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ORIGINAL RESEARCH

The exact solutions of the conformable time fractional version of the generalized Pochhammer–Chree equation

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Abstract
The time-fractional version of the generalized Pochhammer–Chree equation is analyzed. In this paper, the equation is converted into an ordinary differential equation by applying certain real transformation, then the discrimination of polynomials system is used to find exact solutions depending on the fractional order derivative. The obtained solutions are graphically illustrated for different values of the fractional order derivative keeping the other parameters fixed.

Keywords Time fractional differential equation · Generalized Pochhammer–Chree equation · Elastic wave · Exact solutions · The complete discrimination system of a polynomial

Mathematics Subject Classification 34A08 · 34K37 · 35RXX

Introduction

Most physics and engineering real-life problems can be perfectly described using fractional-order systems, these are the dynamical systems that can be modeled using time fractional differential equations [1–3]. For this, and many other reasons, studying the time fractional differential equations have attracted the attention of many researchers. In fact, finding the exact solutions of these equations became an active research topic, and many contributions have been made [4–27]. In this paper, we consider the model of elastic waves, these are the disturbances that propagate in different media under the influence of elastic forces [28]. The standard example of elastic wave is a long rope or rubber tube held at one end. Another example is the pervasion of vibrations in a semi-rod structure onto a moving substrate which is found in many engineering applications, from the fabrication of nanotube to the extension of submarine pipes, and in many other technical and biological processes [29–32]. The mathematical model of elastic rods wave is the generalized Pochhammer–Chree equation [33, 34], and the conformable time fractional version of such equation can be written as

$$D_t^{2\alpha}(\theta - \theta_{xx}) - \left(a_1\theta + a_2\theta^{n+1} + a_3\theta^{2n+1}\right)_{xx} = 0, \qquad n \geq 1, \quad \alpha \in (0, 1] \tag{1}$$

where a_1, a_2 and a_3 are arbitrary real constants, $a_3 \neq 0$, and D^α is an operator of order α representing the conformable fractional derivatives. The classical case where $\alpha = 1$ has been considered in several works for different values of the parameter n, and a number of solutions have been found for special cases [35–46]. In the current work, for the case of $n = 1$ and $\alpha \in (0, 1]$, we find the exact solutions of the Eq. (1) using the complete discrimination system for a polynomial [47, 48]. The complete discrimination system method was a primary tool in solving some differential equations, and in conducting qualitative analysis for others [49–51].

In "Traveling wave reduction of the conformable time fractional Pochhammer–Chree equation" section, the case of $n = 1$ for the Eq. (1) has been reduced to an ordinary equation using the traveling wave substitution. The direct integral of the obtained equation involves a quartic polynomial whose roots will be classified using the complete discrimination system. Section "Exact wave solutions" contains a complete analysis for all possible solutions for the intended equation. To make the article self-contained, a basic introduction to the

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conformable fractional derivatives is provided in "Appendix". For more detailed information about fractional calculus, the reader may refer to [52–54].

Traveling wave reduction of the conformable time fractional Pochhammer–Chree equation

We are interested in constructing a traveling wave solutions for Eq. (1) in which $n = 1$, $\alpha \in (0, 1]$, which will takes the form

$$D_t^{2\alpha}(\theta - \theta_{xx}) - (a_1\theta + a_2\theta^2 + a_3\theta^3)_{xx} = 0. \tag{2}$$

We apply the wave transformation

$$\theta(x, t) = \psi(\zeta), \quad \zeta = x - \frac{\omega}{\alpha}t^\alpha, \tag{3}$$

where ω is the speed of the wave and ζ is the wave variable. Inserting Eq. (3) into Eq. (2) and taking into account the properties of conformable derivitative listed in "Appendix", we get

$$\psi'' - \psi'''' - \frac{1}{\omega^2}(a_1\psi + a_2\psi^2 + a_3\psi^3)'' = 0, \tag{4}$$

where $'$ refers to the derivative with respect to ζ. By integrating both sides of Eq. (4) with respect to ζ twice and setting the first integration constant equal to zero, we have

$$\psi'' = -\frac{a_3}{\omega^2}\psi^3 - \frac{a_2}{\omega^2}\psi^2 + (1 - \frac{a_1}{\omega^2})\psi - m \tag{5}$$

where m is an integration constant. Taking into account $\psi'' = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\psi}(\psi'^2)$ and integrating both sides of the last equation with respect to ζ, we get

$$\psi'^2 = -\frac{a_3}{2\omega^2}\psi^4 - \frac{2a_2}{3\omega^2}\psi^3 + (1 - \frac{a_1}{\omega^2})\psi^2 - m\psi + g \tag{6}$$

where g is the integration constant. To simplify the notations, we set

$$\psi = \phi - \frac{a_2}{3a_3} \tag{7}$$

Substituting the last equation into Eq. (6), we obtain

$$\phi'^2 = \mu(\phi^4 + p_2\phi^2 + p_1\phi + p_0), \tag{8}$$

where

$$\begin{aligned}
\mu =& \frac{-a_3}{2\omega^2}, p_2 = \frac{1}{\mu}\left(1 - \frac{a_1}{\omega^2} + \frac{a_2^2}{3a_3\omega^2}\right),\\
p_1 =& \frac{1}{\mu}\left(\frac{2a_2}{3a_3\omega^2}(a_1 - \omega^2) - \frac{4a_2^3}{27a_3^2\omega^2} - m\right)\\
p_0 =& \frac{1}{\mu}\left(\frac{ma_2}{3a_3} + \frac{a_2^4}{54\omega^2a_3^3} - \frac{a_2^2(a_1 - \omega^2)}{9\omega^2a_3^2} + g\right).
\end{aligned} \tag{9}$$

Separating the variables in Eq. (8), we get the differential form

$$\frac{\mathrm{d}\phi}{\sqrt{\mu P_4(\phi)}} = \mathrm{d}\zeta, \tag{10}$$

where

$$P_4(\phi) = \phi^4 + p_2\phi^2 + p_1\phi + p_0. \tag{11}$$

Our goal is to find the solutions of Eq. (2) by finding the solution of the equation in (10), and using the formula in Eq. (7). To integrate both sides of Eq. (10), the range of the parameters needs to be specified. The reason is that distinct values of the parameters imply different solutions. Many tools are utilized to find these ranges of parameters such as bifurcation theory [55–59] and the complete discrimination system for a polynomial [47]. We use the complete discrimination system for a polynomial to find the ranges of parameters for $P_4(\phi)$. The complete discrimination system is a natural generalization of the discriminant $\Delta = b^2 - 4ac$ for the quadratic polynomial $ax^2 + bx + c$, but it becomes difficult to calculate for the higher degree polynomials. The complete discrimination system for the quartic polynomial in Eq. (11) is given in [47, 48] and has the following form:

$$\begin{aligned}
D_1 =& 4, \quad D_2 = -p_2, \quad D_3 = -2p_2^3 + 8p_2p_0 - 9p_1^2,\\
E_2 =& 9p_2^2 - 32p_2p_0,\\
D_4 =& -p_2^3p_1^2 + 4p_0p_2^4 + 36p_0p_2p_1^2 - 32p_2^2p_0^2 - \frac{27}{4}p_1^2 + 64p_0^3.
\end{aligned} \tag{12}$$

Note that for physical problems, the real propagation is required, consequently, we are going to find the permitted regions of real propagation, or equivalently, we determine certain intervals of ϕ which guarantee that $\mu P_4(\phi)$ is positive. In next section, we consider the nine cases determined by distinct types of the roots for the polynomial in Eq. (11). We will use the following basic fact for the roots of quartic polynomial without mentioning. If $x^4 + p_3x^3 + p_2x^2 + p_1x + p_0$ is a polynomial whose roots are $r_i, i = 1, 2, 3, 4$, then $p_3 = -\sum_i r_i$, $p_2 = \sum_{i<j} r_ir_j$, $p_1 = -\sum_{i<j<k} r_ir_jr_k$, and $p_0 = r_1r_2r_3r_4$. For computations involving elliptic integrals refer to [60].

