Mathematical Sciences (2024) 18:291–303
https://doi.org/10.1007/s40096-022-00501-0

ORIGINAL RESEARCH

Exploration of some novel solutions to a coupled Schrödinger–KdV equations in the interactions of capillary-gravity waves

Dipankar Kumar[1] · Ahmet Yildirim[2] · Mohammed K. A. Kaabar[3] · Hadi Rezazadeh[4] · Mohammad Esmael Samei[5]

Received: 17 December 2021 / Accepted: 4 December 2022 / Published online: 16 December 2022

Abstract

Some novel solutions to a system of coupled Schrödinger–Korteweg–de Vries equations are explored in this work by employing the extended sinh-Gordon equation expansion method to the proposed system. Some novel forms of explicit complex hyperbolic and complex trigonometric function solutions such as singular, combined singular, dark, bright, combined dark–bright, periodic wave, dipole soliton, and other solutions are retrieved and explored into their corresponding system via MAPLE software. Two- and three-dimensional graphs are provided to illustrate this study's novelty. All combined solutions are particularly new in the interactions of capillary-gravity water waves. Extended sinh-Gordon equation expansion method provides an effective tool to explore new precise wave solutions and overcome the difficulties of the ansatz method. All our results in this work play an essential role in explaining various phenomena in ocean and coastal engineering.

Keywords Soliton solutions · Coupled Schrödinger–KdV equations · Extended sinh-Gordon equation expansion method

Introduction

There is no vulnerability that numerous scenarios in nature can be outlined beneath the modeling of nonlinear partial differential equations ($\mathbb{NPDE}$s) with their explanatory or numerical arrangements. To understand such scenarios to $\mathbb{NPDE}$s by their explanatory arrangements (analytic solutions), mathematicians, engineers, and researchers have a common connection in examining the traveling wave solutions to $\mathbb{NPDE}$s by expository strategies due to their critical part and pertinence in liquid mechanics, scientific material science, plasma material science, nonlinear optics, and other related building sciences [1–8]. Many recent studies have employed fractional-order calculus as a tool in studying the fractional-order differential equations and their obtained solutions [9, 10, 40–43, 45].

Coupled nonlinear Schrödinger–KdV equations (Schrödinger–Korteweg–de vries) are considered as one of the vital models related to such of inquire about areas. Colorado [11] investigated the presence of bound and ground states for a framework of coupled nonlinear Schrödinger–KdV equations. A few studies have been conducted on developing modern traveling wave solutions for a system of coupled Schrödinger–KdV equations by utilizing different expository and numerical strategies, such as the strategy of generalized amplified tanh function,

✉ Mohammed K. A. Kaabar
mohammed.kaabar@wsu.edu

Dipankar Kumar
dks.bsmrstu@gmail.com

Ahmet Yildirim
yahmet49ege@gmail.com

Hadi Rezazadeh
h.rezazadeh@ausmt.ac.ir

Mohammad Esmael Samei
mesamei@basu.ac.ir; mesamei@gmail.com

[1] Department of Mathematics, Bangabandhu Sheikh Mujibur Rahman Science and Technology University, Gopalganj 8100, Bangladesh

[2] Department of Mathematics, Faculty of Science, Ege University, 35100 Bornova, Izmir, Turkey

[3] Institute of Mathematical Sciences, Faculty of Science, Universiti Malaya, 50603 Kuala Lumpur, Malaysia

[4] Faculty of Modern Technologies Engineering, Amol University of Special Modern Technologies, Amol, Iran

[5] Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan, Iran

Laplace decomposition, Petrov–Galerkin, homotopy perturbation, Adomian's decomposition, homotopy analysis, Nehari manifold, and solitary wave and Bifurcation [12–16, 6, 18, 46]. Ma provided a brief overview of soliton solutions obtained through the Hirota direct method, discussed bilinear formulation of soliton solutions in both (1+1)-dimensions and (2+1)-dimensions together with applications to various integrable equations and analyzed the Hirota conditions for N-soliton solutions [19]. In [20], they discussed about how to construct and classify nonlocal PT-symmetric integrable equations via nonlocal group reductions of matrix spectral problems which are used to formulate a kind of Riemann–Hilbert problems and thus inverse scattering transforms (for more instance, consider [21, 22, 44]).

Kaya *et al.* and Alomari *et al.* considered the following Schrödinger–KdV equations

$$i\mathfrak{g}_s + \mathfrak{g}_{yy} - \mathfrak{g}\mathfrak{h} = 0,$$
$$\mathfrak{h}_s + 6\mathfrak{g}\mathfrak{h}_y + \mathfrak{h}_{yyy} - \left(|\mathfrak{g}|^2\right)_y = 0,$$

under an initial condition $\mathfrak{g}(y,0) = \rho_1(y)$ and $\mathfrak{h}(y,0) = \rho_2(y)$ [12, 17]. In 2015, Colorado introduced the dimensionless form of a system of coupled nonlinear Schrödinger–KdV equations

$$i\mathfrak{g}_s + \mathfrak{g}_{yy} - \eta\mathfrak{g}\mathfrak{h} + \left(|\mathfrak{g}|^2\right)\mathfrak{g} = 0, \quad (1a)$$

$$i\mathfrak{h}_s + \mathfrak{h}_{yyy} + \mathfrak{h}\mathfrak{h}_y - \frac{1}{2}\eta\left(|\mathfrak{g}|^2\right)_y = 0, \quad (1b)$$

where η is a real coupling constant, $\mathfrak{g}(y,s)$ and $\mathfrak{h}(y,s)$ are complex functions that stand for the short wave profile and real function that represents long wave profile, respectively [11]. Equation (1a) and (b) is well-known models that have been shown up within the marvels of intelligent between brief and long dispersive waves emerging from liquid mechanics, particularly the intuitive of capillary-gravity water waves. Álvarez-Caudevilla *et al.* in [13] analyzed the existence of solutions of the following higher-order system, coupling nonlinear Schrödinger–KdV equations

$$i\mathfrak{g}_s - \mathfrak{g}_{yyyy} + |\mathfrak{g}|^2\mathfrak{g} + \eta\mathfrak{g}\mathfrak{g}\mathfrak{h} = 0,$$
$$\mathfrak{h}_s - \mathfrak{h}_{yyyyy} + \frac{1}{2}(|\mathfrak{h}|\mathfrak{h})_y + \frac{1}{2}\eta\left(|\mathfrak{g}|^2\right)_y = 0,$$

with $\mathfrak{g} = \mathfrak{g}(y,s) \in (C)$, $\mathfrak{h} = \mathfrak{h}(y,s) \in \mathbb{R}$ and $\eta \in \mathbb{R}$ the coupling parameter. Also, Baskonus *et al.* extracted some optical soliton solutions from the following decoupled nonlinear Schrödinger–KdV equation with Kerr law nonlinearity arising in dual-core optical fibers by using the extended sinh-Gordon equation expansion method,

$$i\left(\mathfrak{g}_y + \eta_1\mathfrak{h}_s\right) + \eta_2\mathfrak{g}_{ss} + \eta_3|\mathfrak{g}|^2\mathfrak{g} + \eta_4\mathfrak{h} = 0,$$
$$i\left(\mathfrak{h}_y + \eta_1\mathfrak{g}_s\right) + \eta_2\mathfrak{h}_{ss} + \eta_3|\mathfrak{h}|^2\mathfrak{h} + \eta_4\mathfrak{g} = 0,$$

where $\mathfrak{g}$, $\mathfrak{h}$ are field envelopes, y is the propagation co-ordinate, $\frac{1}{\eta_1}$ is the group velocity mismatch, η_2 is the group velocity dispersion, η_4 is the linear coupling coefficient, and η_3 is defined as $\eta_3 = \frac{2\pi m_2}{kB_{eff}}$, where m_2 is the nonlinear refractive index, k is the wavelength, and B_{eff} is effective mode area of each wavelength [23]. Soliton is characterized as a localized waveform that engenders along the framework with steady speed and undeformed shape [24]. It can be found in all fields of nonlinear dynamics for a variety of shapes, such as singular, combined singular soliton, dark, bright, combined dark–bright, and many other related forms. Such soliton shapes are explored with various nonlinear integer and fractional-order evolution equations via newly generalized or modified integration schemes such as the extended Jacobi's elliptic function approach [25], the sinh-Gordon equation expansion method [26, 27], the generalized $\exp(\varrho(\tau))$-expansion method [28], the modified Kudryashov method [29], and the semi-inverse variational principle [30].

The extended sinh-Gordon equation expansion method has never been considered in the framework of Equ. 1a) and (b). In any case, there are a few investigations that have been considered on understanding other frameworks of ℕℙ𝔻𝔼s by extended sinh-Gordon equation expansion method [23, 31–34]. Extended sinh-Gordon equation expansion method could be considered as a vigorous and capable method that can be effectively connected to both numbers and fragmentary arranged ℙ𝔻𝔼s to build different shapes of soliton arrangements such as particular, singular, combined singular soliton, dark, bright, combined dark–bright, and other related solutions, and to overcome some challenges emerging from utilizing the single wave ansatz method [35, 36]. In addition, Bulut *et al.* [37] applied the extended sinh-Gordon equation expansion method for their formulated space-time fractional nonlinear Schrödinger equation in the sense of conformable derivatives, and they successfully obtained several solitons solutions such as dark, bright, combined dark–bright, singular, combined-singular, and singular periodic wave solutions. For more information about conformable derivative which is a type of local fractional derivative and some approximate-analytical methods for solving second-order wave equation, we refer to [38].

