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ORIGINAL RESEARCH

CAS wavelet quasi-linearization technique for the generalized Burger–Fisher equation

Umer Saeed[1] · Khadija Gilani[1]

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Abstract
In this article, we propose a method for the solution of the generalized Burger–Fisher equation. The method is developed using CAS wavelets in conjunction with quasi-linearization technique. The operational matrices for the CAS wavelets are derived and constructed. Error analysis and procedure of implementation of the method are provided. We compare the results produce by present method with some well known results and show that the present method is more accurate, efficient, and applicable.

Keywords CAS wavelets · Quasi-linearization · Operational matrices · Generalized Burger–Fisher equation

Introduction

The Burger–Fisher equation has important applications in various fields of financial mathematics, gas dynamic, traffic flow, applied mathematics, and physics. This equation shows a prototypical model for describing the interaction between the reaction mechanism, convection effect, and diffusion transport [1]. Consider the generalized Burger–Fisher equation:

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+au^{\gamma}\frac{\partial u}{\partial x}+bu(u^{\gamma}-1)=0, \quad 0\leq x\leq 1,\ t\geq 0, \tag{1}$$

subject to the initial and boundary conditions:

$$u(x,0)=L(x):=\left(\frac{1}{2}-\frac{1}{2}\tan h\left(\frac{a\gamma}{2(1+\gamma)}x\right)\right)^{\frac{1}{\gamma}},$$

$$u(0,t)=E(t):=\left(\frac{1}{2}-\frac{1}{2}\tan h\left(\frac{a\gamma}{2(1+\gamma)}\left[-\left(\frac{a^2+b(1+\gamma^2)}{a(1+\gamma)}\right)t\right]\right)\right)^{\frac{1}{\gamma}},$$

$$u(1,t)=F(t):=\left(\frac{1}{2}-\frac{1}{2}\tan h\left(\frac{a\gamma}{2(1+\gamma)}\left[1-\left(\frac{a^2+b(1+\gamma^2)}{a(1+\gamma)}\right)t\right]\right)\right)^{\frac{1}{\gamma}}.$$

The exact solution is given in Chen and Zhang [2]:

$$u(x,t)=\left(\frac{1}{2}-\frac{1}{2}\tan h\left(\frac{a\gamma}{2(1+\gamma)}\left[x-\left(\frac{a^2+b(1+\gamma^2)}{a(1+\gamma)}\right)t\right]\right)\right)^{\frac{1}{\gamma}}. \tag{2}$$

where a, b, and γ are non-zero parameters. Wavelet analysis is a new development in the area of applied mathematics. Wavelets are a special kind of functions which exhibits oscillatory behavior for a short period of time and then die out. In wavelets, we use a single function and its dilations and translations to generate a set of orthonormal basis functions to represent a function. We define wavelet (mother wavelet) by Radunovic [3]:

$$\psi_{a,b}(x)=\frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right), \qquad a,b\in \mathbf{R},\ a\neq 0, \tag{3}$$

where a and b are called scaling and translation parameter, respectively. If $|a|<1$, the wavelet (3) is the compressed version (smaller support in time domain) of the mother wavelet and corresponds to mainly higher frequencies. On the other hand, when $|a|>1$, the wavelet (3) has larger support in time domain and corresponds to lower frequencies.

Discretizing the parameters via $a=2^{-k}$ and $b=n2^{-k}$, we get the discrete wavelets transform as:

$$\psi_{k,n}(x)=2^{\frac{k}{2}}\psi(2^k x-n). \tag{4}$$

These wavelets for all integers k and n produce an orthogonal basis of $L_2(\mathbb{R})$. It is somewhat surprising that

✉ Umer Saeed
umer.math@gmail.com

Khadija Gilani
khadijagilani100@gmail.com

1 National University of Sciences and Technology (NUST), Sector H-12, Islamabad, Pakistan

among different solution techniques, CAS wavelets method have received rather less attention. We have found some papers [4–7] in which CAS method is used for the solution of integro-differential equations, and CAS wavelets is not implemented for the solution of nonlinear Lane Emden-type equation. In Yi et al. [4], CAS wavelet method is utilized for the solution of integro-differential equations with a weakly singular kernel of fractional order. In addition, error analysis of the CAS wavelets is provided. The CAS wavelets operational matrices are implemented for the numerical solution of nonlinear Volterra integro-differential equations of arbitrary order in Saeedi et al. [5]. CAS wavelet approximation method is presented for the solution of Fredholm integral equations in Yousefi and Banifatemi [6]. The operational matrices are utilized to convert the Fredholm integral equation into a system of algebraic equations. In Shamooshaky et al. [7], authors presented a CAS wavelet method for solving boundary integral equations with logarithmic singular kernels which occur as reformulations of a boundary value problem for Laplace's equation.

The purpose of this article is to propose a numerical method for solving the generalized Burger–Fisher equation using CAS wavelets in conjunction with quasi-linearization technique. The properties of quasi-linearization technique are used to discretize the nonlinear partial differential equation and then utilize the properties of CAS wavelets to convert the obtained discrete partial differential equation into a Sylvester system. The solution of the obtained system provides the values of CAS wavelets coefficients which lead to the solution of the generalized Burger–Fisher equation.

CAS wavelets

The CAS wavelets are defined on interval [0, 1) by Yousefi and Banifatemi [6]

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} CAS_m(2^k x - n + 1), & \dfrac{n-1}{2^k} \le x < \dfrac{n}{2^k}, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\mathrm{CAS}_m(x) = \cos(2m\pi x) + \sin(2m\pi x)$ and $k = 0, 1, 2, \cdots$, is the level of resolution, $n = 0, 1, 2, \cdots, 2^k - 1$, is the translation parameter, $m \in \mathbf{Z}$.

