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TECHNICAL NOTE

Relative Motion Dynamics with Arbitrary Perturbations in the Local-Vertical Local-Horizon Reference Frame

Giovanni Franzini[1,2] · **Mario Innocenti**[1]

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Abstract

Accurate models for spacecraft relative dynamics are essential to the design of high-precision control and long-term estimation. In the literature, numerous analytical models have been proposed that describe, with different degrees of accuracy, the relative dynamics in presence of specific perturbations. These models however may be limited for future missions, like long-baseline formation flying, that will require analytical formulations capable of considering the influence of different perturbations at the same time, in order to meet the high-demanding mission requirements. The aim of this Note is to provide a general and flexible framework for the inclusion of arbitrary perturbations into the equations of relative motion. The framework decouples the perturbations influence from the Keplerian component of the motion, so that designers can include the perturbations of interest according to the mission scenario. In the future, the proposed framework might be used by autonomous spacecraft for real-time reconfiguration of the guidance system in response to a changed operating scenario. The inclusion of arbitrary zonal harmonics perturbations and atmospheric drag into our framework is discussed.

Keywords Relative motion dynamics · Local-vertical local-horizon frame · Perturbation

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✉ Giovanni Franzini
giovanni.franzini@for.unipi.it; franzigi@utrc.utc.com

Mario Innocenti
mario.innocenti@unipi.it

[1] Department of Information Engineering, University of Pisa, Largo Lucio Lazzarino 1, 56122, Pisa, Italy

[2] Present address: United Technologies Research Centre Ireland, 2nd Floor, Penrose Wharf Business Centre, Penrose Quay, Cork City, T23 XN53, Ireland

Introduction

Relative motion dynamic models are fundamental for the study of space operations involving coordination of two or more spacecraft, such as formation flying, rendezvous and in-orbit assembly of orbital structures. Accurate prediction and analysis of the evolution of relative position and velocity, especially in the long term, require accurate models that include several orbital perturbations that may influence the spacecraft dynamics.

Many models describing the relative motion in the *local-vertical local-horizon* (LVLH) frame under the influence of different perturbations have been proposed in the literature. Kechichian [7] presented an exact relative dynamics model under J_2 and atmospheric drag perturbations, applying Newtonian mechanics techniques. A set of equations including a linear model for atmospheric drag was proposed in [5]. A linearized high-fidelity model with J_2 perturbation was presented in [9]. Pluym and Damaren [8] proposed a relative motion model which includes a second order approximation of the J_2 perturbation together with a third order approximation of the differential gravity (i.e. the difference between the two spacecraft gravity acceleration). Xu and Wang [11] used Lagrangian mechanic techniques to develop an exact model of relative motion subject to J_2 perturbation. In [12] arbitrary zonal harmonics perturbation was considered. Xu et al. [13] introduced lunar perturbation in the relative motion dynamics description. A set of relative motion equations taking into account third-body perturbation is also presented in [2], developed by means of Lagrangian mechanics. A recent comprehensive survey on relative motion models, which includes also equation sets developed in frames different from the LVLH, is presented in [10].

Even if some of the previously cited works provide exact models, the authors only considered a specific subset of perturbations. Extension of their results to include other perturbations that may act on the spacecraft and influence the relative motion is not straightforward. To this end a general framework for relative motion analysis in the LVLH frame that can include different perturbations is needed. Casotto [3] recently proposed a general set of equations for describing the relative dynamics in presence of arbitrary perturbations. However, the influence of the orbital perturbation on the dynamics, and in particular on the precession of the LVLH frame, is not evident in the equations developed, since their contribution is not decoupled from the Keplerian component of the motion. As a consequence, perturbations influence cannot be properly analyzed. Moreover, since the author considers perturbations characterized by time-invariant parameters, orbital perturbations with parameters varying with time, as in the case of time-varying atmosphere density or lunar perturbation in lower Earth orbits, cannot be included.

The aim of this Note is to provide a general set of differential equations for the exact description of relative motion dynamics in presence of arbitrary orbital perturbations. Unlike past models, the proposed model offers a structured approach for perturbations inclusion, so that perturbations of interest can be readily introduced according to the mission scenario. Such a capability is possible since the perturbations effect on the relative dynamics is separated from the Keplerian component of the motion, as opposed to [3]. This reconfigurability feature makes the model appealing for motion prediction and design of model-based nonlinear controllers and observers. In particular, different controllers and observers can be designed using the same set of equations but with different sets of perturbations included. In this way, the controllers and observers obtained can address different mission scenario and can be activated depending on the required accuracy, available computational resources, perturbations to be considered in the current mission phase and so on. In addition, in the proposed model perturbations effects can be analyzed separately. Differently to previous works [11–13], the equation set proposed was developed using only vector calculus and simple geometric relations. As examples, we show how arbitrary zonal harmonics perturbation and atmospheric drag can be included in the proposed model. The equation set proposed in this Note can potentially accommodate many other perturbations, like solar radiation pressure, third-body gravitational effects, etc. Inclusion of these perturbations in the equation set developed is not discussed in the following. Nevertheless, general guidelines for inclusion of their expression in the inertial frame are provided.

General Equations of Relative Motion

Consider a passive spacecraft, in the following referred to as *chief*, orbiting a *primary body*, and a spacecraft which is able to maneuver, denoted as *deputy*. The dynamics of the two spacecraft is respectively given by

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{d}_c \tag{1}$$

$$\frac{\mathrm{d}^2\boldsymbol{r}_d}{\mathrm{d}t^2} = -\frac{\mu}{r_d^3}\boldsymbol{r}_d + \boldsymbol{d}_d + \boldsymbol{u} \tag{2}$$

where μ is the primary body *gravitational parameter*, $\boldsymbol{r}$ is the spacecraft position with respect to the primary body center of mass, $\boldsymbol{d}$ quantifies in terms of acceleration the *perturbations* acting on the spacecraft, $\boldsymbol{u}$ is the deputy control vector [6]. The subscript d denotes deputy's parameters. When the subscript is dropped, the parameter is referred to the chief, except for the perturbations. Vectors are denoted using the bold font, e.g. $\boldsymbol{v}$, and their Euclidean norm is denoted using the normal font, e.g. v. Unit vectors are indicated using an hat, e.g. $\hat{\boldsymbol{v}}$.

