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Root Locus of Zeros of Discrete Time Systems as a Function of Sample Rate

Wenxiang Zhou[1] **· Richard W. Longman**[2]

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Abstract
Root locus plots are one of the basic design tools in classical control. They help the designer tune control gains which appear linearly in the coefficients of the closed loop characteristic polynomial. And they give considerable intuition to the designer, based on the simple rules that root loci must follow. When designing a control system, one wants to know where the zeros are, but when designing a digital control system new issues appear. The original zero locations when mapped to discrete time are functions of the new parameter, the sample time T (as well as the pole locations). In addition, new zeros are usually introduced by the discretization process. The purpose of this paper is to give a general understanding of the nature of root loci of discrete time transfer function zeros as a function of this parameter T. We consider the complete range of values from T equal zero to infinity to understand the full plot. Reasonable sample rates will only use part of the plots. The characteristic polynomial coefficients are nonlinear functions of T so the usual root locus rules do not apply. One can be amazed at how the usual root locus rules are repeatedly violated, and what new kinds of unexpected behavior can be observed.

Keywords Sampled data systems · Discrete time systems · Root locus of zeros · Sampling zeros · Intrinsic zeros

✉ Wenxiang Zhou
Wz2321@qq.com

Richard W. Longman
RWL4@columbia.edu

[1] College of Energy and Power Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016 Jiangsu, China

[2] Columbia University, MC4703, 500 West 120th Street, New York, NY 10027, USA

Introduction

Root locus plots are a fundamental tool used in the design of feedback control systems. They describe the motion of the closed loop poles that dictate the decay of transients as a function of a controller gain. The gain appears linearly in the coefficients of the characteristic polynomial, and based on this there is a set of rules that the root locus must satisfy that help the designer understand the effects of changing the gain. One of these rules is that the loci start at the poles of the open loop transfer function when the gain is zero, and there must be one locus ending at each zero of the transfer function as the gain goes to infinity. The remaining root loci must go to infinity as the gain goes from zero to infinity. These rules apply not only to continuous time systems, but also to discrete time systems, provided the parameter being varied appears linearly in the coefficients of the characteristic polynomial.

When the controller of a feedback control system is to be implemented digitally, then the plant is normally fed by a zero order hold. One designs to make the sampled output perform well, after converting the plant Laplace transfer function to its equivalent z-transfer function, or equivalently convert the plant differential equation to an equivalent difference equation that has no approximation, i.e. the difference equation solution is exactly the same as the differential equation solution at the sample times. Poles and zeros in the Laplace transfer function are mapped into poles and zeros in the z-transfer function. In the design process, it is clear that one wants to know the locations of these zeros. For digital systems, the locations of the poles and zeros are now not only a function of any gain one wants to adjust, but also a function of the new variable, the time interval T between samples. This parameter appears in the characteristic polynomial in a nonlinear way, so that the usual root locus rules do not apply. And the behavior as a function of this parameter can be very unexpected and surprising, as illustrated here.

The poles of the Laplace transfer function map in a simple easily understood manner, each pole in the z-plane is only a function of the location of the pole in the s-plane. But the mapping of zero locations in the s-plane to the z-plane is not so simple, the locations are not only a function of the location of the s-plane zeros, but also a function of the poles of the system. This paper studies the root locus plots of the resulting zero locations as a function of the sample time T as it goes from zero to infinity. Only a part of this locus corresponds to sample rates with practical Nyquist frequencies, but we study the behavior of the complete locus. We study the mapping of zero locations not only for zeros located in the left half s-plane, but also the locus of the image of non-minimum phase zeros of the Laplace transfer function, i.e. zeros in the right half of the s-plane. It is observed that these loci can be amazingly complicated, do amazing things, and we present analysis to explain the behavior. In addition, they easily do things that are impossible according the root locus rules that everyone is familiar with. As one mild example of the unexpected behavior, it is observed that a non-minimum phase zero in the s-plane can map inside the unit circle, and be a minimum phase zero in the z-plane.

Vibration suppression is an important problem in many spacecraft. One important use for knowledge of the zero locations is repetitive control [1, 2]. It is one method that can in theory completely cancel vibrations at a location of fine pointing equipment on board spacecraft. Reference [3] performs experiments demonstrating such algorithms.

Reference [4] performs experiments on a floating spacecraft testbed for laser communication between spacecraft or to ground, in order to cancel vibrations of the laser beam caused by slight imbalance in control moment gyros. To make such repetitive controller designs it is fundamental to know where the zeros are outside the unit circle. The repetitive control objective is zero tracking error. Such zeros are ubiquitous and they prevent attempting to invert the system transfer function as a compensator for this purpose, making it necessary to create much more complicated compensators.

Non-minimum phase systems in continuous time plant models present a challenge to control system designers, and such systems are common [5]. The most interesting behavior observed in the mapping of zeros in $G(s)$ to zeros in $G(z)$, occur for such systems. One of the interesting properties of such systems is that initially they go the wrong direction in response to a unit step input, and then reverse direction to reach a steady state response with the right sign. Altitude control of an aircraft represents an example of non-minimum phase behavior. An airplane flying straight and level needs to increase the angle of attack in order to gain altitude, and the plane temporarily looses altitude while increasing the angle of attack. An example that can apply to spacecraft is controlling a sensor at one end of a flexible element by rotating the other end of the element. As the base starts to rotate in one direction, the slope or sensor pointing at the other end of the first vibration mode shape can easily temporarily point in the opposite direction of the commanded base rotation. Minimum phase zeros can arise in physical systems such as vibration dampers. In additions to such zeros, even when there are no zeros in continuous time to map to discrete time, the discretization process for a majority of systems introduces zeros, some of which are non-minimum phase.

The Need for Extra Zeros in Discrete Time Models of Continuous Time Systems

In digital control, the controller usually takes the output error at the sample times kT, where k is an integer and T is the sample time interval, computes an updated control action that goes into a zero order hold that applies this control action to the physical world, holding the most recent control action until the next time step when a new control action arrives and is applied. The output is sampled at the sample times and used to compute the error needed by the control law. The physical world is a differential equation which can be represented by Laplace transfer function $G(s)$. The differential equation can be replaced by a difference equation whose solution at the sample times is the same as that of the differential equation, and this can be represented by a z-transfer function $G(z)$. (We apologize to purists that this should mean $G(s)$ with s replaced by z, instead of a new function.)

The relationship between these two functions is given by

$$G(z) = \left(1 - z^{-1}\right) Z[G(s)/s] \tag{1}$$

This relationship can be derived as follows. The $G(s)/s$ represents the unit step response, and the Z indicates taking the z-transform of the sequence of step response values at the sample times. The zero-order-hold input consists of constant inputs from one time step kT to the next $(k+1)T$. Hence, one can write this time step of input to $G(s)$

using superposition, as the sum of a unit step input at kT multiplied by height $u(kT)$, minus the same input shifted to time step $(k+1)T$. Adding this up for each time step creates two convolution summations, and the transform of a convolution sum is the product of the transforms of the functions involved. The 1 times $Z[G(s)/s]$ gives the first summation in transform space, and the second summation gives $z^{-1}Z[G(s)/s]$. This proof derives the result from the definition of zero order hold which is made in the time domain. Textbooks usually make derivations using less fundamental and intuitive considerations, starting in transform space.

The transfer function consists of a denominator polynomial whose roots are the poles of the transfer function, and a polynomial in the numerator whose roots are the zeros of the transfer function. It is the purpose of this paper to develop an understanding of how the roots of the numerator polynomial move as the sample time interval T is changed from the limiting values of zero to infinity.

