k=0$, e is defined in the interval $\mu=$ 0.03852 … $\leq e \leq \mu=$ 0.5 where $\mu=$ 0.03852 … is the value of critical mass parameter in classical case and for $\alpha=0.01$, $k=0.001$, e is defined in the interval $\mu=$ 0.03457 … $\leq e \leq \mu=$ 0.5. Thus, the domain of e expands as α increases.

For $\alpha=0$ and $e=0$, μ_c = 0.03852 …. is the critical value of mass parameter in classical case [36]. When $\alpha \neq 0$, $e \neq 0$, we suppose that $\mu_c=\mu_o+p_1\,\alpha+p_2\,e^2$ as the root of the Eq. (20), where, μ_o = 0.0385208965 … and p_1, p_2 are to be determined in a manner such that $D=0$. Therefore, we have

$$\begin{aligned} p_1 &= -\frac{k-3k\mu_o+2\mu_o^2+2k\mu_o^2}{9\mu_o(1-2\mu_o)}, \\ p_2 &= \frac{5+39\mu_o-39\mu_o^2}{27(1-2\mu_o)}. \end{aligned} \tag{21}$$

$\therefore\ \mu_c = 0.0385208965 \ldots - (0.00891747 + 2.78224\,k)\,\alpha + 0.258607\,e^2$.

As shown in the Fig. 10, $\Pi_{1,2}<0$ for $\mu \leq \mu_c$. Thus the eigenvalues of the characteristic Eq. (17) are given by $\lambda_{1,2}=\pm i\sqrt{\Pi_1}, \lambda_{3,4}=\pm i\sqrt{\Pi_2}$. Therefore, the non-collinear libration points $L_{4,5}$ are periodic and bounded and hence stable for the critical mass parameter $\mu \leq \mu_c$, where μ_c is defined in Eq. (21). It is also observed that as the eccentricity e increases the value of the critical mass parameter μ_c increases exponentially (Fig. 11) and as α increase, μ_c decreases uniformly (Fig. 12).