We define the local-vertical local-horizon frame $\mathcal{F}_l$, centered in the chief center of mass, as follows:

$$\mathcal{F}_l = \left\{\hat{\boldsymbol{i}},\ \hat{\boldsymbol{j}},\ \hat{\boldsymbol{k}}\right\} = \left\{\frac{\boldsymbol{r}}{r},\ \hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}},\ \frac{\boldsymbol{h}}{h}\right\} \tag{3}$$

where $\boldsymbol{h} = h\hat{\boldsymbol{k}} = \boldsymbol{r} \times \boldsymbol{v}$ is the chief *massless angular momentum* and $\boldsymbol{v}$ its velocity. We also introduce an inertial frame as

$$\mathcal{F}_i = \left\{\hat{\boldsymbol{I}}, \hat{\boldsymbol{J}}, \hat{\boldsymbol{K}}\right\} \tag{4}$$

centered on the primary body center of mass. The unit vector $\hat{\boldsymbol{K}}$ is aligned with the primary body angular velocity vector.

In the following, two different operators will be used for distinguishing between the reference frame in which the time-derivatives are taken. Time-derivative in the inertial frame will be denoted using Leibniz's notation, i.e. with the operator $\mathrm{d}/\mathrm{d}t$, whereas the time-derivative in the LVLH frame will be denoted using Newton's notation, that is using an upper dot.

The coordinate change matrix from the inertial frame to the LVLH frame, $\boldsymbol{C}_i^l : \mathcal{F}_i \rightarrow \mathcal{F}_l$, is a function of the chief orbital elements (i, Ω, θ), respectively the chief orbit *inclination*, *right ascension of the ascending node* and *true latitude* [7],

$$\boldsymbol{C}_i^l(i, \Omega, \theta) = \begin{bmatrix} \mathrm{c}_\Omega \mathrm{c}_\theta - \mathrm{s}_\Omega \mathrm{s}_\theta \mathrm{c}_i & \mathrm{s}_\Omega \mathrm{c}_\theta + \mathrm{c}_\Omega \mathrm{s}_\theta \mathrm{c}_i & \mathrm{s}_\theta \mathrm{s}_i \\ -\mathrm{c}_\Omega \mathrm{s}_\theta - \mathrm{s}_\Omega \mathrm{c}_\theta \mathrm{c}_i & -\mathrm{s}_\Omega \mathrm{s}_\theta + \mathrm{c}_\Omega \mathrm{c}_\theta \mathrm{c}_i & \mathrm{c}_\theta \mathrm{s}_i \\ \mathrm{s}_\Omega \mathrm{s}_i & -\mathrm{c}_\Omega \mathrm{s}_i & \mathrm{c}_i \end{bmatrix} \tag{5}$$

The operator c_α and s_α denote $\cos\alpha$ and $\sin\alpha$, respectively. The LVLH frame angular velocity and acceleration vectors, with respect to the inertial frame, are:

$$\boldsymbol{\omega} = \omega_x \hat{\boldsymbol{i}} + \omega_y \hat{\boldsymbol{j}} + \omega_z \hat{\boldsymbol{k}}, \quad \dot{\boldsymbol{\omega}} = \dot{\omega}_x \hat{\boldsymbol{i}} + \dot{\omega}_y \hat{\boldsymbol{j}} + \dot{\omega}_z \hat{\boldsymbol{k}} \tag{6}$$

The angular velocity components in the LVLH frame are given by [7]:

$$\begin{cases} \omega_x = \dfrac{\mathrm{d}\Omega}{\mathrm{d}t} \sin i \sin\theta + \dfrac{\mathrm{d}i}{\mathrm{d}t} \cos\theta & \text{(7a)} \\ \omega_y = \dfrac{\mathrm{d}\Omega}{\mathrm{d}t} \sin i \cos\theta - \dfrac{\mathrm{d}i}{\mathrm{d}t} \sin\theta & \text{(7b)} \\ \omega_z = \dfrac{\mathrm{d}\Omega}{\mathrm{d}t} \cos i + \dfrac{\mathrm{d}\theta}{\mathrm{d}t} & \text{(7c)} \end{cases}$$

Let us express the disturbance on the chief in the LVLH frame as $\boldsymbol{d}_c = d_{c,x}\hat{\boldsymbol{i}} + d_{c,y}\hat{\boldsymbol{j}} + d_{c,z}\hat{\boldsymbol{k}}$. Orbital parameters time-derivatives are provided by the *Gauss variational equations* [6]:

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos\theta}{h} d_{c,z} \tag{8a}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin\theta}{h\sin i} d_{c,z} \tag{8b}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2} - \frac{r\sin\theta\cos i}{h\sin i} d_{c,z} \tag{8c}$$

Substitution of Eqs. 8 into 7 gives:

$$\omega_x = \frac{r}{h} d_{c,z}, \quad \omega_y = 0, \quad \omega_z = \frac{h}{r^2} \tag{9}$$

The terms ω_x and ω_z are also called *steering rate of the orbital plane* and *orbital rate* [11].

In order to compute the LVLH components of the angular acceleration $\dot{\boldsymbol{\omega}}$, we need the analytical expression of $\dot{h}$. Using Eqs. 1 and 9, the angular momentum time-derivative in the LVLH frame is given by:

$$\begin{aligned}\dot{\boldsymbol{h}} &= \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} - \boldsymbol{\omega} \times \boldsymbol{h} \\ &= \boldsymbol{r} \times \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} - \boldsymbol{\omega} \times \boldsymbol{h} \\ &= \boldsymbol{r} \times \boldsymbol{d}_c - \boldsymbol{\omega} \times \boldsymbol{h} \\ &= r\hat{\boldsymbol{i}} \times \left(d_{c,x}\hat{\boldsymbol{i}} + d_{c,y}\hat{\boldsymbol{j}} + d_{c,z}\hat{\boldsymbol{k}}\right) - \left(\omega_x\hat{\boldsymbol{i}} + \omega_z\hat{\boldsymbol{k}}\right) \times h\hat{\boldsymbol{k}} \\ &= r d_{c,y}\hat{\boldsymbol{k}}\end{aligned} \tag{10}$$