Note that in any usual digital control block diagram, after converting the plant to its equivalent z-transfer function and then finding the closed loop transfer function, the zeros of the converted plant transfer function $G(z)$ become zeros of this closed loop transfer function [6]. Hence, the locations described here are important when making a root locus plot for choosing a proportional control gain, since there must be one root ending at each zero as the gain tends toward infinity. Knowing the locations of zeros outside the unit circle in the z-transfer function is fundamental in designing repetitive control systems as mentioned above. Zeros outside the unit circle prevent using the inverse of the transfer function to try to produce zero tracking error.

Types of Zeros

Sampling Zeros

There are two types of zeros in the z-transfer function after converting $G(s)$ to $G(z)$. One type called sampling zeros is introduced in the conversion process. Consider a third order system that has no zero in continuous time

$$\frac{d^3y}{dt^3}+a_2\frac{d^2y}{dt^2}+a_1\frac{dy}{dt}+a_0y=u \quad G(s)=\frac{1}{s^3+a_2s^2+a_1s+a_0} \tag{2}$$

When transformed, it takes the form

$$\begin{gathered} y((k+3)T)+\alpha_2y((k+2)T)+\alpha_1y((k+1)T)+\alpha_0y(kT)=\beta_2u((k+2)T)+\beta_1u((k+1)T)+\beta_0u(kT) \\ G(z)=\frac{\beta_2z^2+\beta_1z+\beta_0}{z^3+\alpha_2z^2+\alpha_1z+\alpha_0} \end{gathered} \tag{3}$$

The original Laplace transfer function had no zero, but the resulting z-transfer function has two zeros. It must have two zeros because, when you make a change in the input at time kT, you are making the output start moving. You do not expect to see a change instantaneously at kT, but you do expect to see a change in the output at the next sample time $(k + 1)T$. This means that the most up to date input on the right hand side of the difference equation should be one step behind the most up to date output on the left hand side. Hence, in this example, two zeros are introduced in the conversion process.

We will see that there are isolated exceptions to this statement. In the case of non-minimum phase systems in continuous time $G(s)$, the unit step response goes negative initially, and then becomes positive. If the first sample time coincides with the time when the unit step response crosses zero, there will be two time steps delay in the input to output of $G(z)$, i.e. two time steps difference between the most recent entry on the right, and that on the left.

Reference [7] gives the asymptotic locations of these sampling zeros as sample time interval T tends to zero. The development of this result concludes that the zeros introduced approach the locations that are images of $n - 1$ zeros at infinity, given by computing the zeros of

$$G(z) = \left(1 - z^{-1}\right) Z\left[1/s^{n-m-1}\right] \tag{4}$$

n is the number of poles, and m is the number of zeros in continuous time, which is zero in this case. The asymptotic zero locations (as T tends to zero) introduced outside or on the unit circle are given in Table 1, and for every zero outside there is a zero inside introduced at the reciprocal location. References [1, 2] present methods to design a repetitive controller that converges to zero error tracking a periodic trajectory, compensating for the effect of these zeros on the system response by introducing more zeros outside and inside the unit circle.

Consider a third order system $G(s) = 6/[(s + 1)(s + 2)(s + 3)]$. The pole excess $n - m$ is 3 so there are two zeros introduced which asymptotically approach −3.732 and − 1/3.732 as T tends to zero. Figure 1 shows what the zero locations are for sample rates ranging from 0 to 100 Hz. One observes that these zeros are at the origin when the sample interval is so long that the system reaches steady state response to the step input by the end of each time step, and they approach their asymptotic values rather fast as sampling frequency increases. Although not in the usual format, this represents a root locus plot of these zeros as a function of sample rate. All of the introduced zeros lie on the negative real axis in the z-plane, and they all go from the

Table 1 Asymptotic location of sampling zeros

n -m	Zero locations outside and on unit circle
2	−1
3	−3.732
4	−1, −9.899
5	−2.322, −23.20
6	−1, −4.542, −51.22
7	−1.868, −8.160, −109.3
8	−1, −3.138, −13.96, −228.5
9	−1.645, −4.957, −23.14, −471.4

origin to the asymptotic locations given by the table as the sample rate goes from limiting values of zero to infinity.

Intrinsic Zeros

When there are zeros in the continuous time transfer function $G(s)$, there are images of these zeros in $G(z)$ called intrinsic zeros. If there are m intrinsic zeros, then one only needs to introduce $n - m - 1$ zeros in the discretization to have the needed one time step delay through $G(z)$ as described above. The number of sampling zeros introduced is reduced by m as indicated in the Table.

Properties of the Mapping of Intrinsic Zeros Observed from Analytical Solution of a Simple Problem

Consider the transfer function $G(s) = (s - a_1)/[(s - b_1)(s - b_2)]$. The unit step response $S(t)$ can be computed by making a partial fraction expansion and then taking the inverse transform

$$\begin{gathered} G(s)\frac{1}{s} = \frac{(s-a_1)}{(s-b_1)(s-b_2)} = \frac{A}{(s-b_1)} + \frac{B}{(s-b_2)} + \frac{C}{s} \\ S(t) = Ae^{b_1 t} + Be^{b_2 t} + Ce^{0t} \end{gathered} \tag{5}$$

where

$$A = \frac{(-b_1 + a_1)}{(-b_2 + b_1)(-b_1)} \quad B = \frac{(-b_2 + a_1)}{(-b_1 + b_2)(-b_2)} \quad C = \frac{-a_1}{b_1 b_2}$$

Converting each term to its sampled z-transform equivalent ($1/(s - \alpha)$ converts to $z/(z - e^{\alpha T})$) and putting everything over a common denominator produces $Z[G(s)/s]$. Then multiply by $(1 - z^{-1})$ to obtain the equivalent z-transfer function

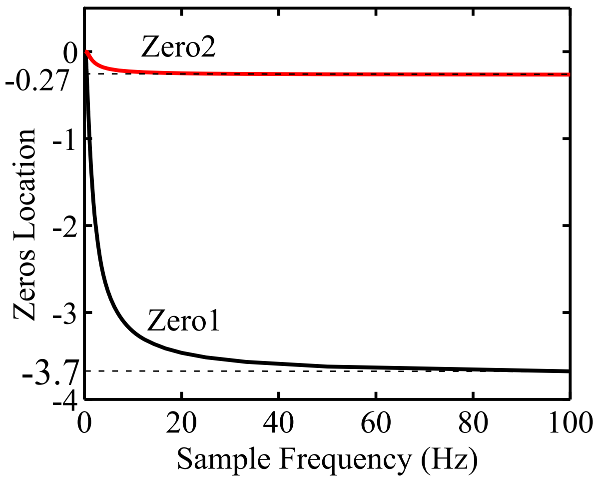

Fig. 1 Sampling zero locations for a third order system as a function of sample frequency

$$G(z) = \frac{A\left(z - e^{b_2T}\right)\left(z - e^{0T}\right) + B\left(z - e^{b_1T}\right)\left(z - e^{0T}\right) + C\left(z - e^{b_1T}\right)\left(z - e^{b_2T}\right)}{\left(z - e^{b_2T}\right)\left(z - e^{b_1T}\right)} \tag{6}$$

