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ORIGINAL RESEARCH

Cubic uncertain linguistic powered Einstein aggregation operators and their application to multi-attribute group decision making

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Abstract

The cubic uncertain fuzzy linguistic variable can easily express the fuzzy information, and the power average (PA) operator is a useful tool which provides more versatility in the information aggregation procedure. In this paper, we will combine the PA operator and Einstein operations to cubic uncertain linguistic environment and propose some new PA operators. Firstly, the definition and some basic operations of cubic uncertain linguistic number, power aggregation (PA) operator and Einstein operations are introduced. Then, we propose cubic uncertain linguistic fuzzy powered Einstein averaging operator, cubic uncertain linguistic fuzzy powered Einstein weighted (CULFPEWA) operator, cubic uncertain linguistic fuzzy Einstein geometric operator and cubic uncertain linguistic fuzzy Einstein weighted geometric (CULFPEWG) operator and discuss some properties of these in detail. Furthermore, we develop the decision-making methods for multi-attribute group decision-making problems with cubic uncertain linguistic information and give the detail decision steps. At last, an illustrate example is given to show the process of decision making and the effectiveness of the proposed method.

Keywords Cubic uncertain linguistic number · Power aggregation (PA) operator · Cubic uncertain linguistic fuzzy powered Einstein weighted (CULFPEWA) operator · Cubic uncertain linguistic fuzzy Einstein weighted geometric (CULFPEWG) operator · Multi-attribute group decision making

Introduction

Fuzzy set (FS) proposed by Zadeh [46] is a very valuable tool to develop the fuzzy information. However, as FS has only a membership function, it is hard to term the more composite fuzzy information. Atanassov [3] further proposed the intuitionistic fuzzy set (IFS) which has a membership function and a non-membership function, so IFS has further advantages than FS on describing the inconsistent information. IFS is with membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$. However, because the membership function and non-membership of IFS are crisp numbers which are hard to be acquired in real decision making, the choices of IFS are further extended [4]. Atanassov [5] proposed the interval-valued intuitionistic fuzzy set (IVIFS) which extended the membership and non-membership to interval numbers. Zhang et al. [47] gave the definition of the triangular intuitionistic fuzzy numbers. Wang [32] defined intuitionistic trapezoidal fuzzy number and interval intuitionistic trapezoidal fuzzy numbers, and then, some decision-making methods had been proposed [31, 34].

In real decision making, sometimes we can use linguistic terms such as 'good' and 'bad' to define the state or performance of a car and cannot use some numbers to express some qualitative information. However, when we use the linguistic variables to express the qualitative information, it only means the membership degree

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belonged to a linguistic term is 1 and the non-membership degree or hesitation degree cannot be expressed. In order to overcome this shortcoming, Wang and Li [33] offered the concept of intuitionistic linguistic set by combining intuitionistic fuzzy set with linguistic variables. For the above-mentioned example, we can give an assessment value 'good' for the state of the car; however, for this evaluation, we have the certainty degree of 80% and negation degree of 10% and then we can use the intuitionistic linguistic set to direct the evaluation result. Of course, it cannot be expressed by IFS or linguistic variables.

Furthermore, the information aggregation operators are chief research direction of decision-making problems and countless research results have been achieved [16, 22–26, 35, 36, 39–43]. In general, they are divided into two types, i.e., the arithmetic aggregation operators and the geometric aggregation operators. About the differences between them, Liu [21] gave the clarifications "The arithmetic aggregation operators emphasize the impact of the whole attribute data and the compensation between the different attribute data and the geometric aggregation operators emphasize the balance of the system and the coordination between the different attribute data". In addition, the whole operators were included in the general concepts of the t-norms and t-conorms [6], which satisfy the requirements of the conjunction and disjunction operators [35]. Einstein operations are a kind of many t-norms and t-conorms families which can be used to perform the corresponding intersections and unions of IFSs. So, based on Einstein operations, Wang and Liu [36] proposed some intuitionistic fuzzy Einstein aggregation operators such as the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator and the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator. Wei and Zhao [37] established intuitionistic fuzzy Einstein hybrid average (IFEHA) operator and intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator and proposed intuitionistic fuzzy MADM methods based on them. Guo et al. [16] proposed some operators which extended Einstein operators to hesitant fuzzy sets counting hesitant fuzzy Einstein weighted geometric (HFEWG) operator, hesitant fuzzy Einstein ordered weighted geometric (HFEOWG) operator, hesitant fuzzy Einstein hybrid geometric (HFEHG) operator and hesitant fuzzy Einstein induced ordered weighted geometric (HFEIOWG) operator.

Yager [45] established a power average operator and a power OWA operator to provide more versatility in the information aggregation process. Based on this, Xu and Yager [44] proposed some new geometric aggregation operators, such as the power geometric operator, weighted PG operator and power-ordered weighted geometric operator. Zhou and Chen [49] presented the generalized power average operator and the generalized power-ordered weighted average operator. Then, they presented the linguistic generalized power average operator and the weighted linguistic generalized power average operator and the linguistic generalized power-ordered weighted average (LGPOWA) operator which extended the GPA operator and the GPOWA operator to linguistic environment. The same character of them is their aggregation functions use linguistic information and generalized mean in the power average (PA) operator. Xu and Cai [43] developed the uncertain power average operators which aggregated interval fuzzy preference relations. Xu and Wang [40] proposed 2-tuple linguistic power average (2TLPA) operator, 2-tuple linguistic weighted PA operator and 2TLPOWA operator. Zhou et al. [50] presented an uncertain generalized power average operator and an uncertain generalized power-ordered weighted average operator to deal with these arguments which take the form of interval numbers. They developed the generalized intuitionistic fuzzy power averaging operator and the generalized intuitionistic fuzzy power-ordered weighted averaging operator which extended the GPA operator and the GPOWA operator to intuitionistic fuzzy environment.

Wei et al. [38] utilized these operators to develop some approaches to solving the picture 2-tuple linguistic multi-attribute decision-making problems. Jiang et al. [19] introduced intuitionistic fuzzy entropy weighted power average aggregation operator. Zhang [48] defined several trapezoidal interval type-2 fuzzy aggregation operators for aggregating trapezoidal interval type-2 fuzzy sets and examine several useful properties of the developed operators. Liu et al. [27] developed general operational laws for the I2L information (I2LI) and propose the I2L generalized aggregation (I2LGA) operator for the I2LI based on extended TN and TC. Liu et al. [28] proposed the q-rung orthopair fuzzy weighted averaging operator and the q-rung orthopair fuzzy weighted geometric operator to deal with the decision information. Liu et al. [29] proposed the linguistic intuitionistic fuzzy partitioned Heronian mean (LIFPHM) operator, the linguistic intuitionistic fuzzy weighted partitioned Heronian mean (LIFWPHM) operator, the linguistic intuitionistic fuzzy partitioned geometric Heronian mean (LIFPGHM) operator and linguistic intuitionistic fuzzy weighted partitioned geometric Heronian mean (LIFWPGHM) operator. Liu et al. [30] proposed the interval-valued intuitionistic fuzzy power Heronian aggregation (IVIFPHA) operator and interval-valued intuitionistic fuzzy power weight Heronian aggregation (IVIFPWHA) operator.

Cubic sets are the generalizations of fuzzy sets and intuitionistic fuzzy sets, in which there are two representations, one is used for the degree of membership and other is used for the degree of non-membership. The membership function is Hold in the form of interval, while non-membership is thought over the normal fuzzy set [20]. Cubic set was introduced by Jun in 2010.

Fahmi et al. [7] developed the Hamming distance for the triangular cubic fuzzy number and weighted averaging operator. Fahmi et al. [8] proposed the cubic TOPSIS method and gray relational analysis set. Fahmi et al. [9] defined the triangular cubic fuzzy number and operational laws. Amin et al. [1] defined the generalized triangular cubic linguistic hesitant fuzzy weighted geometric (GTCHFWG) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted average (GTCLHFOWA) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted geometric (GTCLHFOWG) operator, generalized triangular cubic linguistic hesitant fuzzy hybrid averaging (GTCLHFHA) operator and generalized triangular cubic linguistic hesitant fuzzy hybrid geometric (GTCLHFHG) operator. Fahmi et al. [10] developed trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the MCDM method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method. Fahmi et al. [11] define aggregation operators for triangular cubic linguistic hesitant fuzzy sets which include cubic linguistic fuzzy (geometric) operator, triangular cubic linguistic hesitant fuzzy weighted geometric (TCLHFWG) operator, triangular cubic linguistic hesitant fuzzy ordered weighted geometric (TCHFOWG) operator and triangular cubic linguistic hesitant fuzzy hybrid geometric (TCLHFHG) operator. Fahmi et al. [12] defined the trapezoidal cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator. Expected values, score function and accuracy function of trapezoidal cubic fuzzy numbers are defined. Fahmi et al. [13] developed three arithmetic averaging operators, that is, trapezoidal cubic fuzzy Einstein weighted averaging (TrCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator and trapezoidal cubic fuzzy Einstein hybrid weighted averaging (TrCFEHWA) operator, for aggregating trapezoidal cubic fuzzy information. Fahmi et al. [14] defined some Einstein operations on cubic fuzzy set (CFS) and develop three arithmetic averaging operators, which are cubic fuzzy Einstein weighted averaging (CFEWA) operator, cubic fuzzy Einstein ordered weighted averaging (CFEOWA) operator and cubic fuzzy Einstein hybrid weighted averaging (CFEHWA) operator, for aggregating cubic fuzzy data. Amin et al. [2] introduced the new concept of the trapezoidal cubic hesitant fuzzy TOPSIS method. Fahmi et al. [15] introduced the triangular cubic hesitant Einstein aggregation operators.

Due to the motivation and inspiration of the above discussion, in this paper we generalize the concept of intuitionistic uncertain linguistic powered Einstein aggregation operator, intuitionistic uncertain linguistic fuzzy powered Einstein averaging, intuitionistic uncertain linguistic fuzzy powered Einstein weighted operator, intuitionistic uncertain linguistic fuzzy Einstein geometric operator, intuitionistic uncertain linguistic fuzzy Einstein weighted geometric, interval-valued intuitionistic uncertain linguistic fuzzy number and interval-valued intuitionistic uncertain linguistic fuzzy powered Einstein weighted operator and introduce the concept of fuzzy uncertain linguistic powered Einstein aggregation operator. If we take only one element in the membership degree of the fuzzy uncertain linguistic powered Einstein aggregation operator, i.e., instead of interval we take a fuzzy number, then we get intuitionistic fuzzy uncertain linguistic powered Einstein aggregation operator; similarly, if we take membership degree as fuzzy number and non-membership degree equal to zero, then we get fuzzy uncertain linguistic powered Einstein aggregation operator.

Despite having a bulk of related literature on the problem under consideration, the following aspects related to cubic uncertain linguistic fuzzy numbers (CULFNs) and their aggregation operators motivated the researchers to carry out an in-depth inquiry into the current study.

(1) The main advantages of the proposed operators are: these aggregation operators provided more accurate and precious result as compared to the above-mentioned operators.

(2) We generalize the concept of cubic uncertain linguistic fuzzy numbers and intuitionistic uncertain linguistic fuzzy sets and introduce the concept of cubic uncertain linguistic fuzzy sets. If we take only one element in the membership degree of the cubic uncertain linguistic fuzzy numbers, i.e., instead of interval we take a fuzzy number, then we get intuitionistic uncertain linguistic fuzzy numbers; similarly, if we take membership degree as fuzzy number and non-membership degree equal to zero, then we get cubic uncertain linguistic fuzzy numbers.

(3) The objectives of this study include:

Propose cubic uncertain linguistic fuzzy sets, operational laws, score value and accuracy value of CULFSs.

Propose four aggregation operators, namely cubic uncertain linguistic fuzzy powered Einstein averaging (CULFPEA) operator, cubic uncertain linguistic fuzzy powered Einstein weighted averaging (CULFPEWA) operator, cubic uncertain linguistic fuzzy powered Einstein weighted averaging (CULFPEWA) operator and cubic uncertain linguistic fuzzy powered Einstein weighted geometric (CULFPEWG) operator.

Establish MADM program approach-based cubic uncertain linguistic fuzzy numbers.

Provide illustrative examples of MADM program.
(4) In order to testify the application of the developed method, we apply decision-making methods based on the CULFPEWA operator and CULFPEWG operator.
(5) The initial decision matrix is composed of LVs. In order to fully consider the randomness and ambiguity of linguistic term, we convert LVs into the cubic uncertain linguistic fuzzy sets, and the decision matrix is transformed into the cubic uncertain linguistic fuzzy decision matrix.
(6) The operator can fully express the uncertainty of the qualitative concept, and cubic uncertain linguistic fuzzy operators can capture the interdependencies among any multiple inputs or attributes by a variable parameter. The aggregation operators can take into account the importance of the attribute weights. Nevertheless, sometimes, for some MAGDM problems, the weights of the attributes are important factors for decision process.

In order to achieve this aim, this paper is organized as follows. In "Preliminaries" section, we represent some ideas of the linguistic set and uncertain linguistic numbers. In "The cubic uncertain linguistic set (CULS)" section, we present the cubic uncertain linguistic set (CULS), the power aggregation (PA) operator and Einstein operations of cubic uncertain linguistic numbers. In "Some cubic uncertain linguistic fuzzy powered Einstein operators" section, we propose the some cubic uncertain linguistic fuzzy powered Einstein operators, cubic uncertain linguistic fuzzy powered Einstein averaging operator, cubic uncertain linguistic fuzzy powered Einstein weighted operator, cubic uncertain linguistic fuzzy Einstein geometric (CULFPEG) operator and cubic uncertain linguistic fuzzy Einstein weighted geometric operator and introduce some properties and special cases of them. "The decision-making methods based on the CULFPEWA operator and CULFPEWG operator" section establishes the procedure of the decision-making method based on the CULFPEWA and CULFPEWG operators. Section six gives a numerical example according to our approach. In Section seven, we discuss the comparison analysis. In Section eight, we give the conclusion.

Preliminaries

Definition 1 [46] Let H be a universe of discourse. The idea of fuzzy set was presented by Zadeh and is defined as follows $J = \{h, \Gamma_J(h) | h \in H\}$. A fuzzy set in a set H is defined as $\Gamma_J : H \to I$, which is a membership function. $\Gamma_J(h)$ denotes the degree of membership of the element h to the set H, where $I = [0, 1]$. The collection of all fuzzy subsets in H is denoted by I^H. Define a relation on I^H as follows: $(\forall \Gamma, \eta \in I^H)(\Gamma \leq \eta \Leftrightarrow (\forall h \in H)(\Gamma(h) \leq \eta(h)))$.

Definition 2 [20] Let H be a non-empty set. By a cubic set in H, we mean a structure $F = \{h, \alpha(h), \beta(h) : h \in H\}$ in which α is an IVF set in H and β is a fuzzy set in H. A cubic set $F = \{h, \alpha(h), \beta(h) : h \in H\}$ is simply denoted by $F = \langle \alpha, \beta \rangle$.. The collection of all cubic sets in H is denoted by C^H. A cubic set $F = \langle \alpha, \beta \rangle$ in which $\alpha(h) = 0$ and $\beta(h) = 1(\text{resp.}\alpha(h) = 1$ and $\beta(h) = 0$ for all $h \in H$ is denoted by 0 (resp. 1). A cubic set $D = \langle \lambda, \xi \rangle$ in which $\lambda(h) = 0$ and $\xi(h) = 0$ (resp.$\lambda(h) = 1$ and $\xi(h) = 1$) for all $h \in H$ is denoted by 0 (resp. 1).

Definition 3 [20] Let H be a non-empty set. A cubic set $F = (C, \lambda)$ in H is said to be an internal cubic set if $C^-(h) \leq \lambda(h) \leq C^+(h)$ for all $h \in H$.

Definition 4 [20] Let H be a non-empty set. A cubic set $F = (C, \lambda)$ in H is said to be an external cubic set if $\lambda(h) \notin (C^-(h), C^+(h))$ for all $h \in H$.