Since $\dot{\boldsymbol{h}} = \dot{h}\hat{\boldsymbol{k}}$, we have that

$$\dot{h} = r d_{c,y} \tag{11}$$

The LVLH components of $\dot{\boldsymbol{\omega}}$ are then given by:

$$\begin{cases} \dot{\omega}_x = \dfrac{1}{h}\left(\dot{r}d_{c,z} + r\dot{d}_{c,z} - \omega_x r d_{c,y}\right) & \text{(12a)} \\ \dot{\omega}_z = \dfrac{1}{r}\left(d_{c,y} - 2\omega_z \dot{r}\right) & \text{(12b)} \end{cases}$$

To describe the relative motion, we introduce the vectors $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$, respectively the deputy relative position and velocity with respect to the chief (Fig. 1):

$$\boldsymbol{\rho} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}, \quad \dot{\boldsymbol{\rho}} = \dot{x}\hat{\boldsymbol{i}} + \dot{y}\hat{\boldsymbol{j}} + \dot{z}\hat{\boldsymbol{k}} \tag{13}$$

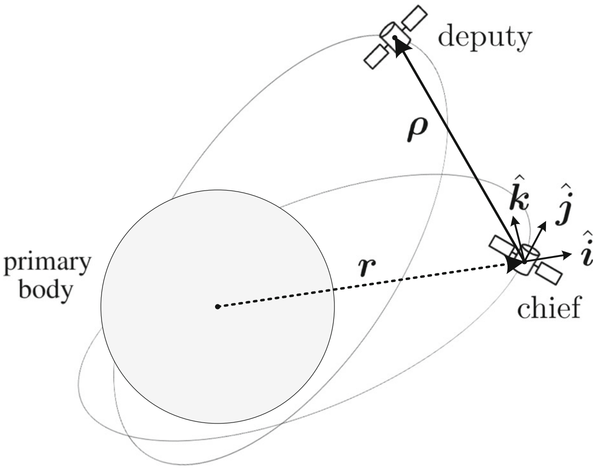

Fig. 1 Local-vertical local-horizon frame

We can write deputy position and distance as

$$\boldsymbol{r}_d = \boldsymbol{r} + \boldsymbol{\rho} = (r + x)\,\hat{\boldsymbol{i}} + y\,\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}} \tag{14}$$

$$r_d = \sqrt{(r + x)^2 + y^2 + z^2} \tag{15}$$

Time-derivative in $\mathcal{F}_i$ of Eq. 14 gives the deputy velocity:

$$\boldsymbol{v}_d = \frac{\mathrm{d}\boldsymbol{r}_d}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{16}$$

Differentiation of Eq. 16 leads to deputy acceleration:

$$\frac{\mathrm{d}^2\boldsymbol{r}_d}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + \ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \tag{17}$$

Introducing (1) and (2) into (17), we obtain the general expression of the relative motion:

$$\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) - \frac{\mu}{r^3}\boldsymbol{r} + \frac{\mu}{r_d^3}(\boldsymbol{r} + \boldsymbol{\rho}) = \Delta\boldsymbol{d} + \boldsymbol{u} \tag{18}$$

where $\Delta\boldsymbol{d}$ is the *differential perturbations acceleration* vector:

$$\Delta\boldsymbol{d} = \boldsymbol{d}_d - \boldsymbol{d}_c = \Delta d_x\hat{\boldsymbol{i}} + \Delta d_y\hat{\boldsymbol{j}} + \Delta d_z\hat{\boldsymbol{k}} \tag{19}$$

In terms of LVLH components of $\boldsymbol{\rho}$, we have the following set of second order differential equations:

$$\begin{cases} \ddot{x} = \left(\omega_z^2 - \dfrac{\mu}{r_d^3}\right) x + \dot{\omega}_z y - \omega_x\omega_z z + 2\omega_z\dot{y} + \mu\left(\dfrac{1}{r^2} - \dfrac{r}{r_d^3}\right) + \Delta d_x + u_x & \text{(20a)} \\ \ddot{y} = -\dot{\omega}_z x + \left(\omega_x^2 + \omega_z^2 - \dfrac{\mu}{r_d^3}\right) y + \dot{\omega}_x z - 2\omega_z\dot{x} + 2\omega_x\dot{z} + \Delta d_y + u_y & \text{(20b)} \\ \ddot{z} = -\omega_x\omega_z x - \dot{\omega}_x y + \left(\omega_x^2 - \dfrac{\mu}{r_d^3}\right) z - 2\omega_x\dot{y} + \Delta d_z + u_z & \text{(20c)} \end{cases}$$

where u_x, u_y and u_z are the components of $\boldsymbol{u}$ along the LVLH unit vectors.

To propagate (20) and to compute $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ components in Eqs. 9 and 12, we need to know the chief distance and speed, i.e. r and $\dot{r}$. Chief velocity in the LVLH frame is given by

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{r} = \dot{r}\hat{\boldsymbol{i}} + \omega_z r\hat{\boldsymbol{j}} \tag{21}$$

Further derivation of Eq. 21 with Eqs. 9 and 12 gives:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{v} = \left(\ddot{r} - \omega_z^2 r\right)\hat{\boldsymbol{i}} + d_{c,y}\hat{\boldsymbol{j}} + d_{c,z}\hat{\boldsymbol{k}} \tag{22}$$

Combining (22) and (1), we obtain

$$\ddot{r} = -\frac{\mu}{r^2} + \omega_z^2 r + d_{c,x} \tag{23}$$

The terms r and $\dot{r}$ can then be obtained by numerical integration of Eq. 23. Note that $d_{c,x}$ is in general a variable quantity, function of the time, of the chief position and/or velocity and of many other variables and parameters according to the perturbations considered. In the following Sections, its expression will be provided for the zonal harmonics perturbation and for the atmospheric drag.

Equations 11, 20 and 23 are the basis of our general framework for relative motion description. Perturbations can be computed separately and then introduced in the equations, in order to obtain an exact model of the relative motion under the influence of the orbital perturbations of interest.

Remark 1 Equations 11, 20 and 23 can be transformed into a set of 9 first order differential equations. From a system theory point of view, we have a system with state vector $\boldsymbol{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}, r, \dot{r}, h]^T$ and input vector $\boldsymbol{u}$. The quantities $\boldsymbol{d}_c$, $\dot{d}_{c,z}$ and $\Delta \boldsymbol{d}$ can be seen as system disturbances. The former two affect the LVLH frame orientation, see Eqs. 9 and 12, as well as chief parameters r, $\dot{r}$ and h. The latter influences the relative motion directly.