Property 1: Examining this procedure shows that a pole at location s in $G(s)$ maps exactly to a pole $z = e^{sT}$ in $G(z)$. It is easy to see that this is a general property. One can also see that this mapping makes the solutions of the homogeneous differential equation and of the homogeneous difference equation match at the sample times. A root of the differential equation characteristic polynomial at a point s, produces a solution e^{st}which when samples is e^{skT}. A root of the difference equation polynomial at point z produces a solution z^k. To make these match one sets $z^k = e^{skT} = (e^{sT})^k$, indicating that to match outputs the difference equation root must be $z = e^{sT}$.
Property 2: Collecting the coefficients in the zeros polynomial gives

$$\begin{aligned}
&(A + B + C)z^2 + \beta_1 z + \beta_0 = 0 \\
&\beta_1 = -\left[A\left(e^{0T} + e^{b_2T}\right) + B\left(e^{0T} + e^{b_1T}\right) + C\left(e^{b_1T} + e^{b_2T}\right)\right] \\
&= -\left[(A + B) + e^{b_1T}(B + C) + e^{b_2T}(C + A)\right] \\
&\beta_0 = Ae^{b_2T} + Be^{b_1T} + Ce^{(b_1+b_2)T}
\end{aligned}$$

The coefficient of the highest power of z is $A + B + C$ which is zero. It must be zero in order to produce the one time step delay needed from input to output. The characteristic polynomial becomes

$$\beta_1 z + \beta_0 = 0 \tag{7}$$

Property 3: Using $A + B = -C$, $B + C = -A$, and $C + A = -B$ in the second form of β_1 produces $\beta_1 = Ae^{b_1T} + Be^{b_2T} + C$. Therefore the coefficient of the highest power in the characteristic polynomial is equal to the unit step response evaluated at the sample time T

$$\beta_1 = S(T) \tag{8}$$

Numerical experience suggests that this is a general property. Note that if the first sample time in a non-minimum phase system happens to coincide with the time at which the unit step response is crossing zero, then the coefficient of z is zero, increasing the time delay through the discrete time system to two time steps.

Property 4: The zero location in$G(z)$, $z = -\beta_0/\beta_1$, is the image of the zero location a_1 in $G(s)$. The value of a_1 appears in A, B, C, and hence in β_0 and β_1, but so do the locations of the poles b_1 and b_2. Furthermore the poles appear in the exponentials

e^{b_1T} and e^{b_2T}. Unlike the mapping of the poles that exactly satisfy $z = e^{sT}$, the zeros in $G(z)$ are functions of both the zero locations and the pole locations.

Property 5: If $G(s)$ is asymptotically stable with all poles in the open left half plane, then the zero location of $G(z)$ will tend to zero as the sample time T tends to infinity. This is clear from the fact that A, B, C are constants, and that the exponentials in β_0 tend to zero, while β_1 will tend to C, the steady state response to a unit step input.

Property 6: Reference [7] presents the following result. A zero s in $G(s)$ approximately satisfies the same mapping as the poles, $z \approx e^{sT}$. For an n^{th} order system $G(s)$, the Taylor series expansion of e^{sT} will match the actual zero location through terms in T to the power n. Terms after that do not match and can involve the pole locations. The expansion for the problem considered here is

$$z_1 = 1 + a_1T + \frac{1}{2!}a_1^2T^2 + \left[\frac{1}{4}a_1^3 - \frac{1}{12}a_1\left[(b_1+b_2)a_1 + 2\left(b_1^2 + b_1b_2 + b_2^2\right)\right]\right]T^3 + \cdots \quad (9)$$

The coefficient of the T^3 term should have been $\frac{1}{3!}a_1^3$. Since T^3 is always positive, taking the difference of the coefficients tells one in which direction the zero location will deviate from the $z = e^{sT}$ mapping as T increases from zero. For example, if b_1 and b_2 approach zero, then a zero in the right half plane will initially be displaced by in the positive direction by $\frac{1}{12}a_1^3T^3$ compared to the expected location $z = e^{sT}$.

Zeros Root Locus as a Function of *T* for Systems with One Non-minimum Phase Zero

Stable System with Real Poles Having established various properties of the mapping of zeros from continuous time to discrete time, let us examine possible forms of the resulting locus of the zeros as a function of sample time interval T. Start with non-minimum phase systems, and consider a stable system with one zero on the positive real axis and two real poles

$$G_1 = -\frac{2(s-9)}{3(s+2)(s+3)} \quad (10)$$

As with all of the future systems considered, the unit step response is unity. This response is shown in Fig. 2 which illustrates that the unit step response initially goes negative, then crosses zero on its way to +1. Figure 3 shows the corresponding zero location as a function of T, and we see that there is a singularity when T crosses the time of the zero crossing. Thus, as the sample time T is increased from zero, the location of the discrete time zero goes to $+\infty$, then appears at $-\infty$ and starts moving to the right, converging to zero as sample time T tends to infinity. Of course, the usual root locus in classical control will never go to infinity and then come back again. This is the first case of the zeros locus behaving in unexpected ways, violating the behavior of the classical root locus for poles. Figure 4 gives the corresponding root locus, although it is not so

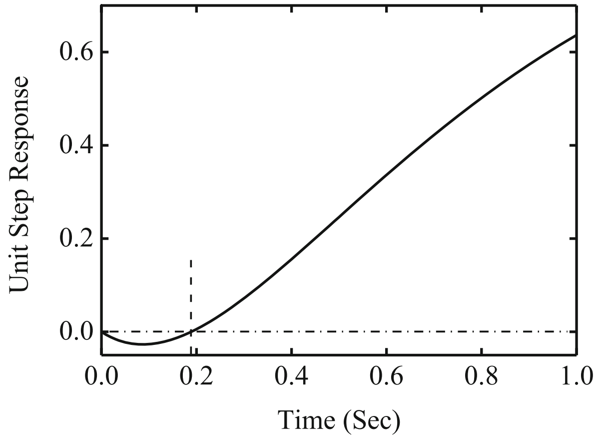

Fig. 2 The unit step response of $G_1(s)$

easy to display it. The plot starts with sample time 0.001 and continues to 1 s. Z1 labels the start point at 1.009 and the end point is at −0.2908. Of course, most of the locus does not correspond to sample time intervals that one would pick for the system, but if the time interval for which the unit step response is negative is very short, it could be reasonable to pick a T that is not small compared to the zero crossing time, in which case the location of the discrete time zero can be very volatile, moving a long distance with a very small change in sample time. If the zero crossing is very fast, one might consider it a reasonable model to have the first time step beyond this small number in which case the discrete time zero is perhaps unexpectedly on the negative real axis. In the sequel we consider the complete root locus for each system, and aim to understand what the range of possible root loci plots can look like.

Lightly Damped System Having understood the role played by zero crossings in the conversion of intrinsic zeros, we consider the following damped system having a damping ratio of 0.4

$$G_2 = -\frac{(6\pi)^2(s-1)}{s^2 + 0.4(6\pi)s + (6\pi)^2} \tag{11}$$

This damping ratio is somewhat small, but not at all extreme although the zero has a large effect on the behavior. The unit step response in Fig. 5 has 5 zero crossings, and each crossing corresponds to the location of the zero going from plus or minus infinity and then appearing at minus or plus infinity. Of course, to do this, the root locus must reverse its direction of travel and back track on top of its previous path – something that no classical root locus would ever do. The points at which the direction of motion of the root reverses is given by the points with zero slope on the plot in Fig. 5. Figure 6 presents the location of the zero as a function of T, and exhibits the expected 5 singularities.