Exact wave solutions

In this section, we aim to construct some traveling wave solutions and study the effect of the fractional order on these solutions. Based on complete discrimination system for a polynomial $P_4(\phi)$, we consider the following cases:

Case 1: If $D_2 = D_3 = D_4 = 0$, then the polynomial $P_4(\phi)$ has the zero as a one real root repeated four times, and can be written as $P_4(\phi) = \phi^4 \geq 0$ for all ϕ. Therefore, $\mu < 0$ gives a complex solution, and we only consider the case where $\mu > 0$ to solve the equation in Eq. (10) which will be in the following form

$$\frac{d\phi}{\phi^2} = \sqrt{\mu}d\zeta \tag{13}$$

Let $-\infty < \phi < 0$, and $\phi(\zeta_0) = -\infty$, by integrating both sides of the above equation, we get

$$\phi = -\frac{1}{\sqrt{\mu}(\zeta - \zeta_0)} \tag{14}$$

Similarly, the case where $0 < \phi < \infty$, and $\phi(\zeta_0) = \infty$, will generate the same solution in Eq. (14). Thus, the solution for Eq. (2) is

$$\psi = -\frac{1}{\sqrt{\mu}(\zeta - \zeta_0)} - \frac{a_2}{3a_3} \tag{15}$$

The solution (15) is a new solution for Eq. (2). Figure 1a, b outlines the 3D- graphic representation of the the solution (15) when $\alpha = 0.4$ and $\alpha = 0.7$. Figure 1c shows the width of the solution decreases when the fractional order increases. Furthermore, when α tends to one, the solution (15) becomes also a solution for the integer order time derivative version of Eq. (2).

Case 2: If $D_3 = D_4$, $E_2 < 0$ and $D_2 > 0$, then the polynomial in Eq (11) has two real roots ϕ_1 and ϕ_2 where $\phi_2 = -3\phi_1$. Therefore, $P_4(\phi) = (\phi - \phi_1)^3(\phi + 3\phi_1)$, where ϕ_1 is postulated to be positive. We consider the following sub-cases based on μ

- If $\mu \in (0, \infty)$, then the permitted region of real propagation is $\phi \in (-\infty, -3\phi_1) \cup (\phi_1, \infty)$.
  - If $\phi \in (-\infty, -3\phi_1)$, then we integrate both sides of Eq. (10), assuming that $\phi(\zeta_0) = -\infty$ to get

$$\int_{-\infty}^{\phi} \frac{d\phi}{(\phi_1 - \phi)\sqrt{(\phi - \phi_1)(\phi + 3\phi_1)}} = \frac{1}{2\phi_1}\left(\sqrt{\frac{\phi + 3\phi_1}{\phi - \phi_1}} - 1\right) = \sqrt{\mu}\int_{\zeta_0}^{\zeta} d\zeta \tag{16}$$

  The last equation implies

$$\phi = \phi_1 - \frac{\phi_1}{\phi_1\sqrt{\mu}(\zeta - \zeta_0) + 1} + \frac{1}{\sqrt{\mu}(\zeta - \zeta_0)}. \tag{17}$$

  - If $\phi \in (\phi_1, \infty)$, then by assuming that $\phi(\zeta_0) = \infty$, the solution of Eq. (10) can be computed similarly and will generate the same solution above. Thus, the solutions of Eq. (2) include

$$\psi = \phi_1 - \frac{\phi_1}{\phi_1\sqrt{\mu}(\zeta - \zeta_0) + 1} + \frac{1}{\sqrt{\mu}(\zeta - \zeta_0)} - \frac{a_2}{3a_3}. \tag{18}$$

  The solutions are graphically clarified for different values of the fractional order α in Fig. 2.
- If $\mu \in (-\infty, 0)$, we obtain the real propagation when $-3\phi_1 < \phi < \phi_1$. So, we postulate $\phi(\zeta_0) = -3\phi_1$, and integrate both side of Eq. (10) to get

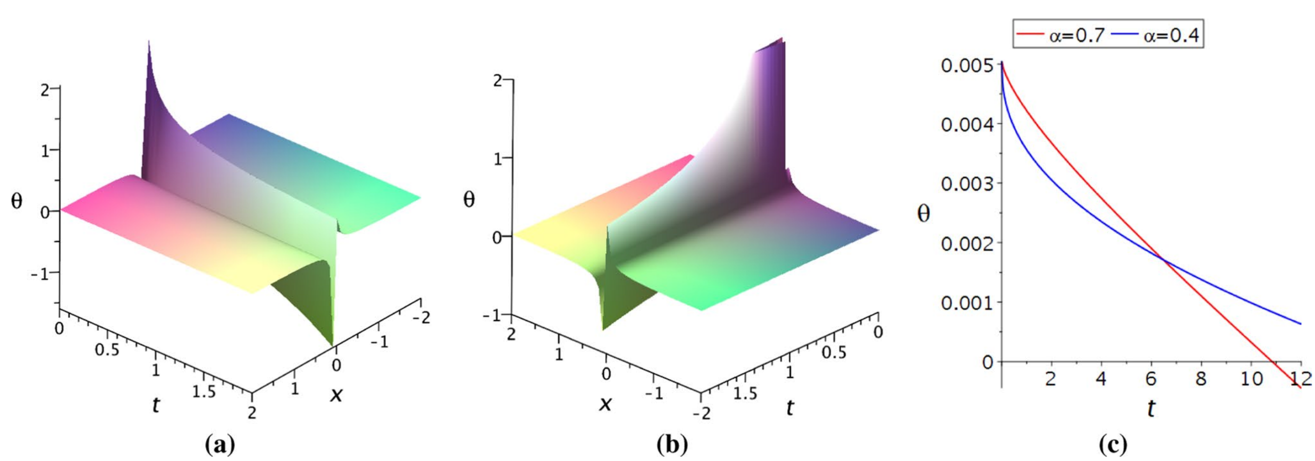

Fig. 1 3D-graph of the solution (15) for $(x, t) \in [-2, 2] \times [0, 2]$ for different values of fractional order α, **a** $\alpha = 0.4$ and **b** $\alpha = 0.7$. While **c** is the 2D-graph of the solution (15) when $x = 1, t \in [0, 12]$

