

# Numerical solutions of fourth-order Volterra integro-differential equations by the Green's function and decomposition method

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**Abstract** We propose a reliable technique based on Adomian decomposition method (ADM) for the numerical solution of fourth-order boundary value problems for Volterra integro-differential equations. We use Green's function technique to convert boundary value problem into the integral equation before establishing the recursive scheme for the solution components of a specific solution. The advantage of the proposed technique over the standard ADM or modified ADM is that it provides not only better numerical results but also avoids solving a sequence of transcendental equations for unknown constant. Approximations of the solutions are obtained in the form of series. Convergence and error analysis is also discussed. The accuracy and generality of the proposed scheme are demonstrated by solving some numerical examples.

**Keywords** Integro-differential equations · Boundary value problems · Adomian decomposition method · Green's function · Approximations

**Mathematics Subject Classification** 34B15 · 34B27 · 34B05 · 65L10 · 65L80

## Introduction

Consider the following class of fourth-order BVPs for Volterra IDEs [1–5]

$$y^{(iv)}(x) = g(x) + \int_0^x K(x,t)f(y(t))dt, \quad x \in [0, b], \quad (1)$$

with the boundary conditions

$$y(0) = \alpha_1, \quad y'(0) = \alpha_2, \quad y(b) = \alpha_3, \quad y'(b) = \alpha_4, \quad (2)$$

where  $\alpha_i, i = 1, 2, 3, 4$  are any finite real constants,  $g(x) \in C[0, b]$ , and  $K(x, t) \in C([0, b] \times [0, b])$ . The IDEs are often involved in the mathematical formulation of physical and engineering phenomena [4–6]. In general, the IDEs with given boundary conditions are difficult to solve analytically. Therefore, these problems must be solved by various approximation and numerical methods. The existence and uniqueness of solutions for such problems can be found in [1].

There is considerable literature on the numerical-approximate treatment of the BVPs for IDEs, for example, the compact finite difference method [7], monotone iterative methods [7, 8], spline collocation method [9], the method of upper and lower solution [10], Haar wavelets [11], and pseudo-spectral method [12]. Though, these numerical techniques have many advantages, a huge amount of computational work is involved that combines some root-finding techniques to obtain an accurate numerical solution especially for nonlinear problems.

Recently, some newly developed semi-numerical methods have also been applied to solve BVPs for IDEs such as, ADM [3], homotopy perturbation method (HPM) [4], and homotopy analysis method (HAM) [13]. In [5], the

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variational iteration method (VIM) was also used for solving the problem (1)–(2). However, in [14] Wazwaz pointed out that VIM gives good approximations only when the problem is linear or nonlinear with the weak nonlinearity of the form  $(y'', yy', y''', \dots)$ , but the VIM suffers when the nonlinearity is of the form  $(e^y, \ln y, \sin y, \dots)$  (for details see [14]).

It is well known that the ADM allows us to solve nonlinear BVPs without restrictive assumptions such as linearization, discretization and perturbation. Many researchers [14–23] have shown interest to study the ADM for different scientific models. According to the ADM, we rewrite the problem (1) in an operator form

$$Ly = g + Ny, \tag{3}$$

where  $L = \frac{d^4}{dx^4}$  is a fourth-order linear differential operator,  $g$  is a function of  $x$  and  $Ny = \int_0^x K(x, t)f(y(t))dt$  is a nonlinear term. Inverse integral operator is usually defined as

$$L^{-1}[\cdot] := \int_0^x \int_0^x \int_0^x \int_0^x [\cdot] dx dx dx dx. \tag{4}$$

Operating with  $L^{-1}$  on both sides of (3) and using the conditions  $y(0) = \alpha_1$  and  $y'(0) = \alpha_2$ , we obtain

$$y(x) = \alpha_1 + \alpha_2x + c_1x^2 + c_2x^3 + L^{-1}[g + Ny], \tag{5}$$

where  $c_1 = \frac{y''(0)}{2!}$  and  $c_2 = \frac{y'''(0)}{3!}$  are unknown constants to be determined.