Figure 7 is the root locus plot for the zero location in this problem as a function of T. Clearly, it is difficult to show an easily understood locus for this problem. The locus was created for T going from 0.001 to 1 s. Considerable description is necessary to

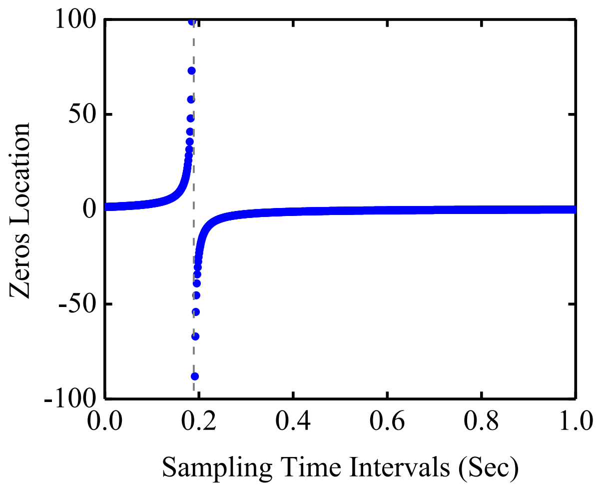

Fig. 3 Discrete time intrinsic zeros location vs. sampling period in $G_1(s)$

understand the locus. The start point corresponding to label Z1 and A is at 1.001. Of course, as T tends to zero this value converges to +1. As sample time T increases going through 0.162 the zero goes to $+\infty$, then reappears at $-\infty$, and starts moving in the direction of the origin along the negative real axis. When the T tends to infinity, the zero must converge to the origin. But the locus this time goes toward the origin, and then goes past it and turns around and returns to $-\infty$. The point at which it turns around is +0.8829 at $T = 0.284$ s indicated by point B. After T reaches 0.347 it is back at $-\infty$ and switches to $+\infty$, and starts moving toward +1. It stops before getting there at +1.2859 at $T = 0.394$ s at which point it reverses direction again, point C. This process continues 5 times. The next time it reverses direction on the positive real axis it does so at location 9.0442 with $T = 0.751$, point E, further from +1 than before. The points at which the direction reverses inside the unit circle decrease from 0.8829 for the first reversal of direction, point B, 0.6700 for the second, point D, and 0.3641 for the third, point F. The

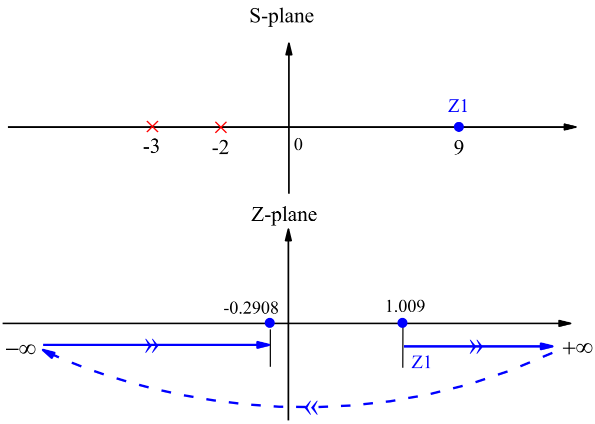

Fig. 4 Root locus plot of the zeros of $G_1(s)$ as a function of sample time T

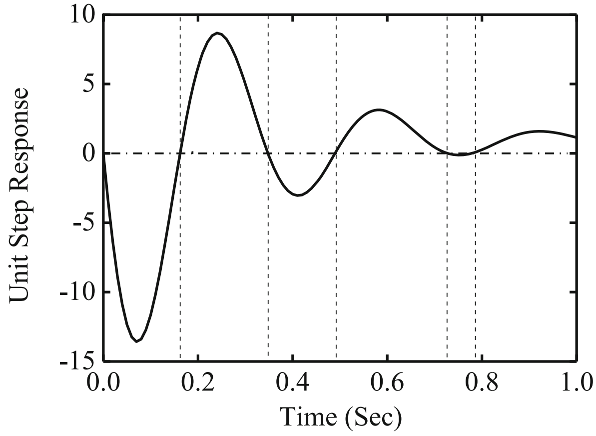

Fig. 5 Unit step response of $G_2(s)$

plot quits at G for the slowest sample rate considered, but if T were increase further, the curve would converge to the origin coming in from 0.3641 on the right.

Undamped System After seeing the results for a somewhat lightly damped system, consider what happens when there is no damping

$$G_3 = -\frac{(6\pi)^2(s-1)}{s^2 + (6\pi)^2} \tag{12}$$

Figure 8 shows the unit step response which has an infinite number of zero crossings. Figure 9 shows the zero location as a function of T. Note that this plot has a fundamentally different character than the corresponding plot with damping, there are no points at which the locus reversed direction. Hence the only points that are

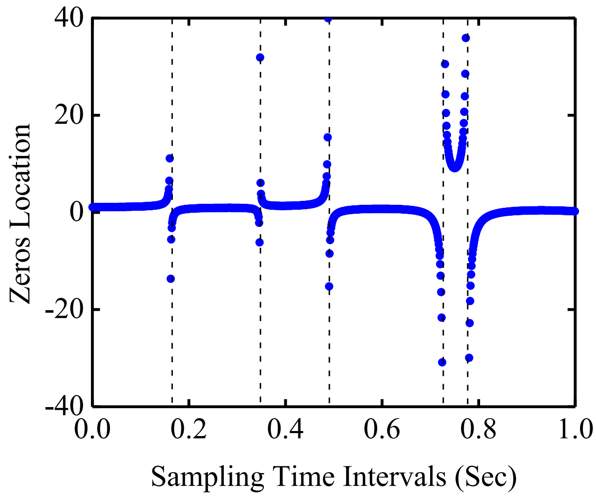

Fig. 6 Discrete time zero location for $G_2(s)$

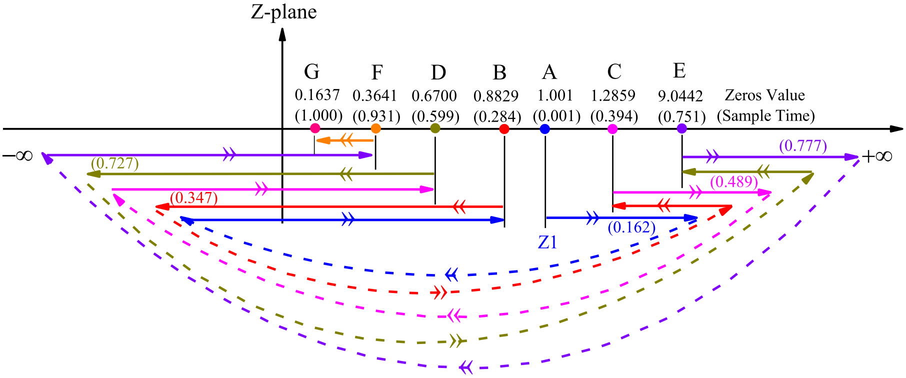

Fig. 7 Root locus plot of the path of the discrete time zero as a function of T for $G_2(s)$

singularities are the zero crossings in the unit step response that go from negative to positive, labeled H, I, J, K, L, M. Figure 10 gives the root locus plot for T from 0.001 to 2 s. Again the sampled values start at 0.001 and go to 2 s, but for $T = 0$ the plot would be at +1. Increasing T through 0.162 the plot goes to +∞ and then switches to −∞ and goes toward the origin. But for this problem the motion does not reverse direction, instead it continues through the origin, then goes through +1 and continues to +∞, and does this repeated path an infinite number of times.