The linguistic set and uncertain linguistic numbers

The linguistic set is regarded as a good tool to express this qualitative information, and we can express the linguistic set by $S = (s_0, s_1, \ldots, s_{l-1})$, and $s_\theta(\theta = 1, 2, \ldots, l-1)$ can be called an linguistic number, where 1 is an odd value which can be the values of 3, 5, 7, 9, etc. Generally, for example, when $l = 9, S = (s_0, s_1, s_2, s_2, s_4, s_5, s_6, s_7, s_8) =$ (extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good). Let s_i and s_j be any two linguistic numbers in linguistic set S, they have the following characteristics [17, 18]:

(i) If $i > j$, then $s_i \succ s_j$.
(ii) There exists negative operator: $\text{neg}(s_i) = s_j$, where $j = l - 1 - i$.
(iii) If $s_i \geq s_j$, $\max(s_i; s_j) = s_i$.
(iv) If $s_i \leq s_j$, $\min(s_i; s_j) = s_i$.

In order to overcome the loss of information in the process of calculations, the original discrete linguistic set $S = (s_0, s_1, \ldots, s_{l-1})$ is extended to the continuous linguistic set $S = \{s_\alpha | \alpha \in R^+\}$ which also meets the strictly monotonically increasing condition [18, 50]. Some operational rules are defined as follows [17, 18]:

(1) $\beta s_i = s_{\beta \times i}; \ \beta \geq 0$
(2) $s_i \oplus s_j = s_{i+j}$;
(3) $s_i \otimes s_j = s_{i \times j}$;
(4) $(s_i)^n = s_{i^n}; \ n \geq 0$.

Definition 5 [41] Suppose $s = [s_a a; s_b b]$, $s_a, s_b \in S$ with $a \leq b$ are the lower limit and the upper limit of s, respectively, then s is called an uncertain linguistic variable. Let S be a set of all uncertain linguistic variables. $s_1 = [s_{a_1}, s_{b_1}]$ and $s_2 = [s_{a_2}, s_{b_2}]$ are any two uncertain linguistic variables, and the operational rules are defined as follows [41, 42]:

(1) $s_1 \oplus s_2 = [s_{a_1}, s_{b_1}] \oplus [s_{a_2}, s_{b_2}] = [s_{a_1+a_2}, s_{b_1+b_2}]$;
(2) $s_1 \otimes s_2 = [s_{a_1}, s_{b_1}] \otimes [s_{a_2}, s_{b_2}] = [s_{a_1 \times a_2}, s_{b_1 \times b_2}]$;
(3) $s_1 = [s_{a_1}, s_{b_1}] = [s_{\lambda * a_1}, s_{\lambda * b_1}]$; $\lambda \geq 0$;
(4) $(s_1)^{\lambda} = [s_{a1}, s_{b1}]^{\lambda} = [s_{a1^{\lambda}}, s_{b1^{\lambda}}]$; $\lambda \geq 0$.

The cubic uncertain linguistic set (CULS)

Definition 6 Let $h_{\theta(x)} \in S$, X be the given discourse domain, then $A = \left\{ \begin{array}{c} \langle \{x \\ [h_{\theta(x)}, \\ [u_A^-(x), \\ u_A^+(x)], \\ v_A(x))] \\ |x \in X)\} \rangle \end{array} \right\}$ is called cubic linguistic set (CLS), where $\langle [u_A^- : X \to [0;1], u_A^+ : X \to [0;1]]$ and $v_A : X \to [0;1] \rangle$.

Definition 7 Let $[s_{\theta(x)}, s_{t(x)}] \in S$, and X be the given discourse domain, then $A = \left\{ \begin{array}{c} \langle \{x[[s_{\theta(x)}, s_{t(x)}], \\ [u_A^-(x), u_A^+(x)], \\ v_A(x))] \; |x \in X)\} \rangle \end{array} \right\}$ is called cubic uncertain linguistic set (CULS) in which $s_{\theta(x)}, s_{t(x)} \in S$, $\langle [u_A^- : X \to [0;1], u_A^+ : X \to [0;1]]$ and $v_A : X \to [0;1] \rangle$.

Definition 8 Let $a_1 = \left\{ \begin{array}{c} \langle \{x[[s_{\theta(a_1)}, s_{t(a_1)}], \\ [u^-(a_1), u^+(a_1)], \\ v(a_1))]\} \rangle \end{array} \right\}$ and $a_2 = \left\{ \begin{array}{c} \langle \{x[[s_{\theta(a_2)}, s_{t(a_2)}], \\ [u^-(a_2), u^+(a_2)], \\ v(a_2))\} \rangle \end{array} \right\}$ be any two cubic uncertain linguistic numbers, the operational laws are defined as follows:

(1) $a_1 + a_2 = \left\langle \begin{array}{c} [s_{\theta(a_1)+\theta(a_2)}, s_{t(a_1)+t(a_2)}], [(1-(1-u^-(a_1))(1-u^-(a_2)), \\ 1-(1-u^+(a_1))(1-u^+(a_2))], v(a_1)v(a_2)) \end{array} \right\rangle$;

(2) $a_1 \otimes a_2 = \left\langle \begin{array}{c} [s_{\theta(a_1)\times\theta(a_2)}, s_{t(a_1)\times t(a_2)}], \\ [u^-(a_1)u^-(a_2)), u^+(a_1))u^+(a_2))], \\ v(a_1) + v(a_2) - v(a_1)v(a_2) \end{array} \right\rangle$;

(3) $\lambda a_1 = \left\langle \begin{array}{c} [s_{\lambda\times\theta(a_1)}, s_{\lambda\times t(a_1)}]; (1-(1-u^-(a_1))^{\lambda}, \\ (1-(1-u^+(a_1))^{\lambda}], (v(a_1))^{\lambda} \end{array} \right\rangle$; $\lambda \geq 0$;

(4) $a_1^{\lambda} = \left\langle \begin{array}{c} [s_{(\theta(a_1))^{\lambda}}, s_{(t(a_1))^{\lambda}}]; [u^-(a_1))^{\lambda}, (u^+(a_1))^{\lambda}], \\ 1-(1-v(a_1))^{\lambda}) \end{array} \right\rangle$; $\lambda \geq 0$.

Example 1 Let $a_1 = \left\{ \begin{array}{c} \langle \{[[s_2, s_4], \\ [0.2, 0.4], \\ 0.3]\} \rangle \end{array} \right\}$ and $a_2 = \left\{ \begin{array}{c} \langle \{x[[s_1, s_3], \\ [0.1, 0.3], \\ 0.2\} \rangle \end{array} \right\}$ be any two cubic uncertain linguistic numbers, the operational laws are defined as follows:

(1) $a_1 + a_2 = \left\{ \begin{array}{c} \langle [s_4, s_7], [(1-(1-0.2)(1-0.1), \\ 1-(1-0.4)(1-0.3)], (0.3)(0.2)) \rangle \\ \langle [s_4, s_7], [0.28, 0.58], 0.06 \rangle \end{array} \right\}$;
(2) $a_1 \otimes a_2 = \{ \langle [s_2, s_{12}], [0.02, 0.12], 0.44 \rangle \}$;
$\lambda = (0.2, 0.3, 0.5)$
(3) $\lambda a_1 = \langle [s_{0.4}, s_{0.8}]; [0.0647, 0.1421], 0.5477 \rangle$; $\lambda \geq 0$;
(4) $a_1^{\lambda} = \langle [s_{1.1486}, s_{1.3195}]; [0.6171, 0.7596], 0.1633 \rangle$; $\lambda \geq 0$.

Theorem 1 *Let* $a_1 = \left\{ \begin{array}{c} \langle \{x[[s_{\theta(a_1)}, s_{t(a_1)}], \\ [u^-(a_1), u^+(a_1)], \\ v(a_1))]\} \rangle \end{array} \right\}$ *and* $a_2 = \left\{ \begin{array}{c} \langle \{x[[s_{\theta(a_2)}, s_{t(a_2)}], \\ [u^-(a_2), u^+(a_2)], \\ v(a_2))\} \rangle \end{array} \right\}$ *be any two cubic uncertain linguistic numbers, the operational laws have the following characteristics.*

(1) $a_1 + a_2 = a_2 + a_1$;
(2) $a_1 \otimes a_2 = a_2 \otimes a_1$;
(3) $\lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2$; $\lambda \geq 0$;
(4) $\lambda_1 a_1 + \lambda_2 a_1 = (\lambda_1 + \lambda_2) a_1$; $\lambda_1, \lambda_2 \geq 0$;
(5) $a_1^{\lambda_1} \otimes a_2^{\lambda_2} = (a_1)^{\lambda_1 + \lambda_2}$, $\lambda_1, \lambda_2 \geq 0$;
(6) $a_1^{\lambda_1} \otimes a_2^{\lambda_2} = (a_1 \otimes a_2)^{\lambda_1}$, $\lambda_1 \geq 0$.

Definition 9 Let $a_1 = \left\{ \begin{array}{c} \langle \{x[[s_{\theta(a_1)}, s_{t(a_1)}], \\ [u^-(a_1), u^+(a_1)], \\ v(a_1))]\} \rangle \end{array} \right\}$ be cubic uncertain linguistic number, then the expectation value $E(a_1)$ of a_1 can be defined as follows: $E(a_1) =$

$$\left\{ \begin{array}{c} \frac{1}{3} \times (u^-(a_1) + 1 - v(a_1)) \\ \times (u^+(a_1) + 1 - v(a_1)) \\ \times s_{\frac{((\theta)(a_1) + t(a_1))}{3}} \end{array} \right\}$$
$$= \left\{ s_{\frac{((\theta)(a_1) + t(a_1)) \times (u^-(a_1) + 1 - v(a_1)) \times (u^+(a_1) + 1 - v(a_1))}{9}} \right\}.$$

Example 2 Let $a_1 = \left\{ \begin{array}{c} \langle [s_2, s_3], \\ [0.8, 0.10], 0.9 \rangle \end{array} \right\}$, $a_2 = \left\{ \begin{array}{c} \langle [s_1, s_3], \\ [0.12, 0.14], 0.13 \rangle \end{array} \right\}$, $a_3 = \left\{ \begin{array}{c} \langle [s_1, s_2], \\ [0.1, 0.3], 0.2 \rangle \end{array} \right\}$ and $a_4 =$

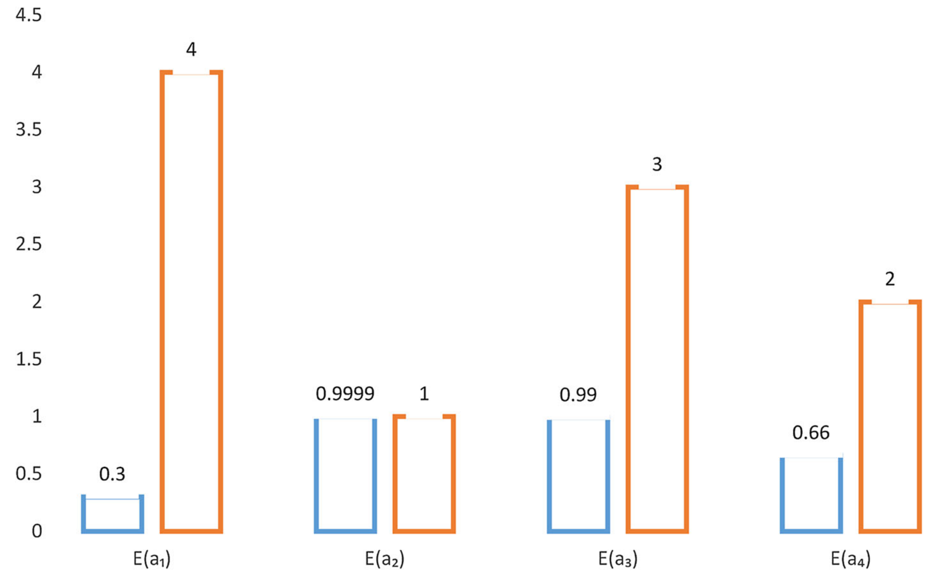


Fig. 1 $E(a_1)$ is the first expectation value, $E(a_4)$ is the second expectation value, $E(a_3)$ is the third expectation value, $E(a_2)$ is the 4th expectation value

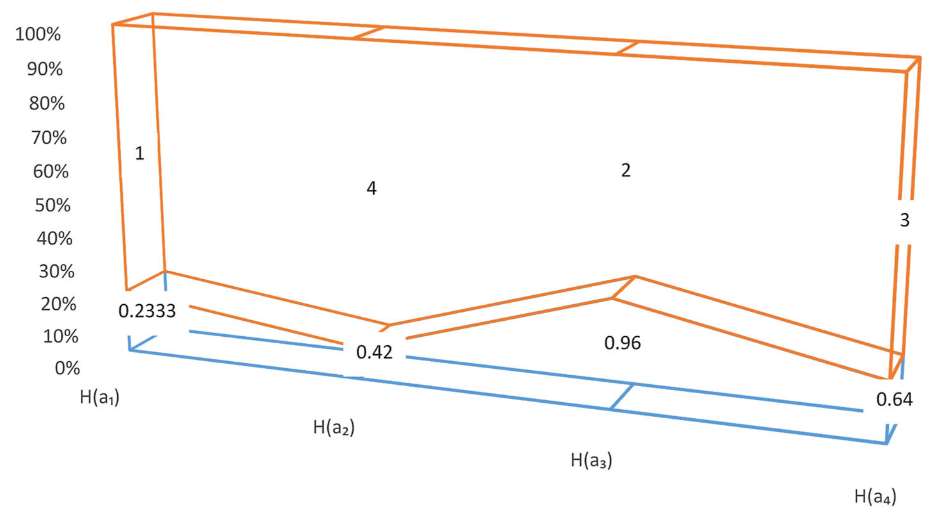


Fig. 2 $H(a_1)$ is the first accuracy value, $H(a_3)$ is the second accuracy value, $H(a_4)$ is the third accuracy value, $H(a_2)$ is the 4th expectation value

$\left\{\begin{matrix}\langle[s_1, s_1],\\ [0.19, 0.21], 0.20\rangle\end{matrix}\right\}$ be the four cubic uncertain linguistic number, then the expectation value E can be defined as follows:
$E(a_1) = s_{0.3}, E(a_2) = s_{0.9999}, E(a_3) = s_{0.99}, E(a_4) = s_{0.6666}$ (Fig. 1).

Definition 10 Let $a_1 = \left\{\begin{matrix}\langle\{x[[s_{\theta(a_1)}, s_{t(a_1)}],\\ [u^-(a_1), u^+(a_1)],\\ v(a_1))]\}\rangle\end{matrix}\right\}$ be cubic uncertain linguistic number, then the accuracy function $H(a_1)$ of a_1 can be defined as follows:

$$H(a_1) = (u^-(a_1) + v(a_1)) \times (u^+(a_1) + v(a_1)) \times s_{\frac{((\theta)(a_1)+t(a_1))}{3}}$$
$$= s_{\frac{((\theta)(a_1)+t(a_1))\times(u^-(a_1)+v(a_1))\times(u^+(a_1)+v(a_1))}{3}}.$$

Example 3 Let $a_1 = \left\{\begin{matrix}\langle[s_1, s_1],\\ [0.2, 0.4],\\ 0.3\rangle\end{matrix}\right\}, a_2 = \left\{\begin{matrix}\langle[s_1, s_1],\\ [0.3, 0.5],\\ 0.4\rangle\end{matrix}\right\},$ $a_3 = \left\{\begin{matrix}\langle[s_1, s_2],\\ [0.3, 0.7],\\ 0.5\rangle\end{matrix}\right\}$ and $a_4 = \left\{\begin{matrix}\langle[s_3, s_1],\\ [0.1, 0.5],\\ 0.3\rangle\end{matrix}\right\}$ be the four cubic uncertain linguistic number, then the accuracy value H can be defined as follows: $H(a_1) = s_{0.2333}, H(a_2) = s_{0.42}, H(a_3) = s_{0.96}, H(a_4) = s_{0.64}$ (Fig. 2).