The ADM relies on decomposing  $y$  by a series of *components* and nonlinear term  $f(y)$  by a series of *Adomian polynomials* as

$$y = \sum_{j=0}^{\infty} y_j(x) \quad \text{and} \quad f(y) = \sum_{j=0}^{\infty} A_j, \tag{6}$$

where  $A_j$  are Adomian’s polynomials [22], which can be computed as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ f \left( \sum_{k=0}^{\infty} y_k \lambda^k \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots, \tag{7}$$

Several algorithms have also been given to generate the Adomian polynomial rapidly in [24–26]. Substituting the series (6) in (5), we get

$$\sum_{j=0}^{\infty} y_j(x) = \alpha_1 + \alpha_2x + c_1x^2 + c_2x^3 + L^{-1}[g] + L^{-1} \left\{ \int_0^x K(x, t) \left[ \sum_{j=0}^{\infty} A_j \right] dt \right\}. \tag{8}$$

On comparing both sides of equation (8), the ADM is given by

$$\left. \begin{aligned} y_0 &= \alpha_1 + \alpha_2x + c_1x^2 + c_2x^3 + L^{-1}(g), \\ y_j &= L^{-1} \left\{ \int_0^x K(x, t) A_{j-1} dt \right\}, \quad j = 1, 2, \dots \end{aligned} \right\} \tag{9}$$

Wazwaz [27] suggested a modified ADM (MADM) which is given by

$$\left. \begin{aligned} y_0 &= \alpha_1, \\ y_1 &= \alpha_2x + c_1x^2 + c_2x^3 + L^{-1}[g] + L^{-1} \left\{ \int_0^x K(x, t) A_0 dt \right\}, \\ y_j &= L^{-1} \left\{ \int_0^x K(x, t) A_{j-1} dt \right\}, \quad j = 2, 3, \dots \end{aligned} \right\} \tag{10}$$

Hence, the  $n$ -term approximate series solution is obtained as

$$\phi_n(x, c_1, c_2) = \sum_{j=0}^n y_j(x, c_1, c_2). \tag{11}$$

We note that the series solution  $\phi_n(x, c_1, c_2)$  depends on the unknown constants  $c_1$  and  $c_2$ . These unknown constants will be determined approximately by imposing the boundary condition at  $x = b$  on  $\phi_n(x, c_1, c_2)$ , which leads a sequence of nonlinear system of equations as

$$\phi_n(b, c_1, c_2) = \alpha_3 \quad \text{and} \quad \phi'_n(b, c_1, c_2) = \alpha_4, \quad n = 1, 2, \dots \tag{12}$$

To determine the unknown constants  $c_1$  and  $c_2$ , we require root finding methods such as Newton–Raphson’s method which requires additional computational work. But solving the nonlinear equation (12) for  $c_1$  and  $c_2$  is a difficult task in general. Moreover, in some cases the unknowns  $c_1$  and  $c_2$  may not be uniquely determined. This may be the main difficulty of the ADM.

In this work, we propose a new recursive scheme which does not involve any unknown constant to be determined. In other words, we introduce a modification of the ADM to overcome the difficulties occurring in ADM or MADM for solving fourth-order BVPs for IDEs.

### The decomposition method with Green’s function

Let us first consider homogeneous version of the problem (1) and (2) as

$$\begin{cases} u^{(iv)}(x) = 0, & x \in [0, b], \\ u(0) = \alpha_1, \quad u'(0) = \alpha_2, \quad u(b) = \alpha_3, \quad u'(b) = \alpha_4. \end{cases} \tag{13}$$

Solving (13) analytically, we obtain

$$u(x) = \alpha_1 + \alpha_2 x - \frac{(3\alpha_1 + 2b\alpha_2 - 3\alpha_3 + b\alpha_4)}{b^2} x^2 + \frac{(2\alpha_1 + b\alpha_2 - 2\alpha_3 + b\alpha_4)}{b^3} x^3. \tag{14}$$

We now construct Green’s function of the following fourth-order boundary value problem

$$\begin{cases} y^{(iv)}(x) = h(x), & x \in [0, b], \\ y(0) = y'(0) = y(b) = y'(b) = 0. \end{cases} \tag{15}$$

The Green’s function of (15) can be easily constructed and it is given by

$$G(x, \xi) = \begin{cases} x^3 \left( -\frac{1}{6} + \frac{\xi^2}{2b^2} - \frac{\xi^3}{3b^3} \right) + x^2 \left( \frac{\xi}{2} - \frac{\xi^2}{b} + \frac{\xi^3}{2b^2} \right), & 0 \leq x \leq \xi, \\ \xi^3 \left( -\frac{1}{6} + \frac{x^2}{2b^2} - \frac{x^3}{3b^3} \right) + \xi^2 \left( \frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2} \right), & \xi \leq x \leq b. \end{cases} \tag{16}$$

Using (14) and (16), we transform BVPs for IDEs (1) and (2) into an integral equation as

$$y(x) = u(x) + \int_0^b G(x, \xi) \left\{ g(\xi) + \int_0^\xi K(\xi, t) f(y(t)) dt \right\} d\xi. \tag{17}$$