Negative Damping To complete the picture, consider what happens when the damping becomes negative making an unstable system

$$G_4 = \frac{-(6\pi)^2(s-1)}{s^2-0.05(6\pi)s+(6\pi)^2} \tag{13}$$

Figure 11 gives the unit step response and Fig. 12 plots the zero location versus sample time to be compared to Figs. 5 and 6 for positive damping and Figs. 8 and 9 for zero

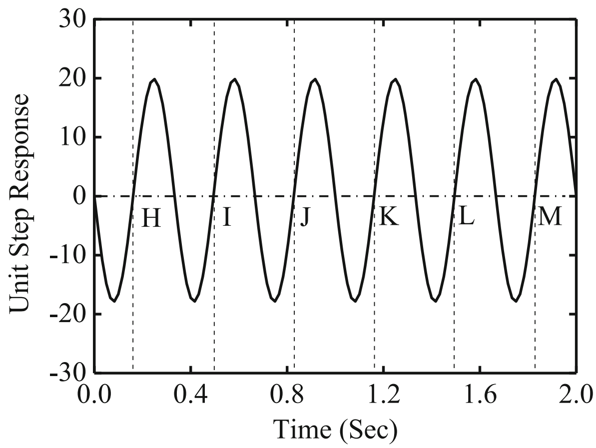

Fig. 8 Unit step response of $G_3(s)$

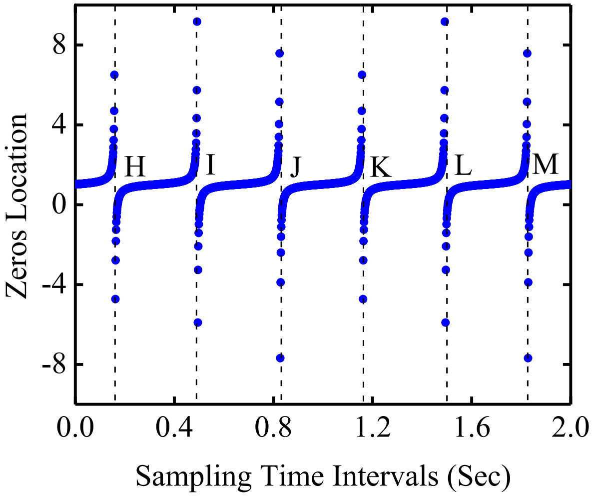

Fig. 9 Plot of zero location vs. T for undamped system $G_3(s)$

damping. Singularities at B, D, F are hard to capture in the sampling but in each case the plot shifts from negative infinity to positive infinity. The plot has the same form as for the lightly damped Fig. 6. In the no damping case, very close examination fails to reveal any singularity at these points, perhaps related to going though a pole on the unit circle which produces a 180 degree sign change in the frequency response. The resulting root locus plot is fundamentally different for zero damping, appearing to have no reversals in directions.

Root Locus of Zeros for Systems with Two Non-mimimum Phase Zeros: Complex Conjugate Pair and Repeated Zeros

Complex Conjugate Pair Consider a pair of complex conjugate non-minimum phase zeros

$$G_5 = \frac{3(s^2 + 4s + 8)}{(s+2)(s+3)(s+4)} \tag{14}$$

The unit step response has a new shape as seen in Fig. 13, first going in the positive direction, then to negative values, and back to positive values, producing two zero

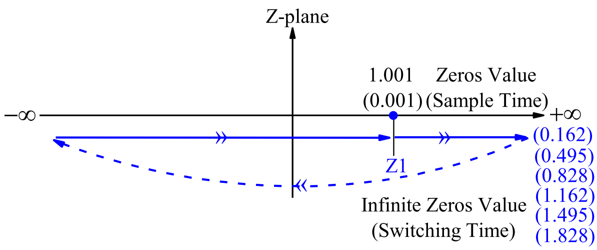

Fig. 10 Root locus plot of the zero for the undamped system $G_3(s)$

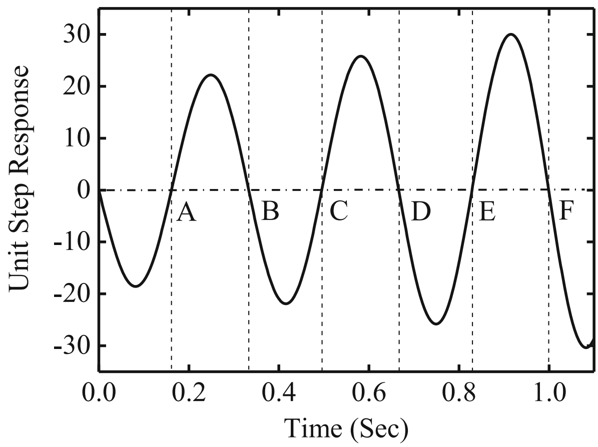

Fig. 11 Unit step response for system $G_4(s)$ having negative damping

crossings. The corresponding root loci are given in Figs. 14 and 15 which are drawn for the range of T from 0.001 to 2 s. At the very small initial T the zero locations both look the same to 5 digits at 1.0020, but of course they are different and slightly complex. Increasing T makes the two loci follow what looks like a circle both ending on the real axis when $T = 0.345$. So what originally were complex conjugate zeros in continuous time and at smaller sample rates, can map to real zeros in discrete time, and also to repeated real zeros, perhaps an unexpected result. Since experience with routine root locus plots tells us that one can easily have loci that make a perfect circle, we examine this and establish that it is not a perfect circle. After reaching the real axis, one root goes to the right getting to infinity very quickly at $T = 0.351$, after a very small change in T. The other root goes left. This is a standard phenomenon in routing root locus plots,

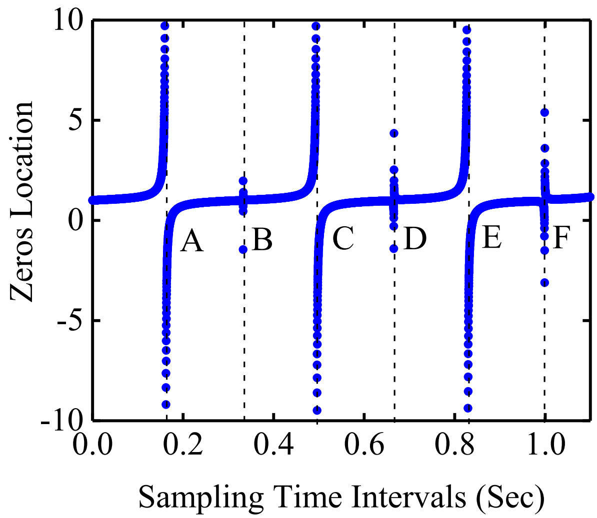

Fig. 12 The zero location as a function of T for negative damping system $G_4(s)$

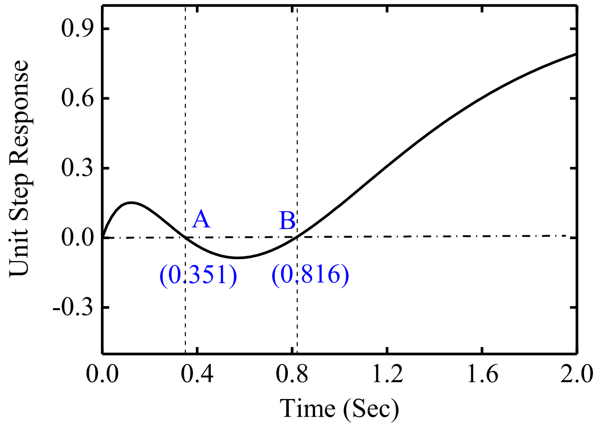

Fig. 13 Unit step response of system with a complex conjugate pair of non-minimum phase zeros $G_5(s)$

arriving on the real axis and one root goes to the right and the other goes to the left. What is not routine is that after progressing to the left to 3.8408 from the entry point at 8.7147, the locus turns around and goes back the other direction to plus infinity, chasing the other root through plus infinity and then chasing the other root toward the origin from negative infinity. When sample time reaches 2, one root is at −0.0028 while the chasing root is at −0.2324.