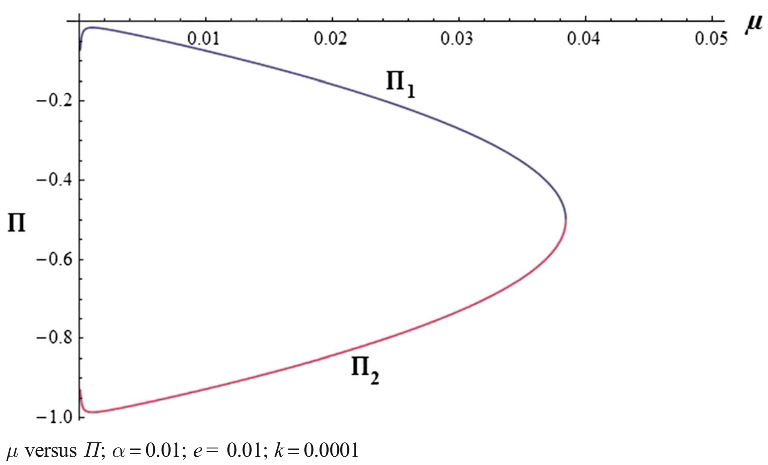

Fig. 10 μ versus Π; $\alpha=0.01$; $e=$ 0.01; $k=0.0001$

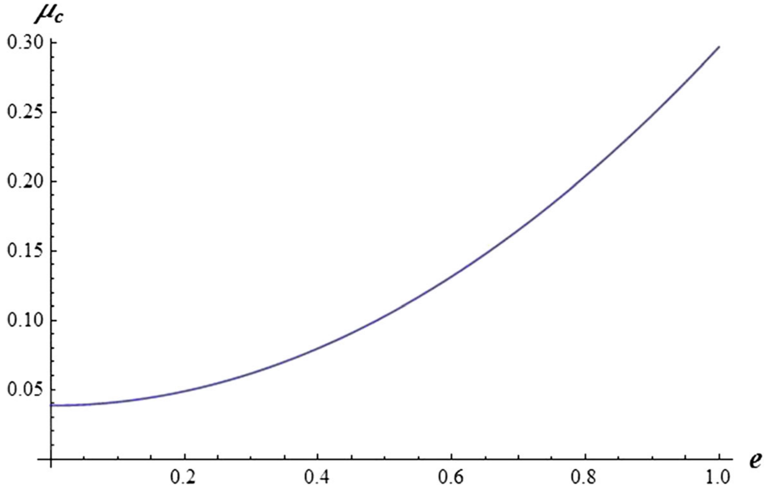

Fig. 11 e versus μ_c; $\alpha = 0.01$; $k = 0.0001$

Stability of Collinear Libration Points

First we consider the point lying in the interval $(\mu - 2, \mu - 1)$. For this point, $r_2 < 1, r_1 > 1$ and

$$\left.\begin{aligned} \Omega^{*0}_{\xi\xi} &= \frac{1}{\sqrt{1-e^2}}\left[1+\frac{1}{n^2}\left\{\frac{2(1-\mu)(1-\alpha)}{r_1^3}+\frac{2\mu(1-\beta)}{r_2^3}\right\}\right] > 0, \\ \Omega^{*0}_{\xi\eta} &= 0, \\ \Omega^{*0}_{\eta\eta} &= \frac{1}{\sqrt{1-e^2}}\left[1-\frac{1}{n^2}\left\{\frac{(1-\mu)(1-\alpha)}{r_1^3}+\frac{\mu(1-\beta)}{r_2^3}\right\}\right] < 0. \end{aligned}\right\} \quad (22)$$

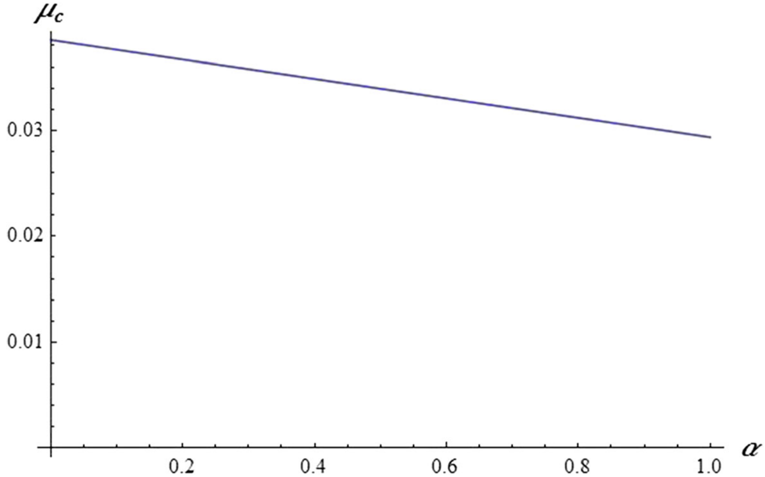

Fig. 12 α versus μ_c; $e = 0.01$; $k = 0.0001$

Similarly, for the points lying in the interval $(\mu - 1{,}0)$ and $(\mu, \mu + 1)$, $\Omega^{*0}_{\xi\xi} > 0, \Omega^{*0}_{\xi\eta} = 0$ and $\Omega^{*0}_{\eta\eta} < 0$. Since the discriminant of Eq. (20) is positive and the four roots of the characteristic Eq. (17) can be written as $\lambda_{1,2} = \pm s$ and $\lambda_{3,4} = \pm\, it$ (s and t are real), the motion around the collinear libration points is unbounded, and consequently, the collinear libration points are unstable.