Substituting the series (6) in (17), we obtain

$$\sum_{j=0}^\infty y_j(x) = u(x) + \int_0^b G(x, \xi) \left\{ g(\xi) + \int_0^\xi K(\xi, t) \left[ \sum_{j=0}^\infty A_j \right] dt \right\} d\xi. \tag{18}$$

Comparing both sides of (18), the decomposition with Green’s function (DMGF) is given by the following recursive scheme as

$$\left. \begin{aligned} y_0 &= u(x) + \int_0^b G(x, \xi) g(\xi) d\xi, \\ y_j &= \int_0^b G(x, \xi) \left\{ \int_0^\xi K(\xi, t) A_{j-1} dt \right\} d\xi, \quad j = 1, 2, 3, \dots \end{aligned} \right\} \tag{19}$$

and the modified decomposition with Green’s function (MDMGF) is given by the following recursive scheme as

$$\left. \begin{aligned} y_0 &= u_0, \\ y_1 &= u_1 + \int_0^b G(x, \xi) \left\{ g(\xi) + \int_0^\xi K(\xi, t) A_0 dt \right\} d\xi, \\ y_j &= \int_0^b G(x, \xi) \left\{ \int_0^\xi K(\xi, t) A_{j-1} dt \right\} d\xi, \quad j = 2, 3, \dots \end{aligned} \right\} \tag{20}$$

where  $u_0 = \alpha_1$  and  $u_1 = \alpha_2 x - \frac{(3\alpha_1 + 2b\alpha_2 - 3\alpha_3 + b\alpha_4)}{b^2} x^2 + \frac{(2\alpha_1 + b\alpha_2 - 2\alpha_3 + b\alpha_4)}{b^3} x^3$ . The  $n$ -terms truncated series solution is obtained as

$$\psi_n = \sum_{j=0}^n y_j. \tag{21}$$

**Convergence and error estimate of the scheme (19) or (20)**

In this section, we shall show that sequence  $\{\psi_n\}$  of the partial sums of series solution defined by (21) converges to the exact solution  $y$  of the problem (1), (2).

**Theorem 3.1** (Convergence theorem) *Suppose that  $\mathbb{X} = C[0, b]$  is a Banach space with the norm  $\|y\| = \max_{x \in I=[0, b]} |y(x)|$ ,  $y \in \mathbb{X}$ . Assume that the function  $f(y)$  satisfies the Lipschitz condition such that  $|f(y) - f(y^*)| \leq l|y - y^*|$  and denote  $\|K\|_\infty = \max |K(\xi, t)|$  and  $\|G\|_\infty = \max |G(x, \xi)|$ . Further, we define  $\delta$  as  $\delta := l\|K\|_\infty \|G\|_\infty b^2$ . Then the sequence  $\{\psi_n\}$  converges to the exact solution whenever  $\delta < 1$  and  $\|y_1\| < \infty$ .*

*Proof* From (19) or (20) and (21), we write

$$\begin{aligned} \psi_n &= y_0 + \sum_{j=1}^n y_j = u(x) + \int_0^b G(x, \xi) g(\xi) d\xi \\ &\quad + \sum_{j=1}^n \left[ \int_0^b G(x, \xi) \left\{ \int_0^\xi K(\xi, t) A_{j-1} dt \right\} d\xi \right] \\ &= u(x) + \int_0^b G(x, \xi) \left\{ g(\xi) + \int_0^\xi K(\xi, t) \left[ \sum_{j=1}^n A_{j-1} \right] dt \right\} d\xi. \end{aligned}$$

For all  $n, m \in \mathbb{N}$ , with  $n > m$ , consider

$$\|\psi_n - \psi_m\| = \max_{x \in I} \left| \int_0^b G(x, \xi) \left\{ \int_0^\xi K(\xi, t) \left[ \sum_{j=0}^{n-1} A_j - \sum_{j=0}^{m-1} A_j \right] dt \right\} d\xi \right|. \tag{22}$$

Using the relation  $\sum_{j=0}^n A_j \leq f(\psi_n)$  (for details see, [28, pp 944–945]) we have

$$\begin{aligned} \|\psi_n - \psi_m\| &\leq \max_{x \in I} \left| \int_0^b G(x, \xi) \left\{ \int_0^\xi K(\xi, t) [f(\psi_{n-1}) - f(\psi_{m-1})] dt \right\} d\xi \right| \\ &\leq \max |G(x, \xi)| \int_0^b \left\{ \max |K(\xi, t)| \int_0^\xi l |\psi_{n-1} - \psi_{m-1}| dt \right\} d\xi \\ &\leq l \|K\|_\infty \|G\|_\infty \|\psi_{n-1} - \psi_{m-1}\| \max_{x \in I} \int_0^b \int_0^\xi dt d\xi \\ &\leq l \|K\|_\infty \|G\|_\infty b^2 \|\psi_{n-1} - \psi_{m-1}\| = \delta \|\psi_{n-1} - \psi_{m-1}\|, \end{aligned}$$

where  $\delta = l\|K\|_\infty\|G\|_\infty b^2$ .