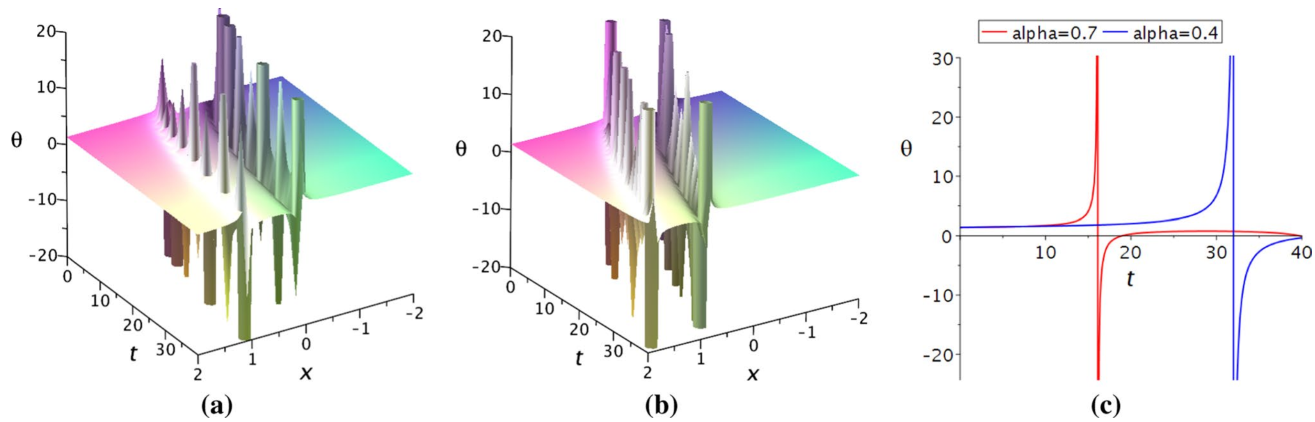

Fig. 2 3D-graph of the solution (18) for $(x, t) \in [-2, 2] \times [0, 40]$ for different values of fractional order α, **a** $\alpha = 0.4$ and **b** $\alpha = 0.7$. While **c** is the 2D-graph of the solution (18) when $x = 1, t \in [0, 40]$

$$\int_{-3\phi_1}^{\phi} \frac{\mathrm{d}\phi}{(\phi_1 - \phi)\sqrt{(\phi_1 - \phi)(\phi + 3\phi_1)}} = \frac{-1}{2\phi_1}\sqrt{\frac{\phi + 3\phi_1}{\phi_1 - \phi}} = \sqrt{-\mu}\int_{\zeta_0}^{\zeta} \mathrm{d}\zeta \tag{19}$$

Therefore,

$$\phi = \phi_1 - \frac{4\phi_1}{1 + 4\mu\phi_1^2(\zeta - \zeta_0)^2} \tag{20}$$

Hence, the Eq. (2) has a solution in the form

$$\psi = \phi_1 - \frac{4\phi_1}{1 + 4\mu\phi_1^2(\zeta - \zeta_0)^2} - \frac{a_2}{3a_3} \tag{21}$$

The solution in (21) is a novel singular solution for Eq. (2). Figure 2a, b are 3D-graphic representation for the solution (21) when $\alpha = 0.4$ and $\alpha = 0.7$, respectively. Figure 2 clarifies the width of the solution decreases as the fractional order increases. Also, when α approaches to one, we obtain a solution for Eq. (2) with integer time derivative.

Case 3: If $D_3 = D_4 = 0, E_2 > 0, D_2 > 0$, then the polynomial $P_4(\phi)$ has two real zeros which are doubled. Moreover, each one of them is the negative of the other which implies that $P_4(\phi)$ can be expressed using one root as

$$P_4(\phi) = (\phi^2 - \phi_1^2)^2. \tag{22}$$

Since $P_4(\phi)$ is non negative, then for $\mu < 0$, the expression $\mu P_4(\phi)$ is always negative for all ϕ, and gives complex solutions for Eq. (10), so we must neglect it. We only consider the case $\mu > 0$ where the real propagation accrues if $\phi \in \mathbb{R}\backslash\{\pm\phi_1\}$ and the equation in (10) will have the following form

$$\frac{\mathrm{d}\phi}{\left|\phi^2 - \phi_1^2\right|} = \sqrt{\mu}\mathrm{d}\zeta \tag{23}$$

We assume that ϕ_1 is positive and we study the following cases:

- If $\phi < -\phi_1$, and $\phi(\zeta_0) = -\infty$, then Eq. (23) implies that

$$\int_{-\infty}^{\phi} \frac{\mathrm{d}\phi}{\phi^2 - \phi_1^2} = \frac{-1}{\phi_1}\mathrm{arcoth}\left(\frac{\phi}{\phi_1}\right) = \sqrt{\mu}(\zeta - \zeta_0) \tag{24}$$

which gives

$$\phi = -\phi_1 \coth(\phi_1\sqrt{\mu}(\zeta - \zeta_0)). \tag{25}$$

Hence, we obtain a new solution for Eq. (2) in the form

$$\psi = -\phi_1 \coth(\phi_1\sqrt{\mu}(\zeta - \zeta_0)) - \frac{a_2}{3a_3}. \tag{26}$$

Notice, when $\alpha \to 1$, this solution will converted into a well known solution for Eq. (2) with $\alpha \to 1$ [61].

- If $\phi > \phi_1$, the corresponding solution can obtained from Eq. (26) by replacing ϕ_1 by $-\phi_1$, which generate the same solution.
- With similar computation, if $\phi \in (-\phi_1, \phi_1)$, then $\phi^2 < \phi_1^2$. So, by letting $\phi(\zeta_0) = 0$, and integrating Eq. (23) from 0 to ϕ, we get that

$$\frac{-1}{\phi_1}\mathrm{arctanh}\left(\frac{\phi}{\phi_1}\right) = \sqrt{\mu}(\zeta - \zeta_0) \tag{27}$$

That is,

$$\phi = -\phi_1 \tanh(\phi_1\sqrt{\mu}(\zeta - \zeta_0)) \tag{28}$$

Hence, the case under consideration will generate the following solution for Eq. (2)

$$\psi = -\phi_1 \tanh(\phi_1 \sqrt{\mu}(\zeta - \zeta_0)) - \frac{a_2}{3a_3}, \tag{29}$$

The solution in (29) is a new solution for Eq. (2). Figure 3a, b illustrate the 3D-graphic representation of the solution (29), like a kink solution. Figure 3c shows the width of the solution increase as α increase. When $\alpha \to 1$, we obtain a well-known solution for equation integer time derivative version of Eq. (2) [61].

Case 4: This case is characterized by $D_2 > 0, D_3 > 0, D_4 > 0$. These conditions guarantee the existence of four real roots for the polynomial $P_4(\phi)$. Assuming that three of them are ϕ_1, ϕ_2, ϕ_3, the fourth one must be $-(\phi_1 + \phi_2 + \phi_3)$. Hence, we write $P_4(\phi) = (\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)(\phi + \phi_1 + \phi_2 + \phi_3)$, where $0 < \phi_1 < \phi_2 < \phi_3$. We consider the following two sub-cases:

- If $\mu \in (0, \infty)$, then there is a real propagation only if $\phi \in (-\infty, -(\phi_1 + \phi_2 + \phi_3)) \cup (\phi_1, \phi_2) \cup (\phi_3, \infty)$, where the Eq. (10) has the form

$$\frac{d\phi}{\sqrt{(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)(\phi + \phi_1 + \phi_2 + \phi_3)}} = \sqrt{\mu}\zeta \tag{30}$$

  - - If we elect $\phi \in (-\infty, -(\phi_1 + \phi_2 + \phi_3))$, postulate $\phi(\zeta_0) = -(\phi_1 + \phi_2 + \phi_3)$, and integrate both sides of Eq. (30), we get

$$\phi = \phi_1 + \frac{(\phi_3 - \phi_1)(2\phi_1 + \phi_2 + \phi_3)}{\phi_1 - \phi_3 + (\phi_1 + \phi_2 + 2\phi_3)\mathrm{sn}^2(\Omega(\zeta - \zeta_0), k)} \tag{31}$$

  where

$$\Omega = \frac{1}{2}\sqrt{\mu(\phi_3 - \phi_1)(\phi_1 + 2\phi_2 + \phi_3)},\quad k = \sqrt{\frac{(\phi_2 - \phi_1)(\phi_1 + \phi_2 + 2\phi_3)}{(\phi_3 - \phi_1)(\phi_1 + 2\phi_2 + \phi_3)}} \tag{32}$$