Repeated Real Zeros or Two Real Zeros Now consider two real zeros outside the unit circle, both at the same location

$$G_6 = \frac{6(s-2)(s-2)}{(s+2)(s+3)(s+4)} \tag{15}$$

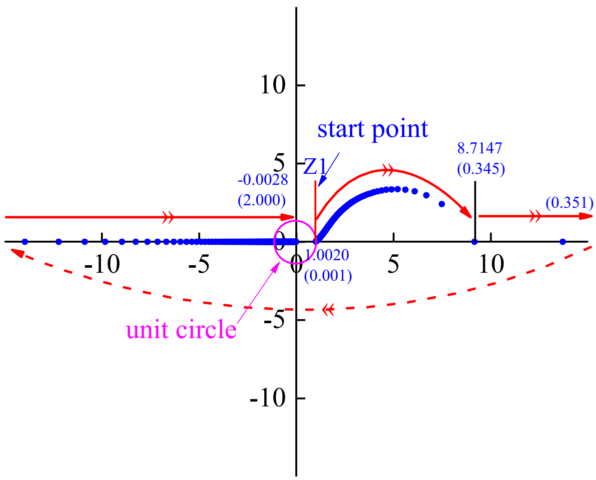

Fig. 14 Root locus plot of one of the zeros of a complex conjugate pair as a function of T for $G_5(s)$

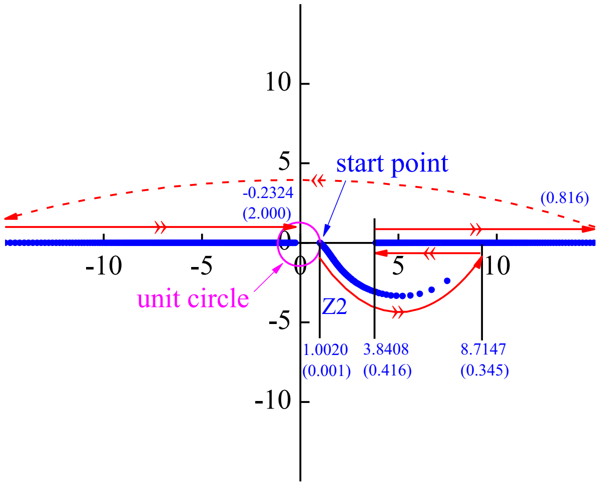

Fig. 15 Root locus of the other zero for $G_5(s)$

The unit step response is given in Fig. 16 showing two zero crossings. The two corresponding root locations are given in Figs. 17 and 18. Obviously the two zeros that were identical in continuous time choose to follow different paths in discrete time. The root locus for T from 0.001 to 2 s is given in Fig. 19, which tries to show the paths of both roots. The values of each discrete time zero at $T = 0.001$ are identical to four digits at 1.002 but are not the same. At the final sample time of 2 s, one of the roots is at −0.0031 and the other is at −0.4451. As T increases the zeros will both end approaching the origin from the left. Meanwhile, all poles are moving monotonically to the origin from the right. So the two loci are in fact the same, but the time history for one root is different than for the other root. We comment that when we separate the zeros in continuous time, so that the zero is no longer repeated, there is no qualitative difference between the results.

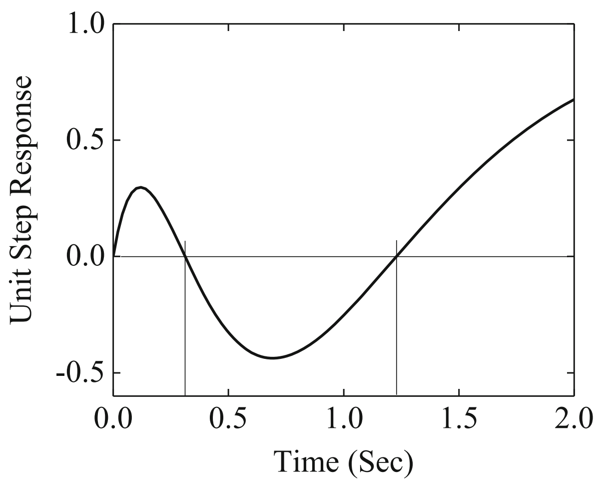

Fig. 16 Unit step response of $G_6(s)$ with a repeated non-minimum phase zero

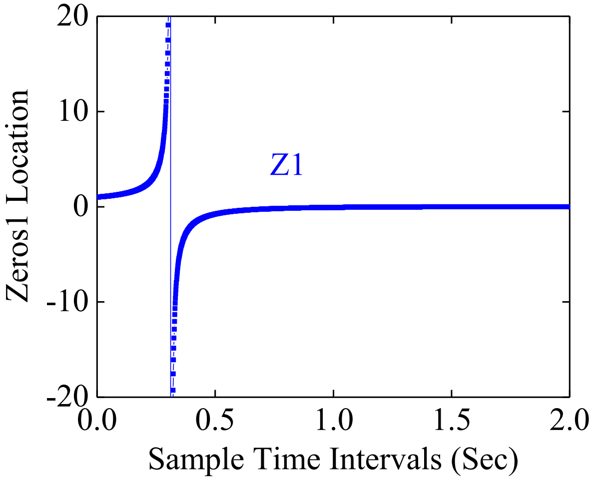

Fig. 17 One of the zero loci for the repeated zero problem $G_6(s)$

Zeros inside the Unit Circle

The above results exhibit many unexpected characteristics for the zeros loci as a function of T for non-minimum phase systems. Now consider corresponding results to minimum phase systems.

Real Zeros The following minimum phase system, has a zero in continuous time that is between the two real poles

$$G_7 = \frac{(s+1)}{(s+2)(s+0.5)} \tag{16}$$

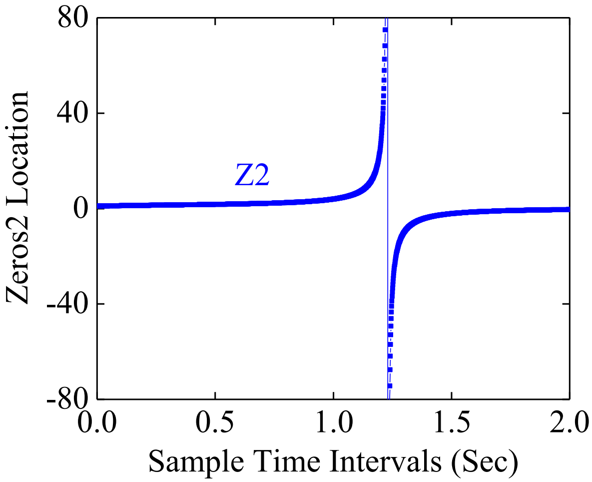

Fig. 18 The locus of the other zero for $G_6(s)$

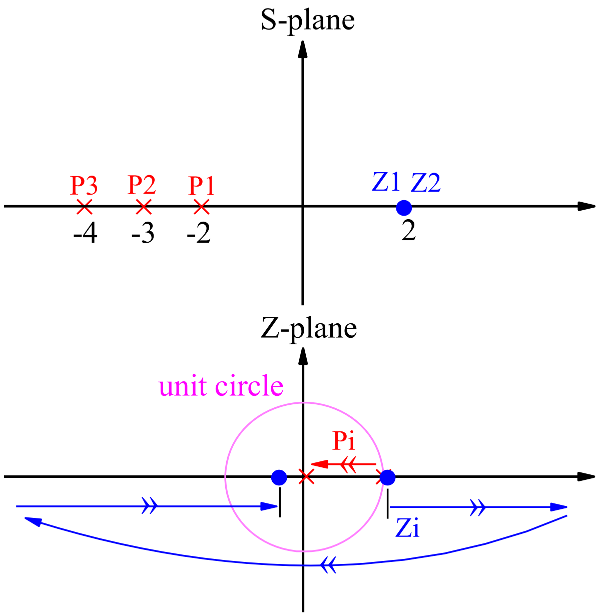

Fig. 19 Root locus plot of the two zeros of $G_6(s)$ as a function of sample time T

The paths of the zero location and the two poles are displayed in Fig. 20. And they all simply move toward the origin as T increases. If the zero is larger than both poles or smaller than both poles, the corresponding plot looks very similar, just reorder the labels on the curves to correspond to the new order.