CAS wavelets have compact support, that is

$$\mathrm{Supp}(\psi_{n,m}(x)) = \overline{\{x : \psi_{n,m}(x) \neq 0\}} = \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right].$$

Function approximations

We can expand any function $y(x) \in L^2[0,1)$ into truncated CAS wavelet series as:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \sum_{m \in \mathbf{Z}} c_{n,m} \psi_{n,m}(x) \\ &\approx \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m} \psi_{n,m}(x) = \mathbf{C}^T \mathbf{\Psi}(x), \end{aligned} \tag{5}$$

where $\mathbf{C}$ and $\mathbf{\Psi}(x)$ are $\hat{m} \times 1$, ($\hat{m} = 2^k(2M+1)$), matrices, given by: $\mathbf{C} = [c_{0,-M}, c_{0,-M+1}, \ldots, c_{0,M}, c_{1,-M}, c_{1,-M+1}, \ldots, c_{1,M}, \ldots, c_{2^k-1,-M}, c_{2^k-1,-M+1}, \ldots, c_{2^k-1,M}]^T$,

$\mathbf{\Psi}(x) = [\psi_{0,-M}(x), \psi_{0,-M+1}(x), \ldots, \psi_{0,M}(x), \psi_{1,-M}(x), \psi_{1,-M+1}(x), \ldots, \psi_{1,M}(x), \ldots, \psi_{2^k-1,-M}(x), \psi_{2^k-1,-M+1}(x), \ldots, \psi_{2^k-1,M}(x)]^T$. Any function of two variables $u(x,t) \in L_2[0,1) \times [0,1)$ can be approximated as:

$$u(x,t) \approx \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t). \tag{6}$$

The collocation points for the CAS wavelets are taken as $x_i = \frac{2i-1}{2\hat{m}}$, where $i = 1, 2, \ldots, \hat{m}$. The CAS wavelets matrix $\mathbf{\Psi}_{\hat{m},\hat{m}}$ is given as:

$$\mathbf{\Psi}_{\hat{m}\times\hat{m}} = \left[\mathbf{\Psi}\left(\frac{1}{2\hat{m}}\right), \mathbf{\Psi}\left(\frac{3}{2\hat{m}}\right), \ldots, \mathbf{\Psi}\left(\frac{2\hat{m}-1}{2\hat{m}}\right).\right] \tag{7}$$

In particular, we fix $k = 2$, $M = 1$, we have $n = 0, 1, 2, 3$; $m = -1, 0, 1$ and $i = 1, 2, \ldots, 12$, the CAS wavelets matrix is given as:

$$\mathbf{\Psi}_{12\times12} = \begin{pmatrix} -0.7321 & -2.0000 & 2.7321 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.7321 & -2.0000 & 2.7321 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.7321 & -2.0000 & 2.7321 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.7321 & -2.0000 & 2.7321 \\ 2.0000 & 2.0000 & 2.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.0000 & 2.0000 & 2.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0000 & 2.0000 & 2.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.0000 & 2.0000 & 2.0000 \\ 2.7321 & -2.0000 & -0.7321 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.7321 & -2.0000 & -0.7321 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.7321 & -2.0000 & -0.7321 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.7321 & -2.0000 & -0.7321 \end{pmatrix}$$

The CAS wavelets operational matrix of integration

For simplicity, we write (5) as:

$$y(x) \approx \sum_{l=1}^{\hat{m}} c_l \psi_l(x) = \mathbf{C}^T \mathbf{\Psi}(x), \tag{8}$$

where $c_l = c_{m,n}$, $\psi_l = \psi_{m,n}(x)$. The index l is determined by the equation $l = M(2n+1) + (n+m+1)$ and $\hat{m} = 2^k(2M+1)$. In addition, $\mathbf{C} = [\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_{\hat{m}}}]^{\mathbf{T}}$, $\mathbf{\Psi}(x) = [\psi_1(x), \psi_2(x), \ldots, \psi_{\hat{m}}(x)]^T$. Equation (6) can be written as:

$$u(x,t) \approx \sum_{l=1}^{\hat{m}} \sum_{p=1}^{\hat{m'}} c_{l,p} \psi_l(x) \psi_p(t) = \mathbf{\Psi}^T(x) \mathbf{C} \mathbf{\Psi}(t),$$

where $\mathbf{C}$ is $\hat{m} \times \hat{m'}$ coefficient matrix and its entries are $c_{l,p} = < \psi_l(x), < u(x,t), \psi_p(t) > >$. The index l and p are determined by the equations $l = M(2n+1) + (n+m+1)$ and $p = M'(2i+1) + (i+j+1)$, respectively. In addition, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

An arbitrary function $u(x,t) \in L_2[0,1) \times [0,1)$, can be expanded into block-pulse functions [8] as:

$$u(x,t) \approx \sum_{i=0}^{\hat{m}-1} \sum_{j=0}^{\hat{m}-1} a_{i,j} b_i(x) b_j(t) = \mathbf{B}^{\mathbf{T}}(\mathbf{x}) \mathbf{a} \mathbf{B}(\mathbf{t}),$$

where $a_{i,j}$ are the coefficients of the block-pulse functions b_i and b_j. The CAS wavelets can be expanded into $\hat{m}$—set of block-pulse functions as:

$$\mathbf{\Psi}(x) = \mathbf{\Psi}_{\hat{m} \times \hat{m}} \mathbf{B}(\mathbf{x}). \tag{9}$$

The qth integral of block-pulse function can be written as:

$$(\mathcal{I}_x^q \mathbf{B})(\mathbf{x}) = \mathbf{F}^{\mathbf{q}}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}} \mathbf{B}(\mathbf{x}), \tag{10}$$

where $q > 0$ and $\mathbf{F}^{\mathbf{q}}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}}$ is given in Kilicman and Al Zhour [8] with

$$\mathbf{P}^{\mathbf{q}}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}} = \mathbf{\Psi}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}} \mathbf{F}^{\mathbf{q}} (\mathbf{\Psi}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}})^{-\mathbf{1}}. \tag{11}$$

The CAS wavelets operational matrix of integration $\mathbf{P}^{\mathbf{q}}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}}$ of integer order q are utilize for solving differential equations.