Definition 11 Let $a_1 = \left\{\begin{matrix}\langle\{x[[s_{\theta(a_1)}, s_{t(a_1)}],\\ [u^-(a_1), u^+(a_1)],\\ v(a_1))]\}\rangle\end{matrix}\right\}$ and $a_2 = \left\{\begin{matrix}\langle\{x[[s_{\theta(a_2)}, s_{t(a_2)}],\\ [u^-(a_2), u^+(a_2)],\\ v(a_2))\rangle\end{matrix}\right\}$ be any two cubic uncertain linguistic numbers, then

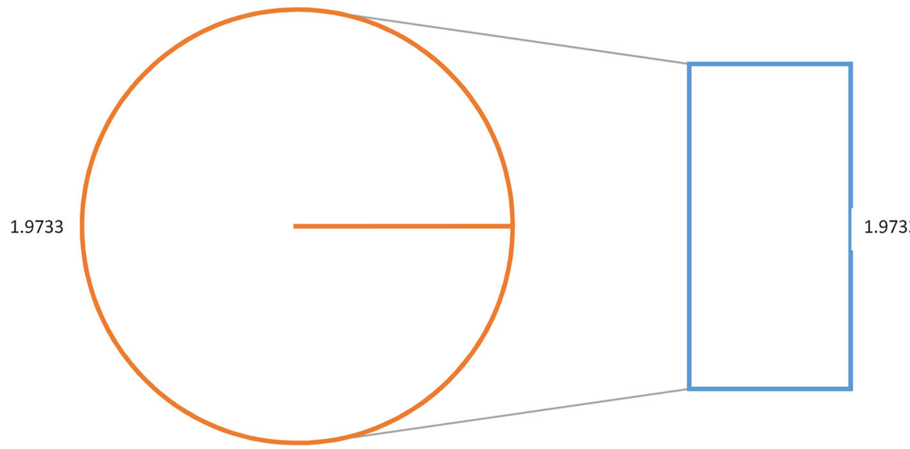


Fig. 3 Ranking of Hamming distance

(1) if $E(a_1) > E(a_2)$, then $a_1 \succ a_2$,
(2) if $E(a_1) = E(a_2)$, then:

  (i) if $H(a_1) > H(a_2)$, then $a_1 \succ a_2$,
  (ii) if $H(a_1) = H(a_2)$, then $a_1 = a_2$.

Definition 12 Let $a_1 = \left\{ \begin{matrix} \langle\{x[[s_{\theta(a_1)}, s_{t(a_1)}], \\ [u^-(a_1), u^+(a_1)], \\ v(a_1))]\}\rangle \end{matrix} \right\}$ and $a_2 = \left\{ \begin{matrix} \langle\{x[[s_{\theta(a_2)}, s_{t(a_2)}], \\ [u^-(a_2), u^+(a_2)], \\ v(a_2))\rangle \end{matrix} \right\}$ be any two cubic uncertain linguistic numbers, then the normalized Hamming distance between a_1 and a_2 can be defined as follows:

$$d(a_1, a_2) = \frac{\left\{ \begin{matrix} |u^-(a_1) - u^-(a_2)| + |u^+(a_1) - u^+(a_2)| \\ +|v(a_1) - v(a_2)| \times |\theta(a_1) + \theta(a_2)| \times |t(a_1) + t(a_2)| \end{matrix} \right\}}{3(l-1)}$$

which meets the following conditions:

(1) $0 \le d(a_1, a_2) \le 1$,
(2) $d(a_1, a_2) = 0$,
(3) $d(a_1, a_2) = d(a_2, a_1)$,
(4) $d(a_1, a_2) + d(a_2, a_3) \ge d(a_1, a_3)$.

Example 4 Let $a_1 = \left\{ \begin{matrix} \langle[s_1, s_1], \\ [0.6, 0.8], 0.7\rangle \end{matrix} \right\}$ and $a_2 = \left\{ \begin{matrix} \langle[s_1, s_1], \\ [0.8, 0.12], 0.10\rangle \end{matrix} \right\}$ be the two cubic uncertain linguistic number, then the normalized Hamming distance between a_1 and a_2 can be defined as follows: $d(a_1, a_2) = s_{1.9733}$ (Fig. 3).

The power aggregation (PA) operator

Definition 13 [45] The power aggregation (PA) operator, which is firstly proposed by Yager, is defined as follows: $PA(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n}(1+T(a_i))a_i}{\sum_{i=1}^{n}(1+T(a_i))}$, where $T(a_i) = \sum_{j=1}^{n} \sup(a_i, a_j)$ and $\sup(a_i a_j)$ means the support for a_i from a_j ,which satisfies the following rules:

(1) $\sup(a_i, a_j) = \sup(a_j, a_i)$;
(2) $\sup(a_i, a_j) \in [0; 1]$;
(3) $\sup(a_i, a_j) \ge \sup(a_m, a_n)$; if $|a_i - a_j| \le |a_m - a_n|$.

Einstein operations of cubic uncertain linguistic numbers

Einstein operations are a kind of the t-norms and t-conorms families which can be used to perform the corresponding intersections and unions of CFSs. Einstein operations are defined in the following, where Einstein product $\otimes_\epsilon$ is a t-norm and Einstein sum $\oplus_\epsilon$ is a t-conorm.

Definition 14 Let $a_1 = \left\{ \begin{matrix} \langle\{x[[s_{\theta(a_1)}, s_{t(a_1)}], \\ [u^-(a_1), u^+(a_1)], \\ v(a_1))]\}\rangle \end{matrix} \right\}$ and $a_2 = \left\{ \begin{matrix} \langle\{x[[s_{\theta(a_2)}, s_{t(a_2)}], \\ [u^-(a_2), u^+(a_2)], \\ v(a_2))\rangle \end{matrix} \right\}$, then we can define the operational laws of cubic uncertain linguistic numbers based on Einstein t-norm and t-conorm shown as follows:

(1)

$$a_1 \otimes_\epsilon a_2 = \left\langle \left[[s_{\theta(a_1)\times\theta(a_2)}, s_{t(a_1)\times t(a_2)}], \left(\left[\frac{u^-_{a_1} \centerdot u^-_{a_2}}{1 + \left(1 - u^-_{a_1}\right)\centerdot\left(1 - u^-_{a_2}\right)}, \frac{u^+_{a_1} \centerdot u^+_{a_2}}{1 + \left(1 - u^+_{a_1}\right)\centerdot\left(1 - u^+_{a_2}\right)} \right], \frac{v_{a_1} + v_{a_2}}{1 + v_{a_1}\centerdot v_{a_2}} \right) \right] \right\rangle.$$

(2)

$$a_1 \oplus_\epsilon a_2 = \left\langle \left[[s_{\theta(a_1)+\theta(a_2)}, s_{t(a_1)+t(a_2)}], \left(\frac{u^-_{a_1} + u^-_{a_2}}{1 + u^-_{a_1}\centerdot u^-_{a_2}}, \left[\frac{u^+_{a_1} + u^+_{a_2}}{1 + u^+_{a_1}\centerdot u^+_{a_2}}\right], \frac{v_{a_1}\centerdot v_{a_2}}{1 - (1 - v_{a_1})\centerdot(1 - v_{a_2})} \right) \right] \right\rangle.$$

(3)

$$\lambda_\epsilon a_1 = \left\langle \left[[s_{\lambda\theta(a_1)}, s_{\lambda t(a_1)}], \left(\frac{(1 + u^-_{a_1})^\lambda - (1 - u^-_{a_1})^\lambda}{(1 + u^-_{a_1})^\lambda + (1 - u^-_{a_1})^\lambda}, \frac{(1 + u^+_{a_1})^\lambda - (1 - u^+_{a_1})^\lambda}{(1 + u^+_{a_1})^\lambda + (1 - u^+_{a_1})^\lambda}, \frac{2v^\lambda_{a_1}}{(2 - v_{a_1})^\lambda + (v_{a_1})^\lambda} \right) \right] \right\rangle.$$

(4)

$$(a_1)^{\wedge^\lambda_\epsilon} = \left\langle [s_{(\theta(a_1))^\lambda}, s_{(t(a_1))^\lambda}] \left[\left(\frac{2u^{-\lambda}_{a_1}}{(2 - u^-_{a_1})^\lambda + u^{-\lambda}_{a_1}}, \frac{2u^{+\lambda}_{a_1}}{(2 - u^+_{a_1})^\lambda + u^{+\lambda}_{a_1}}, \frac{(1 + v_{a_1})^\lambda - (1 - v_{a_1})^\lambda}{(1 + v_{a_1})^\lambda + (1 - v_{a_1})^\lambda} \right) \right] \right\rangle.$$

Theorem 2 *Let* $a_1 = \left\{ \begin{array}{c} \langle\{x[[s_{\theta(a_1)}, s_{t(a_1)}], \\ [u^-(a_1), u^+(a_1)], \\ v(a_1))]\}\rangle \end{array} \right\}$ *and* $a_2 = \left\{ \begin{array}{c} \langle\{x[[s_{\theta(a_2)}, s_{t(a_2)}], \\ [u^-(a_2), u^+(a_2)], \\ v(a_2))\rangle \end{array} \right\}$ *be any two cubic uncertain linguistic numbers, then we have the following operation rules.*

(1) $a_1 \oplus a_2 = a_2 \oplus a_1$;
(2) $a_1 \otimes a_2 = a_2 \otimes a_1$;
(3) $\lambda(a_1 \oplus a_2) = \lambda a_1 \oplus \lambda a_2; \lambda \geq 0$;
(4) $\lambda_1 a_1 + \lambda_2 a_1 = (\lambda_1 + \lambda_2)a_1; \lambda_1, \lambda_2 \geq 0$;
(5) $a_1^{\lambda_1} \otimes a_1^{\lambda_2} = (a_1)^{\lambda_1+\lambda_2}, \lambda_1, \lambda_2 \geq 0$;
(6) $a_1^{\lambda_1} \otimes a_2^{\lambda_2} = (a_1 \otimes a_2)^{\lambda_1}, \lambda_1 \geq 0$.

Proof (3) We have

$$a_1 \oplus_\epsilon a_2 = \left\langle [s_{\theta(a_1)+\theta(a_2)}, s_{t(a_1)+t(a_2)}], \left[\frac{u^-(a_1) + u^-(a_2)}{1 + u^-(a_1)\centerdot u^-(a_2)}, \frac{u^+(a_1) + u^+(a_2)}{1 + u^+(a_1)\centerdot u^+(a_2)} \right], \frac{v(a_1)\centerdot v(a_2)}{1 - (1 - v(a_1))\centerdot(1 - v(a_2))} \right\rangle$$

$$\lambda(a_1 \oplus a_2) = \left\langle [s_{\lambda(\theta(a_1)+\theta(a_2))}, s_{\lambda(t(a_1)+t(a_2))}], \left[\frac{(1 + u^-(a_1))(1 - u^-(a_1))^\lambda(1 + u^-(a_2))(1 - u^-(a_2))^\lambda}{(1 + u^-(a_1))(1 - u^-(a_1))^\lambda(1 + u^-(a_2))(1 - u^-(a_2))^\lambda} \right], \left[\frac{(1 + u^+(a_1))(1 - u^+(a_1))^\lambda(1 + u^+(a_2))(1 - u^+(a_2))^\lambda}{(1 + u^+(a_1))(1 - u^+(a_1))^\lambda(1 + u^+(a_2))(1 - u^+(a_2))^\lambda} \right], \frac{2(v(a_1)v(a_2))^\lambda}{(4 - 2v(a_1) - 2v(a_2) + v(a_1)v(a_2))^\lambda + (v(a_1)v(a_2))^\lambda} \right\rangle$$

and we have

$$\lambda_\epsilon a_1 = \left\langle \begin{array}{c} [s_{\lambda\theta(a_1)}, s_{\lambda t(a_1)}], \left[\begin{array}{c} \frac{(1+u^-(a_1))^\lambda - (1-u^-(a_1))^\lambda}{(1+u^-(a_1))^\lambda + (1-u^-(a_1))^\lambda}, \\ \frac{(1+u^+(a_1))^\lambda - (1-u^+(a_1))^\lambda}{(1+u^+(a_1))^\lambda + (1-u^+(a_1))^\lambda} \end{array} \right], \\ \frac{2v^\lambda(a_1)}{(2-v(a_1))^\lambda + (v(a_1))^\lambda} \end{array} \right\rangle$$

$$\lambda_\epsilon a_2 = \left\langle \begin{array}{c} [s_{\lambda\theta(a_2)}, s_{\lambda t(a_2)}], \left[\begin{array}{c} \frac{(1+u^-(a_2))^\lambda - (1-u^-(a_2))^\lambda}{(1+u^-(a_2))^\lambda + (1-u^-(a_2))^\lambda}, \\ \frac{(1+u^+(a_2))^\lambda - (1-u^+(a_2))^\lambda}{(1+u^+(a_2))^\lambda + (1-u^+(a_2))^\lambda} \end{array} \right], \\ \frac{2v^\lambda_{a_2}}{(2-v(a_2))^\lambda + (v(a_2))^\lambda} \end{array} \right\rangle$$

then

$$\lambda a_1 \oplus \lambda a_2 = \left\langle \begin{array}{c} [s_{\lambda(\theta(a_1)+\theta(a_2))}, s_{\lambda(t(a_1)+t(a_2))}], \\ [\frac{(1+u^-(a_1))(1-u^-(a_1))^\lambda(1+u^-(a_2))(1-u^-(a_2))^\lambda}{(1+u^-(a_1))(1-u^-(a_1))^\lambda(1+u^-(a_2))(1-u^-(a_2))^\lambda}], \\ \left[\frac{(1+u^+(a_1))(1-u^+(a_1))^\lambda(1+u^+(a_2))(1-u^+(a_2))^\lambda}{(1+u^+(a_1))(1-u^+(a_1))^\lambda(1+u^+(a_2))(1-u^+(a_2))^\lambda}\right], \\ \frac{2(v(a_1)v(a_2))^\lambda}{(4-2v(a_1)-2v(a_2)+v(a_1)v(a_2))^\lambda + (v(a_1)v(a_2))^\lambda} \end{array} \right\rangle$$

so, we have $\lambda(a_1 \oplus a_2) = \lambda a_2 \oplus \lambda a_1; \lambda \geq 0$.
(5) We have

$$(a_1)^{\wedge^\lambda_\epsilon} = \left\langle \begin{array}{c} [s_{(\theta(a_1))^{\lambda_1}}, s_{(t(a_1))^{\lambda_1}}], \left[\begin{array}{c} \frac{2u^-(a_1)^{\lambda_1+\lambda_2}}{(2-u^-(a_1))^{\lambda_1} + u^{-\lambda_1}(a_1)}, \\ \frac{2u^{+\lambda_1}(a_1)}{(2-u^+(a_1))^{\lambda_1} + u^{+\lambda_1}(a_1)} \end{array} \right] \\ \left[\left(, \frac{(1+v(a_1))^{\lambda_1} - (1-v(a_1))^{\lambda_1}}{(1+v(a_1))^{\lambda_1} + (1-v(a_1))^{\lambda_1}} \rangle\right)\right] \end{array} \right\rangle$$

$$(a_2)^{\wedge^\lambda_\epsilon} = \left\langle \begin{array}{c} [s_{(\theta(a_2))^{\lambda_1}}, s_{(t(a_2))^{\lambda_1}}], \left[\begin{array}{c} \frac{2u^{-\lambda_2}(a_2)}{(2-u^-(a_2))^{\lambda_2} + u^{-\lambda_2}(a_2)}, \\ \frac{2u^{+\lambda_2}(a_2)}{(2-u^+(a_2))^{\lambda_2} + u^{+\lambda_2}(a_2)} \end{array} \right], \\ \left[\left(\frac{(1+v(a_2))^{\lambda_2} - (1-v(a_2))^{\lambda_2}}{(1+v(a_2))^{\lambda_2} + (1-v(a_2))2} \rangle\right)\right] \end{array} \right\rangle$$

$$a_1^{\lambda_1} \otimes a_1^{\lambda_2} = \left\langle \begin{array}{c} [s_{(\theta(a_1))^{\lambda_1+\lambda_2}}, s_{(t(a_1))^{\lambda_1+\lambda_2}}], \\ \left[\begin{array}{c} \frac{2u^-(a_1)^{\lambda_1+\lambda_2}}{u^-(a_1)^{\lambda_1}(2-u^-(a_1))^{\lambda_2} + u^-(a_1))^{\lambda_2}(2-u^-(a_1))^{\lambda_1}}, \\ \frac{2u^+(a_1)^{\lambda_1+\lambda_2}}{u^+(a_1)^{\lambda_1}(2-u^+(a_1))^{\lambda_2} + u^+(a_1))^{\lambda_2}(2-u^+(a_1))^{\lambda_1}} \end{array} \right], \\ \frac{(1+v(a_1))^{\lambda_1} - (1-v(a_1))^{\lambda_1}}{(1+v(a_1))^{\lambda_1} + (1-v(a_1))^{\lambda_1}} \end{array} \right\rangle$$

$$(a_1)^{\lambda_1+\lambda_2} = \left\langle \begin{array}{c} [s_{(\theta(a_1))^{\lambda_1+\lambda_2}}, s_{(t(a_1))^{\lambda_1+\lambda_2}}], \\ \left[\begin{array}{c} \frac{2u^-(a_1)^{\lambda_1+\lambda_2}}{u^-(a_1)^{\lambda_1}(2-u^-(a_1))^{\lambda_2} + u^-(a_1))^{\lambda_2}(2-u^-(a_1))^{\lambda_1}}, \\ \frac{2u^+(a_1)^{\lambda_1+\lambda_2}}{u^+(a_1)^{\lambda_1}(2-u^+(a_1))^{\lambda_2} + u^+(a_1))^{\lambda_2}(2-u^+(a_1))^{\lambda_1}} \end{array} \right], \\ \frac{(1+v(a_1))^{\lambda_1} - (1-v(a_1))^{\lambda_1}}{((1+v(a_1))^{\lambda_1} + (1-v(a_1))^{\lambda_1})((1+v(a_1))^{\lambda_2} + (1-v(a_1))^{\lambda_2})} \end{array} \right\rangle.$$

□

Some cubic uncertain linguistic fuzzy powered Einstein operators

In this section, we will combine the PA operator and Einstein operations to cubic uncertain linguistic environment and propose cubic uncertain linguistic fuzzy powered Einstein averaging (CULFPEA) operator, cubic uncertain linguistic fuzzy powered Einstein weighted averaging (CULFPEWA) operator, cubic uncertain linguistic fuzzy powered Einstein geometric (CULFPEG) operator and cubic uncertain linguistic fuzzy powered Einstein weighted geometric (CULFPEWG) operator and discuss the properties of them.