Conclusion

In the present paper, the existence and stability of libration points in the elliptic restricted three-body problem has been studied under albedo effect. The equations of motion including albedo effect are derived as Eq. (2). For $\beta = 0$, the problem reduces to the photogravitational elliptic restricted three-body problem where only the more massive primary is a source of radiation. For $\beta = 0$ and $e = 0$, the problem becomes the photogravitational circular restricted three-body problem where the more massive primary is a source of radiation. It is found that there exist five libration points, three collinear and two non-collinear. It is observed that the first collinear libration point L_1 always lies to the left of the primary m_2, the second libration point L_2 lies between the center of mass of the primaries O and m_1, and the third libration point L_3 lies to the right of the primary m_1.It is also observed that the libration point L_1 moves toward m_2, L_2 moves toward the center of mass and L_3 moves toward m_1 in $0 < e < 1$ and $0 < \alpha < 1$ (Fig. 8). There exist two non-collinear libration points $L_{4,5}$, and From Fig. 3, it is observed that as e increases ξ varies little while η is displaced downward. The point L_4 moves toward the ξ-axis (Fig. 5) and hence the shape of the triangle formed by L_4 with the primaries reduces. From Fig. 4, it has been observed that as α increases, ξ rapidly move towards the center of mass, while the value of η decreases uniformly. Thus, the triangle formed by L_4 with the primaries reduces and L_4 moves toward the η-axis (Fig. 6). The non-collinear libration points $L_{4,5}$ are not periodic and bounded and hence stable for the critical mass parameter $\mu \le \mu_c$, where μ_c is defined by $\mu_c = 0.0385208965 \ldots - (0.00891747 + 2.78224\, k)\, \alpha + 0.258607\, e^2$ while the collinear libration points are unstable.

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Compliance with Ethical Standards

Conflict of Interests The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. Abd-El-Salam, F.A.: Stability of triangular equilibrium points in the elliptic restricted three-body problem with oblate and triaxial primaries. Astrophys. Space Sci. **357**, 15 (2015)
2. Ammar, M.K.: The effect of solar radiation pressure on the Lagrangian points in the elliptic restricted three-body problem. Astrophys. Space Sci. **313**(4), 393–408 (2008)
3. Anselmo, L., Farinella, P., Milani, A., Nobili, A.M.: Effects of the earth reflected sunlight on the orbit of the LAGEOs satellite. Astron. Astrophys. **117**(1), 1–3 (1983)