Hence, we get a novel periodic solution for Eq. (2) in the form

$$\psi = \phi_1 + \frac{(\phi_3 - \phi_1)(2\phi_1 + \phi_2 + \phi_3)}{\phi_1 - \phi_3 + (\phi_1 + \phi_2 + 2\phi_3)\mathrm{sn}^2(\Omega(\zeta - \zeta_0), k)} - \frac{a_2}{3a_3}. \tag{33}$$

Figure 4a, b outline the periodicity of the solution (33) for different values of the fractional order α, but the amplitude and the width of the solution are affected. Figure 4c shows the width of the solution increases as the fractional order increases, but the amplitude is approximately unchanged. We also examine the degeneracy of the solution (33). If $\phi_3 = \phi_2$, the modulus of the elliptic function, k, will be reduced to one. Hence, the solution (33) degenerates to

$$\psi = \phi_1 - \frac{2(\phi_2^2 - \phi_1^2)}{(\phi_1 - \phi_2) + (\phi_1 + 3\phi_2)\tanh^2(\Omega(\zeta - \zeta_0)} \tag{34}$$

which is also a new solution. If $\phi_2 = \phi_1$, the modules of the elliptic function becomes zero and hence, the solution (33) degenerates to

$$\psi = \phi_1 + \frac{(\phi_3 - \phi_1)(3\phi_1 + \phi_3)}{(\phi_1 - \phi_3) + 2(\phi_1 + \phi_3)\sin^2\Omega(\zeta - \zeta_0)} \tag{35}$$

which is also a new solution for Eq. (2). Notice, the two solutions (34) and (35) will transform to a

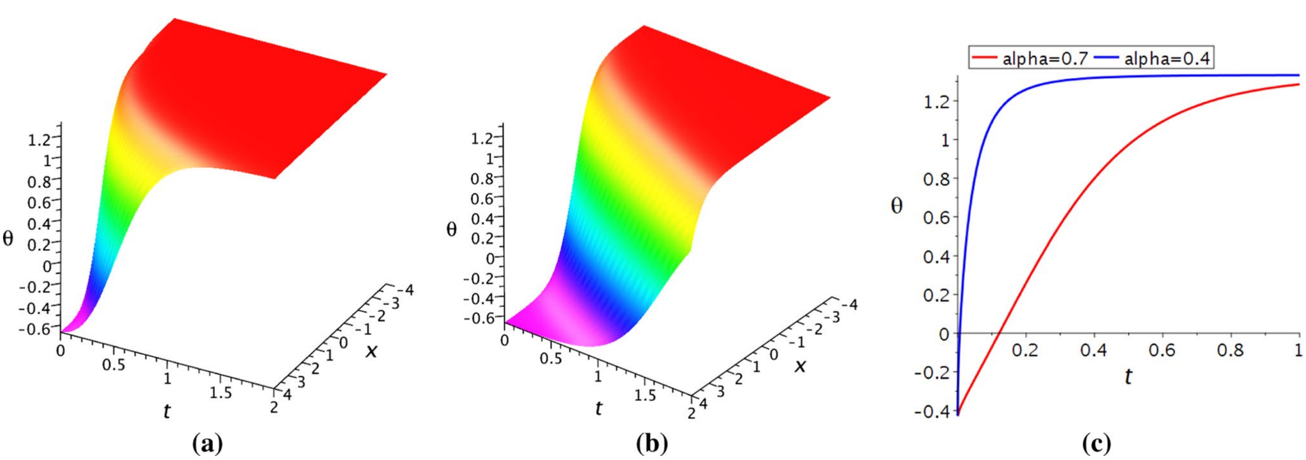

Fig. 3 3D-graph of the solution (29) for $(x, t) \in [-2, 2] \times [0, 2]$ for different values of fractional order α, **a** $\alpha = 0.4$ and **b** $\alpha = 0.7$. While **c** is the 2D-graph of the solution (29) when $x = 1, t \in [0, 1]$

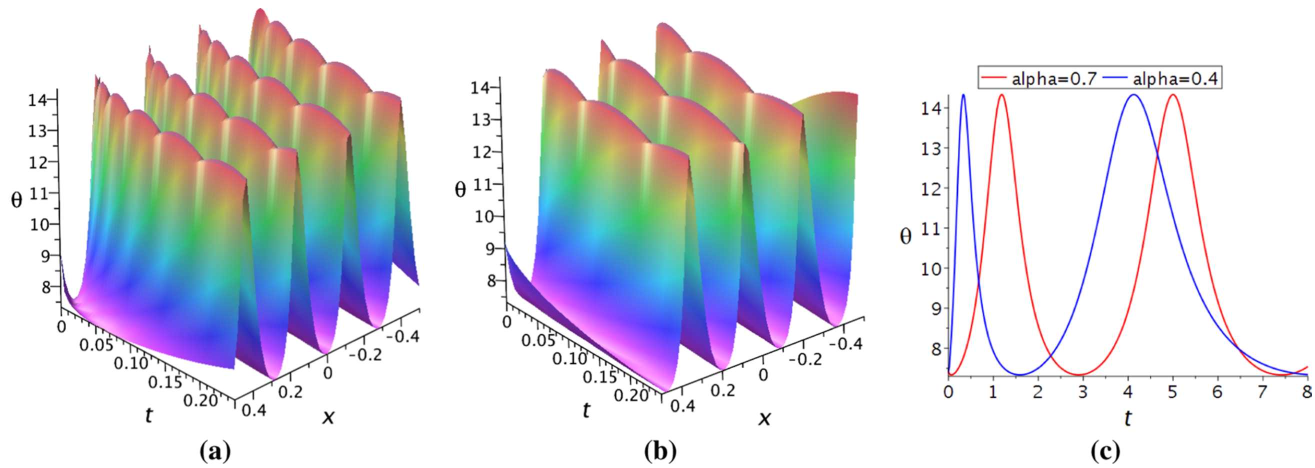

Fig. 4 3D-graph of the solution (33) for $(x, t) \in [-0.5, 0.5] \times [0, 0.25]$ for different values of fractional order α, **a** $\alpha = 0.4$ and **b** $\alpha = 0.7$. While **c** the 2D-graph of the solution (33) when $x = 1, t \in [0, 8]$

well-known solutions for the time integer derivative of Eq. (2) [61].

- - If we choose $\phi \in (\phi_1, \phi_2)$ and set $\phi(\zeta_0) = \phi_1$, then the integral from ϕ_1 to ϕ will generate the following solution

$$\phi = -(\phi_1 + \phi_2 + \phi_3) + \frac{(\phi_1 + 2\phi_2 + \phi_3)(2\phi_1 + \phi_2 + \phi_3)}{\phi_1 + 2\phi_2 + \phi_3 + (\phi_1 - \phi_2)\mathrm{sn}^2(\Omega(\zeta - \zeta_0), k)} \tag{36}$$

Hence, we obtain a novel solution for Eq. (2) in the form

$$\psi = -(\phi_1 + \phi_2 + \phi_3) + \frac{(\phi_1 + 2\phi_2 + \phi_3)(2\phi_1 + \phi_2 + \phi_3)}{\phi_1 + 2\phi_2 + \phi_3 + (\phi_1 - \phi_2)\mathrm{sn}^2(\Omega(\zeta - \zeta_0), k)} - \frac{a_2}{3a_3}. \tag{37}$$