Cubic uncertain linguistic fuzzy powered Einstein averaging (CULFPEA) operator

Definition 15 Let $a_1 = \left\langle \begin{array}{c} \{x[[s_{\theta(a_1)}, s_{t(a_1)}], \\ [u^-(a_1), u^+(a_1)], \\ v(a_1))]\} \end{array} \right\rangle$ and $a_2 = \left\langle \begin{array}{c} \{x[[s_{\theta(a_2)}, s_{t(a_2)}], \\ [u^-(a_2), u^+(a_2)], \\ v(a_2))\} \end{array} \right\rangle$ be a collection of cubic uncertain linguistic fuzzy numbers and CULFPEA: $\Omega_n \to \Omega$. If CULFPEA $(a_1, a_2, \ldots, a_n) = \frac{\bigoplus_{i=1}^n (1+T(a_i))a_i}{\sum_{i=1}^n (1+T(a_i))} = \bigoplus_{i=1}^n \left(\frac{(1+T(a_i))a_i}{\sum_{i=1}^n (1+T(a_i))}\right)$, where Ω is the set of all cubic uncertain linguistic fuzzy numbers and $T(a_i) = \sum_{\substack{j=1 \\ i \neq j}}^n \sup(a_i, a_j)$, in which $\sup(a_i, a_j)$ is the support for a_i from a_j, then CULFPEA is called the cubic uncertain linguistic fuzzy powered Einstein averaging(CULFPEA) operator.

Theorem 3 *Let* $a_i = \langle \{x[[s_{\theta(a_i)}, s_{t(a_i)}], [u^-(a_i), u^+(a_i)], v(a_i))]\} \rangle$ *be a collection of cubic uncertain linguistic fuzzy numbers, then the result aggregated from Definition 15 is still cubic uncertain linguistic fuzzy number and*

$$\mathrm{CULFPEA}(a_1,a_2,\ldots,a_n)=\left\langle\left[s_{\sum_{i=1}^{n}\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)},s_{\sum_{i=1}^{n}\left(\frac{t(a_i)(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}\right],\right.$$

$$\left[\frac{\prod_{i=i}^{n}(1+u^-(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}-\prod_{i=i}^{n}(1-u^-(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}}{\prod_{i=i}^{n}(1+u^-(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}+\prod_{i=i}^{n}(1-u^-(a_1))^{\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}},\frac{\prod_{i=i}^{n}(1+u^+(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}-\prod_{i=i}^{n}(1-u^+(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}}{\prod_{i=i}^{n}(1+u^+(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}+\prod_{i=i}^{n}(1-u^+(a_1))^{\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}}\right],$$

$$\left.\frac{2\prod_{i=i}^{n}(v(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}}{\prod_{i=i}^{n}(2-v(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}+\prod_{i=i}^{n}(v(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}}\right\rangle,$$

where $T(a_i)=\sum_{\substack{j=1\\ i\neq j}}^{n}\sup(a_i,a_j)$, *in which* $\sup(a_i,a_j)$ *is the support for* a_i *from* a_j.

Proof We suppose $c_i=\left(\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)$

$$\mathrm{CULFPEA}(a_1,a_2,\ldots,a_n)=\left\langle\left[s_{\sum_{i=1}^{n}\theta(a_i)c_i},s_{\sum_{i=1}^{n}t(a_i)c_i}\right],\left[\frac{\prod_{i=i}^{n}(1+u^-(a_i))^{c_i}-\prod_{i=i}^{n}(1-u^-(a_1))^{c_i}}{\prod_{i=i}^{n}(1+u^-(a_1))^{c_i}+\prod_{i=i}^{n}(1-u^-(a_1))^{c_i}},\frac{\prod_{i=i}^{n}(1+u^+(a_i))^{c_i}-\prod_{i=i}^{n}(1-u^+(a_1))^{c_i}}{\prod_{i=i}^{n}(1+u^+(a_1))^{c_i}+\prod_{i=i}^{n}(1-u^+(a_1))^{c_i}}\right],\frac{2\prod_{i=i}^{n}(v(a_i))^{c_i}}{\prod_{i=i}^{n}(2-v(a_i))^{c_i}+\prod_{i=i}^{n}(v(a_i))^{c_i}}\right\rangle$$

(i) When $n=1$, the equation above is right obviously.

(ii) Suppose when $n=k$, the equation above is right, i.e.,

$$\mathrm{CULFPEA}(a_1,a_2,\ldots,a_n)=\left\langle\left[s_{\sum_{i=1}^{k}\theta(a_i)c_i},s_{\sum_{i=1}^{k}t(a_i)c_i}\right],\left[\frac{\prod_{i=i}^{k}(1+u^-(a_i))^{c_i}-\prod_{i=i}^{k}(1-u^-(a_1))^{c_i}}{\prod_{i=i}^{k}(1+u^-(a_1))^{c_i}+\prod_{i=i}^{k}(1-u^-(a_1))^{c_i}},\frac{\prod_{i=i}^{k}(1+u^+(a_i))^{c_i}-\prod_{i=i}^{k}(1-u^+(a_1))^{c_i}}{\prod_{i=i}^{k}(1+u^+(a_1))^{c_i}+\prod_{i=i}^{k}(1-u^+(a_1))^{c_i}}\right],\frac{2\prod_{i=i}^{k}(v(a_i))^{c_i}}{\prod_{i=i}^{k}(2-v(a_i))^{c_i}+\prod_{i=i}^{k}(v(a_i))^{c_i}}\right\rangle$$

Then, when $n=k+1$, we have $CULFPEA(a_1,a_2,\ldots,a_{k+1})=\mathrm{CULFPEA}(a_1,a_2,\ldots,a_k)\bigoplus(c_{k+1}a_{k+1})$

$$\mathrm{CULFPEA}(a_1,a_2,\ldots,a_k)\bigoplus\Big\langle[s_{\theta(a_{k+1})c_{k+1}},s_{t(a_{k+1})c_{k+1}}]\\ \left[\frac{(1+u^-(a_{k+1}))^{c_{k+1}}-(1-u^-(a_{k+1}))^{c_{k+1}}}{(1+u^-(a_{k+1}))^{c_{k+1}}+(1-u^-(a_{k+1}))^{c_{k+1}}},\frac{(1+u^+(a_{k+1}))^{c_{k+1}}-(1-u^+(a_{k+1}))^{c_{k+1}}}{(1+u^+(a_{k+1}))^{c_{k+1}}+(1-u^+(a_{k+1}))^{c_{k+1}}}\right],\frac{2(v(a_{k+1}))^{c_{k+1}}}{(2-v(a_{k+1}))^{c_{k+1}}+(v(a_{k+1}))^{c_{k+1}}}\Big\rangle$$

$$=\left\langle\left[s_{\sum_{i=1}^{k+1}\theta(a_i)c_i},s_{\sum_{i=1}^{k+1}t(a_i)c_i}\right],\left[\frac{\prod_{i=i}^{k+1}(1+u^-(a_i))^{c_i}-\prod_{i=i}^{k+1}(1-u^-(a_1))^{c_i}}{\prod_{i=i}^{k+1}(1+u^-(a_1))^{c_i}+\prod_{i=i}^{k+1}(1-u^-(a_1))^{c_i}},\frac{\prod_{i=i}^{k+1}(1+u^+(a_i))^{c_i}-\prod_{i=i}^{k+1}(1-u^+(a_1))^{c_i}}{\prod_{i=i}^{k+1}(1+u^+(a_1))^{c_i}+\prod_{i=i}^{k+1}(1-u^+(a_1))^{c_i}}\right],\frac{2\prod_{i=i}^{k+1}(v(a_i))^{c_i}}{\prod_{i=i}^{k+1}(2-v(a_i))^{c_i}+\prod_{i=i}^{k+1}(v(a_i))^{c_i}}\right\rangle.$$

□.

Theorem 4 (Idempotency) *Let* $a_i=\langle\{x[[s_{\theta(a_i)},s_{t(a_i)}],[u^-(a_i),u^+(a_i)],v(a_i))]\}\rangle$ *, then* $\mathrm{CULFPEA}(a_1,a_2,\ldots,a_k)=a$.

Proof Since $a_i=a_i$ for all i, we have

$$\mathrm{CULFPEA}(a_1,a_2,\ldots,a_k)=\left\langle\left[s_{\sum_{i=1}^n\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)},s_{\sum_{i=1}^n\left(\frac{t(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}\right]\right.$$

$$\left[\frac{\prod_{i=i}^n(1+u^-(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}-\prod_{i=i}^n(1-u^-(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}{\prod_{i=i}^n(1+u^-(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}+\prod_{i=i}^n(1-u^-(a_1))^{\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}\right],$$

$$\left[\frac{\prod_{i=i}^n(1+u^+(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}-\prod_{i=i}^n(1-u^+(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}{\prod_{i=i}^n(1+u^+(a_1))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}+\prod_{i=i}^n(1-u^+(a_1))^{\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}\right],$$

$$\left.\frac{2\prod_{i=i}^n(v(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}{\prod_{i=i}^n(2-v(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}+\prod_{i=i}^n(v(a_i))^{\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}\right\rangle$$

$$\left\langle\left[s_{\sum_{i=1}^n\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)},s_{\sum_{i=1}^n\left(\frac{t(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}\right],\right.$$

$$\left[\frac{\prod_{i=i}^n(1+u^-(a_i))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}-\prod_{i=i}^n(1-u^-(a_1))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}{\prod_{i=i}^n(1+u^-(a_1))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}+\prod_{i=i}^n(1-u^-(a_1))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}\right],$$

$$\left[\frac{\prod_{i=i}^n(1+u^+(a_i))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}-\prod_{i=i}^n(1-u^+(a_1))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}{\prod_{i=i}^n(1+u^+(a_1))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}+\prod_{i=i}^n(1-u^+(a_1))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}\right],$$

$$\left.\frac{2\prod_{i=i}^n(v(a_i))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}{\prod_{i=i}^n(2-v(a_i))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}+\prod_{i=i}^n(v(a_i))^{\sum_{i=1}^n\left(\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}}\right\rangle$$

$$\langle[s_{\theta(a_i)},s_{t(a_i)}],[u^-(a_i),u^+(a_i)],v(a_i))]\rangle=a_i.$$

□

Theorem 5 (Boundary) *The CULFPEA operator lies between the max and min operators:* $a_{\min}=\min(a_1,a_2,\ldots,a_n)$; $a_{\max}=\max(a_1,a_2,\ldots,a_n)$, *then*

$$a_{\min}\le\mathrm{CULFPEA}(a_1,a_2,\ldots,a_n)\le a_{\max}:$$

Theorem 6 (Monotonicity) *Let* $a_1=\left\langle\begin{matrix}[[s_{\theta(a_1)},\\ s_{t(a_1)}],\\ [u^-(a_1),\\ u^+(a_1)],\\ v(a_1))]\}\end{matrix}\right\rangle$ *and* $a_2=\left\langle\begin{matrix}[[s_{\theta(a_2)},\\ s_{t(a_2)}],\\ [u^-(a_2),\\ u^+(a_2)],\\ v(a_2))]\}\end{matrix}\right\rangle$ *be two collections of cubic uncertain linguistic fuzzy number and if* $s_{\theta(a_1)}\le s_{\theta(a_2)},s_{t(a_1)}\le s_{t(a_2)},u^-(a_1)\le u^-(a_2),u^+(a_1)\le u^+(a_2),v(a_1)\le v(a_2)$, *for all* $i;i=1,2,\ldots,n$, *then* $\mathrm{CULFPEA}(a_1,a_2,\ldots,a_n)\le\mathrm{CULFPEA}(a_1,a_2,\ldots,a_n)$.

Proof Since

$$s_{\theta(a_1)}+s_{t(a_1)},s_{\theta(a_2)}+s_{t(a_2)},u^-(a_1)\le u^-(a_2),$$
$$u^+(a_1)\le u^+(a_2),v(a_1)\le v(a_2),$$
$$\sum_{i=1}^n\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))},s_{\theta(a_1)}+s_{t(a_1)},\right)$$
$$\le\sum_{i=1}^n\left(\frac{\theta(a_i)(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))},s_{\theta(a_2)}+s_{t(a_2)},\right),$$
$$s_{\sum_{i=1}^n\left(\theta(a_1)\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)}+s_{\sum_{i=1}^n\left(t(a_1)\frac{(1+T(a_i))}{\sum_{i=1}^n(1+T(a_i))}\right)},$$

i.e.,

$$\leq s_{\sum_{i=1}^{n}\left(\theta(a_2)\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)} + s_{\sum_{i=1}^{n}\left(t(a_2)\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}\right)}.$$

Since

$$\frac{1-u^-(a_1)}{1+u^-(a_1)} \leq \frac{1-u^-(a_2)}{1+u^-(a_2)}, \frac{1-u^+(a_1)}{1+u^+(a_1)} \leq \frac{1-u^+(a_2)}{1+u^+(a_2)}$$

$$\prod_{i=1}^{n}\left(\frac{1-u^-(a_1)}{1+u^-(a_1)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} \leq \prod_{i=1}^{n}\left(\frac{1-u^-(a_2)}{1+u^-(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}},$$

$$\prod_{i=1}^{n}\left(\frac{1-u^+(a_1)}{1+u^+(a_1)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} \leq \prod_{i=1}^{n}\left(\frac{1-u^+(a_2)}{1+u^+(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}$$

$$\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u^-(a_1)}{1+u^-(a_1)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}} \geq \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u^-(a_2)}{1+u^-(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}},$$

$$\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u^+(a_1)}{1+u^+(a_1)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}} \geq \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u^+(a_2)}{1+u^+(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}.$$

We have

$$\frac{\prod_{i=1}^{n}(1+u^-(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} - \prod_{i=1}^{n}(1-u^-(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}{\prod_{i=1}^{n}(1+u^-(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} + \prod_{i=1}^{n}(1-u^-(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}$$

$$\geq \frac{\prod_{i=1}^{n}(1+u^-(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} - \prod_{i=1}^{n}(1-u^-(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}{\prod_{i=1}^{n}(1+u^-(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} + \prod_{i=1}^{n}(1-u^-(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}$$

$$\frac{\prod_{i=1}^{n}(1+u^+(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} - \prod_{i=1}^{n}(1-u^+(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}{\prod_{i=1}^{n}(1+u^+(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} + \prod_{i=1}^{n}(1-u^+(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}$$

$$\geq \frac{\prod_{i=1}^{n}(1+u^+(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} - \prod_{i=1}^{n}(1-u^+(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}{\prod_{i=1}^{n}(1+u^+(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} + \prod_{i=1}^{n}(1-u^+(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}.$$