In particular, for $k = 2$, $M = 1$, $q = 2$, the CAS wavelet operational matrix of integration $\mathbf{P}^{\mathbf{2}}_{\mathbf{12} \times \mathbf{12}}$ is given by:

$$\mathbf{P}^2_{12\times 12} = \begin{pmatrix}
-0.000847281 & -0.0073985 & -0.0208942 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 \\
0 & 0 & 0 & -0.000847281 & -0.0073985 & -0.0208942 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 & -0.0240563 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.000847281 & -0.0073985 & -0.0208942 & -0.0240563 & -0.0240563 & -0.0240563 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.000847281 & -0.0073985 & -0.0208942 \\
0.00231481 & 0.0162037 & 0.0439815 & 0.0833333 & 0.125 & 0.166667 & 0.208333 & 0.25 & 0.291667 & 0.333333 & 0.375 & 0.416667 \\
0 & 0 & 0 & 0.00231481 & 0.0162037 & 0.0439815 & 0.0833333 & 0.125 & 0.166667 & 0.208333 & 0.25 & 0.291667 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.00231481 & 0.0162037 & 0.0439815 & 0.0833333 & 0.125 & 0.166667 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00231481 & 0.0162037 & 0.0439815 \\
0.0031621 & 0.0166578 & 0.023209 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 \\
0 & 0 & 0 & 0.0031621 & 0.0166578 & 0.023209 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 & 0.0240563 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0031621 & 0.0166578 & 0.023209 & 0.0240563 & 0.0240563 & 0.0240563 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0031621 & 0.0166578 & 0.023209
\end{pmatrix}$$

This phenomena makes calculations fast, because the operational matrices $\mathbf{\Psi}_{\hat{m} \times \hat{m}}$ and $\mathbf{P}^{\mathbf{q}}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}}$ contains many zero entries.

CAS wavelets operational matrix of integration for boundary value problems

We need another operational matrix of fractional integration while solving boundary value problems. In this subsection, we drive an operational matrix of integration for dealing with the boundary conditions while solving boundary value problem. Let $g(x) \in L_2[0,1]$ be a given function, then

$$g(x) I^q_{x=1} \psi_{n,m}(x) = \frac{g(x)}{\Gamma(q)} \int_0^1 (1-s)^{q-1} \psi_{n,m}(s) \mathrm{d}s. \tag{12}$$

Since the CAS wavelets are supported on the intervals $\left[\frac{n-1}{2^k}, \frac{n}{2^k}\right)$, therefore

$$\begin{aligned} g(x) I^q_{x=1} \psi_{n,m}(x) &= \frac{g(x) 2^{\frac{k}{2}}}{\Gamma(q)} \int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} (1-s)^{q-1} \mathrm{CAS}_m(2^k s - n + 1) \mathrm{d}s, \\ &= g(x) Q^q_{n,m}, \end{aligned} \tag{13}$$

where $Q^q_{n,m} = \frac{2^{\frac{k}{2}}}{\Gamma(q)} \int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} (1-s)^{q-1} CAS_m(2^k s - n + 1) \mathrm{d}s$.

Expand the Eq. (13) at the collocation points, $x_i = \frac{2i-1}{2\hat{m}}$, where $i = 1, 2, \ldots, \hat{m}$, to obtain

$$\boldsymbol{W}^{g,q}_{\hat{\mathbf{m}} \times \hat{\mathbf{m}}} = \boldsymbol{Q}^{\boldsymbol{q}}_{\hat{\mathbf{m}} \times \boldsymbol{1}} \boldsymbol{G}_{\boldsymbol{1} \times \hat{\mathbf{m}}}, \tag{14}$$

where $\boldsymbol{G}_{\boldsymbol{1} \times \hat{\mathbf{m}}} = [g(x_1), g(x_2), \ldots, g(x_{\hat{m}})]$, $\boldsymbol{Q}^{\boldsymbol{q}}_{\hat{\mathbf{m}} \times \boldsymbol{1}} = [Q^q_{0,-M}, Q^q_{0,-M+1}, \cdots, Q^q_{0,M}, Q^q_{1,-M}, Q^q_{1,-M+1}, \cdots, Q^q_{1,M}, \cdots, Q^q_{2^k-1,-M}, Q^q_{2^k-1,-M+1}, \cdots, Q^q_{2^k-1,M}]^T$. In particular, for $k = 2$, $M = 1$, $q = 2$, and $\mathrm{g(x) = x^2 \sin(x)}$, we have

$$\mathbf{W}^{g,2}_{12\times 12}=\begin{pmatrix}
-0.0000014387 & -0.0000387551 & -0.000178591 & -0.000486649 & -0.0010247 & -0.0018491 & -0.00300938 & -0.00454693 & -0.0064938 & -0.00887167 & -0.0116909 & -0.01495\\
-0.00000143865 & -0.0000387537 & -0.000178585 & -0.000486631 & -0.00102466 & -0.00184904 & -0.00300927 & -0.00454676 & -0.00649356 & -0.00887134 & -0.0116905 & -0.0149495\\
-0.0000014387 & -0.0000387551 & -0.000178591 & -0.000486648 & -0.0010247 & -0.0018491 & -0.00300938 & -0.00454693 & -0.0064938 & -0.00887166 & -0.0116909 & -0.01495\\
-0.00000143875 & -0.0000387565 & -0.000178598 & -0.000486666 & -0.00102474 & -0.00184917 & -0.00300949 & -0.00454709 & -0.00649403 & -0.00887198 & -0.0116914 & -0.0149505\\
0.0000316387 & 0.000852269 & 0.00392743 & 0.010702 & 0.0225343 & 0.0406639 & 0.0661798 & 0.0999922 & 0.142806 & 0.195098 & 0.257097 & 0.328768\\
0.0000225991 & 0.000608763 & 0.00280531 & 0.00764426 & 0.016096 & 0.0290457 & 0.0472713 & 0.071423 & 0.102004 & 0.139356 & 0.183641 & 0.234834\\
0.0000135594 & 0.000365258 & 0.00168318 & 0.00458655 & 0.00965758 & 0.0174274 & 0.0283628 & 0.0428538 & 0.0612026 & 0.0836135 & 0.110184 & 0.1409\\
0.00000451981 & 0.000121753 & 0.000561061 & 0.00152885 & 0.00321919 & 0.00580913 & 0.00945426 & 0.0142846 & 0.0204009 & 0.0278712 & 0.0367282 & 0.0469668\\
0.0000014387 & 0.0000387551 & 0.000178591 & 0.000486649 & 0.0010247 & 0.0018491 & 0.00300938 & 0.00454693 & 0.0064938 & 0.00887167 & 0.0116909 & 0.01495\\
0.0000014387 & 0.0000387551 & 0.000178591 & 0.000486649 & 0.0010247 & 0.0018491 & 0.00300938 & 0.00454693 & 0.0064938 & 0.00887167 & 0.0116909 & 0.01495\\
0.0000014387 & 0.0000387551 & 0.000178591 & 0.000486649 & 0.0010247 & 0.0018491 & 0.00300938 & 0.00454693 & 0.0064938 & 0.00887167 & 0.0116909 & 0.01495\\
0.0000014387 & 0.0000387551 & 0.000178591 & 0.000486649 & 0.0010247 & 0.0018491 & 0.00300938 & 0.00454693 & 0.0064938 & 0.00887167 & 0.0116909 & 0.01495
\end{pmatrix}$$