Setting  $n = m + 1$  we obtain  $\|\psi_{m+1} - \psi_m\| \leq \delta\|\psi_m - \psi_{m-1}\|$ . Thus, we have  $\|\psi_{m+1} - \psi_m\| \leq \delta\|\psi_m - \psi_{m-1}\| \leq \delta^2\|\psi_{m-1} - \psi_{m-2}\| \leq \dots \leq \delta^m\|\psi_1 - \psi_0\|$ . Using triangle inequality for any  $n, m \in \mathbb{N}$ , with  $n > m$  we have

$$\begin{aligned} \|\psi_n - \psi_m\| &= \|(\psi_n - \psi_{n-1}) + (\psi_{n-1} - \psi_{n-2}) + \dots + (\psi_{m+1} - \psi_m)\| \\ &\leq \|\psi_n - \psi_{n-1}\| + \|\psi_{n-1} - \psi_{n-2}\| + \dots + \|\psi_{m+1} - \psi_m\| \\ &\leq [\delta^{n-1} + \delta^{n-2} + \dots + \delta^m]\|\psi_1 - \psi_0\| \\ &= \delta^m[1 + \delta + \delta^2 + \dots + \delta^{n-m-1}]\|\psi_1 - \psi_0\| \\ &= \delta^m\left(\frac{1 - \delta^{n-m}}{1 - \delta}\right)\|y_1\|. \end{aligned}$$

Thus, we obtain

$$\|\psi_n - \psi_m\| \leq \frac{\delta^m}{1 - \delta}\|y_1\|, \tag{23}$$

which converges to zero, i.e.,  $\|\psi_n - \psi_m\| \rightarrow 0$ , as  $m \rightarrow \infty$ . This implies that there exists a  $\psi$  such that  $\lim_{n \rightarrow \infty} \psi_n = \psi$ .  $\square$

In the next theorem, we give the error estimate of the series solution.

**Theorem 3.2** (Error estimate) *The maximum absolute truncation error of the series  $\psi_m$  obtained by the scheme (19) or (20) to problem is given as*

$$\|y - \psi_m\| \leq \frac{\delta^{m+1}}{l(1 - \delta)}\|A_0\|_\infty \tag{24}$$

where  $\|A_0\|_\infty = \max_{x \in I} |A_0|$ ,  $A_0 = f(y_0)$ .

*Proof* Fixing  $m$  and letting  $n \rightarrow \infty$  in the estimate (23) with  $n \geq m$ , we obtain

$$\|y - \psi_m\| \leq \frac{\delta^m}{1 - \delta}\|y_1\|. \tag{25}$$

From the scheme (19) we have  $y_1 = \int_0^b G(x, \xi) \left\{ \int_0^\xi K(\xi, t) A_0 dt \right\} d\xi$ , and following the steps of theorem (3.1), we have

$$\begin{aligned} \|y_1\| &= \max_{x \in I} \left| \int_0^b G(x, \xi) \left\{ \int_0^\xi K(\xi, t) A_0 dt \right\} d\xi \right| \\ &\leq \|K\|_\infty \|G\|_\infty \|A_0\|_\infty b^2. \end{aligned} \tag{26}$$

But we know that  $\delta = l\|K\|_\infty\|G\|_\infty b^2$ . Hence, the inequality (26) becomes

$$\|y_1\| \leq \frac{\delta}{l}\|A_0\|_\infty. \tag{27}$$

Combining the estimates (25) and (27), we get the desired result.  $\square$

### Numerical results

To demonstrate the efficiency and accuracy of the proposed recursive schemes, we consider four fourth-order BVPs for Volterra IDEs. All symbolic and numerical computations are performed using ‘Mathematica’ 8.0 software package.

*Example 4.1* Consider the following linear fourth-order BVP for Volterra IDE [2, 3]

$$\left. \begin{aligned} y^{(iv)}(x) &= g(x) + \int_0^x y(t) dt, \quad x \in [0, 1] \\ y(0) = y'(0) &= 1, \quad y(1) = 1 + e, \quad y'(1) = 2e, \end{aligned} \right\} \tag{28}$$

where  $g(x) = -x + 5e^x - 1$ , and its exact solution is  $y(x) = 1 + xe^x$ .