The most objective of this investigation is to explore some novel shapes of unequivocal complex hyperbolic and complex trigonometric work arrangements by utilizing the extended sinh-Gordon equation expansion method in different shapes such as dull, shinning, combined dark–bright, particular, combined particular optical, occasional wave, dipole soliton and other related arrangements in a framework of coupled nonlinear Schrödinger–KdV Equ. 1).

This work is organized as: In Sect. 2, we step-by-step examine ℕℙ𝔻𝔼s. The arrangement strategy of the overseeing framework of coupled nonlinear Schrödinger–KdV equations is explored in Sect. 3 through the extended sinh-Gordon equation expansion method. In Section 3, we build different shapes of soliton arrangements such as singular, combined singular optical, dark, bright, combined dark–bright, periodic wave, dipole soliton and other arrangements. The found soliton arrangements and their graphical representations are discussed in Sect. 4. In Sect. 5, we conclude our investigation.

Outlines of the extended sinh-Gordon equation expansion method

In this portion, a point-by-point portrayal of the extended sinh-Gordon equation expansion method is displayed. The exp-function method has been introduced to nonlinear equations, particularly KdV equation, and it has been discussed in detail in [39] where generalized solitary and periodic solutions can be resulted from the usage of this method. To utilize our strategy, we require the following steps from [32, 33].

Step 1: Consider the common shape of a coupled ℕℙ𝔻𝔼s:

$$\mathfrak{G}_1\left(\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_{\mathrm{y}}, \mathfrak{h}_{\mathrm{y}}, \mathfrak{g}_{\mathrm{s}}, \mathfrak{h}_{\mathrm{s}}, \mathfrak{g}_{\mathrm{yy}}, \mathfrak{h}_{\mathrm{yy}}, \mathfrak{g}_{\mathrm{ys}}, \mathfrak{h}_{\mathrm{ys}}, \mathfrak{g}_{\mathrm{ss}}, \mathfrak{h}_{\mathrm{ss}}, \dots\right) = 0, \quad (2)$$

$$\mathfrak{G}_2\left(\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_{\mathrm{y}}, \mathfrak{h}_{\mathrm{y}}, \mathfrak{g}_{\mathrm{s}}, \mathfrak{h}_{\mathrm{s}}, \mathfrak{g}_{\mathrm{yy}}, \mathfrak{h}_{\mathrm{yy}}, \mathfrak{g}_{\mathrm{ys}}, \mathfrak{h}_{\mathrm{ys}}, \mathfrak{g}_{\mathrm{ss}}, \mathfrak{h}_{\mathrm{ss}}, \dots\right) = 0, \quad (3)$$

where $\mathfrak{g} = \mathfrak{g}(\mathrm{y}, \mathrm{s})$, $\mathfrak{h} = \mathfrak{h}(\mathrm{y}, \mathrm{s})$ are characterized in Equation (1), and each of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ may be a polynomial work, with respect to a few capacities or indicated factors, which contains the nonlinear terms and most elevated arranged subordinates of $\mathfrak{g}$ and $\mathfrak{h}$. We begin by presenting the complex wave changes:

$$\mathfrak{u}(\mathrm{y}, \mathrm{s}) = \phi(\tau)\exp(\mathrm{i}\varrho(\mathrm{y}, \mathrm{s})), \quad (4)$$

$$\mathfrak{U}(\mathrm{y}, \mathrm{s}) = \Phi(\tau), \quad (5)$$

where $\tau = \mathrm{y} - \nu\mathrm{s}$ and $\varrho(\mathrm{y}, \mathrm{s}) = k\mathrm{y} + \omega\mathrm{s} + \vartheta_0$. Then, by substituting Equs. (4) and (5) into Equs. (2) and (3), we have:

$$\mathfrak{H}_1\left(\phi, \Phi, \frac{\mathrm{d}}{\mathrm{d}\tau}\phi, \frac{\mathrm{d}}{\mathrm{d}\tau}\Phi, \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\phi, \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\Phi, \dots\right) = 0, \quad (6)$$

$$\mathfrak{H}_2\left(\phi, \Phi, \frac{\mathrm{d}}{\mathrm{d}\tau}\phi, \frac{\mathrm{d}}{\mathrm{d}\tau}\Phi, \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\phi, \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\Phi, \dots\right) = 0, \quad (7)$$

here each of $\mathfrak{H}_\iota$ $(\iota = 1, 2)$ is a polynomial of $\phi(\tau)$ and $\Phi(\tau)$. The delineations of k, ν, ω and ϑ are shown in Sect. 3.

Step 2: Presently, we consider the formal arrangements of Equs. (6) and (7) as takes after:

$$\phi(\wp) = \sum_{\iota=1}^{\mathrm{N}_1} \cosh^{\iota-1}\wp\left[\bar{b}_\iota \sinh\wp + \bar{a}_\iota \cosh\wp\right] + \bar{a}_0, \quad (8)$$

$$\Phi(\wp) = \sum_{\iota=1}^{\mathrm{N}_2} \cosh^{\iota-1}\wp\left[\bar{B}_\iota \sinh\wp + \bar{A}_\iota \cosh\wp\right] + \bar{A}_0, \quad (9)$$

here $\wp(\tau)$ satisfies the following equation [32]:

$$\wp' = \sqrt{\mu + \lambda \sinh^2\wp}. \quad (10)$$

This is a transformed form of sinh-Gordon equation. The parameters μ and λ have different values in the following two cases:

Case I. (Equ.10) is reduced to

$$\wp' = \sinh(\wp), \quad (11)$$

whenever $\mu = 0$ and $\lambda = 1$. This can be a streamlined shape of the sinh-Gordon equation. Thus, Equ. (11) concedes the following solutions [32]

$$\sinh(\wp) = \pm\mathrm{i}\,\mathrm{sech}\,\tau, \qquad \cosh\wp = -\tanh\tau, \quad (12)$$

here $(\mathrm{i} = \sqrt{-1})$, and

$$\sinh\wp = \pm\,\mathrm{csch}\,\tau, \qquad \cosh\wp = -\coth\tau. \quad (13)$$

Therefore, from $\phi(\tau)$ and $\Phi(\tau)$ in Equs. (8) and (9), we obtain:

$$\phi(\tau) = \sum_{\iota=1}^{\mathrm{N}_1} (-\tanh\tau)^{\iota-1}\left[\pm\mathrm{i}\,\overline{b}_\iota\,\mathrm{sech}\,\tau - \overline{a}_\iota \tanh\tau\right] + \overline{a}_0, \quad (14)$$

$$\Phi(\tau) = \sum_{\iota=1}^{\mathrm{N}_2} (-\tanh\tau)^{\iota-1}\left[\pm\mathrm{i}\,\overline{B}_\iota\,\mathrm{sech}\,\tau - \overline{A}_\iota \tanh\tau\right] + \overline{A}_0, \quad (15)$$

and

$$\phi(\tau) = \sum_{\iota=1}^{\mathrm{N}_1} (-\coth\tau)^{\iota-1}\left[\pm\overline{b}_\iota\,\mathrm{csch}\,\tau - \overline{a}_\iota \coth\tau\right] + \overline{a}_0, \quad (16a)$$

$$\Phi(\tau) = \sum_{\iota=1}^{\mathrm{N}_2} (-\coth\tau)^{\iota-1}\left[\pm\overline{B}_\iota\,\mathrm{csch}\,\tau - \overline{A}_\iota \coth\tau\right] + \overline{A}_0. \quad (16b)$$

Case II. Equ. (10) becomes

$$\wp' = \cosh\wp, \quad (17)$$

whenever $\mu = 1$ and $\lambda = 1$. Typically too a streamlined shape of the sinh-Gordon equation. Essentially, Equation (17) concedes the following solutions [32]:

$$\sinh \wp = \tan \tau, \qquad \cosh \wp = \pm \sec \tau, \tag{18}$$

and

$$\sinh \wp = -\cot \tau, \qquad \cosh \wp = \pm \csc \tau. \tag{19}$$

So, from Equs.8) and (9), we obtain

$$\phi(\tau) = \sum_{\iota=1}^{N_1} (\pm \sec \tau)^{\iota-1} \left[\overline{b}_\iota \tan \tau \pm \overline{a}_\iota \sec \tau\right] + \overline{a}_0, \tag{20}$$

$$\Phi(\tau) = \sum_{\iota=1}^{N_2} (\pm \sec \tau)^{\iota-1} \left[\overline{B}_\iota \tan \tau \pm \overline{A}_\iota \sec \tau\right] + \overline{A}_0, \tag{21}$$

and

$$\phi(\tau) = \sum_{\iota=1}^{N_1} (\pm \csc \tau)^{\iota-1} \left[-\overline{b}_\iota \cot \tau \pm \overline{a}_\iota \csc \tau\right] + \overline{a}_0, \tag{22}$$

$$\Phi(\tau) = \sum_{\iota=1}^{N_2} (\pm \csc \tau)^{\iota-1} \left[-\overline{B}_\iota \cot \tau \pm \overline{A}_j \csc \tau\right] + \overline{A}_0. \tag{23}$$

Step 3: By substituting the values of N_1 and N_2 into Equs. (8) and (9), which are determined by using the homogeneous balance principle, along with Equation (11), we have a nonlinear framework of conditions in terms of $\sinh^\iota(\wp)$ and $\cosh^\iota(\wp)$. Setting up the coefficients of $\sinh^\iota(\wp)$ and $\cosh^\iota(\wp)$ to zero, we have frameworks of conditions and the values of $\overline{A}_\iota$, $\overline{B}_\iota$, $\overline{a}_\iota$, $\overline{b}_\iota$ and ω, ν are found. Now, by substituting the results into Equs. (14) and (15), we can retrieve the solitary wave solutions of Equs. (2) and (3) (as in Case I). As a result, one may continue the same way for Case II and can get the particular and occasional wave arrangements of Equs. (2) and (3).