Remark 2 A more general expression for the LVLH angular velocity and acceleration components is provided in [3], and repeated below for clarity's sake:

$$\omega_x = \frac{r}{h}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{k}}\right), \quad \omega_y = 0, \quad \omega_z = \frac{1}{r}\left(\boldsymbol{v}\cdot\hat{\boldsymbol{j}}\right) \tag{24}$$

$$\begin{cases} \dot{\omega}_x = \dfrac{r}{h}\left[\dfrac{\dot{r}}{r}\left(\dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{k}}\right) - 2\dfrac{r}{h}\left(\dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{j}}\right)\left(\dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{k}}\right) + \left(\dfrac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}\cdot\hat{\boldsymbol{k}}\right)\right] & \text{(25a)} \\ \dot{\omega}_z = \dfrac{1}{r}\left[\left(\dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{j}}\right) - 2\dfrac{\dot{r}}{r}\left(\boldsymbol{v}\cdot\hat{\boldsymbol{j}}\right)\right] & \text{(25b)} \end{cases}$$

The third time derivative of the position vector (*jerk*) is given by

$$\frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3} = \frac{\partial}{\partial t}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right) + \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)\boldsymbol{v} + \frac{\partial}{\partial \boldsymbol{v}}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \tag{26}$$

where the term $\partial\left(\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2\right)/\partial t$ accounts for time variation of the gravitational parameter and of the perturbations' parameters. In [3], the author considers this term as being equal to zero under the assumption of constant mass distribution and conservative force fields. This results in the following simplified expression for $\dot{\omega}_x$:

$$\dot{\omega}_x = \frac{r}{h}\left[\frac{\dot{r}}{r}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{k}}\right) - 2\frac{r}{h}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{j}}\right)\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{k}}\right) + \hat{\boldsymbol{k}}^T\frac{\partial}{\partial \boldsymbol{r}}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)\boldsymbol{v}\right] \tag{27}$$

used for later derivation in the reference.

Conversely, in our formulation only the gravitational parameter is assumed time-invariant. Hence, the formulation in the present paper can take into account orbital perturbations characterized by time-varying parameters, such as atmospheric drag with time-dependent atmosphere density (see Remark 8), and the presence of a second primary body (e.g. lunar perturbation in lower Earth orbits, see [13]). In addition, in Eqs. 9 and 12 perturbations influence is explicit, as opposed to Eqs. 24 and 25. Note also that Eqs. 9 and 12 can be obtained from Eqs. 24 and 25, observing that $\mathrm{d}^3\boldsymbol{r}/\mathrm{d}t^3\cdot\hat{\boldsymbol{k}} = \dot{d}_{c,z}$.

Zonal Harmonics Perturbation

General Analytical Expression

Primary body gravity potential can be modeled using spherical harmonic series as

$$U(\boldsymbol{r}, \psi) = -\frac{\mu}{r}\left[1 - \sum_{k=2}^{+\infty}\left(\frac{R_{\text{eq}}}{r}\right)^k J_k P_k(\cos\psi)\right] \tag{28}$$

where R_{eq} is the primary body *equatorial radius*, J_k are the *zonal harmonics*, $P_k(\xi)$ is the *Legendre polynomial of order k*

$$P_k(\xi) = \frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\left(\xi^2 - 1\right)^k \tag{29}$$

and ψ is the angle between the primary body polar direction $\hat{\boldsymbol{K}}$ and the spacecraft position $\boldsymbol{r}$, see Fig. 2 [6]. To include the effect of these perturbations in Eq. 20, we must isolate the potential of a spherical body, i.e. $-\mu/r$ (whose effect is already included in Eq. 20), from the zonal harmonics potential, that is

$$U_{J_k}(\boldsymbol{r}, \psi) = \frac{\mu}{r}\sum_{k=2}^{+\infty}\left(\frac{R_{\text{eq}}}{r}\right)^k J_k P_k(\cos\psi) \tag{30}$$

Zonal harmonics perturbation acceleration is given by minus the gradient of U_{J_k} in the spherical coordinates (r, ψ) [1],

$$\boldsymbol{d}_{J_k} = -\nabla U_{J_k} = -\frac{\partial U_{J_k}}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial U_{J_k}}{\partial \psi}\hat{\boldsymbol{\psi}} \tag{31}$$

Using the geometric relation

$$\hat{\boldsymbol{K}} = \cos\psi\,\hat{\boldsymbol{r}} - \sin\psi\,\hat{\boldsymbol{\psi}} \tag{32}$$

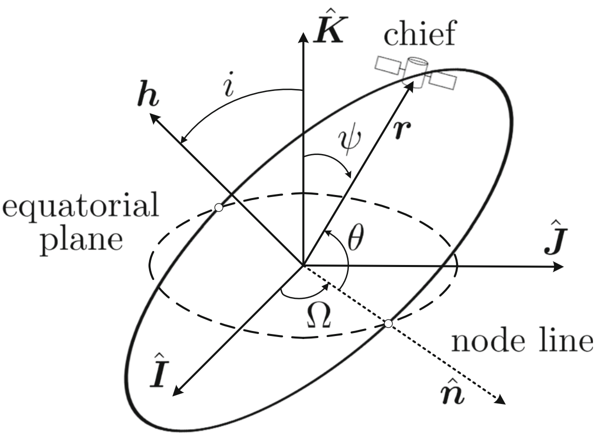

Fig. 2 Chief orbital parameters

and the chain rule, we can write (31) as follows,

$$
\begin{aligned}
\boldsymbol{d}_{J_k} &= -\frac{\partial U_{J_k}}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial U_{J_k}}{\partial \cos\psi}\frac{\partial \cos\psi}{\partial \psi}\hat{\boldsymbol{\psi}} \\
&= -\frac{\partial U_{J_k}}{\partial r}\hat{\boldsymbol{r}} + \frac{\sin\psi}{r}\frac{\partial U_{J_k}}{\partial \cos\psi}\hat{\boldsymbol{\psi}} \\
&= -\frac{\partial U_{J_k}}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial U_{J_k}}{\partial \cos\psi}\left(\hat{\boldsymbol{K}} - \cos\psi\hat{\boldsymbol{r}}\right) \\
&= \alpha(r,\psi)\hat{\boldsymbol{r}} + \beta(r,\psi)\left(\hat{\boldsymbol{K}} - \cos\psi\hat{\boldsymbol{r}}\right)
\end{aligned}
\tag{33}
$$