Quasi-linearization [9]

The quasi-linearization approach is a generalized Newton–Raphson technique for functional equations. It converges quadratically to the exact solution, if there is convergence at all, and it has monotone convergence.

Quasi-linearization for the nonlinear partial differential equations is as follows. Given the problem of the form:

$$\frac{\partial u}{\partial t}=u_{xx}+g(u,u_x),\qquad 0<x<1,\ t\geq 0, \tag{15}$$

with the initial condition

$$u(x,0)=h(x),$$

and boundary conditions of the form:

$$u(0,t)=f_1(t),\ u(1,t)=f_2(t),$$

where g is the nonlinear function of u and u_x. Quasi-linearization approach for Eq. (15) implies:

$$\frac{\partial u_{r+1}}{\partial t}=(u_{r+1})_{xx}+g(u_r,(u_r)_x)+(u_{r+1}-u_r)g_u(u_r,(u_r)_x)+\\ ((u_{r+1})_x-(u_r)_x)g_{u_x}(u_r,(u_r)_x),\qquad r\geq 0, \tag{16}$$

with the initial and boundary conditions of the form:

$$u_{r+1}(x,0)=h(x),\qquad 0<x<1,$$

$$u_{r+1}(0,t)=f_1(t),\ u_{r+1}(1,t)=f_2(t),\qquad t\geq 0.$$

Starting with an initial approximation $u_0(x,t)$, we have a linear equation for each $u_{r+1}, r\geq 0$.

Procedure of implementation

In this section, the procedure of implementing the method for nonlinear partial differential equation is explained. The procedure begins with the conversion of nonlinear partial differential equation into discretize form by quasi-linearization technique, explained in Sect. 3. Next the discretized nonlinear partial differential equation is solved by CAS wavelet operational matrix method.

Consider the following discretized nonlinear partial differential equation:

$$\frac{\partial^2 u_{r+1}}{\partial t^2}-a(x)\frac{\partial^2 u_{r+1}}{\partial t^2}+b(x)\frac{\partial u_{r+1}}{\partial x}+d(x)u_{r+1}=f(x,t),\qquad r>0, \tag{17}$$

with initial and boundary conditions as

$$u_{r+1}(x,0)=g_1(x),\quad \frac{\partial u_{r+1}}{\partial t}(x,0)=g_2(x),\quad u_{r+1}(0,t)\\ =h_1(t),\quad u_{r+1}(1,t)=h_2(t).$$

Approximate the highest order term by CAS wavelet quasi-linearization method as:

$$\frac{\partial^2 u_{r+1}}{\partial x^2}=\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'}c^{r+1}_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t)\\ =\Psi^T(x)C^{r+1}\Psi(t).$$

Applying the integral operator on above equation, we have

$$\frac{\partial u_{r+1}}{\partial x}=(I^1_x\Psi^T(x))C^{r+1}\Psi(t)+p(t)$$

$$u_{r+1}(x,t)=(I^2_x\Psi^T(x))C^{r+1}\Psi(t)+p(t)x+q(t),$$

where $p(t)$ and $q(t)$ are

$$p(t)=h_2(t)-h_1(t)-(I^2_{x=1}\Psi^T(x))C^{r+1}\Psi(t)$$

$$q(t)=h_1(t).$$

By putting the values of $p(t)$ and $q(t)$ in $u_{r+1}(x,t)$, we get

$$u_{r+1}(x,t)=(I^2_x\Psi^T(x))C^{r+1}\Psi(t)+(h_2(t)-h_1(t))x\\ -(I^2_{x=1}\Psi^T(x))C^{r+1}\Psi(t)x+h_1(t). \tag{18}$$

Equation (17) implies that

$$\frac{\partial^2 u_{r+1}}{\partial t^2}-a(x)\Psi^T(x)C^{r+1}\Psi(t)+b(x)(I^1_x\Psi^T(x))C^{r+1}\Psi(t)\\ +b(x)(h_2(t)-h_1(t))-b(x)(I^2_{x=1}\Psi^T(x))C^{r+1}\Psi(t)\\ +d(x)(I^2_x\Psi^T(x))C^{r+1}\Psi(t)+d(x)(h_2(t)-h_1(t))x\\ -d(x)(I^2_{x=1}\Psi^T(x))C^{r+1}\Psi(t)x+d(x)h_1(t)=f(x,t).$$

We make substitution as: $G=f(x,t)-d(x)h_1(t)-d(x)(h_2(t)-h_1(t))x-b(x)(h_2(t)-h_1(t))$ and $G=\Psi^T(x)M\Psi(t)$ for simplification and get

$$\frac{\partial^2 u_{r+1}}{\partial t^2} - a(x)\Psi^T(x)C^{r+1}\Psi(t) + b(x)(I_x^1\Psi^T(x))C^{r+1}\Psi(t) - b(x)(I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) + d(x)(I_x^2\Psi^T(x))C^{r+1}\Psi(t) - d(x)(I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t)x = \Psi^T(x)M\Psi(t).$$