Figure 5a, b shows that the solution (37) is periodic for distinct values of the fractional order α, but its width and its amplitude are influenced. Figure 5c clarifies the width of the solution (37) decreases as the fractional order α increases while the amplitude is approximately unaltered. Let us now study the degeneracy of the solution (37). When $\phi_3 = \phi_2$, the modules of the elliptic function, k, becomes one, and the solution (37) degenerates to

$$\psi = -\phi_1 - 2\phi_2 + \frac{2(\phi_1 + 3\phi_2)(\phi_1 + \phi_2)}{\phi_1 + 3\phi_2 + (\phi_1 - \phi_2)\tanh^2\Omega(\zeta - \zeta_0)}, \tag{38}$$

which is also a new solution for Eq. (2). When $\phi_2 = \phi_1$, the modules of the elliptic function, k, equals to zero and the solution (37) is reduced to $\psi = \phi_1$ which is a trivial solution for Eq. (2). Notice, when $\alpha \to 1$, the solution (38) will be converted to a

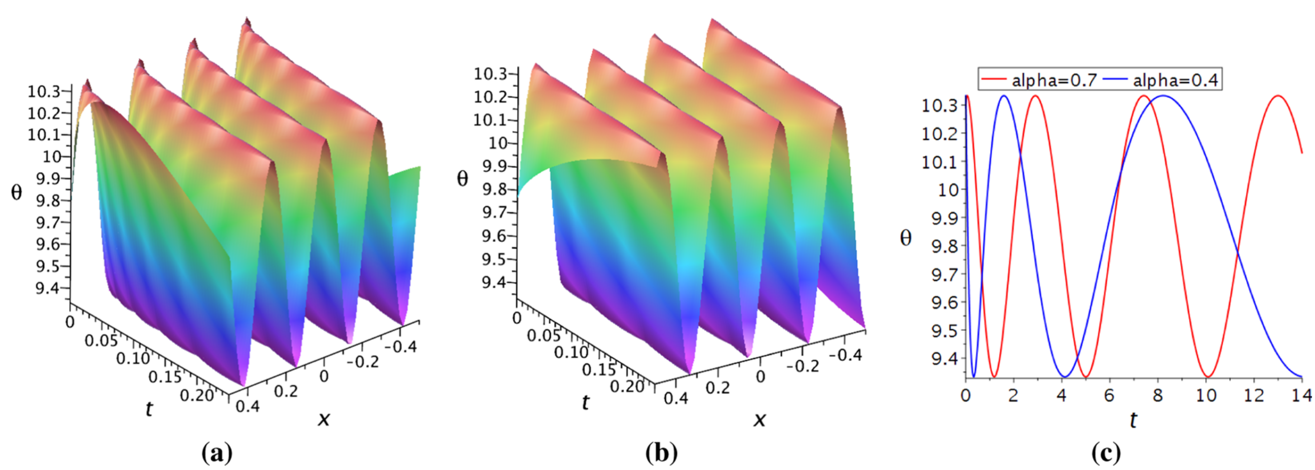

Fig. 5 3D-graph of the solution (37) for $(x, t) \in [-0.5, 0.5] \times [0, 0.25]$ for different values of fractional order α, **a** $\alpha = 0.4$ and **b** $\alpha = 0.7$. While **c** is the 2D-graph of the solution (37) when $x = 1, t \in [0, 14]$

well known solution for the time integer derivative for Eq. (2).

- - If we choose $\phi \in (\phi_3, \infty)$ with assumption $\phi(\zeta_0) = \phi_3$, and integrate both sides of Eq. (30), we obtain a new solution for Eq. (2)

$$\phi = \phi_2 - \frac{(\phi_2 - \phi_3)(\phi_1 + 2\phi_2 + \phi_3)}{(\phi_1 + 2\phi_2 + \phi_3) - (\phi_1 + \phi_2 + 2\phi_3)\mathrm{sn}^2(\Omega(\zeta - \zeta_0), k)} \tag{39}$$

where k and Ω are as defined in Eq. (32) above. Hence, we obtain a new solution for Eq. (2) in the form

$$\psi = \phi_2 - \frac{(\phi_2 - \phi_3)(\phi_1 + 2\phi_2 + \phi_3)}{(\phi_1 + 2\phi_2 + \phi_3) - (\phi_1 + \phi_2 + 2\phi_3)\mathrm{sn}^2(\Omega(\zeta - \zeta_0), k)} - \frac{a_2}{3a_3} \tag{40}$$

Similarly, we can study the degeneracy of the solution (40). Notice, when $\alpha \to 1$, the solution (40) is also a new solution of the time integer order derivative for Eq. (2).

- If $\mu \in (-\infty, 0)$, then there is a real propagation only if $\phi \in (-\phi_1 - \phi_2 - \phi_3, \phi_1) \cup (\phi_2, \phi_3)$, where the Eq. (10) will be the form

$$\frac{\mathrm{d}\phi}{\sqrt{(\phi_1 - \phi)(\phi_2 - \phi)(\phi_3 - \phi)(\phi + \phi_1 + \phi_2 + \phi_3)}} = \sqrt{-\mu}\zeta \tag{41}$$

– If we elect $\phi \in (-\phi_1 - \phi_2 - \phi_3, \phi_1)$ with assumption $\phi(\zeta_0) = -\phi_1 - \phi_2 - \phi_3$, and integrate both sides of Eq. (41), then Eq. (10) possesses a new solution in the form

$$\phi = \phi_3 + \frac{(\phi_1 - \phi_3)(\phi_1 + \phi_2 + 2\phi_3)}{(\phi_3 - \phi_1) + (2\phi_1 + \phi_2 + \phi_3)\mathrm{sn}^2(\Omega_1(\zeta - \zeta_0), k_1)} \tag{42}$$

where

$$\Omega_1 = \frac{1}{2}\sqrt{-\mu(\phi_3 - \phi_1)(\phi_1 + 2\phi_2 + \phi_3)}, k_1 = \sqrt{\frac{(\phi_3 - \phi_2)(2\phi_1 + \phi_2 + \phi_3)}{(\phi_3 - \phi_1)(\phi_1 + 2\phi_2 + \phi_3)}} \tag{43}$$

Hence, Eq. (2) has a new wave solution in the form

$$\psi = \phi_3 + \frac{(\phi_1 - \phi_3)(\phi_1 + \phi_2 + 2\phi_3)}{(\phi_3 - \phi_1) + (2\phi_1 + \phi_2 + \phi_3)\mathrm{sn}^2(\Omega_1(\zeta - \zeta_0), k_1)} - \frac{a_2}{3a_3}. \tag{44}$$

If $\phi_2 = \phi_1$, the modules $k_1 = 1$. Therefore, the solution (44) degenerates to

$$\psi = \phi_3 + \frac{2(\phi_1^2 - \phi_3^2)}{\phi_3 - \phi_1 + (3\phi_1 + \phi_3)\tanh^2\Omega_1(\zeta - \zeta_0)} \tag{45}$$

which is also a new solution for Eq. (2). When $\phi_3 = \phi_2$, the modules $k_1 = 0$, and the solution (44) degenerates to

$$\psi = \phi_2 + \frac{(\phi_1 - \phi_2)(\phi_1 + 3\phi_2)}{\phi_2 - \phi_1 + 2(\phi_1 + \phi_2)\sin^2(\Omega_1(\zeta - \zeta_0)} \tag{46}$$

which is also a novel solution for Eq. (2). Notice, when $\alpha \to 1$, the solution (44) reduces to a new wave solution for the Eq. (2) with $\alpha = 1$.

– With similar computations, we can select $\phi \in (\phi_2, \phi_3)$ with $\phi(\zeta_0) = \phi_2$, and integrate both side of Eq. (2) to get the following novel solution of Eq. (10)

$$\psi = \phi_1 - \frac{(\phi_1 - \phi_2)(\phi_1 - \phi_3)}{\phi_1 - \phi_3 + (\phi_3 - \phi_2)\mathrm{sn}^2(\Omega_1(\zeta - \zeta_0), k_1)} - \frac{a_2}{3a_3} \tag{47}$$

where k_1 and Ω_1 are as defined in Eq. (43) above.