For the reader convenience, we defined the following accelerations

$$
\alpha(r,\psi) = -\frac{\partial U_{J_k}}{\partial r} = \frac{\mu}{r^2}\sum_{k=2}^{+\infty}(k+1)\left(\frac{R_{\text{eq}}}{r}\right)^k J_k P_k(\cos\psi) \tag{34}
$$

$$
\beta(r,\psi) = -\frac{1}{r}\frac{\partial U_{J_k}}{\partial \cos\psi} = -\frac{\mu}{r^2}\sum_{k=2}^{+\infty}\left(\frac{R_{\text{eq}}}{r}\right)^k J_k \frac{\partial P_k(\cos\psi)}{\partial \cos\psi} \tag{35}
$$

where

$$
\frac{\partial P_k(\xi)}{\partial \xi} = \frac{1}{2^k k!}\frac{\mathrm{d}^{k+1}}{\mathrm{d}\xi^{k+1}}\left(\xi^2 - 1\right)^k \tag{36}
$$

Remark 3 Note that zonal harmonics perturbation is independent of Ω, since the zonal harmonics gravity potential is axisymmetric and depends only on r and ψ.

Remark 4 In Eq. 33 it is easy to see that $\beta(r,\psi)$ is the gravity component pointing to the equatorial plane due to zonal harmonics perturbations. The perturbation component pointing to the primary body center of mass is given by $\alpha(r,\psi) - \beta(r,\psi)\cos\psi$ instead.

Remark 5 If we denote with $\boldsymbol{r}^i = r_X\hat{\boldsymbol{I}} + r_Y\hat{\boldsymbol{J}} + r_Z\hat{\boldsymbol{K}}$ the spacecraft position in the inertial frame $\mathcal{F}_i$, we have that $\cos\psi = r_Z/r$. We can use this geometric relation to simplify the expressions above and to avoid the use of the angle ψ.

Differential Zonal Harmonics Perturbation

Using Eq. 33, we can obtain the expressions in the LVLH frame of the zonal harmonics perturbation for the chief and the deputy.

For the chief, we know that $\hat{\boldsymbol{r}} = \hat{\boldsymbol{i}}$ and

$$
r\cos\psi = r\sin\theta\sin i \tag{37}
$$

where (r,ψ) are chief spherical coordinates (see Fig. 2). Thus, Eq. 33 for the chief in the LVLH frame simplifies in

$$
\boldsymbol{d}_{J_k,c} = \alpha(r,\psi)\hat{\boldsymbol{i}} + \beta(r,\psi)\left(\sin i\cos\theta\hat{\boldsymbol{j}} + \cos i\hat{\boldsymbol{k}}\right) \tag{38}
$$

since, given (5),

$$
\hat{\boldsymbol{K}} = \sin i\sin\theta\hat{\boldsymbol{i}} + \sin i\cos\theta\hat{\boldsymbol{j}} + \cos i\hat{\boldsymbol{k}} \tag{39}
$$

For the deputy, we write (33) introducing deputy spherical coordinates (r_d, ψ_d),

$$\begin{aligned} \boldsymbol{d}_{J_k,d} &= \alpha(r_d,\psi_d)\hat{\boldsymbol{r}}_d + \beta(r_d,\psi_d)\left(\hat{\boldsymbol{K}} - \cos\psi_d\hat{\boldsymbol{r}}_d\right) \\ &= \left(\alpha(r_d,\psi_d) - \beta(r_d,\psi_d)\cos\psi_d\right)\hat{\boldsymbol{r}}_d + \beta(r_d,\psi_d)\hat{\boldsymbol{K}} \end{aligned} \tag{40}$$

The unit vector $\hat{\boldsymbol{r}}_d$ is given by,

$$\hat{\boldsymbol{r}}_d = \frac{1}{r_d}\left((r+x)\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}\right) \tag{41}$$

Therefore, we can write the zonal harmonic perturbations acting on the deputy in the LVLH frame as follows,

$$\begin{aligned} \boldsymbol{d}_{J_k,d} = &\left[\left(\alpha(r_d,\psi_d) - \beta(r_d,\psi_d)\cos\psi_d\right)\tfrac{r+x}{r_d} + \beta(r_d,\psi_d)\sin i\sin\theta\right]\hat{\boldsymbol{i}} \\ &+ \left[\left(\alpha(r_d,\psi_d) - \beta(r_d,\psi_d)\cos\psi_d\right)\tfrac{y}{r_d} + \beta(r_d,\psi_d)\sin i\cos\theta\right]\hat{\boldsymbol{j}} \\ &+ \left[\left(\alpha(r_d,\psi_d) - \beta(r_d,\psi_d)\cos\psi_d\right)\tfrac{z}{r_d} + \beta(r_d,\psi_d)\cos i\right]\hat{\boldsymbol{k}} \end{aligned} \tag{42}$$

The differential zonal harmonics perturbation is finally given by,

$$\begin{aligned} \Delta\boldsymbol{d}_{J_k} &= \boldsymbol{d}_{J_k,d} - \boldsymbol{d}_{J_k,c} \\ &= \left[\left(\alpha_d - \beta_d\cos\psi_d\right)\frac{r+x}{r_d} - \alpha + \beta_d\sin i\sin\theta\right]\hat{\boldsymbol{i}} \\ &\quad + \left[\left(\alpha_d - \beta_d\cos\psi_d\right)\frac{y}{r_d} + (\beta_d - \beta)\sin i\cos\theta\right]\hat{\boldsymbol{j}} \\ &\quad + \left[\left(\alpha_d - \beta_d\cos\psi_d\right)\frac{z}{r_d} + (\beta_d - \beta)\cos i\right]\hat{\boldsymbol{k}} \end{aligned} \tag{43}$$

where $\alpha = \alpha(r,\psi)$, $\beta = \beta(r,\psi)$, $\alpha_d = \alpha(r_d,\psi_d)$ and $\beta_d = \beta(r_d,\psi_d)$.