Apply second-order integral on above equation to get

$$u_{r+1}(x,t) = a(x)\Psi^T(x)C^{r+1}(I_t^2\Psi(t)) - b(x)(I_x^1\Psi^T(x))C^{r+1}I_t^2\Psi(t) + b(x)(I_{x=1}^2\Psi^T(x))C^{r+1}(I_t^2\Psi(t)) - d(x)(I_x^2\Psi^T(x))C^{r+1}(I_t^2\Psi(t)) + d(x)(I_{x=1}^2\Psi^T(x))C^{r+1}(I_t^2\Psi(t))x + \Psi^T(x)MI_t^2\Psi(t) + g_2(x)t + g_1(x). \tag{19}$$

Now, by equating Eqs. (18) and (19) and simplification, it is

$$(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))(C^{r+1}) - ((a(x)\Psi^T(x) - b(x)I_x^1\Psi^T(x) + b(x)I_{x=1}^2\Psi^T(x) - d(x)I_x^2\Psi^T(x) + d(x)I_{x=1}^2\Psi^T(x)x)) (C^{r+1}I_t^2\Psi(t))(\Psi(t)^{-1}) = (\Psi^T(x)MI_t^2\Psi(t) + g_2(x)t + g_1(x) - (h_2(t) - h_1(t))x - h_1(t))(\Psi(t)^{-1}).$$

For simplification let $k(x,t) = g_2(x)t + g_1(x) - (h_2(t) - h_1(t))x - h_1(t)$ above equation at collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m'}}$, where $i = 1,2,3,\ldots,\hat{m}$, $j = 1,2,3,\ldots,\hat{m'}$ $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$(I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i))(C^{r+1}) - (a(x_i)\Psi^T(x_i) - b(x_i)I_x^1\Psi^T(x_i) + b(x_i)I_{x=1}^2\Psi^T(x_i) - d(x_i)I_x^2\Psi^T(x_i) + d(x_i)x_iI_{x=1}^2\Psi^T(x_i)))C^{r+1}I_t^2\Psi(t_j)(\Psi(t_j)^{-1}) = (\Psi^T(x_i)MI_t^2\Psi(t_j) + k(x_i,t_j))(\Psi(t_j)^{-1}),$$

which can be written in matrix form as:

$$(P^{2,x}_{\hat{m}\times\hat{m'}} - W^{x,2}_{\hat{m}\times\hat{m'}})(C^{r+1}) - ((A)\Psi^T - BP^{1,x}_{\hat{m}\times\hat{m'}} + BW^{1,2}_{\hat{m}\times\hat{m'}} - DP^{2,x}_{\hat{m}\times\hat{m'}} + DW^{x,2}_{\hat{m}\times\hat{m'}})C^{r+1}P^{2,t}_{\hat{m}\times\hat{m'}}(\Psi^{-1}) = (\Psi^TMP^{2,x}_{\hat{m}\times\hat{m'}} + K)(\Psi^{-1}). \tag{20}$$

After simplification, we obtain the sylvester equation:

$$\mathbf{vQC}^{r+1} - \mathbf{C}^{r+1}\mathbf{R} = \mathbf{vS}, \tag{21}$$

where $\mathbf{v} = ((\mathbf{A})\Psi^T - \mathbf{BP}^{1,x}_{\hat{m}\times\hat{m'}} + \mathbf{BW}^{1,2}_{\hat{m}\times\hat{m'}} - \mathbf{DP}^{2,x}_{\hat{m}\times\hat{m'}} + \mathbf{DW}^{x,2}_{\hat{m}\times\hat{m'}})^{-1}$,

$\mathbf{Q} = (\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})$,

$\mathbf{R} = \mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$ and

$\mathbf{S} = (\Psi^T\mathbf{MP}^{2,x}_{\hat{m}\times\hat{m'}} + \mathbf{K})(\Psi^{-1})$,

and, **A**, **B** and **D** are diagonal matrices, which are given by:

$$\mathbf{A} = \begin{pmatrix} a(x_1) & 0 & \cdots & 0 \\ 0 & a(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a(x_j) \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} b(x_1) & 0 & \cdots & 0 \\ 0 & b(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b(x_j) \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} d(x_1) & 0 & \cdots & 0 \\ 0 & d(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d(x_j) \end{pmatrix}.$$

The matrix **K** is defined as

$$\mathbf{K} = \begin{pmatrix} k(x_1,y_1) & k(x_1,y_2) & \cdots & k(x_1,y_n) \\ k(x_2,y_1) & k(x_2,y_2) & \cdots & k(x_2,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ k(x_n,y_1) & k(x_n,y_2) & \cdots & k(x_n,y_n) \end{pmatrix}.$$

From Eq. (21), we get $\mathbf{C}^{r+1}$ which is used in Eq. (18) to get the solution u_{r+1} at the collocation points.

Error analysis

Lemma *If the CAS wavelets expansion of a continuous function $u_{r+1}(x,t)$ converges uniformly, then the CAS wavelets expansion converges to the function $u_{r+1}(x,t)$.*

Proof Let

$$v_{r+1}(x,t) = \sum_{i=0}^{\infty}\sum_{j\epsilon Z}\sum_{m=0}^{\infty}\sum_{n\epsilon Z} c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t). \tag{22}$$

Multiply both sides of Eq. (22) by $\psi_{p,q}(t)$ and $\psi_{r,s}(x)$, then integrating from 0 to 1 with respect to x as well as t, we obtain (23) using orthonormality of CAS wavelet:

$$\int_0^1\int_0^1 v_{r+1}(x,t)\psi_{p,q}(t)\psi_{r,s}(x)\mathrm{d}t\mathrm{d}x = c_{pq,rs}^{r+1}. \tag{23}$$

Thus, $c_{pq,rs}^{r+1} = \langle\langle v_{r+1}(x,t),\psi_{p,q}(x)\rangle,\psi_{r,s}(t)\rangle$ for $p,r\ \epsilon\ N$, $q,s\ \epsilon\ Z$. This implies that $u_{r+1}(x,t) = v_{r+1}(x,t)$.