Investigation of single and other wave arrangements

We consider Equs. (1a) and (b) as traveling wave transformations such that k is the soliton frequency, while ω is the wave number of the soliton and ϑ_0 is a phase constant [11]. On the other hand, ν is the speed of the soliton [11]. By part the real and imaginary parts of Equ. 1a) and (b), individually, we have

$$\phi'' - \left(k^2 + \omega\right)\phi - \sigma\phi\Phi + \phi^3 = 0, \tag{24}$$

$$\Phi'' - 2k\Phi - \frac{1}{2}\sigma\phi^2 + \frac{1}{2}\Phi^2 = 0, \tag{25}$$

and $\nu\phi' - 2k\phi' = 0$. Indeed, $\nu = 2k$. It gives the speed of the soliton in terms of the soliton recurrence.

Here, a recently compelling adaptation of the extended sinh-Gordon equation expansion method is utilized to investigate modern single wave and other arrangements of Equs. (24) and (25).

For case I: $\wp' = \sinh \wp$

If we take $N_1 = 1$ and $N_2 = 2$ in Equs. (8)–(9), (14)–(15) and (16a)–(b), at that point, we get the formal arrangements of Equs. (24) and (25) as taken after:

$$\phi(\wp) = \overline{b}_1 \sinh \wp + \overline{a}_1 \cosh \wp + \overline{a}_0, \tag{26}$$

$$\begin{aligned}\Phi(\wp) = {} & \overline{B}_1 \sinh \wp + \overline{B}_2 \sinh \wp \cosh \wp \\ & + \overline{A}_1 \cosh \wp + \overline{A}_2 \cosh^2 \wp + \overline{A}_0,\end{aligned} \tag{27}$$

and

$$\phi(\tau) = \mathrm{i}\,\overline{b}_1 \operatorname{sech} \tau - \overline{b}_1 \tanh \tau + \overline{b}_0, \tag{28}$$

$$\begin{aligned}\Phi(\tau) = {} & \mathrm{i}\,\overline{B}_1 \operatorname{sech} \tau - \mathrm{i}\,\overline{B}_2 \tanh \tau \operatorname{sech} \\ & - \overline{A}_1 \tanh \tau + \overline{A}_2 \tanh^2 \overline{A} + \overline{A}_0,\end{aligned} \tag{29}$$

$$\phi(\tau) = \overline{b}_1 \operatorname{csch} \tau - \overline{a}_1 \coth \tau + \tau_0, \tag{30}$$

$$\begin{aligned}\Phi(\tau) = {} & \overline{B}_1 \operatorname{csch} \overline{B} - \overline{B}_2 \coth \tau \operatorname{csch} \tau \\ & - \overline{A}_1 \coth \tau + \overline{A}_2 \coth^2 \tau + \overline{A}_0,\end{aligned} \tag{31}$$

where either $\overline{A}_1$ or $\overline{A}_2$ or $\overline{B}_1$ or $\overline{B}_2$ or $\overline{a}_1$ or $\overline{b}_1$ may be zero, but not all of them ended up zero at the same time. By substituting Equs. (26) and (27) into Equs. (24) and (25), we obtain a system of nonlinear algebraic equation ($\mathbb{NAE}$). Then, by solving the system, we have the following results:

Set 1:

$$-\omega = \frac{1}{2}\sigma^3 + k^2 + 2k\sigma + \frac{1}{12}\sigma^2 + 8\sigma + 2,$$
$$\overline{A}_0 = \frac{1}{2}\sigma^2 + 2k + \frac{1}{12}\sigma + 8,$$

where

$$\overline{A}_1 = \overline{B}_1 = \overline{B}_2 = \overline{a}_0 = \overline{b}_1 = 0,$$

and $\overline{A}_2 = -12, \overline{a}_1 = \pm \mathrm{i}\sqrt{12\sigma + 2}$. By substituting Set 1 into Equations (28)–(29) and (30)–(31), the following solitary wave solutions for Equ. (1a)–(b) which can be determined as:

$$\mathfrak{u}_{1,2}(\mathrm{y},\mathrm{s}) = \pm\mathrm{i}\left(\sqrt{12\sigma+2}\tanh(\mathrm{y}-2k\mathrm{s})\right) \times \exp\left(\mathrm{i}\left(k\mathrm{y} - \left(\frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2+8\sigma+2\right)\mathrm{s}+\vartheta_0\right)\right), \tag{32}$$

$$\mathfrak{U}_{1,2}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+8-12\tanh^2(\mathrm{y}-2k\mathrm{s}), \tag{33}$$

and

$$\mathfrak{u}_{3,4}(\mathrm{y},\mathrm{s}) = \pm\mathrm{i}\left(\sqrt{12\sigma+2}\coth(\mathrm{y}-2k\mathrm{s})\right) \times \exp\left(\mathrm{i}\left(k\mathrm{y} - \left(\frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2+8\sigma+2\right)\mathrm{s}+\vartheta_0\right)\right), \tag{34}$$

$$\mathfrak{U}_{3,4}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+8-12\coth^2(\mathrm{y}-2k\mathrm{s}). \tag{35}$$

Set 2:

$$-\omega = \frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2-4\sigma-1,$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+8,$$
$$\overline{A}_1 = \overline{B}_1 = \overline{B}_2 = \overline{a}_0 = \overline{a}_1 = 0,$$

and $\overline{A}_2 = -12, \overline{b}_1 = \pm\mathrm{i}\sqrt{12\sigma+2}$. Again by substituting Set 2 into Equs. (28)–(29) and (30)–(31), we obtain here wave solutions

$$\mathfrak{u}_{5,6}(\mathrm{y},\mathrm{s}) = \mp\mathrm{i}\left(\sqrt{12\sigma+2}\,\mathrm{sech}\,(\mathrm{y}-2k\mathrm{s})\right) \times \exp\left(\mathrm{i}\left(k\mathrm{y} - \left(\frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2-4\sigma-1\right)\mathrm{s}+\theta_0\right)\right), \tag{36}$$

$$\mathfrak{U}_{5,6}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+8-12\tanh^2(\mathrm{y}-2k\mathrm{s}), \tag{37}$$

and

$$\mathfrak{u}_{7,8}(\mathrm{y},\mathrm{s}) = \mp\mathrm{i}\left(\sqrt{12\sigma+2}\,\mathrm{csch}\,(\mathrm{y}-2k\mathrm{s})\right) \times \exp\left(\mathrm{i}\left(k\mathrm{y} - \left(\frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2-4\sigma-1\right)\mathrm{s}+\vartheta_0\right)\right), \tag{38}$$

$$\mathfrak{U}_{7,8}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+8-12\coth^2(\mathrm{y}-2k\mathrm{s}). \tag{39}$$

Set 3-a:

$$-\omega = \frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2+2\sigma+\frac{1}{2},$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+5,$$
$$\overline{A}_1 = \overline{B}_1 = \overline{a}_0 = 0, \overline{A}_2 = -6, \overline{B}_2 = 6 \text{ and}$$
$$\overline{a}_1 = \mp\mathrm{i}\sqrt{3\sigma+0.5}, \qquad \overline{b}_1 = \pm\mathrm{i}\sqrt{3\sigma+0.5}.$$

After substituting the arrangement in Set 3-a into Equs. 28)–(29) and (30)–(31), we construct the following wave solutions:

$$\mathfrak{u}_{9,10}(\mathrm{y},\mathrm{s}) = \pm\mathrm{i}\left(\sqrt{3\sigma+\frac{1}{2}}\tanh(\mathrm{y}-2k\mathrm{s}) - \sqrt{3\sigma+\frac{1}{2}}\,\mathrm{sech}\,(\mathrm{y}-2k\mathrm{s})\right) \times \exp\left(\mathrm{i}\left(k\mathrm{y} - \left(\frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2+2\sigma+\frac{1}{2}\right)\mathrm{s}+\vartheta_0\right)\right), \tag{40}$$

$$\mathfrak{U}_{9,10}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+5 - 6\tanh^2(\mathrm{y}-2k\mathrm{s}) + 6\mathrm{i}\tanh(\mathrm{y}-2k\mathrm{s})\,\mathrm{sech}\,(\mathrm{y}-2k\mathrm{s}), \tag{41}$$

and

$$\mathfrak{u}_{11,12}(\mathrm{y},\mathrm{s}) = \pm\mathrm{i}\left(\sqrt{3\sigma+\frac{1}{2}}\coth(\mathrm{y}-2k\mathrm{s}) - \sqrt{3\sigma+\frac{1}{2}}\,\mathrm{csch}\,(\mathrm{y}-2k\mathrm{s})\right) \times \exp\left(\mathrm{i}\left(k\mathrm{y} - \left(\frac{1}{2}\sigma^3+k^2+2k\sigma+\frac{1}{12}\sigma^2+2\sigma+\frac{1}{2}\right)\mathrm{s}+\vartheta_0\right)\right), \tag{42}$$

$$\mathfrak{U}_{11,12}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+2k+\frac{1}{12}\sigma+5 - 6\coth^2(\mathrm{y}-2k\mathrm{s}) - 6\coth(\mathrm{y}-2k\mathrm{s})\,\mathrm{csch}\,(\mathrm{y}-2k\mathrm{s}). \tag{43}$$