Time-Derivative of Zonal Harmonics Perturbation Acting on the Chief Along $\hat{k}$

To compute the angular acceleration $\dot{\omega}_x$ in Eq. 12a, we must find the time-derivative in the LVLH frame of the component along $\hat{\boldsymbol{k}}$ of $\boldsymbol{d}_{J_k,c}$, i.e. of the term

$$d_{J_k,c_z} = \beta(r,\psi)\cos i \tag{44}$$

Derivation with respect to time of Eq. 44 gives

$$\dot{d}_{J_k,c_z} = \frac{\mathrm{d}\beta(r,\psi)}{\mathrm{d}t}\cos i + \beta(r,\psi)\frac{\mathrm{d}\cos i}{\mathrm{d}t} \tag{45}$$

The time-derivative of $\beta(r,\psi)$ is

$$\frac{\mathrm{d}\beta(r,\psi)}{\mathrm{d}t} = -\frac{\mu}{r^2}\sum_{k=2}^{+\infty} J_k\left(\frac{R_{\mathrm{eq}}}{r}\right)^k\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P_k(\cos\psi)}{\partial\cos\psi}\right) - (k+2)\frac{\dot{r}}{r}\frac{\partial P_k(\cos\psi)}{\partial\cos\psi}\right] \tag{46}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P_k(\cos\psi)}{\partial\cos\psi}\right) = \frac{\partial^2 P_k(\cos\psi)}{\partial\cos^2\psi}\frac{\mathrm{d}\cos\psi}{\mathrm{d}t} \tag{47}$$

and

$$\frac{\partial^2 P_k(\xi)}{\partial \xi^2} = \frac{1}{2^k k!} \frac{\mathrm{d}^{k+2}}{\mathrm{d}\xi^{k+2}} \left(\xi^2 - 1\right)^k \tag{48}$$

Using Eq. 37 along with Eqs. 8 and 9, we write the time-derivative of $\cos\psi$ as follows:

$$\begin{aligned}\frac{\mathrm{d}\cos\psi}{\mathrm{d}t} &= \frac{\mathrm{d}i}{\mathrm{d}t}\cos i \sin\theta + \frac{\mathrm{d}\theta}{\mathrm{d}t}\sin i\cos\theta \\ &= \omega_z \sin i \cos\theta\end{aligned} \tag{49}$$

Introducing (47) and (49) into (46), we obtain

$$\frac{\mathrm{d}\beta(r,\psi)}{\mathrm{d}t} = -\frac{\mu}{r^2}\sum_{k=2}^{+\infty} J_k \left(\frac{R_{\mathrm{eq}}}{r}\right)^k \left[\frac{\partial^2 P_k(\cos\psi)}{\partial\cos^2\psi}\omega_z \sin i\cos\theta - (k+2)\frac{\dot{r}}{r}\frac{\partial P_k(\cos\psi)}{\partial\cos\psi}\right] \tag{50}$$

Finally, the time-derivative of $\cos i$ can be expressed using Eqs. 8 and 9,

$$\frac{\mathrm{d}\cos i}{\mathrm{d}t} = -\frac{\mathrm{d}i}{\mathrm{d}t}\sin i = -\omega_x \sin i\cos\theta \tag{51}$$

Introducing (50) and (51) in Eq. 45, we finally obtain the expression of $\dot{d}_{J_k,c_z}$.

Remark 6 Notice that in Eq. 51 the angular velocity ω_x appears. As expected, the time-derivative of $\cos i$ is influenced by all the perturbations acting on the chief, not only the zonal harmonics.

Remark 7 In order to compute $\Delta \boldsymbol{d}_{J_k}$ and $\dot{d}_{J_k,c_z}$, the parameters i, θ, $\cos\psi$ and $\cos\psi_d$ are needed. The former two can be obtained by propagation of Eqs. 8a and 8c. The term $\cos\psi$ can be computed using Eq. 37 or as described in Remark 5. In the latter case,

$$r_Z = \boldsymbol{r}\cdot\hat{\boldsymbol{K}} = r\sin i\sin\theta \tag{52}$$

For the deputy, using again (37), we have that

$$\cos\psi_d = \frac{\boldsymbol{r}_d\cdot\hat{\boldsymbol{K}}}{r_d} = \frac{1}{r_d}\left[(r+x)\sin i\sin\theta + y\sin i\cos\theta + z\cos i\right] \tag{53}$$

Therefore, to include zonal harmonics perturbation we must propagate two additional first order differential equation for i and θ, i.e. Eqs. 8a and 8c.

Atmospheric Drag

Atmospheric drag (for chief and deputy) can be quantified using the following formulae,

$$\boldsymbol{d}_{a,c} = -\frac{1}{2}\rho_a(\boldsymbol{r})C_{b,c}^{-1} v_{r,c}\boldsymbol{v}_{r,c} \tag{54}$$

$$\boldsymbol{d}_{a,d} = -\frac{1}{2}\rho_a(\boldsymbol{r}_d)C_{b,d}^{-1} v_{r,d}\boldsymbol{v}_{r,d} \tag{55}$$

where $\rho_a(\boldsymbol{r})$ is the *atmosphere density* in $\boldsymbol{r}$ according to the model adopted, $\boldsymbol{v}_r$ is the velocity of the spacecraft relative to the atmosphere, $C_b = m/(C_d A)$ is the *spacecraft ballistic coefficient* with m denoting the spacecraft mass, C_d its *drag coefficient* and A its *average transversal section area* [6]. If we assume that the atmosphere rotates with the primary body, the spacecraft relative velocity $\boldsymbol{v}_r$ can be approximated as follows,

$$\boldsymbol{v}_r = \boldsymbol{v} - \boldsymbol{\omega}_p \times \boldsymbol{r} \tag{56}$$

with $\boldsymbol{\omega}_p = \omega_p \hat{\boldsymbol{K}}$ denoting the primary body angular velocity.