Notice that the solution (47) degenerates to a trivial solution for Eq. (2) whether $k_1 = 1(\phi_1 = \phi_2)$, or $k_1 = 0(\phi_3 = \phi_2)$. When $\alpha \to 1$, the solution (47) becomes also a new solution for Eq. (2) with $\alpha \to 1$.

Case 5: This case is determined by the constrains $D_4 = 0$ and $D_2 D_3 < 0$. Based on these conditions, $P_4(\phi)$ is written as $P_4(\phi) = (\phi - \phi_1)^2(\phi - \phi_2)(\phi - \overline{\phi_2})$ where the bar indicates the complex conjugate and $\phi_1 = -\mathrm{Re}\phi_2$. Since $(\phi - \phi_2)(\phi - \overline{\phi_2}) = (\phi - \mathrm{Re}\phi_2)^2 + \mathrm{Im}\phi_2^2 \geq 0$, then $P_4(\phi) \geq 0$, and the conditions for real propagation are $\mu \in (0, \infty)$ and $\phi \in \mathbb{R}$, in which case the Eq. (10) becomes

$$\frac{\mathrm{d}\phi}{|\phi - \phi_1|\sqrt{(\phi + \phi_1)^2 + \mathrm{Im}\phi_2^2}} = \sqrt{\mu}\mathrm{d}\zeta \tag{48}$$

Notice that the case in which $\mu \in (-\infty, 0)$ is excluded since it does not give real propagation. Thus, for $\mu \in (0, \infty)$, we select ϕ such that $\phi < \phi_1$, assume $\phi(\zeta_0) = -\infty$, and integrate both sides of the above equation using the substitution $\phi = \phi_1 - \frac{1}{y}$, the Eq. (2) has a new solution in the form

$$\phi = \phi_1 + \frac{4\phi_1^2 + \mathrm{Im}\phi_2}{-2\phi_1 + \mathrm{Im}\phi_2 \sinh\sqrt{\mu(4\phi_1^2 + \mathrm{Im}^2\phi_2)}(\zeta - \epsilon)} \tag{49}$$

where $\epsilon = \zeta_0 - \frac{1}{\sqrt{\mu(4\phi_1^2 + \mathrm{Im}^2\phi_2)}}\sinh^{-1}(-2\phi_1/\mathrm{Im}\phi_2)$.

If $\alpha \to 1$, the solution (49) is a well known solution for time integer derivative for Eq. (2). Similarly, we can find the solution when $\phi > \phi_1$.

Case 6: This case is determined by $D_2 D_3 \leq 0$, and $D_4 > 0$. These conditions guarantee the existence of two complex conjugate roots, namely $\phi_1, \overline{\phi}_1, \phi_2, \overline{\phi}_2$ for $P_4(\phi)$. This means, $P_4(\phi)$ takes the form $P_4(\phi) = (\phi - \phi_1)(\phi - \overline{\phi}_1)(\phi - \phi_2)(\phi - \overline{\phi}_2)$, where $\mathrm{Re}\phi_1 = -\mathrm{Re}\phi_2$. Based on $(\phi - \phi_k)(\phi - \overline{\phi}_k) = (\phi - \mathrm{Re}\phi_k)^2 + \mathrm{Im}\phi_k^2$ for $k = 1, 2$, then $P_4(\phi) \geq 0$. Hence, there is a real propagation for Eq. (2) if $\mu \in (0, \infty)$, $\phi \in \mathbb{R}$, where the equation in Eq. (10) becomes

$$\frac{\mathrm{d}\phi}{\sqrt{(\phi - \phi_1)(\phi - \overline{\phi}_1)(\phi - \phi_2)(\phi - \overline{\phi}_2)}} = \sqrt{\mu}\mathrm{d}\zeta \tag{50}$$

Let

$$\begin{aligned} A^2 &= 4(\mathrm{Re}\phi_1)^2 + (\mathrm{Im}\phi_1 + \mathrm{Im}\phi_2)^2, \\ B^2 &= 4(\mathrm{Re}\phi_1)^2 + (\mathrm{Im}\phi_1 - \mathrm{Im}\phi_2)^2, \end{aligned} \tag{51}$$

and

$$\delta^2 = \frac{4(\mathrm{Im}\phi_1)^2 - (A - B)^2}{(A + B)^2 - 4(\mathrm{Im}\phi_1)^2} \tag{52}$$

By evaluating the integrals of both side of Eq. (50), we get

$$\phi = \frac{\mathrm{Re}\phi_1 \mp \mathrm{Im}\phi_1 \delta + (\mathrm{Re}\phi_1 \delta \pm \mathrm{Im}\phi_1)\mathrm{tn}\left(\frac{\sqrt{\mu}}{2}(A + B)(\zeta_- \zeta_0), k_2\right)}{1 + \delta \mathrm{tn}(\frac{\sqrt{\mu}}{2}(A + B)(\zeta - \zeta_0), k_2)} \tag{53}$$

where $k_2 = \sqrt{\frac{4AB}{(A+B)^2}}$.

Therefore, we obtain a new solution for Eq. (2) in the form

$$\psi = \frac{\mathrm{Re}\phi_1 \mp \mathrm{Im}\phi_1 \delta + (\mathrm{Re}\phi_1 \delta \pm \mathrm{Im}\phi_1)\mathrm{tn}\left(\frac{\sqrt{\mu}}{2}(A + B)(\zeta_- \zeta_0), k_2\right)}{1 + \delta \mathrm{tn}(\frac{\sqrt{\mu}}{2}(A + B)(\zeta - \zeta_0), k_2)} - \frac{a_2}{3a_3}. \tag{54}$$

It is obvious that when $\alpha \to 1$, the solution (54) is also a new solution for the Eq. (2) with $\alpha \to 1$. Now, let us investigate the degeneracy of the solution (54). It is easy to show that $k_2 = 1$ if either one of the complex roots ϕ_1 or ϕ_2 is real, i.e., $\mathrm{Im}\phi_1 = 0$ or $\mathrm{Im}\phi_2 = 0$. Therefore, the solution (54) reduces to $\psi = \mathrm{Re}\phi_1 - \frac{a_2}{3a_3}$ which is trivial solution for Eq. (2).

Case 7: This case is characterized by $D_4 < 0$ and $D_2 D_3 \leq 0$, which implies that $P_4(\phi)$ has two real roots and two complex conjugate roots. That is,

$$P_4(\phi) = (\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)(\phi - \overline{\phi_3}) \tag{55}$$

where $\phi_1 < \phi_2$ and $\mathrm{Re}\phi_3 = -\frac{1}{2}(\phi_1 + \phi_2)$. Let $A_1^2 = \frac{1}{4}(\phi_1 + 3\phi_1)^2 + (\mathrm{Im}\phi_3)^2$, $B_1^2 = \frac{1}{4}(3\phi_1 + \phi_2)^2 + (\mathrm{Im}\phi_3)^2$. Thus, we consider the following two cases

- If $\mu \in (0, \infty)$, then the choice $\phi \in (-\infty, \phi_1) \cup (\phi_2, \infty)$ gives a real propagation where the Eq. (10) has the following form

$$\frac{\mathrm{d}\phi}{\sqrt{(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)(\phi - \overline{\phi_3})}} = \sqrt{\mu}\mathrm{d}\zeta \tag{56}$$

  - If $\phi \in (\phi_2, \infty)$, we choose $\phi(\zeta_0) = \phi_2$ and integrate both sides of Eq. (56), we get

$$\phi = \frac{\phi_1 A_1 + \phi_2 B_1}{A_1 + B_1} - \frac{2A_1 B_1(\phi_1 - \phi_2)}{(A_1 + B_1)[(A_1 + B_1)\mathrm{cn}(\sqrt{\mu A_1 B_1}(\zeta - \zeta_0), k_3) - A_1 + B_1]} \tag{57}$$

where $k_3 = \sqrt{\frac{(A_1+B_1)^2-(\phi_2-\phi_1)^2}{4A_1B_1}}$ and A_1,B_1 as defined above.