Set 3-b:

$$-\omega = \frac{1}{2}\sigma^3 + k^2 + 2k\sigma + \frac{1}{12}\sigma^2 + 2\sigma + \frac{1}{2},$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2 + 2k + \frac{1}{12}\sigma + 5,$$

$$\overline{A}_1 = 0, \overline{A}_2 = -6, \overline{B}_1 = 0, \overline{B}_2 = -6, \overline{a}_0 = 0 \text{ and}$$

$$\overline{a}_1 = \mp i\sqrt{3\sigma + \frac{1}{2}}, \qquad \overline{b}_1 = \pm i\sqrt{3\sigma + \frac{1}{2}}.$$

By substituting Set 3-b into Equs. 28)–(29) and (30)–(31), we extract the following wave solutions:

$$\mathfrak{u}_{13,14}(y,s) = \pm\left(i\sqrt{3\sigma + \frac{1}{2}}\tanh(y - 2ks) - \sqrt{3\sigma + \frac{1}{2}}\,\text{sech}\,(y - 2ks)\right) \times \exp\Big(i\Big(ky - \Big(\frac{1}{2}\sigma^3 + k^2 + 2k\sigma + \frac{1}{12}\sigma^2 + 2\sigma + \frac{1}{2}\Big)s + \vartheta_0\Big)\Big), \tag{44}$$

$$\mathfrak{U}_{13,14}(y,s) = \frac{1}{2}\sigma^2 + 2k + \frac{1}{12}\sigma + 5 - 6\tanh^2(y - 2ks) - 6\tanh(y - 2ks)\,\text{sech}\,(y - 2ks), \tag{45}$$

and

$$\mathfrak{u}_{15,16}(x,t) = \pm i\left(\sqrt{3\sigma + \frac{1}{2}}\coth(y - 2ks) - \sqrt{3\sigma + \frac{1}{2}}\,\text{csch}\,(y - 2kts)\right) \times \exp\Big(i\Big(ky - \Big(\frac{1}{2}\sigma^3 + k^2 + 2k\sigma + \frac{1}{12}\sigma^2 + 2\sigma + \frac{1}{2}\Big)s + \vartheta_0\Big)\Big), \tag{46}$$

$$\mathfrak{U}_{15,16}(y,s) = \frac{1}{2}\sigma^2 + 2k + \frac{1}{12}\sigma + 5 - 6\coth^2(y - 2ks) + 6\coth(y - 2ks)\,\text{csch}\,(y - 2ks), \tag{47}$$

For case-II: $\wp' = \cosh(\wp)$

If we take $N_1 = 1$ and $N_2 = 2$ in Equs. 8)–(9), (20)–(21) and (22)–(23), we obtain the following formal solutions of Equs. 24) and (25):

$$\phi(\wp) = \overline{b}_1\sinh\wp + \overline{a}_1\cosh\wp + \overline{a}_0, \tag{48}$$

$$\Phi(\wp) = \overline{B}_1\sinh\wp + \overline{B}_2\sinh\wp\cosh\wp + \overline{A}_1\cosh\wp + \overline{A}_2\cosh^2\wp + \overline{A}_0, \tag{49}$$

and

$$\phi(\tau) = \overline{b}_1\sec\tau + \overline{a}_1\tan\tau + \overline{a}_0, \tag{50}$$

$$\Phi(\tau) = \overline{B}_1\sec\tau + \overline{B}_2\tan\tau\sec\tau + \overline{A}_1\tan\tau + \overline{A}_2\tan^2\tau + \overline{A}_0, \tag{51}$$

$$\phi(\tau) = -\overline{b}_1\cot\wp + \overline{a}_1\csc\tau + \overline{a}_0, \tag{52}$$

$$\Phi(\tau) = -\tau_1\cot\tau - \overline{B}_2\cot\tau\csc\tau + \overline{A}_1\csc\tau + \overline{A}_2\csc^2\tau + \overline{A}_0, \tag{53}$$

where either $\overline{A}_1$ or $\overline{A}_2$ or $\overline{B}_1$ or $\overline{B}_2$ or $\overline{a}_1$ or $\overline{b}_1$ may be zero, but not all of them ended up zero at the same time. By substituting Equs. 46)–(47) into Equs. 24)–(25), we get a framework of $\mathbb{NAE}$ and by tackling the frameworks we have the taking after three sets.

Set 1:

$$-k = \frac{1}{4}\sigma^2 + \frac{1}{24}\sigma + 2,$$

$$-\omega = \frac{1}{16}\sigma^4 + \frac{1}{48}\sigma^3 + \frac{577}{576}\sigma^2 + \frac{1}{6}\sigma + 5,$$

where

$$\overline{A}_0 = \overline{A}_1 = \overline{B}_1 = \overline{B}_2 = \overline{a}_0 = \overline{b}_1 = 0,$$

and $\overline{A}_2 = -12$, $\overline{a}_1 = \pm i\sqrt{12\sigma + 2}$. Again by substituting Set 1 into Equs. 48)–(49) and (50)–(51), we can determine the following trigonometric work arrangements for Equs. 1a)–(b):

$$\mathfrak{u}_{15,16}(y,s) = \pm i\left(\sqrt{12\sigma + 2}\sec\left(y + 2\Big(\frac{1}{4}\sigma^2 + \frac{1}{24}\sigma + 2\Big)s\right)\right) \times \exp\left(i\left(-\Big(\frac{1}{4}\sigma^2 + \frac{1}{24}\sigma + 2\Big)y - \Big(\frac{1}{16}\sigma^4 + \frac{1}{48}\sigma^3 + \frac{577}{576}\sigma^2 + \frac{1}{6}\sigma + 5\Big)s + \vartheta_0\right)\right), \tag{54}$$

$$\mathfrak{U}_{15,16}(y,s) = -12\sec^2\left(y + 2\Big(\frac{1}{4}\sigma^2 + \frac{1}{24}\sigma + 2\Big)s\right), \tag{55}$$

and

$$\mathfrak{u}_{17,18}(\mathrm{y},\mathrm{s}) = \pm \mathrm{i}\left(\sqrt{12\sigma+2}\csc\left(\mathrm{y}+2\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)\mathrm{s}\right)\right) \times \exp\left(\mathrm{i}\left(-\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)\mathrm{y} -\Big(\frac{1}{16}\sigma^4+\frac{1}{48}\sigma^3+\frac{577}{576}\sigma^2+\frac{1}{6}\sigma+5\Big)\mathrm{s}+\vartheta_0\right)\right), \tag{56}$$

$$\mathfrak{U}_{17,18}(\mathrm{y},\mathrm{s}) = -12\csc^2\left(\mathrm{y}+2\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)\mathrm{s}\right). \tag{57}$$

Set 2:

$$k=\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2,$$
$$-\omega=\frac{1}{16}\sigma^4+\frac{49}{48}\sigma^3+\frac{673}{576}\sigma^2+\frac{49}{6}\sigma+5,$$

where

$$\overline{A}_0=\sigma^2+\frac{1}{6}\sigma+8,$$
$$\overline{A}_1=\overline{B}_1=\overline{B}_2=\overline{a}_0=\overline{b}_1=0,$$

and $\overline{A}_2=-12, \overline{a}_1=\pm\mathrm{i}\sqrt{12\sigma+2}$. By substituting Set 2 into Equs. 48)–(49) and (50)–(51), again, we determine the following trigonometric work arrangements:

$$\mathfrak{u}_{19,20}(\mathrm{y},\mathrm{s}) = \pm \mathrm{i}\left(\sqrt{12\sigma+2}\sec\left(\mathrm{y}+2\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)t\right)\right) \times \exp\left(\mathrm{i}\left(-\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)\mathrm{y} -\Big(\frac{1}{16}\sigma^4+\frac{49}{48}\sigma^3+\frac{673}{576}\sigma^2+\frac{49}{6}\sigma+5\Big)\mathrm{s}+\vartheta_0\right)\right), \tag{58}$$

$$\mathfrak{U}_{19,20}(\mathrm{y},\mathrm{s}) = \sigma^2+\frac{1}{6}\sigma+8-12\sec^2\left(\mathrm{y}+2\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)\mathrm{s}\right), \tag{59}$$

and

$$\mathfrak{u}_{21,22}(\mathrm{y},\mathrm{s}) = \pm \mathrm{i}\left(\sqrt{12\sigma+2}\csc\left(\mathrm{y}+2\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)t\right)\right) \times \exp\left(\mathrm{i}\left(-\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)\mathrm{y} -\Big(\frac{1}{16}\sigma^4+\frac{49}{48}\sigma^3+\frac{673}{576}\sigma^2+\frac{49}{6}\sigma+5\Big)\mathrm{s}+\vartheta_0\right)\right), \tag{60}$$

$$\mathfrak{U}_{21,22}(\mathrm{y},\mathrm{s}) = \sigma^2+\frac{1}{6}\sigma+8-12\csc^2\left(\mathrm{y}+2\Big(\frac{1}{4}\sigma^2+\frac{1}{24}\sigma+2\Big)\mathrm{s}\right). \tag{61}$$