Using Eq. 21, the chief relative velocity and speed with respect to the atmosphere are

$$\begin{aligned} \boldsymbol{v}_{r,c} &= \dot{\boldsymbol{r}} + \left(\boldsymbol{\omega} - \boldsymbol{\omega}_p\right) \times \boldsymbol{r} \\ &= \dot{r}\hat{\boldsymbol{i}} + \Delta\omega_z r \hat{\boldsymbol{j}} - \Delta\omega_y r \hat{\boldsymbol{k}} \end{aligned} \tag{57}$$

$$v_{r,c} = \sqrt{\dot{r}^2 + r^2\left(\Delta\omega_y^2 + \Delta\omega_z^2\right)} \tag{58}$$

where we introduced the relative angular velocity,

$$\Delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_p = \Delta\omega_x \hat{\boldsymbol{i}} + \Delta\omega_y \hat{\boldsymbol{j}} + \Delta\omega_z \hat{\boldsymbol{k}} \tag{59}$$

Using Eq. 39, we have

$$\Delta\boldsymbol{\omega} = \left(\omega_x - \omega_p \sin i \sin\theta\right)\hat{\boldsymbol{i}} - \omega_p \sin i \cos\theta \hat{\boldsymbol{j}} + \left(\omega_z - \omega_p \cos i\right)\hat{\boldsymbol{k}} \tag{60}$$

The relative velocity and speed of the deputy are

$$\begin{aligned} \boldsymbol{v}_{r,d} &= \boldsymbol{v}_d - \boldsymbol{\omega}_p \times \boldsymbol{r}_d \\ &= \boldsymbol{v}_{r,c} + \dot{\boldsymbol{\rho}} + \Delta\boldsymbol{\omega} \times \boldsymbol{\rho} \end{aligned} \tag{61}$$

$$\begin{aligned} v_{r,d} = \Big[&\left(\dot{r} + \dot{x} + \Delta\omega_y z - \Delta\omega_z y\right)^2 \\ &+ \left(\dot{y} + \Delta\omega_z(r+x) - \Delta\omega_x z\right)^2 \\ &+ \left(\dot{z} + \Delta\omega_x y - \Delta\omega_y(r+x)\right)^2\Big]^{\frac{1}{2}} \end{aligned} \tag{62}$$

where we used Eqs. 16, 21 and 57. By means of Eq. 61 and defining the coefficients,

$$\gamma_c = -\frac{1}{2}\rho_a(\boldsymbol{r})C_{b,c}^{-1} \tag{63}$$

$$\gamma_d = -\frac{1}{2}\rho_a(\boldsymbol{r}_d)C_{b,d}^{-1} \tag{64}$$

we write the differential atmospheric drag perturbation as,

$$\Delta\boldsymbol{d}_a = \boldsymbol{d}_{a,d} - \boldsymbol{d}_{a,c} = \left(\gamma_d v_{r,d} - \gamma_c v_{r,c}\right)\boldsymbol{v}_{r,c} + \gamma_d v_{r,d}\left(\dot{\boldsymbol{\rho}} + \Delta\boldsymbol{\omega} \times \boldsymbol{\rho}\right) \tag{65}$$

or, using Eq. 57, in terms of LVLH components as,

$$\begin{aligned} \Delta\boldsymbol{d}_a = &\left[\left(\gamma_d v_{r,d} - \gamma_c v_{r,c}\right)\dot{r} + \gamma_d v_{r,d}\left(\dot{x} + \Delta\omega_y z - \Delta\omega_z y\right)\right]\hat{\boldsymbol{i}} \\ &+ \left[\left(\gamma_d v_{r,d} - \gamma_c v_{r,c}\right)\Delta\omega_z r + \gamma_d v_{r,d}\left(\dot{y} + \Delta\omega_z x - \Delta\omega_x z\right)\right]\hat{\boldsymbol{j}} \\ &- \left[\left(\gamma_d v_{r,d} - \gamma_c v_{r,c}\right)\Delta\omega_y r - \gamma_d v_{r,d}\left(\dot{z} + \Delta\omega_x y - \Delta\omega_y x\right)\right]\hat{\boldsymbol{k}} \end{aligned} \tag{66}$$

To compute $\dot{\omega}_x$, Eq. 12a, we need the time-derivative of d_{a,c_z}. From Eqs. 54, 57 and 60, we have

$$d_{a,c_z} = -\gamma_c v_{r,c} \Delta\omega_y r = \omega_p \gamma_c r v_{r,c} \sin i \cos\theta \tag{67}$$

Given Eqs. 8a, 8c and 9, the time-derivative of Eq. 67 is

$$\begin{aligned} \dot{d}_{a,c_z} = \omega_p \sin i \cos\theta \left[\tfrac{\mathrm{d}\gamma_c}{\mathrm{d}t} r v_{r,c} + \gamma_c \left(\dot{r} v_{r,c} + r \dot{v}_{r,c}\right)\right] \\ +\omega_p \gamma_c v_{r,c} r \left[\omega_x \cos i - \omega_z \sin i \sin\theta\right] \end{aligned} \tag{68}$$

with

$$\frac{\mathrm{d}\gamma_c}{\mathrm{d}t} = -\frac{1}{2} C_{b,c}^{-1} \frac{\mathrm{d}\rho_a(\boldsymbol{r})}{\mathrm{d}t} = -\frac{1}{2} C_{b,c}^{-1} \frac{\partial \rho_a(\boldsymbol{r})}{\partial \boldsymbol{r}} \boldsymbol{v} \tag{69}$$

Note that in Eq. 69 the time-derivative of the ballistic coefficients does not appear, since we implicitly assumed that the chief does not change its mass.

Remark 8 The atmosphere density changes also with time, for example due to solar activity cycles. Hence, the coefficients γ_c and γ_d are functions of time as well as of spacecraft position. If we consider atmosphere time-variability, Eq. 69 becomes

$$\frac{\mathrm{d}\gamma_c}{\mathrm{d}t} = -\frac{1}{2} C_{b,c}^{-1} \frac{\mathrm{d}\rho_a(t, \boldsymbol{r})}{\mathrm{d}t} = -\frac{1}{2} C_{b,c}^{-1} \left(\frac{\partial \rho_a(t, \boldsymbol{r})}{\partial \boldsymbol{t}} + \frac{\partial \rho_a(t, \boldsymbol{r})}{\partial \boldsymbol{r}} \boldsymbol{v}\right) \tag{70}$$

However, time-variability of the atmosphere is obviously difficult to estimate and may be taken in account only for high fidelity simulations.