Consequently, Eq. (2) has a new solution in the form

$$\psi = \frac{\phi_1 A_1 + \phi_2 B_1}{A_1 + B_1} - \frac{2A_1 B_1(\phi_1 - \phi_2)}{(A_1 + B_1)[(A_1 + B_1)\mathrm{cn}(\sqrt{\mu A_1 B_1}(\zeta - \zeta_0), k_3) - A_1 + B_1]} - \frac{a_2}{3a_3}. \tag{58}$$

It is easy to show that if $A_1 + B_1 = \phi_2 - \phi_1$, then $k_3 = 0$ and the solution (58) degenerates to

$$\psi = \phi_2 - A_1 + \frac{2A_1(A_1 - \phi_2 + \phi_1)}{(\phi_1 - \phi_2)[\cos\sqrt{\mu_1 A_1 B_1}(\zeta - \zeta_0)]} \tag{59}$$

which also is a new solution for Eq. (2). Similarly, $k_3 = 1$ when $A_1 - B_1 = \phi_2 - \phi_1$ and the solution (58) degenerates to

$$\psi = \frac{\phi_2^2 + (A_1 - \phi_1)\phi_2 + \phi_1 A_1}{2A_1 + \phi_2 - \phi_1} - \frac{2A_1(A_1 + \phi_2 - \phi_1)(\phi_1 - \phi_2)}{(2A_1 + \phi_2 - \phi_1)^2 \mathrm{sech}\sqrt{\mu_1 A_1 B_1}(\zeta - \zeta_0) + (2A_1 + \phi_2 - \phi_1)(\phi_2 - \phi_1)} \tag{60}$$

which is also a novel solution for Eq. (2). Furthermore, when $\alpha \to 1$, the solution (58) reduces to a new solution for integer time derivative of Eq. (2).

– If $\phi \in (-\infty, \phi_1)$, then by choosing $\phi(\zeta_0) = -\infty$, we get

$$\phi = \frac{\phi_1 B_1 - \phi_2 A_1 + (\phi_1 B_1 + \phi_2 A_1)\mathrm{cn}(\sqrt{\mu A_1 B_1}(\zeta - \zeta_0), k_3)}{(B_1 - A_1) + (A_1 + B_1)\mathrm{cn}(\sqrt{\mu A_1 B_1}(\zeta - \zeta_0), k_3)} \quad (61)$$

Thus, we get a novel solution for Eq. (2) in the form

$$\psi = \frac{\phi_1 B_1 - \phi_2 A_1 + (\phi_1 B_1 + \phi_2 A_1)\mathrm{cn}(\sqrt{\mu A_1 B_1}(\zeta - \zeta_0), k_3)}{(B_1 - A_1) + (A_1 + B_1)\mathrm{cn}(\sqrt{\mu A_1 B_1}(\zeta - \zeta_0), k_3)} - \frac{a_2}{3a_3}. \quad (62)$$

- If $\mu \in (-\infty, 0)$, then the allowed interval for real propagation is (ϕ_1, ϕ_2) in which case the Eq. (10) has the form of

$$\frac{\mathrm{d}\phi}{\sqrt{(\phi - \phi_1)(\phi_2 - \phi)(\phi - \phi_3)(\phi - \overline{\phi_3})}} = \sqrt{-\mu}\mathrm{d}\zeta \quad (63)$$

By choosing $\phi \in (\phi_1, \phi_2)$ with the assumption $\phi(\zeta_0) = \phi_1$, and integrating both side of Eq. (63), we obtain

$$\phi = \frac{\phi_2 B_1 + \phi_1 A_1 - (\phi_1 A_1 + \phi_2 B_1)\mathrm{cn}(\sqrt{-\mu A_1 B_1}(\zeta - \zeta_0), k_4)}{B_1 + A_1 - (B_1 - A_1)\mathrm{cn}(\sqrt{-\mu A_1 B_1}(\zeta - \zeta_0), k_4)} \quad (64)$$

where $k_4 = \sqrt{\frac{(\phi_2 - \phi_1)^2 - (A_1 - B_1)^2}{4A_1 B_1}}$ and A_1, B_1 as defined above

Consequently, A new solution for Eq. (2) is written as

$$\psi = \frac{\phi_2 B_1 + \phi_1 A_1 - (\phi_1 A_1 + \phi_2 B_1)\mathrm{cn}(\sqrt{-\mu A_1 B_1}(\zeta - \zeta_0), k_4)}{B_1 + A_1 - (B_1 - A_1)\mathrm{cn}(\sqrt{-\mu A_1 B_1}(\zeta - \zeta_0), k_4)} - \frac{a_2}{3a_3}. \quad (65)$$

Case 8: If $D_3 > 0$ $D_2 > 0$, and $D_4 = 0$ then $P_4(\phi)$ has four zeros which one of them is doubled and the others are simple. That is,

$$P_4(\phi) = (\phi - \phi_1)^2(\phi - \phi_2)(\phi - \phi_3) \quad (66)$$

where $\phi_1 < \phi_2 < 0 < \phi_3$ and $\phi_3 = -(2\phi_1 + \phi_2)$. We consider the following cases:

- If $\mu \in (0, \infty)$, then the real propagation occurs if $\phi \in (-\infty, \phi_1) \cup (\phi_1, \phi_2) \cup (\phi_3, \infty)$, in which case the Eq. (10) becomes

$$\frac{\mathrm{d}\phi}{|\phi - \phi_1|\sqrt{(\phi - \phi_2)(\phi + 2\phi_1 + \phi_2)}} = \sqrt{\mu}\mathrm{d}\zeta \quad (67)$$

By choosing $\phi \in (-2\phi_1 - \phi_2, \infty)$ and assuming that $\phi(\zeta_0) = -(2\phi_1 + \phi_2)$, we have

$$\phi = \phi_1 + \frac{(3\phi_1 + \phi_2)(\phi_2 - \phi_1)}{\phi_1 + \phi_2 + 2\phi_1 \cos\sqrt{\mu(\phi_2 - \phi_1)(3\phi_1 + \phi_2)}(\xi - \epsilon)}. \quad (68)$$

Hence, Eq. (2) has a new solution in the form

$$\psi = \phi_1 + \frac{(3\phi_1 + \phi_2)(\phi_2 - \phi_1)}{\phi_1 + \phi_2 + 2\phi_1 \cos\sqrt{\mu(\phi_2 - \phi_1)(3\phi_1 + \phi_2)}(\xi - \epsilon)} - \frac{a_2}{3a_3}. \quad (69)$$

Similarly, we can find the solution for $\phi \in (-\infty, \phi_1) \cup (\phi_1, \phi_2)$.