Set 3-a:

$$k=\pm\sqrt{\Delta},$$
$$\omega=-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{287}{576}\sigma^2 -\sigma\Big(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta}\Big)+\frac{37}{12}\sigma+\frac{1}{4},$$

where

$$\overline{A}_0=\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta},$$
$$\overline{A}_1=\overline{B}_1=\overline{a}_0=0, \overline{A}_2=-6, \overline{B}_2=6,$$
$$\overline{a}_1=-\mathrm{i}\sqrt{3\sigma+\frac{1}{2}}, \qquad \overline{b}_1=\mathrm{i}\sqrt{3\sigma+\frac{1}{2}},$$

and

$$\Delta=36\sigma^4+12\sigma^3-287\sigma^2-48\sigma+144. \tag{62}$$

After substituting Set 3-a into Equs. 48)–(49) and (50)–(51), we produce the following trigonometric work arrangements:

$$\mathfrak{u}_{23,24}(\mathrm{y},\mathrm{s}) = -\mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\tan\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{s}\right)+\mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\sec\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{y}\right) \times\exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y}+\Big(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{287}{576}\sigma^2 -\sigma\Big(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta}\Big)+\frac{37}{12}\sigma+\frac{1}{4}\Big)\mathrm{s}+\vartheta_0\right)\right), \tag{63}$$

$$\mathfrak{U}_{23,24}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta} -6\tan^2\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{s}\right)+6\tan\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{y}\right)\sec\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{s}\right), \tag{64}$$

and

$$\mathfrak{u}_{25,26}(\mathrm{y},\mathrm{s}) = -\mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\csc\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{s}\right)-\mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\cot\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{s}\right) \times\exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y}+\Big(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{287}{576}\sigma^2 -\sigma\Big(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta}\Big)+\frac{37}{12}\sigma+\frac{1}{4}\Big)\mathrm{s}+\vartheta_0\right)\right), \tag{65}$$

$$\mathfrak{u}_{27,28}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta} -6\csc^2\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{y}\right)-6\cot\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{s}\right)\csc\left(\mathrm{y}\pm2\sqrt{\Delta}\mathrm{s}\right). \tag{66}$$

Set 3-b:

$$k = \pm\sqrt{\Delta},$$
$$\omega = -\frac{1}{16}\sigma^4 - \frac{1}{48}\sigma^3 + \frac{287}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta}\right) + \frac{37}{12}\sigma + \frac{1}{4},$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta},$$

$\overline{A}_1 = 0, \overline{A}_2 = -6, \overline{B}_1 = 0, \overline{B}_2 = 6, \overline{A}_0 = 0,$ and

$$\overline{a}_1 = \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}, \qquad \overline{b}_1 = -\mathrm{i}\sqrt{3\sigma + \frac{1}{2}}.$$

After substituting Set 3-b into Equs. 48)–(49) and (50)–(51), we extract the following trigonometric work arrangements:

$$\mathfrak{u}_{29,30}(\mathrm{y},\mathrm{s}) = \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\tan\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) - \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\sec\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{y}\right) \times \exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y} + \left(-\frac{1}{16}\sigma^4 - \frac{1}{48}\sigma^3 + \frac{287}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta}\right) + \frac{37}{12}\sigma + \frac{1}{4}\right)\mathrm{s} + \vartheta_0\right)\right), \quad (67)$$

$$\mathfrak{U}_{29,30}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta} - 6\tan^2\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{S}\right) + 6\tan\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{y}\right)\sec\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right), \quad (68)$$

and

$$\mathfrak{u}_{31,32}(\mathrm{y},\mathrm{s}) = \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\csc\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{y}\right) + \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\cot\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) \times \exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y} + \left(-\frac{1}{16}\sigma^4 - \frac{1}{48}\sigma^3 + \frac{287}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta}\right) + \frac{37}{12}\sigma + \frac{1}{4}\right)\mathrm{s} + \vartheta_0\right)\right), \quad (69)$$

$$\mathfrak{U}_{31,32}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta} - 6\csc^2\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) - 6\cot\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right)\csc\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right). \quad (70)$$

Set 3-c:

$$k = \pm\sqrt{\Delta},$$
$$\omega = -\frac{1}{16}\sigma^4 - \frac{1}{48}\sigma^3 + \frac{287}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta}\right) + \frac{37}{12}\sigma + \frac{1}{4},$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta},$$

$\overline{A}_1 = 0, \overline{A}_2 = -6, \overline{B}_1 = 0, \overline{B}_2 = -6, \overline{A}_0 = 0$ and

$$\overline{a}_1 = -\mathrm{i}\sqrt{3\sigma + \frac{1}{2}}, \qquad \overline{b}_1 = \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}.$$

By substituting the Set 3-c into Equs. 48)–(49) and (50)–(51), we obtain the following trigonometric work arrangements:

$$\mathfrak{u}_{33,34}(\mathrm{y},\mathrm{s}) = -\mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\tan\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) + \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\sec\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) \times \exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y} + \left(-\frac{1}{16}\sigma^4 - \frac{1}{48}\sigma^3 + \frac{287}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta}\right) + \frac{37}{12}\sigma + \frac{1}{4}\right)\mathrm{s} + \vartheta_0\right)\right), \quad (71)$$

$$\mathfrak{U}_{33,34}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta} - 6\tan^2\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) - 6\tan\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right)\sec\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right), \quad (72)$$

and

$$\mathfrak{u}_{35,36}(\mathrm{y},\mathrm{s}) = -\mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\csc\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) - \mathrm{i}\sqrt{3\sigma + \frac{1}{2}}\cot\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) \times \exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y} + \left(-\frac{1}{16}\sigma^4 - \frac{1}{48}\sigma^3 + \frac{287}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta}\right) + \frac{37}{12}\sigma + \frac{1}{4}\right)\mathrm{s} + \vartheta_0\right)\right), \quad (73)$$

$$\mathfrak{U}_{35,36}(\mathrm{y},\mathrm{s}) = \frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta} - 6\csc^2\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right) - 6\cot\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right)\csc\left(\mathrm{y} \pm 2\sqrt{\Delta}\mathrm{s}\right). \quad (74)$$

Set 3-d:

$$k = \pm\sqrt{\Delta},$$
$$\omega = -\frac{1}{16}\sigma^4 - \frac{1}{48}\sigma^3 + \frac{287}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta}\right) + \frac{37}{12}\sigma + \frac{1}{4},$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2 + \frac{1}{12}\sigma + 1 + \frac{1}{12}\sqrt{\Delta},$$

$\overline{A}_1 = \overline{B}_1 = \overline{a}_0 = 0, \overline{A}_2 = -6, \overline{B}_2 = -6$ and

$$\overline{a}_1 = \mathrm{i}\sqrt{3\sigma+\frac{1}{2}}, \qquad \overline{b}_1 = -\mathrm{i}\sqrt{3\sigma+\frac{1}{2}}.$$

By substituting the Set 3-d into Equations (48)–(49) and (50)–(51), we extract the following trigonometric work arrangements:

$$\begin{aligned}\mathfrak{u}_{37,38}(\mathrm{y},\mathrm{s}) &= \mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\tan\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right) - \mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\sec\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right)\\ &\times\exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y}+\left(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{287}{576}\sigma^2\right.\right.\right.\\ &\left.\left.\left.-\sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta}\right)+\frac{37}{12}\sigma+\frac{1}{4}\right)\mathrm{s}+\vartheta_0\right)\right),\end{aligned} \tag{75}$$

$$\begin{aligned}\mathfrak{U}_{37,38}(\mathrm{y},\mathrm{s}) &= \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta}\\ &-6\tan^2\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right)-6\tan\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right)\sec\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right),\end{aligned} \tag{76}$$

and

$$\begin{aligned}\mathfrak{u}_{39,40}(\mathrm{y},\mathrm{s}) &= \mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\csc\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right) + \mathrm{i}\sqrt{3\sigma+\frac{1}{2}}\cot\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right)\\ &\times\exp\left(\mathrm{i}\left(\pm\sqrt{\Delta}\mathrm{y}+\left(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{287}{576}\sigma^2\right.\right.\right.\\ &\left.\left.\left.-\sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta}\right)+\frac{37}{12}\sigma+\frac{1}{4}\right)\mathrm{s}+\vartheta_0\right)\right),\end{aligned} \tag{77}$$

$$\begin{aligned}\mathfrak{U}_{39,40}(\mathrm{y},\mathrm{s}) &= \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+1+\frac{1}{12}\sqrt{\Delta}\\ &-6\csc^2\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right)-6\cot\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right)\csc\left(\mathrm{y}\pm 2\sqrt{\Delta}\mathrm{s}\right),\end{aligned} \tag{78}$$

Set 4-a:

$$k = \pm\sqrt{\Lambda},$$
$$\omega = -\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{1151}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\right)+\frac{37}{3}\sigma-2,$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda},$$
$$\overline{A}_1 = \overline{B}_1 = \overline{B}_2 = \overline{a}_0 = \overline{a}_1 = 0,$$
$$\overline{A}_2 = -12,\ \overline{b}_1 = \mathrm{i}\sqrt{12\sigma+2} \text{ and}$$
$$\Lambda = 36\sigma^4+12\sigma^3-1151\sigma^2-192\sigma+2304. \tag{79}$$