Remark 9 If we consider a *static atmosphere* model, i.e. the atmosphere does not rotate with the primary body (as in [4]), we can significantly simplify the previous results. In this case, chief and deputy relative velocity with respect to the atmosphere are, respectively, $\boldsymbol{v}_{r,c} = \boldsymbol{v}$ and $\boldsymbol{v}_{r,d} = \boldsymbol{v}_d$. Using Eq. 21, the differential perturbation simplifies in

$$\begin{aligned} \Delta \boldsymbol{d}_a &= (\gamma_d v_d - \gamma_c v)\, \boldsymbol{v} + \gamma_d v_d\, (\dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times \boldsymbol{\rho}) \\ &= \left[(\gamma_d v_d - \gamma_c v)\, \dot{r} + \gamma_d v_d\, (\dot{x} - \omega_z y)\right] \hat{\boldsymbol{i}} \\ &\quad + \left[(\gamma_d v_d - \gamma_c v_c)\, \omega_z r + \gamma_d v_d\, (\dot{y} + \omega_z x - \omega_x z)\right] \hat{\boldsymbol{j}} \\ &\quad + \gamma_d v_d\, (\dot{z} + \omega_x y)\, \hat{\boldsymbol{k}} \end{aligned} \tag{71}$$

In addition, given (21), the atmospheric drag on the chief is equal to,

$$\boldsymbol{d}_{a,c} = \gamma_c v \boldsymbol{v} = \gamma_c v \left(\dot{r} \hat{\boldsymbol{i}} + \omega_z r \hat{\boldsymbol{j}}\right) \tag{72}$$

Therefore, the term $\dot{d}_{a,c_z}$ is equal to zero.

Direct Inclusion of Perturbations Expressed in the Inertial Frame

It is possible to avoid direct calculation of perturbations in the LVLH frame, using the coordinate change matrix $\boldsymbol{C}_i^l$, Eq. 5. Given the perturbation in the inertial frame, $\boldsymbol{d}^i$, the expression in the LVLH frame is simply $\boldsymbol{d}^l = \boldsymbol{C}_i^l \boldsymbol{d}^i$. Consequently, the differential perturbations acceleration vector in $\mathcal{F}_l$ is equal to

$$\Delta \boldsymbol{d}^l = \boldsymbol{C}_i^l \left(\boldsymbol{d}_d^i - \boldsymbol{d}_c^i \right) \tag{73}$$

The angular acceleration $\dot{\omega}_x$ depends on the time-derivative of $d_{c,z}$. This term can be computed using the expression of the disturbance in the inertial frame, $\boldsymbol{d}_c^i$, by means of the following relation,

$$\begin{aligned} \dot{\boldsymbol{d}}_c^l &= \dot{d}_{c,x}\hat{\boldsymbol{i}} + \dot{d}_{c,y}\hat{\boldsymbol{j}} + \dot{d}_{c,z}\hat{\boldsymbol{k}} \\ &= \boldsymbol{C}_i^l \frac{\mathrm{d}\boldsymbol{d}_c^i}{\mathrm{d}t} - \boldsymbol{\omega}^l \times \boldsymbol{C}_i^l \boldsymbol{d}_c^i \end{aligned} \tag{74}$$

Equations 73 and 74 may be useful for motion prediction, since one can avoid the analytical computation of perturbations in the LVLH frame. However, they do not provide any physical insight into the relative motion dynamics, unlike the derivation that led to the results in the previous sections.

Conclusions

A general framework for relative motion description in the LVLH frame was developed using simple geometric relations and vector calculus. The equation set proposed can be easily extended to include the perturbations that characterize the mission scenario. We also derived the analytical expression of zonal harmonics perturbation and atmosphere drag for their introduction in the proposed equations set. Inclusion of general perturbations was also addressed. Since no approximations were made, all the results presented here are exact and can be used for accurate relative motion analysis.

References

1. Arfken, G.B., Weber, H.J., Harris, F.E.: Mathematical Methods for Physicists, 7th edn. Elsevier, Amsterdam (2013)
2. Bakhtiari, M., Daneshjou, K., Abbasali, E.: A new approach to derive a formation flying model in the presence of a perturbing body in inclined elliptical orbit: relative hovering analysis. Astrophys. Space Sci. **362**(2), 36 (2017)
3. Casotto, S.: The equations of relative motion in the orbital reference frame. Celest. Mech. Dyn. Astron. **124**(3), 215–234 (2016)
4. Wy, C., Wx, J.: Differential Equations of Relative Motion under the Influence of J2 Perturbation and Air Drag. In: Proceedings of 2010 AIAA SPACE Conference & Exposition. AIAA, Anaheim (2010)
5. Humi, M., Carter, T.: Rendezvous equations in a central-force field with linear drag. J. Guid.Control Dyn. **25**(1), 74–79 (2002)
6. Junkins, J., Schaub, H.: Analytical Mechanics of Space Systems, 2nd edn. American Institute of Aeronautics and Astronautics, Anaheim (2009)
7. Kechichian, J.A.: Motion in general elliptic orbit with respect to a dragging and presessing coordinate frame. J. Astronaut. Sci. **46**(1), 25–45 (1998)
8. Pluym, J.P., Damaren, C.J.: Second Order Relative Motion Model for Spacecraft under J2 Perturbations. In: Proceedings of 2006 AIAA/AASAstrodynamics Specialist Conference and Exhibit. AIAA, Keystone (2006)
9. Schweighart, S.A., Sedwick, R.J.: High-fidelity linearized J2 model for satellite formation flight. J. Guid. Control Dyn. **25**(6), 1073–1080 (2002)

10. Sullivan, J., Grimberg, S., D'Amico, S.: Comprehensive survey and assessment of spacecraft relative motion dynamics models. J. Guid. Control Dyn. **40**(8), 1837–1859 (2017)
11. Xu, G., Wang, D.: Nonlinear dynamic equations of satellite relative motion around an oblate Earth. J. Guid. Control Dyn. **31**(5), 1521–1524 (2008)
12. Xu, G., Xiang, F., Chen, Y.: Exact Dynamic of Satellite Relative Motion under Arbitrary Zonal Harmonic Perturbations. In: Proceedings of 24th Chinese Control and Decision Conference, pp. 3676–3681. IEEE, Taiyuan (2012)
13. Xu, G., Luo, J., Li, Z., Chen, X.: Equations of Satellite Relative Motion in Low Earth Orbit under Lunar Perturbation. In: Proceedings of 2014 IEEE Chinese Guidance, Navigation and Control Conference, pp. 113-120. IEEE, Yantai (2014)