Theorem Assume that $u_{r+1}(x,t)\epsilon L^2([0,1]\times[0,1])$ is a differentiable function with bounded partial derivative on $([0,1]\times[0,1])$ that is $\exists\ \gamma>0;\forall\ (x,t)\ \epsilon\ ([0,1]\times[0,1]):$ $|\frac{\partial^4 u_{r+1}}{\partial^2 x\partial^2 t}|\le\gamma$. The function $u_{r+1}(x,t)$ is expanded as an infinite sum of the CAS wavelets and the series converges uniformly to $u_{r+1}(x,t)$, that is $u_{r+1}(x,t)=\sum_{n=0}^{\infty}\sum_{m\epsilon Z}\sum_{i=0}^{\infty}\sum_{j\epsilon Z} c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t)$. Furthermore, $u_{r+1}^{k,k',M,M'}(x,t)=\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t)$, we have

$$|u_{r+1}^{k,k',M,M'}-u_{r+1}(x,t)|\le\frac{\gamma}{\pi^4}\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\sum_{i=2^k}^{\infty}\sum_{j=M'+1}^{\infty}\frac{1}{(mj)^2(n+1)^{\frac{5}{2}}(i+1)^{\frac{5}{2}}},$$

$u_{r+1}^{k,k',M,M'}$ converges to $u_{r+1}(x,t)$ as k,k',M and $M'\to\infty$ and $u_{r+1}(x,t)$ converges to $u(x,\,t)$ as $r\to\infty$.

Proof Since $u_{r+1}^{k,k',M,M'}(x,t)=\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t)$ and

$$c_{nm,ij}^{r+1}=\int_0^1\int_0^1 u_{r+1}(x,t)\psi_{n,m}(x)\psi_{i,j}(t)\mathrm{d}x\mathrm{d}t$$

$$=\int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}}\int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} 2^{\frac{k}{2}}2^{\frac{k'}{2}}u_{r+1}(x,t)\mathrm{CAS}_m(2^kx-n+1)\,CAS_j(2^{k'}t-n+1)\mathrm{d}x\mathrm{d}t.$$

Let $2^kx-n+1=p$ and $2^{k'}t-i+1=q$ then we have

$$c_{nm,ij}^{r+1}=\int_0^1\int_0^1\frac{1}{2^{\frac{k}{2}}2^{\frac{k'}{2}}}u\left(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}}\right)\mathrm{CAS}_m(p)\mathrm{CAS_j}(q)\mathrm{d}p\mathrm{d}q,$$

$$c_{nm,ij}^{r+1}=\int_0^1\int_0^1\frac{1}{2^{\frac{k}{2}}2^{\frac{k'}{2}}}u\left(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}}\right)(\cos(2m\pi p)+\sin(2m\pi p))(\cos(2j\pi q)+\sin(2j\pi q))\mathrm{d}p\mathrm{d}q.$$

Use integration with respect to p to get

$$c_{nm,ij}^{r+1}=-\frac{1}{2m\pi 2^{\frac{3k}{2}}2^{\frac{k'}{2}}}\int_0^1\int_0^1\frac{\partial u}{\partial p}\left(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}}\right)(\sin(2m\pi p)-\cos(2m\pi p))(\cos(2j\pi q)+\sin(2j\pi q))\mathrm{d}p\mathrm{d}q.$$

Now, applying integration with respect to q, we obtain

$$c_{nm,ij}^{r+1}=\frac{1}{(2m\pi)(2j\pi)2^{\frac{3k}{2}}2^{\frac{3k'}{2}}}\int_0^1\int_0^1\frac{\partial^2 u}{\partial p\partial q}\left(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}}\right)(\sin(2m\pi p)-\cos(2m\pi p))(\sin(2j\pi q)-\cos(2j\pi q))\mathrm{d}p\mathrm{d}q.$$

Again, integrating with respect to p and q, we obtain

$$c_{nm,ij}^{r+1}=\frac{1}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}}2^{\frac{5k'}{2}}}\int_0^1\int_0^1\frac{\partial^4 u}{\partial^2 p\partial^2 q}\left(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}}\right)(-\cos(2m\pi p)-\sin(2m\pi p))(-\cos(2j\pi q)-\sin(2j\pi q))\mathrm{d}p\mathrm{d}q,$$

or

Table 1 Comparison of the approximate solutions of generalized Burger–Fisher equation with reduced differential transform method and variational iteration method

x	t	E_{RTDM} [10]	E_{VIM} [10]	E_{CAS}
0.01	0.02	0.4999e−05	2.5031e−03	1.9435e−07
0.01	0.04	0.4999e−05	2.5081e−03	2.7604e−07
0.01	0.06	1.4999e−05	2.5131e−03	3.3814e−07
0.01	0.08	1.9999e−05	2.5181e−03	3.8724e−07
0.04	0.02	0.4997e−05	9.9962e−03	7.1200e−07
0.04	0.04	0.9997e−05	1.0001e−02	1.0346e−06
0.04	0.06	1.4997e−05	1.0006e−02	1.2805e−06
0.04	0.08	1.9997e−05	1.0011e−02	1.4781e−06
0.08	0.02	0.4995e−05	1.9979e−02	1.2555e−06
0.08	0.04	0.9995e−05	1.9984e−02	1.8928e−06
0.08	0.06	1.4995e−05	1.9989e−02	2.3807e−06
0.08	0.08	1.9995e−05	1.9994e−02	2.7727e−06

Table 2 Comparison of the approximate solution of Burger–Fisher equation by present method and reduced differential transform method.

x	t	E_{RTDM} [10]	E_{CAS}
0.01	0.02	4.7133e−06	3.17037e−08
0.01	0.04	9.4271e−06	2.72883e−08
0.01	0.06	1.4142e−05	2.64518e−08
0.01	0.08	1.8855e−05	2.63137e−08
0.04	0.02	4.7117e−06	1.20316e−07
0.04	0.04	9.4260e−06	1.05412e−07
0.04	0.06	1.4140e−05	1.02628e−07
0.04	0.08	1.8854e−05	1.02080e−07
0.08	0.02	4.7104e−06	2.27121e−07
0.08	0.04	9.4241e−06	2.01289e−07
0.08	0.06	1.4138e−05	1.96390e−07
0.08	0.08	1.8852e−05	1.95436e−07