By substituting the Set 4-a into Equs. (48)–(49) and (50)–(51), we explore the following trigonometric work arrangements:

$$\begin{aligned}\mathfrak{u}_{41,42}(\mathrm{y},\mathrm{s}) &= \mathrm{i}\sqrt{12\sigma+2}\sec\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{s}\right)\\ &\times\exp\left(\mathrm{i}\left(\pm\sqrt{\Lambda}\mathrm{y}+\left(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{1151}{576}\sigma^2\right.\right.\right.\\ &\left.\left.\left.-\sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\right)+\frac{37}{3}\sigma-2\right)\mathrm{s}+\vartheta_0\right)\right),\end{aligned} \tag{80}$$

$$\begin{aligned}\mathfrak{U}_{41,42}(\mathrm{y},\mathrm{s}) &= \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\\ &-12\tan^2\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{s}\right),\end{aligned} \tag{81}$$

and

$$\begin{aligned}\mathfrak{u}_{43,44}(\mathrm{y},\mathrm{s}) &= \mathrm{i}\sqrt{12\sigma+2}\cot\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{s}\right)\\ &\times\exp\left(\mathrm{i}\left(\pm\sqrt{\Lambda}\mathrm{y}+\left(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{1151}{576}\sigma^2\right.\right.\right.\\ &\left.\left.\left.-\sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\right)+\frac{37}{3}\sigma-2\right)\mathrm{s}+\vartheta_0\right)\right),\end{aligned} \tag{82}$$

$$\begin{aligned}\mathfrak{U}_{43,44}(\mathrm{y},\mathrm{s}) &= \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\\ &-12\csc^2\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{s}\right).\end{aligned} \tag{83}$$

Set 4-b:

$$k = \pm\sqrt{\Lambda},$$
$$\omega = -\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{1151}{576}\sigma^2 - \sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\right)+\frac{37}{3}\sigma-2,$$

where

$$\overline{A}_0 = \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda},$$

$\overline{A}_1 = 0$, $\overline{A}_2 = -12$, $\overline{B}_1 = 0$, $\overline{B}_2 = 0$, $\overline{a}_0 = 0$, $\overline{a}_1 = 0$ and $\overline{b}_1 = -\mathrm{i}\sqrt{12\sigma+2}$. By substituting the Set 4-b into Equs. (48)–(49) and (50)–(51), we obtain the following trigonometric work arrangements:

$$
\begin{aligned}
\mathfrak{u}_{41,42}(\mathrm{y},\mathrm{s}) &= -\mathrm{i}\sqrt{12\sigma+2}\,\sec\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{S}\right)\\
&\quad\times\exp\Bigg(\mathrm{i}\Bigg(\pm\sqrt{\Lambda}\mathrm{y}+\Bigg(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{1151}{576}\sigma^2\\
&\quad-\sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\right)+\frac{37}{3}\sigma-2\Bigg)\mathrm{S}+\vartheta_0\Bigg)\Bigg),
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
\mathfrak{U}_{41,42}(\mathrm{y},\mathrm{s}) &= \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\\
&\quad-12\tan^2\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{s}\right),
\end{aligned}
\tag{85}
$$

and

$$
\begin{aligned}
\mathfrak{u}_{43,44}(\mathrm{y},\mathrm{s}) &= -\mathrm{i}\sqrt{12\sigma+2}\,\cot\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{s}\right)\\
&\quad\times\exp\Bigg(\mathrm{i}\Bigg(\pm\sqrt{\Lambda}\mathrm{y}+\Bigg(-\frac{1}{16}\sigma^4-\frac{1}{48}\sigma^3+\frac{1151}{576}\sigma^2\\
&\quad-\sigma\left(\frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\right)+\frac{37}{3}\sigma-2\Bigg)\mathrm{s}+\vartheta_0\Bigg)\Bigg),
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
\mathfrak{U}_{43,44}(\mathrm{y},\mathrm{s}) &= \frac{1}{2}\sigma^2+\frac{1}{12}\sigma+4+\frac{1}{12}\sqrt{\Lambda}\\
&\quad-12\csc^2\left(\mathrm{y}\mp 2\sqrt{\Lambda}\mathrm{s}\right),
\end{aligned}
\tag{87}
$$

Discussion and graphical illustration of the attained solutions

This section discusses our attained results and their three-dimensional (3D) and two-dimensional (2D) graphical representations that can be helpful in understanding the proper physical meaning of the coupled Schrödinger–KdV equations. The extended sinh-Gordon equation expansion method is successfully employed by constructing various types of wave solutions to coupled Schrödinger–KdV equations. The specified governing equations of (1a)–(1b) have been previously investigated by diverse methods [12, 13, 11, 14, 16, 6].

According to all previous studies, singular, dark, bright, and periodic waves solutions have not been constructed yet. In our work, we have developed some particular, singular, combined singular, dark, bright, combined dark–bright, periodic, singular periodic wave solutions, and other solutions to the coupled Schrödinger–KdV equations through the extended sinh-Gordon equation expansion method. For the first time ever, this research paper can be assured that all singular, combined-singular, combined dark–bright, periodic, and singular periodic wave solutions have been

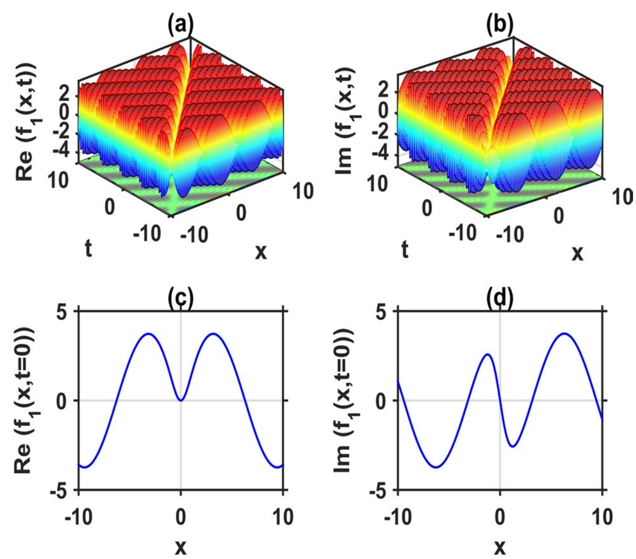

Fig. 1 3D plot $\mathfrak{u}(\mathrm{x},\mathrm{t})$ with forms for Equation (32): **a** real portion, **b** imaginary portion with $k = 0.5$, $\sigma = 1$, $\vartheta_0 = 0$, and **c**, **d**: 2D line plot of **a** and **b** individually at $t = 0$

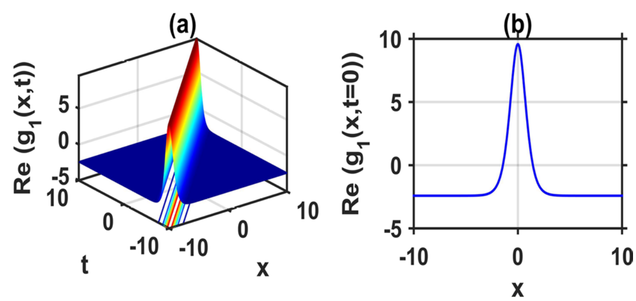

Fig. 2 3D plot $\mathfrak{U}(\mathrm{x},\mathrm{t})$ with forms for (33): **a** real part with $k = 0.5$, $\sigma = 1$, and **b**: 2D line plot of **a** at $t = 0$

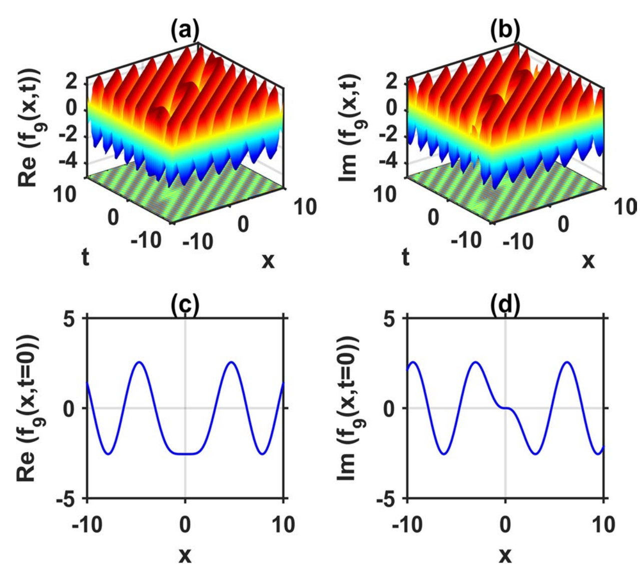

Fig. 3 3D plot $\mathfrak{u}(\mathrm{x},\mathrm{t})$ with forms for Equation (40): **a** real part, **b** imaginary part with $k = 1$, $\sigma = 2$, $\vartheta_0 = 0$, and **c**, **d**: 2D line plot of **a** and **b** individually at $t = 0$

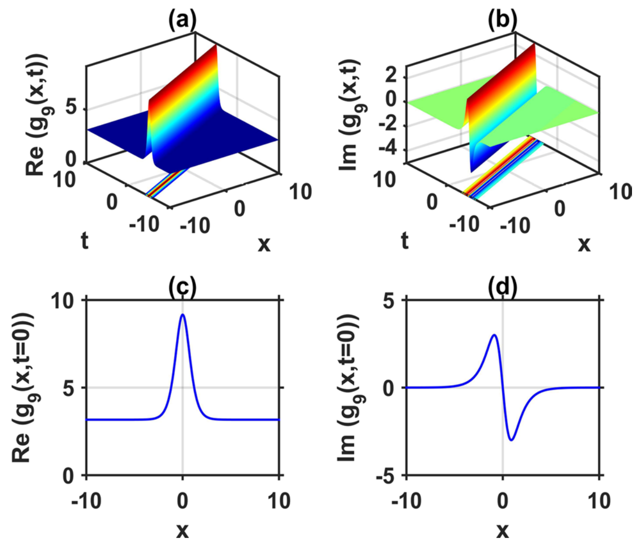

Fig. 4 3D plot $\mathfrak{U}(x,t)$ with forms for Equation (41): **a** real part, **b** imaginary part with $k = 1$, $\sigma = 2$, and **c**, **d**: (2D) line plot of **a** and **b** individually at $t = 0$

discussed and investigated. It too ought to be specified that the legitimacy of the extricated solutions is examined by substituting each of the precise solutions back into its comparing equation.