Since $\frac{2-v(a_2)}{v(a_2)} \geq \frac{2-v(a_1)}{v(a_1)}$, then

$$\left(\frac{2-v(a_2)}{v(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} \geq \left(\frac{2-v(a_1)}{v(a_1)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} \prod_{i=1}^{n}\left(\frac{2-v(a_2)}{v(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}$$

$$\geq \prod_{i=1}^{n}\left(\frac{2-v(a_1)}{v(a_1)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} \frac{1}{\prod_{i=1}^{n}\left(\frac{2-v(a_2)}{v(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}+1} \frac{1}{\prod_{i=1}^{n}\left(\frac{2-v(a_2)}{v(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}+1},$$

$$\frac{2}{\prod_{i=1}^{n}\left(\frac{2-v(a_2)}{v(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}+1} \leq \frac{2}{\prod_{i=1}^{n}\left(\frac{2-v(a_2)}{v(a_2)}\right)^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}+1},$$

so

$$\frac{2\prod_{i=1}^{n}(v(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}{\prod_{i=1}^{n}(2-v(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} + \prod_{i=1}^{n}(2-v(a_2))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}$$

$$\leq \frac{2\prod_{i=1}^{n}(v(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}{\prod_{i=1}^{n}(2-v(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} + \prod_{i=1}^{n}(2-v(a_1))^{\frac{(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}.$$

We can get $\text{CULFPEA}(a_1, a_2, \ldots, a_n) \leq \text{CULFPEA}(a_1, a_2, \ldots, a_n)$, which complete the proof. □

Cubic uncertain linguistic fuzzy powered Einstein weighted averaging (CULFPEWA) operator

Definition 16 Let $a_i = \left\langle \begin{matrix} [s_{\theta(a_i)}, \\ s_{t(a_i)}], \\ [u^-(a_i), \\ u^+(a_i)], \\ v(a_i))] \end{matrix} \right\rangle$ be a collection of cubic uncertain linguistic fuzzy numbers, and CULFPEWA: $\Omega n \to \Omega$. If $\text{CULFPEWA}(a_1, a_2, \ldots, a_n) = \frac{\bigoplus_{i=1}^n w_i(1+T(a_i))a_i}{\sum_{i=1}^n w_i(1+T(a_i))} = \bigoplus_{i=1}^n \left(\frac{w_i(1+T(a_i))a_i}{\sum_{i=1}^n w_i(1+T(a_i))} \right)$, where Ω is the set of all cubic uncertain linguistic fuzzy numbers, $T(a_i) = \sum_{\substack{j=1 \\ i \neq j}}^n \sup(a_i, a_j)$ in which $\sup(a_i, a_j)$ is the support for a_i from $a_j$and $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of the $(a_1, a_2, \ldots, a_n)$ such that $w_i \in [0; 1], \sum_{i=1}^n w_i = 1$. Then, CULFPEWA is called the cubic uncertain linguistic fuzzy powered Einstein weighted averaging (CULFPEWA) operator.

Theorem 7 *Let* $a_i = \langle \{x[[s_{\theta(a_i)}, s_{t(a_i)}], [u^-(a_i), u^+(a_i)], v(a_i))]\}\rangle$ *be a collection of cubic uncertain linguistic fuzzy numbers, then the result aggregated from Definition 16 is still cubic uncertain linguistic fuzzy number and*

$$\text{CULFPEWA}(a_1, a_2, \ldots, a_n) = \bigoplus_{i=1}^n \left(\frac{w_i(1+T(a_i))a_i}{\sum_{i=1}^n w_i(1+T(a_i))} \right)$$

$$= \Bigg\langle s_{\sum_{i=1}^n \left(\frac{\theta(a_1)w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))} \right)} + s_{\sum_{i=1}^n \left(\frac{t(a_1)w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))} \right)},$$

$$\left[\frac{\prod_{i=1}^n (1+u^-(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}} - \prod_{i=1}^n (1-u^-(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}}}{\prod_{i=1}^n (1+u^-(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}} + \prod_{i=1}^n (1-u^-(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}}} \right],$$

$$\left[\frac{\prod_{i=1}^n (1+u^+(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}} - \prod_{i=1}^n (1-u^+(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}}}{\prod_{i=1}^n (1+u^+(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}} + \prod_{i=1}^n (1-u^+(a_i))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}}} \right],$$

$$\frac{2\prod_{i=1}^n (v(a_2)^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}}}{\prod_{i=1}^n (2-v(a_2))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}} + \prod_{i=1}^n (2-v(a_2))^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^n (1+T(a_i))}}} \Bigg\rangle,$$

where $T(a_i) = \sum_{\substack{j=1 \\ i \neq j}}^n \sup(a_i, a_j)$, *in which* $\sup(a_i, a_j)$ *is the support for* a_i *from* a_j $w = (w_1, w_2, \ldots, w_n)^T$ *is the weighting vector of the* $(a_1, a_2, \ldots, a_n)$ *such that* $w_i \in [0; 1], \sum_{i=1}^n w_i = 1$.

Theorem 8 (Idempotency) *Let* $a_i = a$ *for all* i, *then* $\text{CULFPEWA}(a_1, a_2, \ldots, a_n) = a$.

Theorem 9 (Boundary) *The CULFPEWA operator lies between the max and min operators:* $a_{\min} = \min(a_1, a_2, \ldots, a_n)$; $a_{\max} = \max(a_1, a_2, \ldots, a_n)$, *then* $a_{\min} \leq \text{CULFPEWA}(a_1, a_2, \ldots, a_n) \leq a_{\max}$.

Cubic uncertain linguistic fuzzy powered Einstein geometric (CULFPEG) operator

Definition 17 Let $a_i = \left\langle \begin{matrix} [[s_{\theta(a_i)}, \\ s_{t(a_i)}], \\ [u^-(a_i), \\ u^+(a_i)], \\ v(a_i))] \end{matrix} \right\rangle$ be a collection of cubic uncertain linguistic fuzzy number, then cubic uncertain linguistic fuzzy powered Einstein geometric (CULFPEG) operator of dimension n is a mapping and is

$$\text{CULFPEG}(a_1, a_2, \ldots, a_n) = \bigoplus_{i=1}^n (a_i)^{\frac{(1+T(a_i))a_i}{\sum_{i=1}^n w_i(1+T(a_i))}},$$

where $T(a_i) = \sum_{\substack{j=1 \\ i \neq j}}^n \sup(a_i, a_j)$ in which $\sup(a_i, a_j)$ is the support for a_i from a_j and $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of the $(a_1, a_2, \ldots, a_n)$ such that $w_i \in [0; 1], \sum_{i=1}^n w_i = 1$.

Theorem 10 *Let* $a_i = \left\langle \begin{matrix} [[s_{\theta(a_i)}, s_{t(a_i)}], \\ [u^-(a_i), u^+(a_i)], \\ v(a_i))] \end{matrix} \right\rangle$ *be a collection of cubic uncertain linguistic fuzzy number, then the result aggregated from Definition 17 is still cubic uncertain linguistic fuzzy numbers and*

$$\text{CULFPEG}(a_1, a_2, \ldots, a_n) = \bigotimes_{i=1}^n (a_i)^{\frac{(1+T(a_i))a_i}{\sum_{i=1}^n w_i(1+T(a_i))}}$$

$$\text{CULFPEG}(a_1, a_2, \ldots, a_n) =$$

$$\bigotimes_{i=1}^n (a_i)^{\frac{(1+T(a_i))a_i}{\sum_{i=1}^n w_i(1+T(a_i))}} \frac{\prod_{i=1}^n (1+v(a_i))^{bi} - \prod_{i=1}^n (1-v(a_i))^{b_i}}{\prod_{i=1}^n (1+v(a_i))^{b_i} + \prod_{i=1}^n (1-v(a_i))^{b_i}} \rangle,$$

where $b_i = \frac{(1+T(a_i))a_i}{\sum_{i=1}^n w_i(1+T(a_i))}$ *and* $T(a_i) = \sum_{\substack{j=1 \\ i \neq j}}^n \sup(a_i, a_j)$, *in which* $\sup(a_i, a_j)$ *is the support for* a_i *from* a_j.

Theorem 11 (Idempotency) *Let* $a_i = a$ *for all* i, *then* $\text{CULFEG}(a_1, a_2, \ldots, a_n) = a$.

Theorem 12 (Boundary) *The CULFPEWA operator lies between the max and min operators:* $a_{\min} = \min(a_1, a_2, \ldots, a_n)$; $a_{\max} = \max(a_1, a_2, \ldots, a_n)$, *and then* $a_{\min} \leq \text{CULFEWA}(a_1, a_2, \ldots, a_n) \leq a_{\max}$.

Theorem 13 (Monotonicity) *Let* $a_1 = \left\langle \begin{matrix} [[s_{\theta(a_1)}, \\ s_{t(a_1)}], \\ [u^-(a_1), \\ u^+(a_1)], \\ v(a_1))] \end{matrix} \right\rangle$ *and* $b_2 = \left\langle \begin{matrix} [[s_{\theta(b_2)}, \\ s_{t(b_2)}], \\ [u^-(b_2), \\ u^+(b_2)], \\ v(b_2))] \end{matrix} \right\rangle$ *be two collections of cubic uncertain linguistic fuzzy number and if* $s_{\theta(a_1)} \le s_{\theta(b_2)}, s_{t(a_1)} \le s_{t(b_2)}$, $u^-(a_1) \le u^-(b_2), u^+(a_1) \le u^+(b_2), v(a_1) \le v(b_2)$, *for all* $i; i = 1, 2, \ldots, n$, *then* $\text{CULFEG}(a_1, a_2, \ldots, a_n) \le \text{CULFEG}(b_1, b_2, \ldots, b_n)$.

Cubic uncertain linguistic fuzzy powered Einstein weighted geometric (CULFPEWG) operator

Definition 18 Let $a_i = \left\langle \begin{matrix} [[s_{\theta(a_i)}, \\ s_{t(a_i)}], \\ [u^-(a_i), \\ u^+(a_i)], \\ v(a_i))] \end{matrix} \right\rangle$ be a collection of cubic uncertain linguistic fuzzy numbers, and CULFPEWG : $\Omega n \to \Omega$. If

$$\text{CULFPEWG}(a_1, a_2, \ldots, a_n) \bigoplus\nolimits_{i=1}^{n} (a_i)^{\frac{(1+T(a_i))a_i}{\sum_{i=1}^{n} w_i(1+T(a_i))}},$$

where Ω is the set of all cubic uncertain linguistic fuzzy numbers, and $T(a_i) = \sum_{\substack{j=1 \\ i \ne j}}^{n} \sup(a_i, a_j)$, and $\sup(a_i, a_j)$ is the support for a_i from $a_j$and $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of the $(a_1, a_2, \ldots, a_n)$ such that $w_i \in [0; 1], \sum_{i=1}^{n} w_i = 1$. Then, CULFPEWG is called the cubic uncertain linguistic fuzzy powered Einstein weighted geometric (CULFPEWG) operator.

Theorem 14 *Let* $a_i = \left\langle \begin{matrix} [s_{\theta(a_i)}, s_{t(a_i)}], \\ [u^-(a_i), u^+(a_i)], \\ v(a_i))] \end{matrix} \right\rangle$ *be a collection of cubic uncertain linguistic fuzzy number, then the result aggregated from Definition 18 is still cubic uncertain linguistic fuzzy numbers and*

$$\text{CULFPEWG}(a_1, a_2, \ldots, a_n) = \bigotimes\nolimits_{i=1}^{n} (a_i)^{\frac{(1+T(a_i))a_i}{\sum_{i=1}^{n} w_i(1+T(a_i))}}$$

$$= \left\langle s_{\prod_{i=1}^{n} (\theta(a_i)^{b_i})}, s_{\prod_{i=1}^{n} (t(a_i)^{b_i})}, \left[\begin{matrix} \dfrac{2\prod_{i=1}^{n} (u^-(a_i))^{b_i}}{\prod_{i=1}^{n} (2-u^-(a_i))^{b_i} + \prod_{i=1}^{n} (2-u^-(a_i))^{b_i}}, \\ \dfrac{2\prod_{i=1}^{n} (u^+(a_i))^{b_i}}{\prod_{i=1}^{n} (2-u^+(a_i))^{b_i} + \prod_{i=1}^{n} (2-u^+(a_i))^{b_i}} \end{matrix} \right], \right.$$

$$\left. \frac{\prod_{i=1}^{n} (1+v(a_i))^{bi} - \prod_{i=1}^{n} (1-v(a_i))^{b_i}}{\prod_{i=1}^{n} (1+v(a_i))^{b_i} + \prod_{i=1}^{n} (1-v(a_i))^{b_i}} \right\rangle,$$

where $b_i = \frac{(1+T(a_i))a_i}{\sum_{i=1}^{n} w_i(1+T(a_i))}$ *and* $T(a_i) = \sum_{\substack{j=1 \\ i \ne j}}^{n} \sup(a_i, a_j)$, *in which* $\sup(a_i, a_j)$ *is the support for* a_i *from* a_j.

Theorem 15 (Idempotency) *Let all* $a_i = a$ *for all* i, *then* $\text{CULFEWG}(a_1, a_2, \ldots, a_n) = a$.

Theorem 16 (Boundary) *The CULFPEWG operator lies between the max and min operators:* $a_{\min} = \min(a_1, a_2, \ldots, a_n)$; $a_{\max} = \max(a_1, a_2, \ldots, a_n)$, *and then* $a_{\min} \le \text{CULFEWG}(a_1, a_2, \ldots, a_n) \le a_{\max}$.

The decision-making methods based on the CULFPEWA operator and CULFPEWG operator

In order to strengthen the efficiency of this decision making, we can make several experts participate in the decision making under cubic uncertain linguistic fuzzy environment. Considering the multi-attribute group decision-making problems with cubic uncertain linguistic fuzzy information described as follows.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ be the set of attributes and $W = \{w_1, w_2, \ldots, w_n\}$ be the weight vector of the attribute $C_j(j = 1; 2; ::::; n)$, where $w_j \ge 0$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} w_j = 1$. Let $D = \{D_1, D_2, \ldots, D_p\}$ be the set of decision makers and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T$ be the weight vector of decision makers $D_q(q = 1, 2, \ldots, p)$, where $q \ge 0, \sum_{j=1}^{n} \lambda_q = 1$.

Suppose $H_{ij}^{q} = [h_{ij}^{q}]_{m \times n}$ are the decision matrices where $h(q) = \langle [s_{\theta(q)_{(h_{ij})}}, s_{t(q)_{(h_{ij})}}]$,

$[u^-(h_{ij}^{(q)}), u^+(h_{ij}^{(q)})], v(h_{ij}^{(q)})\rangle$ takes the form of the cubic uncertain linguistic variables given by the decision maker D_q for alternative A_i with respect to attribute C_j, $u^-(h_{ij}^{(q)}) \ge 0, u^+(h_{ij}^{(q)}) \ge 0, v(h_{ij}^{(q)}) \ge 0, s_{\theta(q)_{(h_{ij})}}, s_{t(q)_{(h_{ij})}} \in s$.

Then, the ranking of alternatives is finally acquired. The methods involve the following steps:

Step 1 Calculate the supports.

$\sup([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)}) = 1 - d([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)})$, which satisfies the support conditions expressed definition 23, where $d([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)})$ is the distance between two CULFNs $d([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)})$.

Step 2 Calculate $T(h_{ij}^{(q)})$. $T(h_{ij}^{(q)}) = \sum_{t=1, t\neq q}^{p} \sup([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)})$.