Complex Conjugate Pair As with the non-minimum phase zero loci, the root locus for complex conjugate minimum phase zeros exhibits interesting behavior. Figure 21 gives the locus of the discrete time zeros associated with

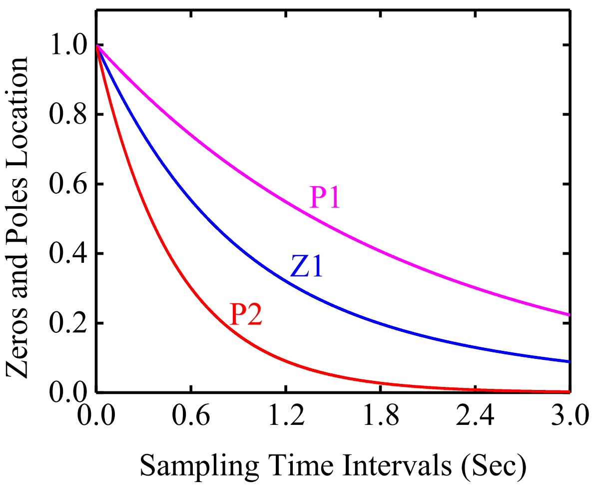

Fig. 20 Path of both zero and poles vs. T for $G_7(s)$ having one real zero inside the unit circle

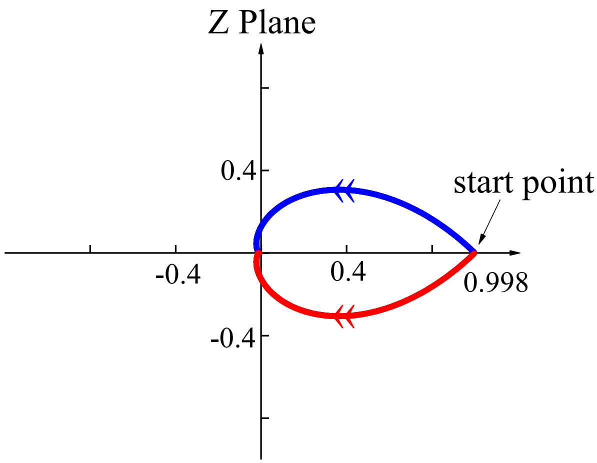

Fig. 21 Root locus plot of the two minimum phase complex conjugate zeros of $G_8(s)$

$$G_8 = \frac{3(s^2 + 4s + 8)}{(s+2)(s+3)(s+4)} \tag{17}$$

for T going from 0.001 to 2.000 s. The locus starts very near +1, with two complex roots that are indistinguishable to 3 digit accuracy, appearing as if real and located at 0.998. This time the curves are clearly not parts of a circle. Again, two complex conjugate zeros can map to real zeros as seen in the more detailed plot of Fig. 22, which exhibits the two zeros joining the real axis at −0.0132. One root goes toward the origin reaching −0.0068 when T is 2. The other root goes away from the origin reaching −0.0166 when T is 2. Of course, as T increases further it must reverse direction and go to the origin as T tends to infinity. Figure 23 gives a still more detailed view, and shows that the loci enter the real axis perpendicularly.

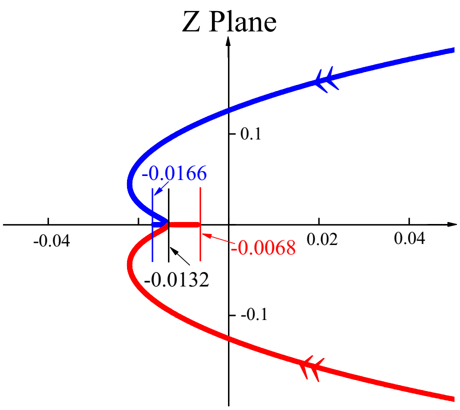

Fig. 22 Detail of the root locus of Fig. 21 for $G_8(s)$

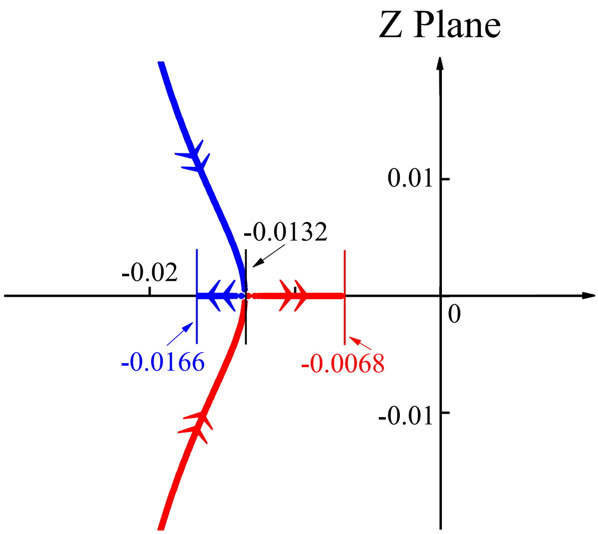

Fig. 23 A further close up of the root locus of Fig. 21 for $G_8(s)$

Sampling Zeros Together with Intrinsic Zeros

All of the above examples included only intrinsic zeros. If there are only sampling zeros, then the zeros locus plot starts at the locations given in the Table, and move toward the origin along the negative real axis as T increases. Figure 1 is a plot of the loci for a third order system with only sampling zeros, with one outside the unit circle, and one inside at the reciprocal location asymptotically as T tends to zero. One might ask, when there are both sampling zeros and intrinsic zeros, are there any new phenomena to be discovered. The unit step response of the following system

$$G_9 = \frac{24(0.1s-1)}{(s+1)(s+2)(s+3)(s+4)} \tag{18}$$

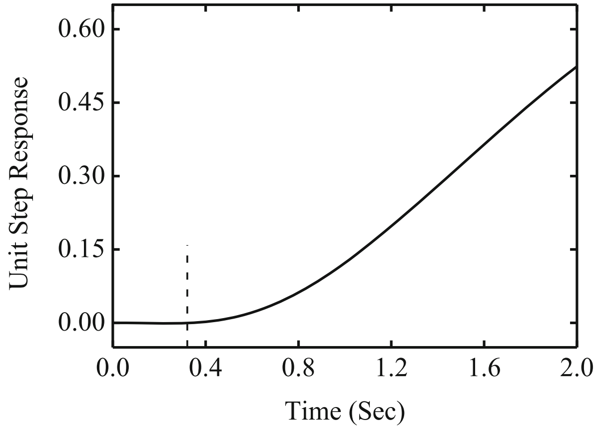

Fig. 24 The unit step response of $G_9(s)$ which has both sampling and intrinsic zeros

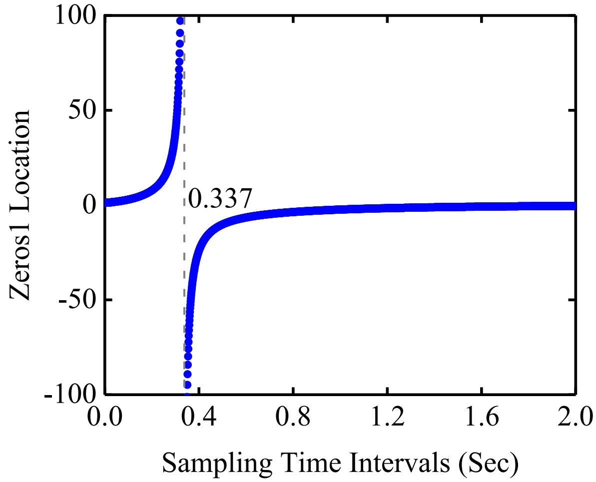

Fig. 25 The discrete time intrinsic zero location as a function of T for $G_9(s)$

is given in Fig. 24, and the sampling zero as a function of T is given in Fig. 25 with one singularity to transfer the zero from the positive real axis to the negative real axis. After this transfer, it proceeds monotonically toward the origin, while the sampling zeros do the same, Fig. 26. We comment that the two kinds of zeros never cross over each other, and that the form of the plot is just the combination of the separate behaviors for each type of zero without any apparent interaction between the two (Fig. 27).