Table 3 Comparison of the approximate solution of Burger–Fisher equation with solution obtained by present method and Adomian decomposition method

x	t	E_{ADM} [11]	E_{CAS}
0.1	0.005	9.68763e–06	5.70883e–07
0.1	0.01	1.93752e–05	9.29559e–07
0.5	0.005	9.68691e–06	6.87261e–07
0.5	0.01	1.93738e–05	1.31043e–06
0.9	0.005	9.68619e–06	5.71285e–07
0.9	0.01	1.93724e–05	9.30207e–07

$$|c_{nm,ij}^{r+1}|^2 \le \left|\frac{1}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}} 2^{\frac{5k'}{2}}}\right|^2 \int_0^1\int_0^1 \left|\frac{\partial^4 u}{\partial^2 p \partial^2 q}\left(\frac{p+n-1}{2^k}, \frac{q+i-1}{2^{k'}}\right)\right|^2 |(-\cos(2m\pi p) - \sin(2m\pi p))|^2 |(-\cos(2j\pi q) - \sin(2j\pi q))|^2 \mathrm{d}p^2\mathrm{d}q^2.$$

Since $\left|\frac{\partial^4 u}{\partial^2 p \partial^2 q}\right| \le \gamma$, we have

$$|c_{nm,ij}^{r+1}|^2 \le \left(\frac{\gamma}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}} 2^{\frac{5k'}{2}}}\right)^2 \int_0^1\int_0^1 |(-\cos(2m\pi p) - \sin(2m\pi p))|^2 \mathrm{d}p\mathrm{d}q \int_0^1\int_0^1 |(-\cos(2j\pi q) - \sin(2j\pi q))|^2 \mathrm{d}p\mathrm{d}q.$$

By orthogonality of CAS wavelet as $\int_0^1 (\mathrm{CAS}_m(x)\,\mathrm{CAS}_m(x))\mathrm{d}x = 1$, so

$$|c_{nm,ij}^{r+1}| \le \left|\frac{\gamma}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}} 2^{\frac{5k'}{2}}}\right|.$$

Using above Lemma, the series $2^k \ge n+1$ and $2^{k'} \ge i+1$, we get

$$|c_{nm,ij}^{r+1}| \le \left|\frac{\gamma}{(2m\pi)^2(2j\pi)^2 (n+1)^{\frac{5}{2}} (i+1)^{\frac{5}{2}}}\right|.$$

Hence, the series $\sum_{n=0}^{\infty}\sum_{m\epsilon Z}\sum_{i=0}^{\infty}\sum_{j\epsilon Z} c_{nm,ij}^{r+1}$ is absolutely convergent. In addition, we can obtain

$$\left|\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t)\right| \le \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} |c_{nm,ij}||\psi_{n,m}(x)||\psi_{i,j}(t)| \le 4\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} |c_{nm,ij}|$$

as $\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t)$ converges to $u_{r+1}(x,t)$, so we have

$$|u_{r+1}^{k,k',M,M'} - u_{r+1}(x,t)| \le 4\left|\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\sum_{i=2^{k'}}^{\infty}\sum_{j=M'+1}^{\infty} c_{n,m,i,j}\psi_{n,m}(x)\psi_{i,j}(t)\right|,$$

or

$$|u_{r+1}^{k,k',M,M'} - u_{r+1}(x,t)| \le \frac{\gamma}{\pi^4}\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\sum_{i=2^{k'}}^{\infty}\sum_{j=M'+1}^{\infty} \frac{1}{(mj)^2(n+1)^{\frac{5}{2}}(i+1)^{\frac{5}{2}}}. \tag{24}$$

Inequality (24) exhibits that the absolute error at the $(r+1)$th iteration is inversely proportional to k, k', M and M'. This implies that $u_{r+1}^{k,k',M,M'}(x,t)$ converges to $u_{r+1}(x,t)$ as $k, k', M, M' \longrightarrow \infty$. Since $u_{r+1}(x,t)$ is obtained at $(r+1)$th iteration of quasi-linearization technique so according to the convergence analysis of quasi-linearization technique [9] which states that $u_{r+1}(x,t)$ converges to $u(x, t)$ as $r \longrightarrow \infty$, if there is convergence at all. This suggest that solution by CAS wavelet quasi-linearization technique $u_{r+1}^{k,k',M,M'}(x,t)$ converges to $u(x, t)$ when k, k', M, M' and $r \longrightarrow \infty$.

Applications of CAS wavelet quasi-linearization technique

Consider the generalized Burgers–Fisher equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au^{\gamma}\frac{\partial u}{\partial x} + bu(u^{\gamma} - 1) = 0, \quad 0 \le x \le 1,\ t \ge 0, \tag{25}$$

subject to the initial and boundary conditions:

$$u(x,0) = \left(\frac{1}{2} - \frac{1}{2}\tan h\left(\frac{a\gamma}{2(1+\gamma)}x\right)\right)^{\frac{1}{\gamma}},$$