Step 3 Utilize the CULFPEWA operator or the CULFPEWG operator to aggregate the individual cubic uncertain linguistic fuzzy decision matrices $H^{(q)} = [h_{ij}^{q}]_{m\times n}$ into the collective cubic uncertain linguistic fuzzy decision matrix $H = (h_{ij})_{m\times n}$ where $h_{ij} = \langle [s_{\theta(q)_{(h_{ij})}}, s_{t(q)_{(h_{ij})}}], [u^-(h_{ij}^{(q)}), u^+(h_{ij}^{(q)})], v(h_{ij}^{(q)})\rangle$

$$h_{ij} = \mathrm{CULFPEWA}(h_{ij}^1, h_{ij}^2, \ldots h_{ij}^p)$$

$$= \left\langle s_{\sum_{q=1}^{p}\left(\frac{\theta^q(h_{ij})\lambda_q\left(1+T(h_{ij}^q)\right)}{\sum_{i=1}^{n}\left(1+T(h_{ij}^q)\right)}\right)} + s_{\sum_{i=1}^{n}\left(\frac{t^q(h_{ij})\lambda_q\left(1+T(h_{ij}^q)\right)}{\sum_{i=1}^{n}\left(1+T(h_{ij}^q)\right)}\right)}, \right.$$

$$\left[\frac{\prod_{i=1}^n (1+u^-(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}} - \prod_{i=1}^n (1-u^-(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}}}{\prod_{i=1}^n (1+u^-(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}} + \prod_{i=1}^n (1-u^-(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}}}\right],$$

$$\left[\frac{\prod_{i=1}^n (1+u^+(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}} - \prod_{i=1}^n (1-u^+(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}}}{\prod_{i=1}^n (1+u^+(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}} + \prod_{i=1}^n (1-u^+(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}}}\right],$$

$$\left.\frac{2\prod_{i=1}^n (v(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}}}{\prod_{i=1}^n (2-v(h_{ij}^q))^{\frac{\lambda_q(1+T(a_i))}{\sum_{i=1}^n \lambda_q(1+T(a_i))}} + \prod_{i=1}^n (2-v(h_{ij}^q))^{\frac{\lambda_q(1+T(h_{ij}^q))}{\sum_{i=1}^n \lambda_q(1+T(h_{ij}^q))}}}\right\rangle,$$

where $T(a_i) = \sum_{j=1, i\neq j}^{n} \sup(a_i, a_j)$, in which $\sup(a_i, a_j)$ is the support for a_i from a_j, $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of the $(a_1, a_2, \ldots, a_n)$ such that $w_i \in [0;1], \sum_{i=1}^n w_i = 1$.

$$h_{ij} = \mathrm{CULFPEWG}(h_{ij}^1, h_{ij}^2, \ldots, h_{ij}^p)$$

$$= \left\langle s_{\prod_{q=1}^p (\theta(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}, s_{\prod_{q=1}^p (t(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}, \right.$$

$$\left[\frac{2\prod_{q=1}^p (u^-(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}{\prod_{q=1}^p (2-u^-(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}} + \prod_{q=1}^p (2-u^-(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}\right],$$

$$\left[\frac{2\prod_{q=1}^p (u^+(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}{\prod_{q=1}^p (2-u^+(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}} + \prod_{q=1}^p (2-u^+(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}\right],$$

$$\left.\frac{\prod_{q=1}^p (1+v(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}} - \prod_{q=1}^p (1-v(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}{\prod_{q=1}^p (1+v(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}} + \prod_{q=1}^p (1-v(h_{ij}^q))^{\frac{(1+T(h_{ij}^q))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}\right\rangle.$$

Step 4 Calculate the supports

$$\sup([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)}) = 1 - d([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)}),$$

Step 5 Calculate

$$T(h_{ij}).T(h_{ij}^{(q)}) = \sum_{t=1, t\neq q}^{p} \sup([h_{ij}^{-q}, h_{ij}^{+q}], h_{ij}^{(f)}).$$

Step 6 Aggregate the cubic uncertain linguistic fuzzy numbers for each alternative by the CULFPEWA (or CULFPEWG) operator:

$$h_{ij} = \mathrm{CULFPEWA}(h_{i1}, h_{i2}, \ldots h_{in})$$

$$= \left\langle s_{\sum_{j=1}^{n}\left(\frac{\theta(h_{ij})w_j(1+T(h))}{\sum_{j=1}^{n} w_j(1+T(h_{ij}))}\right)} + s_{\sum_{j=1}^{n}\left(\frac{t(h_{ij})w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j(1+T(h_{ij}))}\right)}, \right.$$

$$\left[\frac{\prod_{j=1}^n (1+u^-(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}} - \prod_{j=1}^n (1-u^-(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}}}{\prod_{j=1}^n (1+u^-(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}} + \prod_{j=1}^n (1-u^-(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}}}\right],$$

$$\left[\frac{\prod_{j=1}^n (1+u^+(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}} - \prod_{j=1}^n (1-u^+(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}}}{\prod_{j=1}^n (1+u^+(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}} + \prod_{j=1}^n (1-u^+(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}}}\right],$$

$$\left.\frac{2\prod_{j=1}^n (v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}}}{\prod_{j=1}^n (2-v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}} + \prod_{j=1}^n (2-v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^n w_j(1+T(h_{ij}))}}}\right\rangle.$$

$$h_{ij} = \mathrm{CULFPEWG}(h_{i1}, h_{i2}, \ldots h_{in})$$

$$= \left\langle \left[= \langle s_{\prod_{j=1}^n (\theta(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^n w_j(1+T(h_{ij}^q))}}}, s_{\prod_{q=1}^{p} (t(h_{ij}))^{\frac{(1+T(h_{ij}))\lambda_q}{\sum_{q=1}^p \lambda_q(1+T(h_{ij}^q))}}}, \right], \right.$$

$$\left[\frac{2\prod_{j=1}^n (u^-(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^n w_j(1+T(h_{ij}^q))}}}{\prod_{j=1}^n (2-u^-(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^n w_j(1+T(h_{ij}^q))}} + \prod_{j=1}^n (2-u^-(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^n w_j(1+T(h_{ij}^q))}}}\right],$$

$$\left[\frac{2\prod_{j=1}^{n}(u^{+}(h_{ij}^{q}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^{n} w_j\left(1+T(h_{ij}^{q})\right)}}}{\prod_{j=1}^{n}(2-u^{+}(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^{n} w_j\left(1+T(h_{ij}^{q})\right)}}+\prod_{j=1}^{n}(2-u^{+}(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^{n} w_j\left(1+T(h_{ij}^{q})\right)}}}\right],$$
$$\left.\frac{\prod_{j=1}^{n}(1+v(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^{n} w_j\left(1+T(h_{ij}^{q})\right)}}-\prod_{j=1}^{n}(1-v(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^{n} w_j\left(1+T(h_{ij}^{q})\right)}}}{\prod_{j=1}^{n}(1+v(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^{n} w_j\left(1+T(h_{ij}^{q})\right)}}+\prod_{j=1}^{n}(1-v(h_{ij}))^{\frac{(1+T(h_{ij}))w_j}{\sum_{j=1}^{n} w_j\left(1+T(h_{ij}^{q})\right)}}}\right\rangle.$$

Step 7 Calculate the value $E(h_i)$ of h_i.

Step 8 Rank $h_i(i = 1; 2, \ldots, m)$ in descending order according to the comparison method of CULFNs

Step 9 End (Fig. 4).

An numerical example

In this section, we provide an example to illustrate the application of CULFPEWA and CULFPEWG operators. Suppose that an investment company wants to invest an amount of money to a company. There are four candidate companies $A_i(i = 1, 2, 3, 4)$ evaluated by three decision makers $\{D_1, D_2, D_3\}$. The weight vector of the decision makers is $\lambda = (0.5, 0.45, 0.50)^T$ and the attributes considered include: C_1 (the risk index), C_2 (the growth index), C_3 (the social–political impact index) and C_4 (the environmental impact index). Suppose the attribute weight vector is $w = (0.30, 0.28, 0.20, 0.22)^T$. The three decision makers $\{D_1, D_2, D_3\}$ evaluate the four companies $A_i(i = 1, 2, 3, 4)$ with respect to the attributes $C_j(j = 1, 2, 3, 4)$ by using the cubic uncertain linguistic variables (suppose that the decision makers use linguistic term set $S = (s_0, s_1, s_2, s_3, s_4)$ to express their evaluation results) and construct the following decision matrices $H^{(q)} = [h_{ij}^{(q)}]$ $(q = 1, 2, 3)$ listed in Tables 1, 2 and 3.

Ranking four candidate companies by the CULFPEWA operator

Step 1 Calculate the supports $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$.

$$\sup(h_{ij}^1, h_{ij}^2) = \begin{pmatrix} 0.7894 & 0.6148 & 0.5516 & 0.7053 \\ 0.8527 & 0.9579 & 0.9263 & 0.9579 \\ 0.7789 & 0.9631 & 0.8158 & 0.9824 \\ 0.9368 & 0.9263 & 0.9807 & 0.9579 \end{pmatrix},$$

$$\sup(h_{ij}^2, h_{ij}^3) = \begin{pmatrix} 0.9264 & 0.7222 & 0.7664 & 0.5789 \\ 0.8316 & 0.8246 & 0.9369 & 0.9338 \\ 0.9853 & 0.7474 & 0.9859 & 0.8737 \\ 0.6922 & 0.9369 & 0.8948 & 0.7474 \end{pmatrix},$$

$$\sup(h_{ij}^1, h_{ij}^3) = \begin{pmatrix} 0.8737 & 0.9719 & 0.8316 & 0.8246 \\ 0.7895 & 0.9438 & 0.9508 & 0.9561 \\ 0.9508 & 0.8878 & 0.8685 & 0.9473 \\ 0.4106 & 0.8422 & 0.7895 & 0.9210 \end{pmatrix}.$$

Step 2 Calculate $T(h_{ij}^{(q)})(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$.

$$T(h_{ij}^1) = \begin{pmatrix} 1.9894 & 2.4148 & 2.6516 & 2.5053 \\ 2.7189 & 2.3579 & 1.8263 & 2.4579 \\ 1.7189 & 2.7631 & 2.5158 & 2.2824 \\ 1.6368 & 2.2263 & 2.7807 & 3.0579 \end{pmatrix},$$

$$T(h_{ij}^2) = \begin{pmatrix} 2.7264 & 2.5022 & 2.1569 & 1.3789 \\ 2.1316 & 2.0246 & 2.1369 & 1.3338 \\ 2.1653 & 2.6474 & 2.3859 & 2.0737 \\ 1.5922 & 2.4369 & 2.7948 & 2.5474 \end{pmatrix},$$

$$T(h_{ij}^3) = \begin{pmatrix} 2.6737 & 2.7316 & 2.5316 & 2.6246 \\ 1.8895 & 2.5438 & 2.0508 & 2.5561 \\ 2.0908 & 2.4878 & 2.4685 & 2.4473 \\ 2.3106 & 2.6422 & 2.1895 & 2.721 \end{pmatrix}.$$

Step 3 Utilize the CULFPEWA operator to aggregate all the three decision matrices mentioned above into the following decision matrix given in Table 4

Step 4 Calculate the supports according to $(k, j = 1, 2, 3, 4)$:

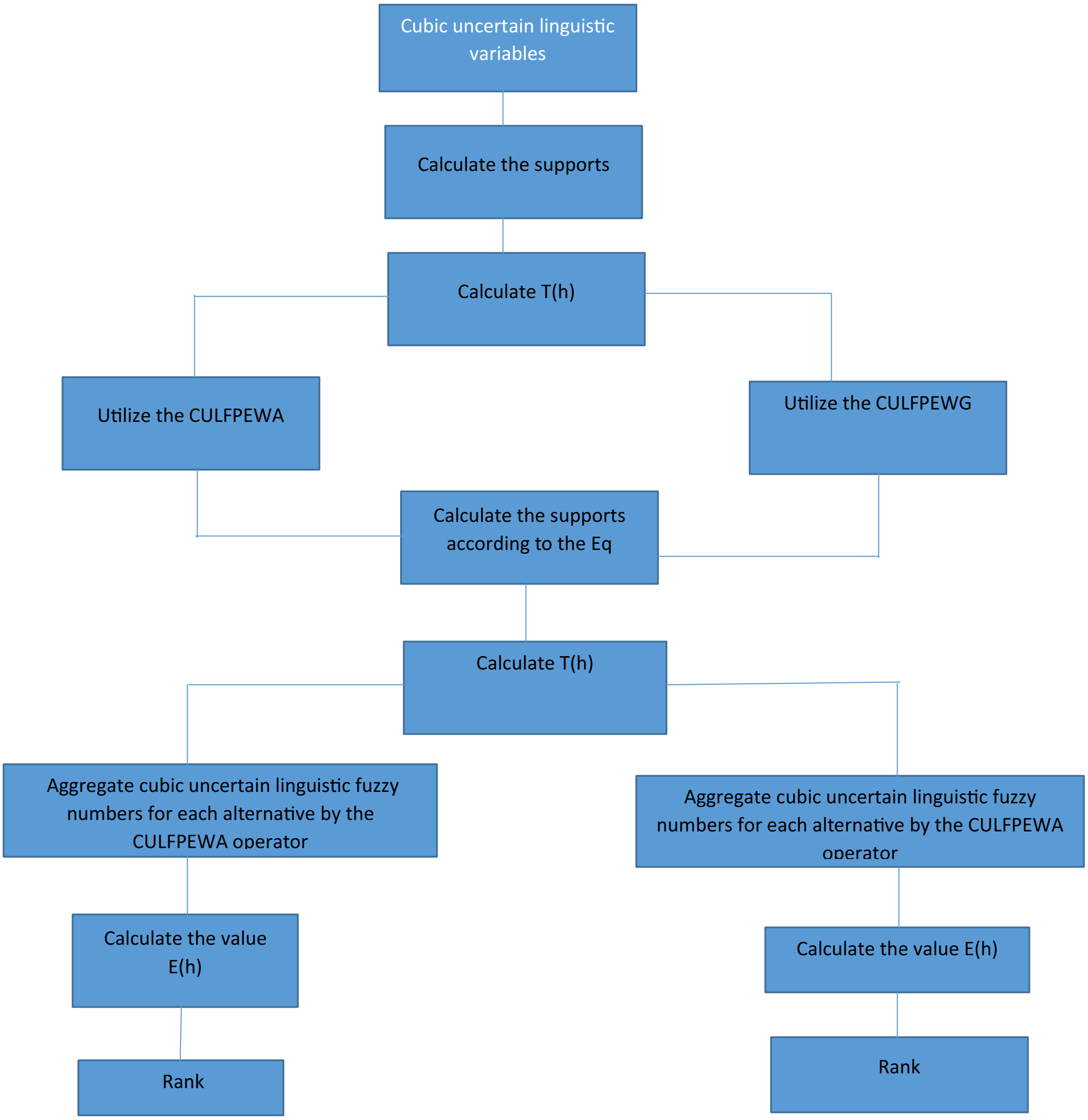


Fig. 4 Proposed method

Table 1 Decision matrix H^1

C_1	C_2	C_3	C_4
$\langle [s_1, s_2], [0.2, 0.6], 0.4 \rangle$	$\langle [s_1, s_2], [0.4, 0.8], 0.6 \rangle$	$\langle [s_2, s_2], [0.5, 0.9], 0.7 \rangle$	$\langle [s_1, s_4], [0.4, 0.9], 0.5 \rangle$
$\langle [s_1, s_3], [0.4, 0.9], 0.6 \rangle$	$\langle [s_1, s_1], [0.2, 0.8], 0.4 \rangle$	$\langle [s_1, s_4], [0.1, 0.5], 0.3 \rangle$	$\langle [s_1, s_4], [0.3, 0.7], 0.5 \rangle$
$\langle [s_1, s_3], [0.7, 0.13], 0.11 \rangle$	$\langle [s_2, s_4], [0.4, 0.8], 0.6 \rangle$	$\langle [s_4, s_4], [0.4, 0.8], 0.5 \rangle$	$\langle [s_1, s_2], [0.3, 0.6], 0.4 \rangle$
$\langle [s_2, s_4], [0.1, 0.4], 0.2 \rangle$	$\langle [s_2, s_3], [0.3, 0.6], 0.4 \rangle$	$\langle [s_3, s_4], [0.4, 0.8], 0.6 \rangle$	$\langle [s_1, s_2], [0.5, 0.9], 0.7 \rangle$