All of our generated solutions have some physical significance. To ensure such types of realization, we have displayed some 3D graphs with contour and 2D line graphs among the detected singular solitons,

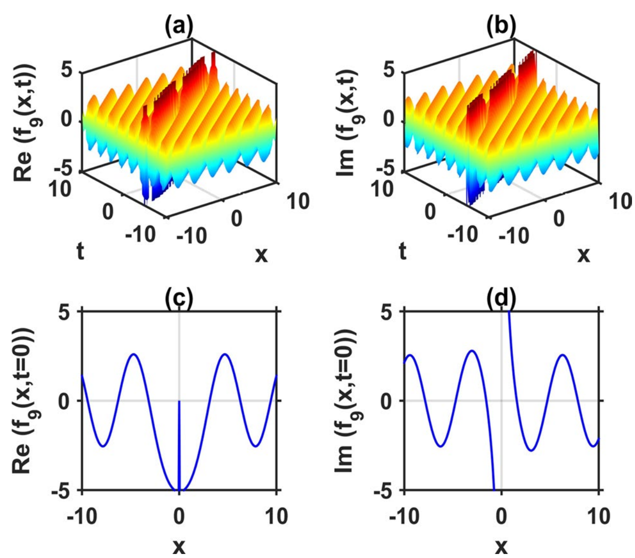

Fig. 5 3D plot $\mathfrak{u}(x,t)$ with contours for Equation (42): **a** real part, **b** imaginary part with $k = 1$, $\sigma = 2$, $\vartheta_0 = 0$, and **c**, **d**: 2D line plot of **a** and **b** individually at $t = 0$

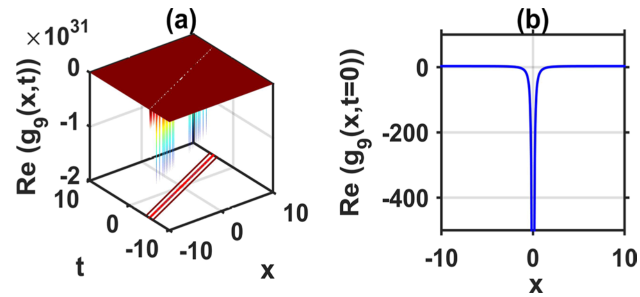

Fig. 6 3D plot $\mathfrak{u}(x,t)$ with forms for (43): **a** real part with $k = 1$, $\sigma = 2$, and **b**: 2D line plot of **a** at $t = 0$

combined-singular solitons, dark solitons, bright solitons, combined dark–bright solitons, periodic, and singular- periodic wave solutions for different values of their arbitrary parameters, which are indicated in Figs. 1, 2, 3, 4, 5, 6, 7, 8.

For Equs. (1a) and (b), dark solitons, bright solitons, combined dark–bright solitons, singular solitons, combined singular solitons, and singular periodic wave solutions are reported in Equs. (32)–(33), (46)–(47) and Equs. (54)–(55), (86)–(87). In order to have a good understanding of the physical properties, the 3D graphs with contour and 2D line graphs are included among the obtained solutions under the choice of suitable values of arbitrary parameters. The perspective view of the obtained solutions prescribed by Equs. (32)–(33), (40)–(41), (42)–(43) and (54)–(55) can be seen in the 3D and 2D graphs at $t = 0$, which appear in Figs. 1, 2, 3, 4, 5, 6, 78, respectively.

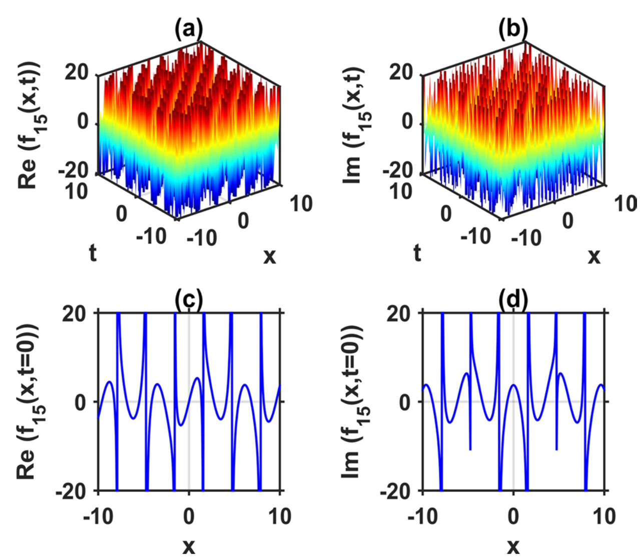

Fig. 7 3D plot $\mathfrak{u}(x,t)$ with forms for (54): **a** real part, **b** imaginary part with $\sigma = 1$, $\vartheta_0 = 0$, and **c**, **d**: 2D line plot of **a** and **b** individually at $t = 0$

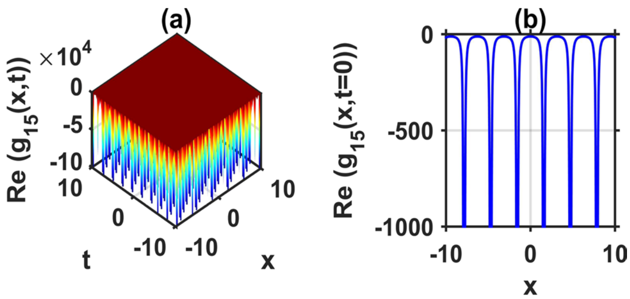

Fig. 8 3D plot $\mathfrak{U}(x, t)$ with forms for (55): **a** real part with $\sigma = 1$, and **b**: 2D line plot of **a** at $t = 0$

Therefore, it is clear from the graphical outputs that the procedure of the extended sinh-Gordon equation expansion method will contribute for other related equations for obtaining more new soliton and other solutions.