- If $\mu \in (-\infty, 0)$, then the allowed interval for real propagation is (ϕ_2, ϕ_3). By choosing $\phi \in (\phi_2, \phi_3)$ with the assumption $\phi(\zeta_0) = \phi_2$, and where Eq. (10) becomes

$$\frac{\mathrm{d}\phi}{(\phi - \phi_1)\sqrt{(\phi - \phi_2)(-\phi + 2\phi_1 + \phi_2)}} = \sqrt{-\mu}\mathrm{d}\zeta \quad (70)$$

By integrating both side of Eq. (70), we obtain

$$\phi = \phi_1 + \frac{(\phi_2 - \phi_1)(3\phi_1 + \phi_2)}{2\phi_1 + (\phi_1 + \phi_2)\cosh\sqrt{-\mu(3\phi_1 + \phi_2)(\phi_1 - \phi_2)}(\zeta - \epsilon)} \quad (71)$$

Thus, the Eq. (2) has a new solution in the form

$$\psi = \phi_1 + \frac{(\phi_2 - \phi_1)(3\phi_1 + \phi_2)}{2\phi_1 + (\phi_1 + \phi_2)\cosh\sqrt{-\mu(3\phi_1 + \phi_2)(\phi_1 - \phi_2)}(\zeta - \epsilon)} - \frac{a_2}{3a_3}. \quad (72)$$

Case 9: If $D_2 < 0$ and $D_3 = D_4 = 0$, then $P_4(\phi)$ has two doubled imaginary roots along with their conjugates. i.e., $P_4(\phi) = (\phi - \phi_1)^2(\phi - \overline{\phi_1})^2 \geq 0$. The real propagation occurs only if $\mu > 0$. Therefore, Eq. (10) has the following form

$$\frac{\mathrm{d}\phi}{\phi^2 + \mathrm{Im}\phi_1^2} = \sqrt{\mu}\mathrm{d}\zeta \quad (73)$$

We postulate $\phi(\zeta_0) = 0$ and integrate both sides of the last equation to obtain

$$\phi = \mathrm{Im}\phi_1 \tan(\mathrm{Im}\phi_1 \sqrt{\mu}(\zeta - \zeta_0)) \quad (74)$$

Hence, Eq. (2) has a novel solution in the form

$$\psi = \mathrm{Im}\phi_1 \tan(\mathrm{Im}\phi_1 \sqrt{\mu}(\zeta - \zeta_0)) - \frac{a_2}{3a_3}. \quad (75)$$

Figure 6a, b outlines the 3D graphic representation for diverse values of α. Figure 6c illustrates the width of the solution (75) increases as the fractional order α increases. Notice, when $\alpha \to 1$, the solution (75) is reduced to a well known solution for the integer order derivative [61].

Conclusions

This work has endeavored to study the problem for constructing wave solutions for conformable time fractional version of the generalized Pochhammer–Chree equation. A certain transformation has been applied to transform the

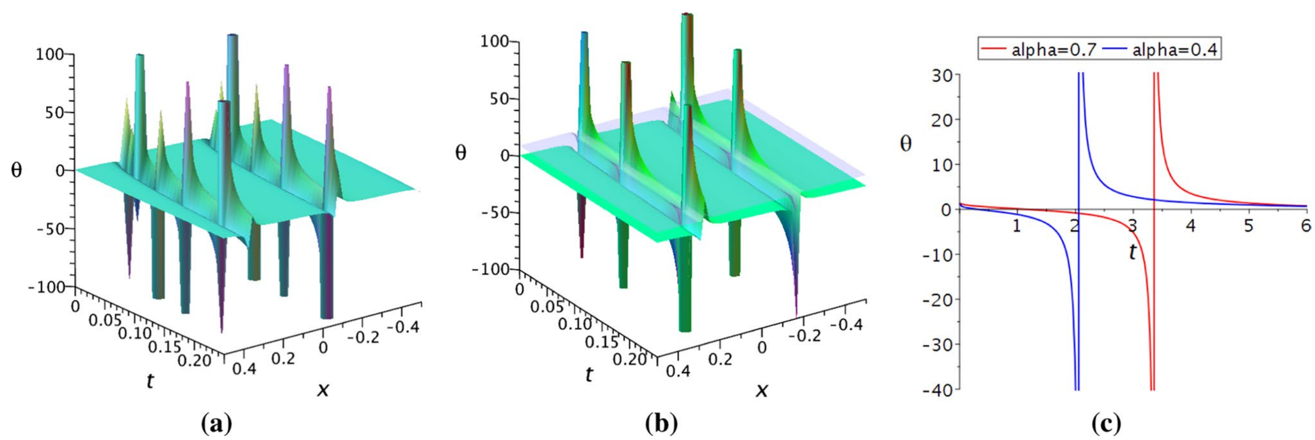

Fig. 6 3D-graph of the solution (75) for $(x, t) \in [-0.5, 0.5] \times [0, 0.25]$ for different values of fractional order α, **a** $\alpha = 0.4$ and **b** $\alpha = 0.7$. While **c** is the 2D-graph for the solution (75) when $x = 1, t \in [0, 6]$

equation under consideration into an a second order ordinary differential equation which is integrated once to give the differential form (10). The key step to integrate this differential form is knowing the types of the roots of the the polynomial $P_4(\phi)$. The complete discrimination system of this polynomial has been employed and has implied about nine cases. For each case, we determined the intervals of real propagation and integrated the differential form of Eq. (10) along these intervals. There are several intervals of real propagation corresponding to each type of the zeros of $P_4(\phi)$ which have been enabled us to construct more than one solution for the equation under consideration. Finally, we have illustrated some of these solutions graphically for different values of the fractional order derivatives. Some of these solutions will be reduced to new wave solutions for the generalized Pochhammer- Chree equation when the fractional order derivative approaches one. We also investigate the degeneracy of some solutions involving Jacobi-elliptic functions.

Appendix: Conformable derivatives

As we know the fractional calculus is more suitable in describing the real world problems appearing in engineering and physical science. Recently, scholars study the fractional calculus and introduced new operators such as Caputo, Riemann Liouville and conformable fractional operator. The usage of the conformable fractional operator overcomes some restrictions of the different fractional operator's properties such as the chain rule, the derivative of the quotient of two functions, product of two functions, and mean value theorem. Thus, it becomes more interesting in describing many physical problems.

Definition 1 [52] Let $g : (0, \infty) \to \mathbb{R}$ be a function, then the conformable fractional derivative of order α is defined as

$$T_\alpha(g)(t) = \lim_{\sigma \to 0} \frac{g(t + \sigma t^{1-\alpha}) - g(t)}{\sigma}, \tag{76}$$

for all $t > 0$ and $0 < \alpha \leq 1$.

We present some significant properties of the conformable derivatives. Let the two functions g_1, g_2 are $\alpha-$ conformable differential for $t > 0$ and a, b are two constants. We have the following properties

1. $T_\alpha(ag_1 + bg_2) = aT_\alpha(g_1) + bT_\alpha(g_2)$,
2. $T_\alpha(t^\rho) = \rho t^{\rho-\alpha}$, for all $\rho \in \mathbb{R}$,
3. $T_\alpha(g_1 g_2) = g_1 T_\alpha(g_2) + g_2 T_\alpha(g_1)$.
4. $T_\alpha\left(\frac{g_1}{g_2}\right) = \frac{1}{g_2^2}\left(g_2 T_\alpha(g_1) - g_1 T_\alpha(g_1)\right)$
5. $T_\alpha(g)(t) = t^{1-\alpha} \frac{dg}{dt}(t)$.
6. If $g : (0, \infty) \to \mathbb{R}$ is a map which is differentiable and $\alpha-$ differentiable and f is another function which is defined in the range of g, then $T_\alpha(g \circ f) = t^{1-\alpha} f'(t) g'(f(t))$.

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Data availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The author declares that the research was conducted in the absence of any conflict of interest.

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