Intrinsic Zero Location as a Function of Pole Locations

The locations of the discrete time zeros were shown in Eq. (9) to be a function of the pole locations. We examine this dependence for a stable non-minimum phase system

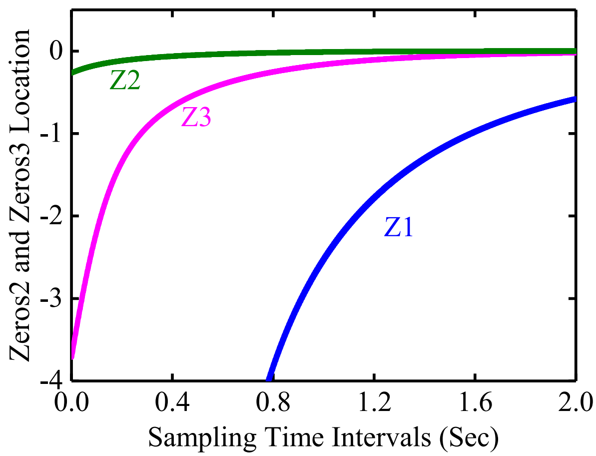

Fig. 26 Path of the intrinsic zero (Z1) and the sampling zeros (Z2 and Z3) vs. T for $G_9(s)$

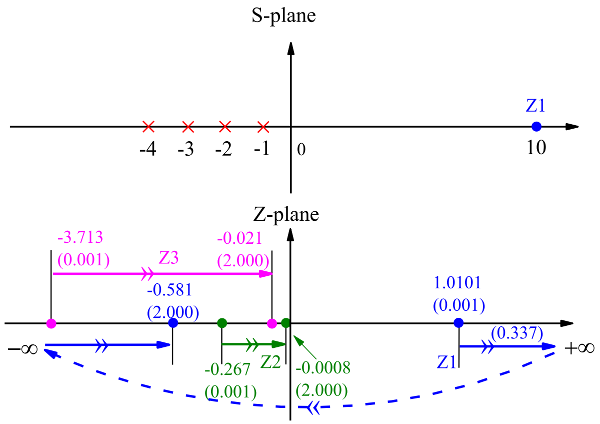

Fig. 27 Root locus plot of the intrinsic and sampling zeros for $G_9(s)$

$$G_{10} = \frac{2b_1^2(s+9)}{9(s+b_1)(s+2b_1)} \tag{19}$$

Figure 28 shows the discrete time zero locus as a function of b_1 going from 0.1 to 10 with a fixed sample time $T = 0.01$ s. The image in the z-plane of the b_1 pole moves to the left from 0.999 to 0.905. The image of the minimum phase zero as a function of b_1 is given in Fig. 29. The motion of the zero is small going between 0.91387 to 0.91394. An interesting property is that when the zero reaches that location, it reverses direction and finally ends at 0.91392. Another interesting property is that clearly there is some sample time T for which the intrinsic zero is mapped on top of one of the poles. With two poles in the system there can be two such points. Therefore, it is possible for a continuous time system with distinct zero and pole locations to map to discrete time and produce in a system with pole-zero cancellation.

If the $(s+9)$ is changed to $(s-9)$ no new properties are introduced, the pole inside the unit circle moves monotonically toward the origin between the same values, while

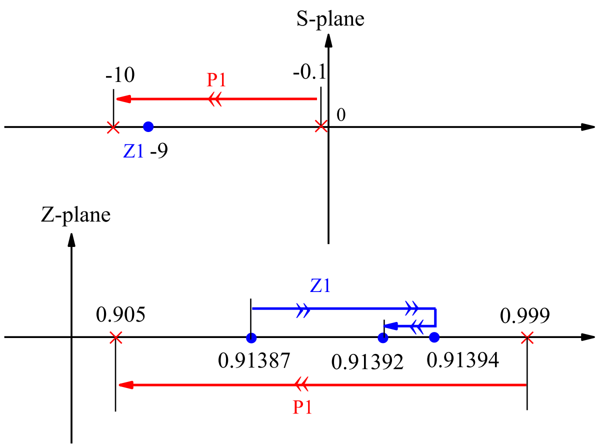

Fig. 28 Locus of the discrete time zero location of $G_{10}(s)$ as a function of the pole location $-b_1$

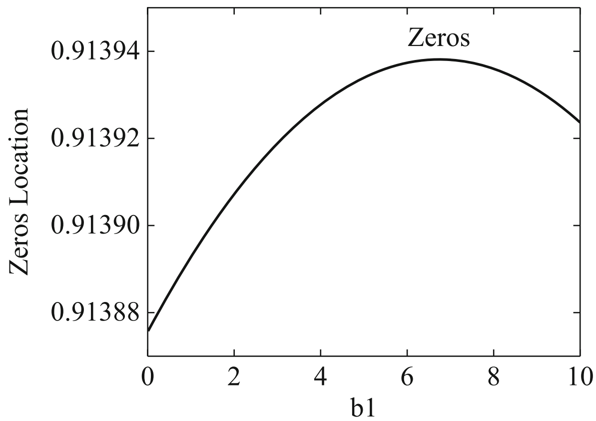

Fig. 29 The discrete time zero location of $G_{10}(s)$ as pole $-b_1$ moves from −0.1 to −10

the zero outside the unit circle moves monotonically in the opposite direction from 1.0942 to 1.0946. If one considers a non-minimum phase system that is unstable, the same interesting properties observed for Eq. (19) occur again.

Conclusions

The locus of the zero locations of discrete time transfer functions as a function of the sample rate can exhibit unexpected behavior:

(1) A non-minimum phase system in continuous time can be a minimum phase system in discrete time.
(2) In particular, a zero on the positive real axis of the s-plane can map not only to the expected positive real axis in the z-plane outside the unit circle, but can map to: (i) a zero on the negative real axis outside the unit circle, (ii) a zero on the negative real axis inside the unit circle, and (iii) a zero on the positive real axis inside the unit circle.
(3) A complex conjugate pair of zeros can map not only to complex conjugate zeros in discrete time, but also to: (i) distinct real valued zeros, (ii) and to repeated real zeros.
(4) Poles and zeros in continuous time that are distinct, can result in pole zero cancellation in discrete time.
(5) Loci of zeros can easily move in one direction and then decide to go back the direction they came.
(6) There are singularities in the zero location as sample time interval T is increased for non-minimum phase systems. This can be of practical importance. For a system that crosses the real axis very quickly, the zero location can be vary volatile, changing by a large amount for a very small change in T.
(7) One can create a system where a non-minimum phase zero mapped to discrete time goes to plus infinity, re-appears at minus infinity, comes in toward zero or

even past zero, turns around and goes out to negative infinity, then appears at positive infinity, comes in, turns around – and repeats this an infinite number of times as T goes to infinity.

This paper develops an understanding of how these unexpected results can happen. And gives a general idea of what the possible zero loci can look like for the whole locus from T equal zero to T equal infinity.

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Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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