Table 2 Decision matrix H^2

C_1	C_2	C_3	C_4
$\langle [s_2, s_3], [0.4, 0.8], 0.6 \rangle$	$\langle [s_1, s_2], [0.7, 0.18], 0.9 \rangle$	$\langle [s_2, s_3], [0.5, 0.19], 0.7 \rangle$	$\langle [s_2, s_4], [0.1, 0.4], 0.3 \rangle$
$\langle [s_3, s_4], [0.5, 0.10], 0.7 \rangle$	$\langle [s_2, s_2], [0.2, 0.6], 0.4 \rangle$	$\langle [s_1, s_3], [0.1, 0.6], 0.5 \rangle$	$\langle [s_1, s_4], [0.3, 0.7], 0.4 \rangle$
$\langle [s_2, s_3], [0.9, 0.15], 0.13 \rangle$	$\langle [s_2, s_4], [0.4, 0.9], 0.6 \rangle$	$\langle [s_1, s_1], [0.2, 0.7], 0.5 \rangle$	$\langle [s_1, s_3], [0.2, 0.6], 0.4 \rangle$
$\langle [s_1, s_2], [0.2, 0.4], 0.3 \rangle$	$\langle [s_1, s_3], [0.3, 0.8], 0.4 \rangle$	$\langle [s_2, s_2], [0.4, 0.9], 0.6 \rangle$	$\langle [s_3, s_3], [0.4, 0.9], 0.5 \rangle$

Table 3 Decision matrix H^3

C_1	C_2	C_3	C_4
$\langle [s_1, s_4], [0.5, 0.7], 0.6 \rangle$	$\langle [s_1, s_4], [0.4, 0.9], 0.6 \rangle$	$\langle [s_2, s_4], [0.1, 0.9], 0.7 \rangle$	$\langle [s_1, s_4], [0.2, 0.9], 0.7 \rangle$
$\langle [s_1, s_2], [0.2, 0.10], 0.8 \rangle$	$\langle [s_3, s_3], [0.3, 0.8], 0.5 \rangle$	$\langle [s_1, s_3], [0.1, 0.7], 0.3 \rangle$	$\langle [s_1, s_4], [0.3, 0.7], 0.6 \rangle$
$\langle [s_1, s_4], [0.9, 0.13], 0.11 \rangle$	$\langle [s_2, s_4], [0.3, 0.8], 0.5 \rangle$	$\langle [s_1, s_1], [0.2, 0.9], 0.5 \rangle$	$\langle [s_3, s_3], [0.4, 0.6], 0.5 \rangle$
$\langle [s_2, s_3], [0.3, 0.9], 0.7 \rangle$	$\langle [s_1, s_3], [0.4, 0.8], 0.6 \rangle$	$\langle [s_1, s_2], [0.4, 0.6], 0.4 \rangle$	$\langle [s_3, s_3], [0.2, 0.9], 0.7 \rangle$

$$\sup(h_{1j}, h_{1j}) = \begin{pmatrix} 1.000 & 0.915 & 0.904 & 0.679 \\ 0.915 & 1 & 0.489 & 0.916 \\ 0.904 & 0.489 & 1 & 0.958 \\ 0.678 & 0.916 & 0.958 & 1 \end{pmatrix},$$

$$\sup(h_{2j}, h_{2j}) = \begin{pmatrix} 1.000 & 0.929 & 0.832 & 0.141 \\ 0.929 & 1 & 0.901 & 0.614 \\ 0.832 & 0.901 & 1 & 0.824 \\ 0.141 & 0.614 & 0.824 & 1 \end{pmatrix},$$

$$\sup(h_{3j}, h_{3j}) = \begin{pmatrix} 1.000 & 0.463 & 0.789 & 0.156 \\ 0.463 & 1 & 0.302 & 0.972 \\ 0.789 & 0.302 & 1 & 0.859 \\ 0.156 & 0.972 & 0.859 & 1 \end{pmatrix},$$

$$\sup(h_{4j}, h_{4j}) = \begin{pmatrix} 1.000 & 0.706 & 0.748 & 0.894 \\ 0.706 & 1 & 0.957 & 0.906 \\ 0.748 & 0.957 & 1 & 0.755 \\ 0.894 & 0.906 & 0.755 & 1 \end{pmatrix}.$$

Step 5 Calculate $T(h_{ij})(i, j = 1, 2, 3, 4)$.

$$T(h_{ij}) = \begin{pmatrix} 4.000 & 3.013 & 3.273 & 1.869 \\ 3.013 & 4.000 & 2.648 & 3.408 \\ 3.273 & 2.649 & 4.000 & 3.396 \\ 1.869 & 3.408 & 3.396 & 4.000 \end{pmatrix}.$$

Step 6 Aggregate cubic uncertain linguistic fuzzy numbers for each alternative by the CULFPEWA operator:

$$h_1 = \langle (s_{5.3164}, s_{5.6706}), [0.1236, 0.2095]; 0.8802 \rangle,$$
$$h_2 = \langle (s_{5.3094}, s_{5.4055}), [0.0858, 0.2334]; 0.6836 \rangle,$$
$$h_3 = \langle (s_{5.3193}, s_{5.4011}), [0.1993, 0.2157]; 0.7497 \rangle,$$
$$h_4 = \langle (s_{5.3213}, s_{5.3856}), [0.0908, 0.6370]; 0.8491 \rangle.$$

Step 7 Calculate the value $E(h_i)$ of h_i (Fig. 5).

$$E(h_1) = s_{0.0978};\ E(h_2) = s_{0.2632};\ E(h_3) = s_{0.2495};\ E(h_4) = s_{0.2265}.$$

Table 4 Decision matrix

C_1	C_2	C_3	C_4
$\langle [s_{3.0912}, s_{3.4375}], [0.1491, 0.3277], 0.8042 \rangle$	$\langle [s_{3.000}, s_{4.5134}], [0.1327, 0.2204], 0.9239 \rangle$	$\langle [s_{3.1795}, s_{3.2781}], [0.0995, 0.1761], 0.9925 \rangle$	$\langle [s_{3.0555}, s_{3.4662}], [0.0804, 0.3606], 0.8339 \rangle$
$\langle [s_{3.1075}, s_{3.3167}], [0.1251, 0.1697], 0.9109 \rangle$	$\langle [s_{3.1565}, s_{3.1565}], [0.0707, 0.2618], 0.8252 \rangle$	$\langle [s_{3.000}, s_{3.4205}], [0.0331, 0.2298], 0.7546 \rangle$	$\langle [s_{3.000}, s_{3.4856}], [0.0999, 0.2738], 0.8275 \rangle$
$\langle [s_{3.1026}, s_{3.484}], [0.4488, 0.1377], 0.5184 \rangle$	$\langle [s_{3.1606}, s_{3.3301}], [0.0866, 0.2698], 0.8955 \rangle$	$\langle [s_{3.1250}, s_{3.1250}], [0.0695, 0.2805], 0.8629 \rangle$	$\langle [s_{3.1148}, s_{3.2836}], [0.0886, 0.1924], 0.8213 \rangle$
$\langle [s_{3.1901}, s_{3.4305}], [0.0797, 0.3026], 0.7349 \rangle$	$\langle [s_{3.0562}, s_{3.2923}], [0.0886, 0.2421], 0.8493 \rangle$	$\langle [s_{3.1526}, s_{3.2371}], [0.1038, 0.2614], 0.8743 \rangle$	$\langle [s_{3.1598}, s_{3.2072}], [0.0829, 0.2997], 0.9194 \rangle$

Step 8 Rank $E(h_i)$ in descending order; we can get the best alternative.

$E(h_2) > E(h_3) > E(h_4) > E(h_1)$. A_2 is the best choice.

Step 9 End.

Ranking four candidate companies by the CULFPEWG operator

Step 1 ′ Calculate the supports and the result is same with Step 1.

$$\sup(h_{ij}^1, h_{ij}^2) = \begin{pmatrix} 0.7894 & 0.6148 & 0.5516 & 0.7053 \\ 0.8527 & 0.9579 & 0.9263 & 0.9579 \\ 0.7789 & 0.9631 & 0.8158 & 0.9824 \\ 0.9368 & 0.9263 & 0.9807 & 0.9579 \end{pmatrix},$$

$$\sup(h_{ij}^2, h_{ij}^3) = \begin{pmatrix} 0.9264 & 0.7222 & 0.7664 & 0.5789 \\ 0.8316 & 0.8246 & 0.9369 & 0.9338 \\ 0.9853 & 0.7474 & 0.9859 & 0.8737 \\ 0.6922 & 0.9369 & 0.8948 & 0.7474 \end{pmatrix},$$

$$\sup(h_{ij}^1, h_{ij}^3) = \begin{pmatrix} 0.8737 & 0.9719 & 0.8316 & 0.8246 \\ 0.7895 & 0.9438 & 0.9508 & 0.9561 \\ 0.9508 & 0.8878 & 0.8685 & 0.9473 \\ 0.4106 & 0.8422 & 0.7895 & 0.9210 \end{pmatrix}.$$

Step 2 Calculate $T(h_{ij}^{(q)})\ (i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$.

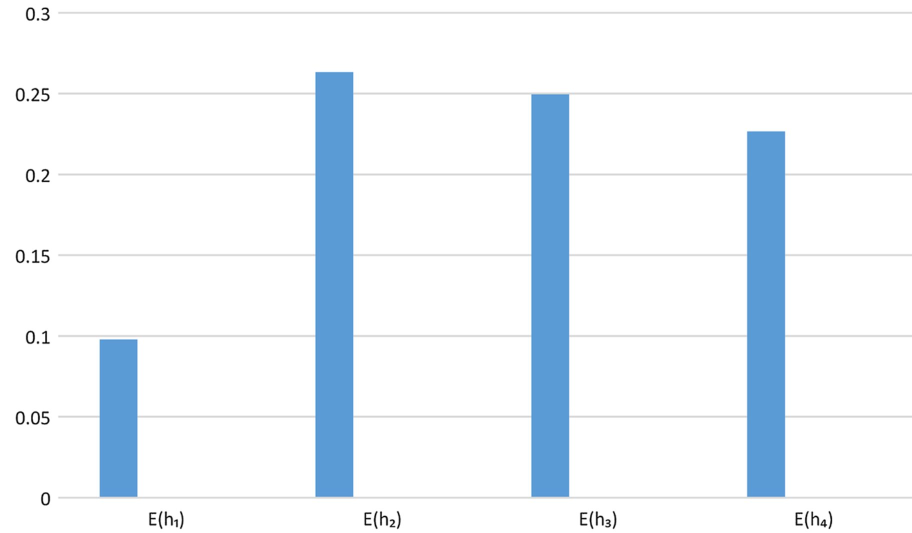


Fig. 5 $E(h_2)$ is the first expectation value, $E(h_3)$ is the second expectation value, $E(h_4)$ is the third expectation value, $E(h_1)$ is the 4th expectation value

$$T(h_{ij}^1) = \begin{pmatrix} 1.9894 & 2.4148 & 2.6516 & 2.5053 \\ 2.7189 & 2.3579 & 1.8263 & 2.4579 \\ 1.7189 & 2.7631 & 2.5158 & 2.2824 \\ 1.6368 & 2.2263 & 2.7807 & 3.0579 \end{pmatrix},$$

$$T(h_{ij}^2) = \begin{pmatrix} 2.7264 & 2.5022 & 2.1569 & 1.3789 \\ 2.1316 & 2.0246 & 2.1369 & 1.3338 \\ 2.1653 & 2.6474 & 2.3859 & 2.0737 \\ 1.5922 & 2.4369 & 2.7948 & 2.5474 \end{pmatrix},$$

$$T(h_{ij}^3) = \begin{pmatrix} 2.6737 & 2.7316 & 2.5316 & 2.6246 \\ 1.8895 & 2.5438 & 2.0508 & 2.5561 \\ 2.0908 & 2.4878 & 2.4685 & 2.4473 \\ 2.3106 & 2.6422 & 2.1895 & 2.721 \end{pmatrix}.$$

Table 5 Decision matrix

C_1	C_2	C_3	C_4
$\langle [s_{1.0912}, s_{1.5015}], [0.7156, 0.8704], 0.2275 \rangle$	$\langle [s_{1.000}, s_{3.2306}], [0.8650, 0.8886], 0.2206 \rangle$	$\langle [s_{1.1904}, s_{1.3035}], [0.7856, 0.8960], 0.2152 \rangle$	$\langle [s_{1.0555}, s_{1.5408}], [0.6857, 0.9226], 0.1882 \rangle$
$\langle [s_{1.1075}, s_{1.3493}], [0.6516, 0.7395], 0.2374 \rangle$	$\langle [s_{1.1622}, s_{1.1622}], [0.7275, 0.8977], 0.1280 \rangle$	$\langle [s_{1.000}, s_{1.4820}], [0.5479, 0.8604], 0.1292 \rangle$	$\langle [s_{1.000}, s_{1.5668}], [0.7261, 0.9000], 0.1834 \rangle$
$\langle [s_{1.1026}, s_{1.5659}], [0.9368, 0.5433], 0.0445 \rangle$	$\langle [s_{1.1693}, s_{1.3677}], [0.8324, 0.9620], 0.1452 \rangle$	$\langle [s_{1.1250}, s_{1.1250}], [0.7632, 0.9490], 0.1370 \rangle$	$\langle [s_{1.1148}, s_{1.3105}], [0.7581, 0.8815], 0.1307 \rangle$
$\langle [s_{1.1990}, s_{1.4919}], [0.6008, 0.8276], 0.1833 \rangle$	$\langle [s_{1.1056}, s_{1.3216}], [0.7981, 0.9311], 0.1310 \rangle$	$\langle [s_{1.1581}, s_{1.2549}], [0.8311, 0.9405], 0.1473 \rangle$	$\langle [s_{1.1598}, s_{1.2216}], [0.8386, 0.9789], 0.1597 \rangle$

Step 3' Utilize the CULFPEWG operator to aggregate all the three decision matrices mentioned above into the following decision matrix given in Table 5.

Step 4' Calculate the supports according to $(k;\ j = 1, 2, 3, 4)$.

$$\sup(h_{1j}, h_{1j}) = \begin{pmatrix} 1.000 & 0.915 & 0.904 & 0.679 \\ 0.915 & 1 & 0.489 & 0.916 \\ 0.904 & 0.489 & 1 & 0.958 \\ 0.678 & 0.916 & 0.958 & 1 \end{pmatrix},$$

$$\sup(h_{2j}, h_{2j}) = \begin{pmatrix} 1.000 & 0.929 & 0.832 & 0.141 \\ 0.929 & 1 & 0.901 & 0.614 \\ 0.832 & 0.901 & 1 & 0.824 \\ 0.141 & 0.614 & 0.824 & 1 \end{pmatrix},$$

$$\sup(h_{3j}, h_{3j}) = \begin{pmatrix} 1.000 & 0.463 & 0.789 & 0.156 \\ 0.463 & 1 & 0.302 & 0.972 \\ 0.789 & 0.302 & 1 & 0.859 \\ 0.156 & 0.972 & 0.859 & 1 \end{pmatrix},$$

$$\sup(h_{4j}, h_{4j}) = \begin{pmatrix} 1.000 & 0.706 & 0.748 & 0.894 \\ 0.706 & 1 & 0.957 & 0.906 \\ 0.748 & 0.957 & 1 & 0.755 \\ 0.894 & 0.906 & 0.755 & 1 \end{pmatrix}.$$

Step 5 Calculate $T(h_{ij})\ (i, j = 1, 2, 3, 4)$.

$$T(h_{ij}) = \begin{pmatrix} 4.000 & 3.013 & 3.273 & 1.869 \\ 3.013 & 4.000 & 2.648 & 3.408 \\ 3.273 & 2.649 & 4.000 & 3.396 \\ 1.869 & 3.408 & 3.396 & 4.000 \end{pmatrix}.$$

Step 6' Aggregate the cubic uncertain linguistic fuzzy numbers for each alternative by the CULFPEWG operator.