4. Bennett, A.: Characteristic exponents of the five equilibrium solutions in the elliptically restricted problem. Icarus. **4**(2), 177–187 (1965)
5. Bhanderi, D.D.V., Bak, T.: Modeling Earth Albedo Satellites in Earth Orbit. AIAA Guidance, Navigation and Control Proceedings, San Franciso (2005)
6. Bhatnagar, K.B., Chawla, J.M.: A study of Lagrangian points in the photogravitational restricted three-body problem. Indian J. Pure Appl. Math. **10**(11), 1443–1451 (1979)
7. Chernikov, Y.A.: The photogravitational restricted three body problem. Sov. Astron. AJ. **14**, 176–181 (1970)
8. Danby, J.M.A.: Stability of the triangular points in the elliptic restricted problem of three bodies. Astron. J. **69**(2), 165–172 (1964)
9. Douskos, C.N.: Collinear equilibrium points of Hill's problem with radiation and oblateness and their fractal basins of attraction. Astrophys. Space Sci. **326**, 263–271 (2010)
10. Ershkov, S.V.: The Yarkovsky effect in generalized photogravitational 3-body problem. Planet. Space Sci. **70**(1), 221–223 (2012)
11. Grün, E., Gustafson, B.A.S., Dermott, S., Fechtig, H.: Interplanetary Dust. Springer, Berlin (2001)
12. Gyorgyey, J.: On the nonlinear stability of motion around L3 in the elliptic restricted problem of three bodies. Celest. Mech. Dyn. Astron. **36**(3), 281–285 (1985)
13. Harris, M., Lyle, R.: Spacecraft Radiation Torque. NASA Space Vehicle Design Criteria (Guidance and Control) NASA SP-8027, 1–37 (1969)
14. Idrisi, M.J.: A study of libration points in CR3BP under albedo effect. Int. J. Adv. Astron. **5**(1), 01–06 (2017a)
15. Idrisi, M. Javed, Ullah, M. Shahbaz: Restricted three-body problem with albedo effect when smaller primary is an oblate spheroid. The Global Journal of Science Frontier Research, 17(5), 09-24 (2017).
16. Katour, D.A., Abd El-Salam, F.A., Shaker, M.O.: Relativistic restricted three body problem with oblateness and photo-gravitational corrections to triangular equilibrium points. Astrophys. Space Sci. **351**, 143–149 (2014)
17. Kumar, V., Choudhry, R.K.: Non linear stability of the triangular libration points for the photogravitational elliptic restricted problem of three bodies. Celest. Mech. Dyn. Astron. **48**(4), 299–317 (1990)
18. Kumar, S., Ishwar, B.: Solutions of generalized Photogravitational elliptic restricted three-body problem. AIP Conference Proceedings. **1146**, 456 (2009)
19. Kunitsyn, A.L., Tureshbaev, A.T.: On the collinear libration points in the photogravitational three-body problem. Celest. Mech. **35**(2), 105–112 (1985)
20. Lukyanov, L.G.: On the family of the libration points in the restricted photogravitational three-body problem. Astron. Zh. **65**, 422–432 (1988)
21. MacDonald, M., McInnes, C.: Solar sail science mission applications and advancement. Adv. Space Res. **48**, 1702–1716 (2011)
22. Markeev, A.P.: Libration points in celestial mechanics and astrodynamics. Moscow, Izdatl'stvo Nauka, 312 p. in Russian (1978)
23. Markellos, V.V., Perdios, E., Labrapoulou, P.: Linear stability of the triangular equilibrium points in the photogravitational elliptic restricted problem. Astrophys. Space Sci. **194**(2), 207–213 (1992)
24. McInnes, A.I.S.: Strategies for Solar Sail Mission Design in the Circular Restricted Three-Body Problem. Master's Thesis, Purdue University (2000)
25. Narayan, A., Kumar, R.C.: Effects of Photogravitational and Oblantensss on the triangular Lagrangian points in the elliptical restricted three body problem. Int. J. Pure Appl. Math. **68**(2), 201–224 (2011)
26. Narayan, A., Usha, T.: Stability of triangular equilibrium points in the elliptic restricted problem of three bodies with radiating and triaxial primaries. Astrophys. Space Sci. **351**(1), 135–142 (2014)
27. Narayan, A., Pandey, K.K., Shrivastava, S.K.: Characteristic exponents of the triangular solution in the elliptical restricted three-body problem under radiating and triaxial primaries. Int. J. Adv. Astron. **5**(1), 12–18 (2017)
28. Nuss, J.S.: The Use of Solar Sails in the Circular Restricted Problem of Three Bodies. Master's Thesis, Purdue University (1998)
29. Pontus, A.: Attitude estimation from magnetometer and earth-albedo-corrected coarse Sun sensor measurements. J. Acta. Astonaut. **56**(1–2), (2005)
30. Radzievskii, V.V.: The restricted problem of three bodies taking account of light pressure. Akad. Nauk. USSR, Astron. Journal, **27**, 250 (1950)
31. Rocco, E.M.: Evaluation of the Terrestial albedo applied to some scientific missions. Space Sci. Rev. **151**(1–3), 135–147 (2009)

32. Schuerman, D.W.: The restricted three body problem including radiation pressure. Astrophys. J. **238**, 337–342 (1980)
33. Sharma, R.K.: The linear stability of libration points of the photogravitational restricted three-body problem when the smaller primary is an oblate spheroid. Astrophys. Space Sci. **135**(2), 271–281 (1987)
34. Simmons, J.F.L., McDonald, A.J.C., Brown, J.C.: The restricted 3-body problem with radiation pressure. Celest. Mech. Dyn. Astron. **35**, 145–187 (1985)
35. Singh, J., Umar, A.: On the stability of triangular equilibrium points in the elliptic R3BP under radiating and oblate primaries. Astrophys. Space Sci. **341**, 349–358 (2012)
36. Szebehely, V.: Theory of Orbits, the Restricted Problem of Three Bodies. Academic Press, New York and London (1967)
37. Xuetang, Z.H.E.N.G., Lizhong, Y.U.: Photogravitationally restricted three-body problem and coplanar libration point. Chin. Phys. Lett. **10**(1), 61–64 (1993)
38. Zimovshchikov, A.S., Tkhai, V.N.: Instability of Libration points and resonance phenomena in the Photogravitational elliptic restricted three-body problem. Sol. Syst. Res. **38**, 155–164 (2004)