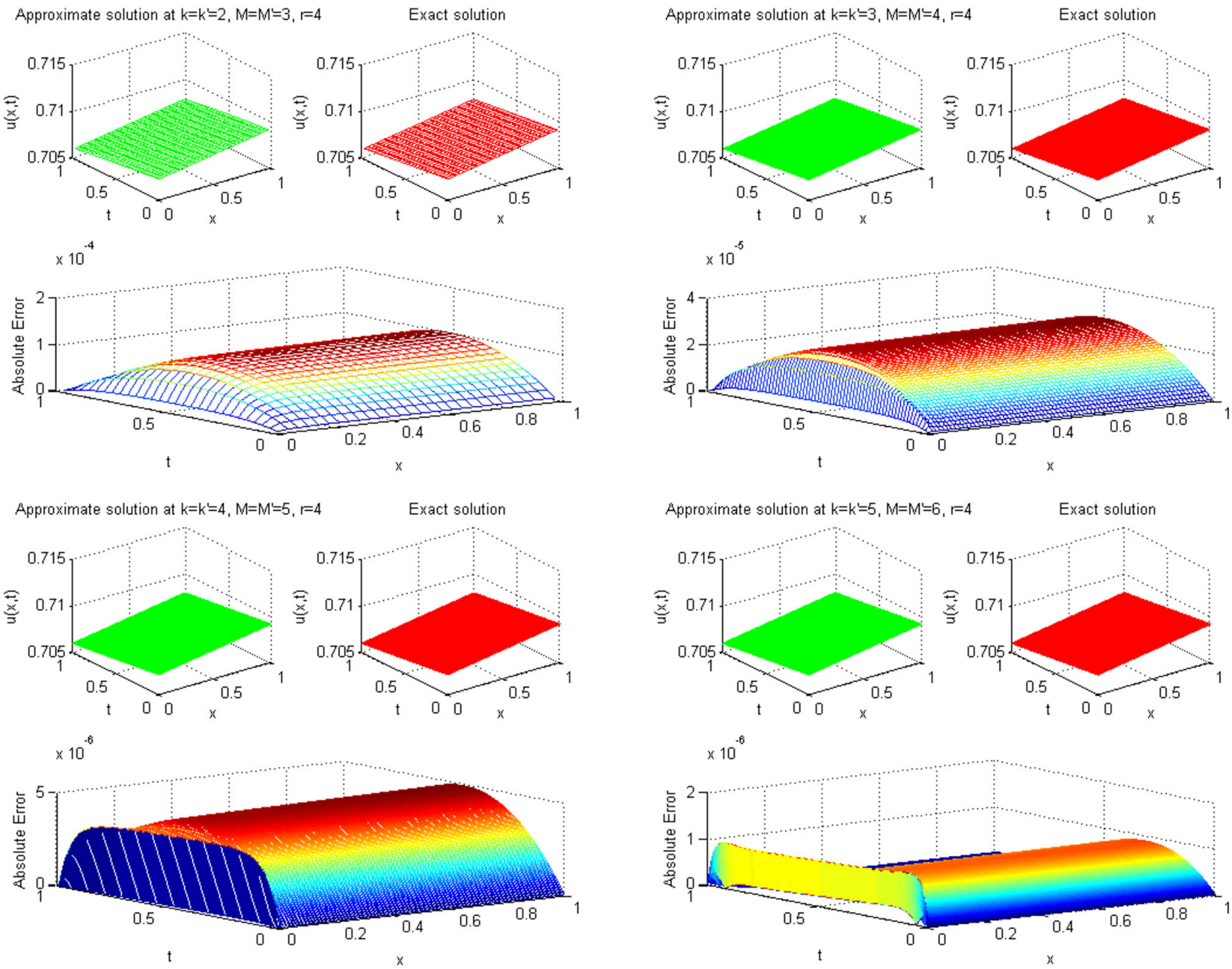

Fig. 1 Comparison of the numerical results by present method at $r = 4$ and different values of k, k', M, M' with exact solution of generalized Burger–Fisher equation

$$u(0,t) = \left(\frac{1}{2} - \frac{1}{2}\tan h\left(\frac{a\gamma}{2(1+\gamma)}\left[-\left(\frac{a^2 + b(1+\gamma^2)}{a(1+\gamma)}\right)t\right]\right)\right)^{\frac{1}{\gamma}},$$

$$u(1,t) = \left(\frac{1}{2} - \frac{1}{2}\tan h\left(\frac{a\gamma}{2(1+\gamma)}\left[1-\left(\frac{a^2 + b(1+\gamma^2)}{a(1+\gamma)}\right)t\right]\right)\right)^{\frac{1}{\gamma}}.$$

Implement the CAS wavelet quasi-linearization technique on Eq. (25), as described in Sect. 4, we get the following results as given in Tables 1, 2, and 3, and Fig. 1. We consider the three different forms of Eq. (25) using different values of a, b and γ. E_{RTDM}, E_{VIM}, E_{ADM} and $\mathrm{E}_{\mathrm{CAS}}$ represents the absolute error by reduced differential transform method, variational iteration method, Adomian decomposition method, and present method, respectively.

Solution of generalize Burger–Fisher equation for $a = 0.001, b = 0.001$ and $\gamma = 1$ by present method at $M = M' = 5,\ k = k' = 4$ and $\mathrm{r} = 4$ is given in Table 1. The obtained results are compared with the results obtained from reduced differential transform method (RDTM) [10] and variational iteration method (VIM) [10].

Table 2 is used to list the results of generalized Burger–Fisher equation at $a = 0.001,\ b = 0.001$ and $\gamma = 2$. We implement the proposed method at $M = M' = 7,\ k = k' = 5$ and $r = 3$. We compared our results with the results obtained from reduced differential transform method (RDTM) [10].

Present method at $k = k' = 5,\ M = M' = 7,\ r = 4$ is implemented on generalized Burger–Fisher equation with $a = 0.001,\ b = 0.001$ and $\gamma = 1$. The obtained results are listed in Table 3.

Figure 1 is used to plot the exact solution of equation (1.1) with $a = 0.01$ $b = 0.01$ and $\gamma = 2$, solution by CAS wavelet quasi-linearization technique at $r = 4$ and different values of k, k', M, and M'.

Conclusion

We have derived and constructed the CAS wavelets matrix, $\Psi_{\hat{m}\times\hat{m}}$, and the CAS wavelets operational matrix of q^{th} order integration, $\mathbf{P}^{\mathbf{q}}_{\hat{m}\times\hat{m}}$, and CAS wavelets operational matrix of integration for boundary value problems, $W^{g,q}_{\hat{m}\times\hat{m}}$. These matrices are successfully utilized to solve the generalized Burger–Fisher equation.

According to Tables 1, 2, and 3, our results are more accurate as compared to reduced differential transform method, variational iteration method and Adomian decomposition method. Figure 1 shows that our results converge to the exact solution while increasing k, k', M and M', when $r = 4$.

It is shown that present method gives excellent results when applied to generalized Burger–Fisher equation. The different types of non-linearities can easily be handled by the present method.

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Compliance with ethical standards

Conflict of interest We, Umer Saeed and Khadija Gilani, declares that there is no conflict of interests regarding the publication of this article.

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