Conclusions

Extended sinh-Gordon equation expansion method has been applied in this study to generate novel soliton and other solutions of the coupled Schrödinger–KdV equations. As a result, unused unequivocal complex hyperbolic and complex trigonometric work arrangements have been found, which are communicated by singular, combined singular, dark, bright, combined dark–bright, periodic wave soliton, and other shapes. All obtained arrangements fulfill their comparing condition. The 3D charts with form and 2D charts have been appeared for a few of the found solutions. All our results are fundamental in clarifying the physical meaning of some nonlinear models emerging from nonlinear sciences. The constructed solitons have never been discussed in any of the previous studies [12, 13, 11, 14, 16, 6] in Schrödinger–KdV systems. In this manner, it is very apparent that this work gives a great contribution to this field of research due to its significance in numerous areas of material science.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Debnath, L.: Nonlinear partial differential equations for scientists and engineers. Springer Science & Business Media , Berlin (2011)
2. Kumar, A., Pankaj, R.D.: Laplace decomposition method to study solitary wave solutions of coupled nonlinear partial differential equations. ISRN Computational Math. (2012). https://doi.org/10.5402/2012/423469
3. Kumar, D., Hosseini, K., Kaabar, M.K.A., Kaplan, M., Salahshour, S.: On some novel soliton solutions to the generalized Schrödinger-Boussinesq equations for the interaction between complex short wave and real long wave envelope. J. Ocean Eng. Sci. 2021 https://doi.org/10.1016/j.joes.2021.09.008
4. Kaabar, M.K.A., Kaplan, M., Siri, Z.: New Exact Soliton Solutions of the (3+1)-Dimensional Conformable Wazwaz-Benjamin-Bona-Mahony Equation via Two Novel Techniques. J. Funct. Spaces. Article ID 4659905 (2021). https://doi.org/10.1155/2021/4659905
5. Ray, S.: Nonlinear differential equations in Physics. Springer, Berlin (2020). https://doi.org/10.1007/978-981-15-1656-6
6. Seadawy, A.R.: El-Rashidy K. Classification of multiply travelling wave solutions for coupled burgers, Combined KdV-Modified KdV, and Schrödinger-KdV Equations. Abstr. Appl. Anal. 369294 (2015). https://doi.org/10.1155/2015/369294
7. Samei, M.E., Hedayati, V., Rezapour, S.: Existence results for a fraction hybrid differential inclusion with Caputo-Hadamard type fractional derivative. Adv. Differ. Equ. **2019**, 163 (2019)
8. Subramanian, M., Alzabut, J., Dumitru, D., Samei, M.E., Zadaf, A.: Existence, uniqueness and stability analysis of a coupled fractional-order differential systems involving Hadamard derivatives and associated with multi-point boundary conditions. Adv. Differ. Equ. **2021**, 267 (2021)
9. Abu-Shady, M., Kaabar, M.K.A.: A generalized definition of the fractional derivative with applications. Math. Prob. Eng. 2021. https://doi.org/10.1155/2021/9444803
10. Kaabar, M.K.A., Martínez, F., Gómez-Aguilar, J.F., Ghanbari, B., Kaplan, M., Günerhan, H.: New approximate analytical solutions for the nonlinear fractional Schrödinger equation with second-order spatio-temporal dispersion via double laplace transform method. Math. Method. Apllied Sci. **44**(14), 11138–11156 (2021)
11. Colorado, E.: Existence of bound and ground states for a system of coupled nonlinear Schrödinger-KdV equations. Comptes Rendus Mathematique. **353**(6), 511–6 (2015)
12. Alomari, A.K., Noorani, M.S.M., Nazar, R.: Comparison between the homotopy analysis method and homotopy perturbation method to solve coupled Schrödinger-KdV equation. J. Appl. Math. Comput. **31**(1–2), 1–12 (2009). https://doi.org/10.1007/s12190-008-0187-4
13. Álvarez-Caudevilla, P., Colorado, E., Fabelo, R.: A higher order system of some coupled nonlinear Schrödinger and Korteweg-de Vries equations. J. Math. Phys. **58**(11), 111503 (2017)
14. Geng, Q., Liao, M., Wang, J., Xiao, L.: Existence and bifurcation of nontrivial solutions for the coupled nonlinear Schrödinger-Korteweg-de Vries system. Zeitschrift für angewandte Mathematik und Physik. **71**(1), 33 (2020)
15. He, J.H., He, C.H., Saeed, T.: A fractal modification of Chen-Lee-Liu equation and its fractal variational principle. Int. J. Mod. Phys. B. **35**(21), 2150214 (2021). https://doi.org/10.1142/S0217979221502143
16. Ismail, M.S., Mosally, F.M., Alamoudi, K.M.: Petrov-Galerkin method for the Schrödinger-KdV equation. Abstr. Appl. Anal. **2014**, 705204 (2014)
17. Kaya, D., El-Sayed, S.M.: On the solution of the coupled Schrödinger-KdV equation by the decomposition method. Phys. Lett. A. **313**(1–2), 82–88 (2003)
18. Küçükarslan, S.: Homotopy perturbation method for coupled Schrödinger-KdV equation. Nonlinear Analysis: Real World Applications. **10**(4), 2264–2271 (2009)
19. Ma, W.X.: Soliton solutions by means of Hirota bilinear forms. Partial Differ. Equ. Appl. Math. **5**, 100220 (2022). https://doi.org/10.1016/j.padiff.2021.100220

20. Ma, W.X.: Nonlocal PT-symmetric integrable equations and related Riemann-Hilbert problems. Partial Differ. Equ. Appl. Math. **4**, 100190 (2021). https://doi.org/10.1016/j.padiff.2021.100190
21. Ma, W.X.: Riemann-Hilbert problems and soliton solutions of nonlocal reverse-time NLS hierarchies. Acta Mathematica Scientia. **42**, 127–140 (2022). https://doi.org/10.1007/s10473-022-0106-z
22. Ma, W.X.: Reduced nonlocal integrable mKdV equations of type $(-\lambda, \lambda)$ and their exact soliton solutions. Communications in Theoretical Physics. **74**(6), 104522 (2022). https://doi.org/10.1088/1572-9494/ac75e0
23. Baskonus, H.M., Sulaiman, T.A., Bulut, H.: Dark, bright and other optical solitons to the decoupled nonlinear Schrödinger equation arising in dual-core optical fibers. Optical Quantum Electron. **50**(4), 165 (2018). https://doi.org/10.1007/s11082-018-1433-0
24. Marin, F.: Solitons: Historical and physical introduction. In: Mathematics of complexity and dynamical systems (2009)
25. Biswas, A., Ekici, M., Sonmezoglu, A., Triki, H., Majid, F.B., Zhou, Q., Moshokoa, S.P., Mirzazadeh, M., Belic, M.: Optical solitons with Lakshmanan-Porsezian-Daniel model using a couple of integration schemes. Optik. **158**, 705–711 (2018)
26. Hosseini, K., Kumar, D., Kaplan, M., Bejarbaneh, E.Y.: New exact traveling wave solutions of the unstable nonlinear Schrödinger equations. Commun. Theor. Phys. **68**(6), 761 (2017)
27. Kumar, D., Hosseini, K., Samadani, F.: The sine-Gordon expansion method to look for the traveling wave solutions of the Tzitzéica type equations in nonlinear optics. Optik. **149**, 439–446 (2017)
28. Kumar, D., Kaplan, M., Haque, M., Osman, M.S., Baleanu, D.: A variety of novel exact solutions for different models with the conformable derivative in shallow water. Front. Phys. **8**, 177 (2020)
29. Kumar, D., Darvishi, M.T., Joardar, A.K.: Modified Kudryashov method and its application to the fractional version of the variety of Boussinesq-like equations in shallow water. Optical Quantum Electron. **50**(3), 128 (2018)
30. Biswas, A., Alqahtani, R.T.: Chirp-free bright optical solitons for perturbed Gerdjikov-Ivanov equation by semi-inverse variational principle. Optik. **147**, 72–76 (2017)
31. Bulut, H., Sulaiman, T.A., Baskonus, H.M.: Optical solitons to the resonant nonlinear Schrödinger equation with both spatio-temporal and inter-modal dispersions under Kerr law nonlinearity. Optik. **163**, 49–55 (2018)
32. Kumar, D., Manafian, J., Hawlader, F., Ranjbaran, A.: New closed form soliton and other solutions of the Kundu-Eckhaus equation via the extended sinh-Gordon equation expansion method. Optik. **160**, 159–167 (2018)
33. Seadawy, A.R., Kumar, D., Chakrabarty, A.K.: Dispersive optical soliton solutions for the hyperbolic and cubic-quintic nonlinear Schrödinger equations via the extended sinh-Gordon equation expansion method. Eur. Phys. J. Plus. **133**(5), 182 (2018)
34. Xian-Lin, Y., Jia-Shi, T.: Travelling wave solutions for Konopelchenko-Dubrovsky equation using an extended sinh-Gordon equation expansion method. Commun. Theor. Phys. **50**(5), 1047–1051 (2008)
35. Biswas, A., Milovic, D.: Bright and dark solitons of the generalized nonlinear Schrödinger's equation. Commun. Nonlinear Sci. Numer. Simul. **15**, 1473–1484 (2010)
36. Zayed, E.M., Al-Nowehy, A.G.: The solitary wave ansatz method for finding the exact bright and dark soliton solutions of two nonlinear Schrödinger equations. J.Assoc. Arab Univ. Basic Appl. Sci. **24**, 184–90 (2017)
37. Bulut, H., Sulaiman, T.A., Baskonus, H.M.: Dark, bright optical and other solitons with conformable space-time fractional second-order spatiotemporal dispersion. Optik **163**, 1–7 (2018)
38. Kaabar, M.: Novel Methods for Solving the Conformable Wave Equation. J. New Theory **31**, 56–85 (2020)
39. He, J.H., Wu, X.H.: Exp-function method for nonlinear wave equations. Chaos, Solitons & Fractals **30**(3), 700–708 (2006)
40. Baitiche, Z., Derbazi, C., Alzabut, J., Samei, M.E., Kaabar, M.K.A., Siri, Z.: Monotone iterative method for ψ-Caputo fractional differential equation with nonlinear boundary conditions. Fractal Fract. **5**(3), 81 (2021)
41. Samei, M.E., Karimi, L., Kaabar, M.K.A.: To investigate a class of multi-singular pointwise defined fractional q–integro-differential equation with applications. AIMS Math. **7**(5), 7781–7816 (2022)
42. Matar, M.M., Abbas, M.I., Alzabut, J., Kaabar, M.K.A., Etemad, S., Rezapour, S.: Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. Adv. Differ. Equ. **68** (2021). https://doi.org/10.1186/s13662-021-03228-9
43. Achar, S.J., Baishya, C., Kaabar, M.K.A.: Dynamics of the worm transmission in wireless sensor network in the framework of fractional derivatives. Math. Method. Applied Sci. **45**(8), 4278–4294 (2022)
44. Ma, W.X.: Nonlocal integrable mKdV equations by two nonlocal reductions and their soliton solutions. Journal of Geometry and Physics **177**,104522 (2022). https://doi.org/10.1016/j.geomphys.2022.104522
45. Abu-Shady, M., Kaabar, M.K.A.: A novel computational tool for the fractional-order special functions arising from modeling scientific phenomena via Abu-Shady–Kaabar fractional derivative. Computational and Mathematical Methods in Medicine 2022 (2022). https://doi.org/10.1155/2022/2138775
46. He, J.H., Qie, N., He, C.H.: Solitary waves travelling along an unsmooth boundary. Results in Physics **24**(3-4),104104 (2021). https://doi.org/10.1016/j.rinp.2021.104104