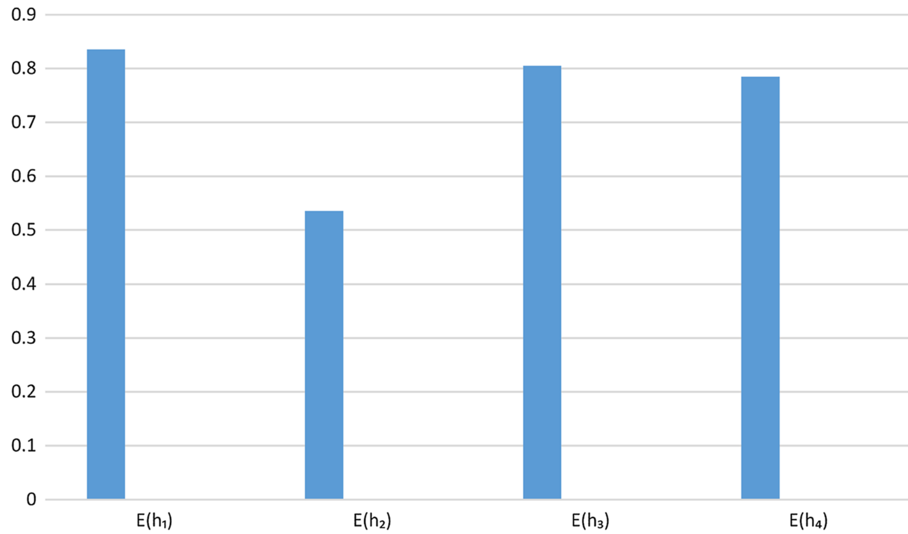


Fig. 6 $E(h_1)$ is the first expectation value, $E(h_3)$ is the second expectation value, $E(h_4)$ is the third expectation value, $E(h_1)$ is the 4th expectation value in the CULFPEWG operator

$$h_1 = \langle (s_{1.0791}, s_{1.8030}), [0.7657, 0.9006]; 0.2166 \rangle,$$
$$h_2 = \langle (s_{0.5636}, s_{1.3497}), [0.6743, 0.8472]; 0.1713 \rangle,$$
$$h_3 = \langle (s_{1.1261}, s_{1.3472}), [0.8367, 0.8065]; 0.1097 \rangle,$$
$$h_4 = \langle (s_{1.1503}, s_{1.3116}), [0.7725, 0.9241]; 0.1529 \rangle.$$

Step 7′ Calculate the value $E(h_i)$ of h_i (Fig. 6).

$E(h_1) = s_{0.8353}$; $E(h_2) = s_{0.5354}$; $E(h_3) = s_{0.8053}$; $E(h_4) = s_{0.7847}$.

Step 8′ Rank $E(h_i)$ in descending order; we can get the best alternative.

$E(h_1) > E(h_3) > E(h_4) > E(h_2)$. A_1 is the best choice.

Step 9 End.

Results and discussion

See Table 6.

Comparison analyses

In order to verify the validity and effectiveness of the proposed approach, a comparative study is conducted using the methods of intuitionistic linguistic fuzzy number [24] and intuitionistic uncertain linguistic fuzzy number [21], which are special cases of CULFNs, to the same illustrative example (Fig. 7).

A comparison analysis with the existing MCDM method intuitionistic linguistic fuzzy number

The intuitionistic linguistic fuzzy number can be considered as a special case of cubic uncertain linguistic fuzzy numbers (CULFNs) when there is the element in membership and non-membership degrees. For comparison, the intuitionistic linguistic fuzzy number (ITrFNs) can be transformed to CULFNs by calculating the average value of the membership and non-membership degrees [24]. After transformation, the intuitionistic linguistic fuzzy number (ILFNs) information is given in Table 7.

Step 1 Calculate the ILFWA operator and $w = (0.30, 0.28, 0.20, 0.22)^T$.

Step 2 Calculate the score function $E(h_1) = -s_{0.7165}$; $E(h_2) = -s_{1.7251}$; $E(h_3) = -s_{0.3632}$; $E(h_4) = -s_{0.9801}$.

Step 3 Find the ranking $E(h_2) > E(h_4) > E(h_1) > E(h_3)$ and $E(h_2)$ (Table 8).

A comparison analysis with the existing MCDM method intuitionistic uncertain linguistic fuzzy number

The intuitionistic uncertain linguistic fuzzy number can be considered as a special case of cubic uncertain linguistic fuzzy numbers (CULFNs) when there is the element in membership and non-membership degrees. For

Table 6 Results and discussion of above values

Score value	CULFPEWA operator	Ranking 1	CULFPEWG operator	Ranking 2	Final ranking
$E(h_1)$	$s_{0.0978}$	4	$s_{0.8353}$	1	1
$E(h_2)$	$s_{0.2632}$	1	$s_{0.5354}$	4	4
$E(h_3)$	$s_{0.2495}$	2	$s_{0.8053}$	2	2
$E(h_4)$	$s_{0.2265}$	3	$s_{0.7847}$	3	3

Fig. 7 Comparison analysis

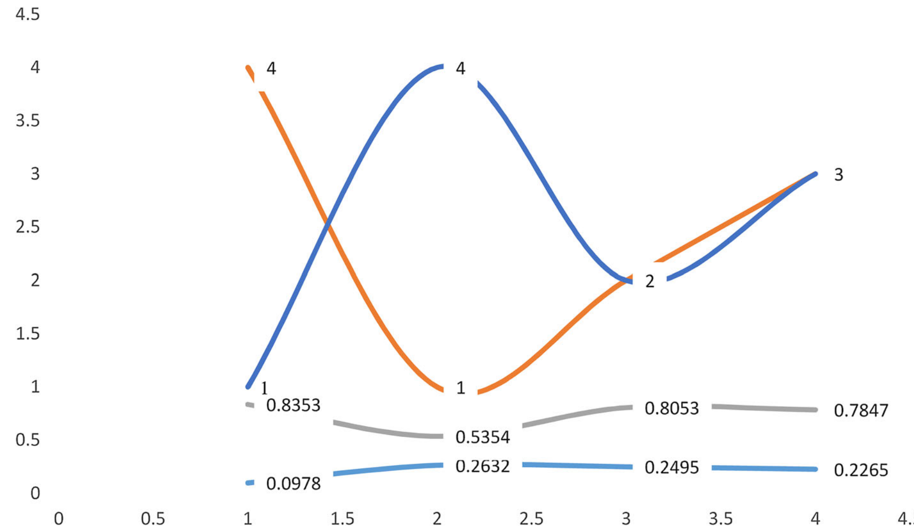


Table 7 Intuitionistic linguistic fuzzy decision matrix

C_1	C_2	C_3	C_4
$\langle s_1, [0.2, 0.6] \rangle$	$\langle s_2, [0.4, 0.8] \rangle$	$\langle s_2, [0.5, 0.9] \rangle$	$\langle s_4\ [0.4, 0.9] \rangle$
$\langle s_3, [0.4, 0.9] \rangle$	$\langle s_1, [0.2, 0.8] \rangle$	$\langle s_4, [0.1, 0.5] \rangle$	$\langle s_4, [0.3, 0.7] \rangle$
$\langle s_3, [0.7, 0.13] \rangle$	$\langle s_4, [0.4, 0.8] \rangle$	$\langle s_4, [0.4, 0.8] \rangle$	$\langle s_2, [0.3, 0.6] \rangle$
$\langle s_4, [0.1, 0.4] \rangle$	$\langle s_3, [0.3, 0.6] \rangle$	$\langle s_3, [0.4, 0.8] \rangle$	$\langle s_2, [0.5, 0.9] \rangle$

Table 8 ILFWA operator

$\langle s_{2.2973}, [0.4408, 0.7532] \rangle$
$\langle s_{2.9562}, [0.2845, 0.6798] \rangle$
$\langle s_{2.4914}, [0.4033, 0.5491] \rangle$
$\langle s_{2.5622}, [0.2971, 0.6796] \rangle$

comparison, the intuitionistic uncertain linguistic fuzzy number can be transformed to CULFNs by calculating the average value of the membership and non-membership degrees [21]. After transformation, the intuitionistic uncertain linguistic fuzzy number information is given in Table 9.

Step 1 Calculate the IULFWA operator and $w = (0.30, 0.28, 0.20, 0.22)^T$ (Table 10).

Step 2 Calculate the score function $E(h_1) = s_{1.8276}$; $E(h_2) = -s_{0.2169}$; $E(h_3) = -s_{0.1437}$; $E(h_4) = -s_{1.4547}$.

Step 3 Find the ranking $E(h_1) > E(h_4) > E(h_2) > E(h_3)$ and $E(h_1)$.

The ranking values of the above discussion are given in Table 11.

The following advantages of our proposal can be summarized on the basis of the above comparison analyses.

Table 9 Intuitionistic uncertain linguistic fuzzy decision matrix

C_1	C_2	C_3	C_4
$\langle [s_2, s_3], [0.4, 0.8] \rangle$	$\langle [s_1, s_2], [0.7, 0.18] \rangle$	$\langle [s_2, s_3], [0.5, 0.19] \rangle$	$\langle [s_2, s_4], [0.1, 0.4] \rangle$
$\langle [s_3, s_4], [0.5, 0.10] \rangle$	$\langle [s_2, s_2], [0.2, 0.6] \rangle$	$\langle [s_1, s_3], [0.1, 0.6] \rangle$	$\langle [s_1, s_4], [0.3, 0.7] \rangle$
$\langle [s_2, s_3], [0.9, 0.15] \rangle$	$\langle [s_2, s_4], [0.4, 0.9] \rangle$	$\langle [s_1, s_1], [0.2, 0.7] \rangle$	$\langle [s_1, s_3], [0.2, 0.6] \rangle$
$\langle [s_1, s_2], [0.2, 0.4] \rangle$	$\langle [s_1, s_3], [0.3, 0.8] \rangle$	$\langle [s_2, s_2], [0.4, 0.9] \rangle$	$\langle [s_3, s_3], [0.4, 0.9] \rangle$

Table 10 IULFWA operator

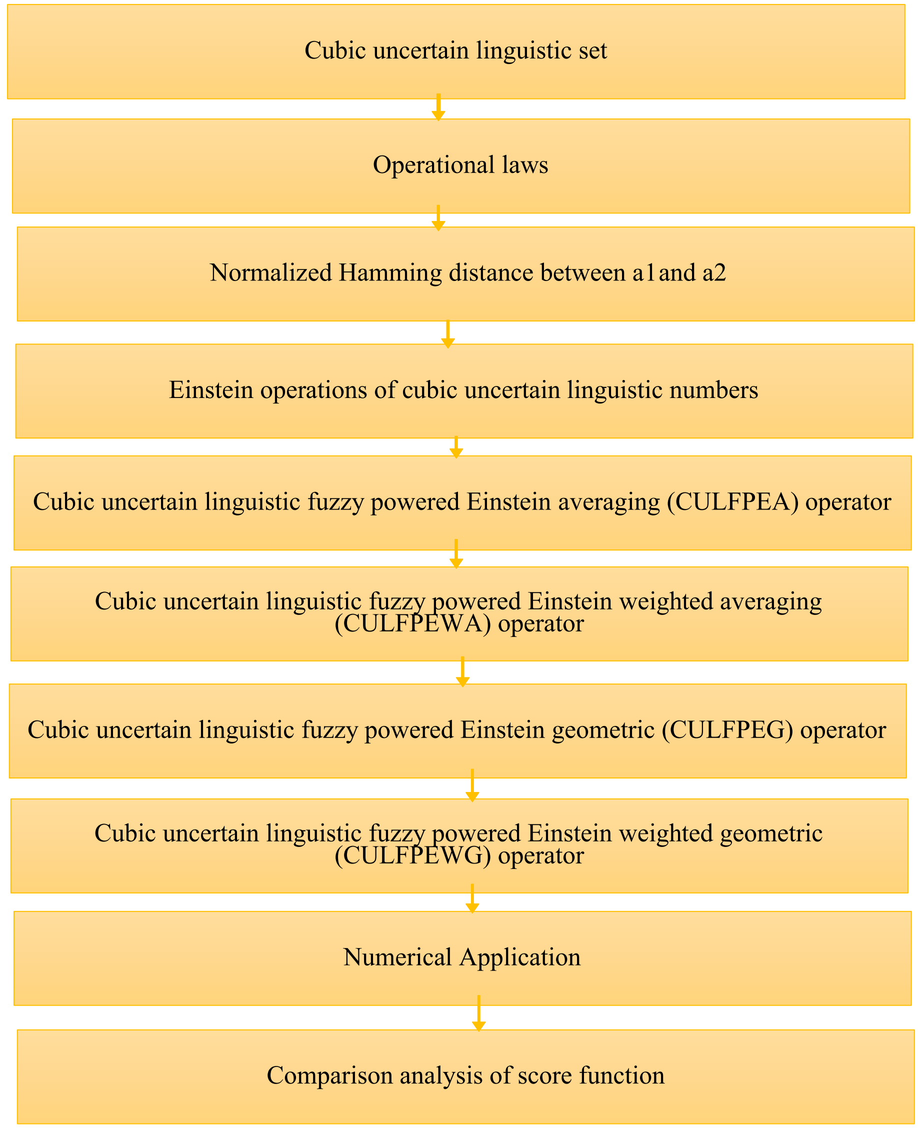
$\langle [s_{1.8661}, s_{3.6074}], [0.5295, 0.2580] \rangle$
$\langle [s_{1.6515}, s_{3.5895}], [0.3201, 0.3567] \rangle$
$\langle [s_{1.3195}, s_{1.2919}], [0.4789, 0.5632] \rangle$
$\langle [s_{1.4831}, s_{2.1998}], [0.2969, 0.7431] \rangle$

Cubic uncertain linguistic fuzzy number (CULFN) is very suitable for illustrating uncertain or fuzzy information in MCDM problems because the membership and non-membership degrees can be two sets of several possible values, which cannot be achieved by Intuitionistic linguistic fuzzy numbers (ILFNs) and intuitionistic uncertain linguistic

Table 11 The ranking values of the above all given methods

Method	Ranking
CULFPEWA operator	$E(h_2) > E(h_3) > E(h_4) > E(h_1)$
CULFPEWG operator	$E(h_1) > E(h_3) > E(h_4) > E(h_2)$
ILFWA operator [24]	$E(h_2) > E(h_4) > E(h_1) > E(h_3)$
IULFWA operator [21]	$E(h_1) > E(h_4) > E(h_2) > E(h_3)$

Fig. 8 Flowchart of whole paper



fuzzy numbers (IULFNs). On the bases of basic operations, aggregation operators and comparison method of cubic uncertain linguistic fuzzy number can be also used to process intuitionistic linguistic fuzzy numbers and intuitionistic uncertain linguistic fuzzy number after slight adjustments, because cubic uncertain linguistic fuzzy number (CULFNs) can be considered as the generalized form of intuitionistic linguistic fuzzy numbers (ILFNs) and intuitionistic uncertain linguistic fuzzy numbers (IULFNs). The defined operations of cubic uncertain linguistic fuzzy numbers (CULFNs) give us more accurate than the existing operators.

Conclusion

In this paper, we define the idea of cubic uncertain linguistic set and operational laws of cubic uncertain linguistic set. We initiated the concept of cubic uncertain linguistic set. The concept of cubic uncertain linguistic set is the generalization of cubic number, intuitionistic uncertain linguistic fuzzy numbers, cubic uncertain linguistic fuzzy set and interval-valued uncertain linguistic fuzzy numbers. We proposed a new decision method to solve the MCDM problems. The cubic uncertain linguistic numbers are a valuable instrument to convey the fuzzy information. This paper focuses on multi-attribute group decision-making (MAGDM) problems in which the attribute values are expressed by cubic uncertain linguistic numbers. The definition and some basic operations of cubic uncertain linguistic numbers, power aggregation (PA) operators and Einstein operations are introduced. Then, we apply the Einstein operations to the PA operators under cubic uncertain linguistic environment and put forward some new aggregation operators such as cubic uncertain linguistic fuzzy powered Einstein averaging operator, cubic uncertain linguistic fuzzy powered Einstein weighted averaging operator, cubic uncertain linguistic fuzzy Einstein geometric operator and cubic uncertain linguistic fuzzy Einstein weighted geometric operator. We also discuss some properties of them in detail. Further, we propose the decision making for MAGDM problems with cubic uncertain linguistic information and show the detail decision steps. In the future, we should try our best to use the proposed operators to extend the scope of application (Fig. 8).

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