Here,  $b = 1$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$ ,  $\alpha_3 = 1 + e$ ,  $\alpha_4 = 2e$ ,  $K(x, t) = 1$ ,  $f(y) = y$ , and  $g(\xi) = -\xi + 5e^\xi - 1$ . According to the MDMGF (20), the problem (28) is transformed into the following recursive scheme

$$\left. \begin{aligned} y_0 &= 1, \\ y_1 &= x + (e - 2)x^2 + x^3 + \int_0^1 G(x, \xi) \left\{ g(\xi) + \int_0^\xi K(\xi, t) y_0 dt \right\} d\xi, \\ y_j &= \int_0^1 G(x, \xi) \left\{ \int_0^\xi K(\xi, t) y_{j-1} dt \right\} d\xi, \quad j = 2, 3, \dots \end{aligned} \right\} \tag{29}$$

$G(x, \xi)$  is given by

$$G(x, \xi) = \begin{cases} x^3 \left( -\frac{1}{6} + \frac{\xi^2}{2} - \frac{\xi^3}{3} \right) + x^2 \left( \frac{\xi}{2} - \frac{\xi^2}{1} + \frac{\xi^3}{2} \right), & 0 \leq x \leq \xi, \\ \xi^3 \left( -\frac{1}{6} + \frac{x^2}{2} - \frac{x^3}{3} \right) + \xi^2 \left( \frac{x}{2} - \frac{x^2}{1} + \frac{x^3}{2} \right), & \xi \leq x \leq 1. \end{cases} \tag{30}$$

Using (29) and (30), we compute the solution components  $y_j$  as

$$\begin{aligned} y_0 &= 1; \\ y_1 &= -5 + 5e^x - 4x - 1.5062x^2 - 0.325258x^3 - 0.0416667x^4; \\ y_2 &= -5 + 5e^x - 5x - 2.4938x^2 - 0.84140x^3 - 0.20833x^4 \\ &\quad - 0.041666x^5 - 0.00555x^6 + \dots \\ &\vdots \end{aligned}$$

To check the accuracy and efficiency of the proposed methods, the absolute error function is defined as

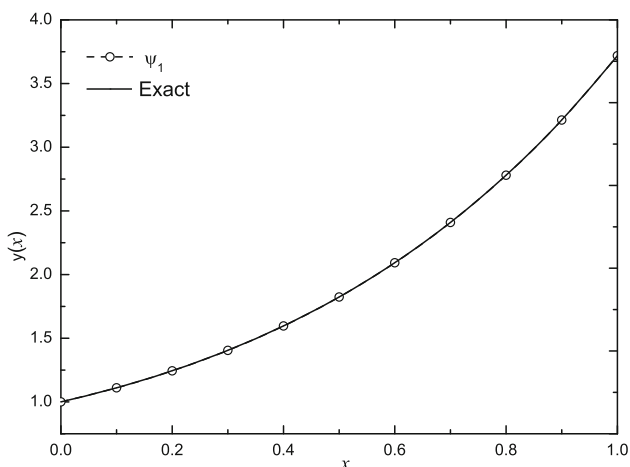
$$E_n(x) = |\psi_n(x) - y(x)|, \quad n = 1, 2, \dots$$

where  $y$  is the exact solution and  $\psi_n$  is the  $n$ th-stage approximation obtained by the proposed (19) or (20).

In Table 1, we list the numerical results of the absolute errors  $|\psi_n - y|$  (obtained by the proposed MDMGF (20) [27])

**Table 1** The absolute error  $|\psi_n - y|$  and  $|\phi_n - y|$  for  $n = 1, 2, 3$  of Example 4.1

$x$	MDMGF			MADM [27]		
	$ \psi_1 - y $	$ \psi_2 - y $	$ \psi_3 - y $	$ \phi_1 - y $	$ \phi_2 - y $	$ \phi_3 - y $
0.1	5.39E-05	3.54E-08	2.29E-11	4.40E-03	4.54E-05	2.29E-08
0.3	3.41E-04	2.23E-07	1.44E-10	8.41E-03	4.23E-05	4.74E-07
0.5	5.66E-04	3.63E-07	2.34E-10	8.66E-03	6.63E-05	3.37E-07
0.7	4.70E-04	2.92E-07	1.88E-10	7.70E-03	6.92E-05	4.78E-07
0.9	1.02E-04	5.97E-08	3.83E-11	2.02E-03	7.97E-06	5.81E-08



**Fig. 1** Plots of the exact and the approximate solution of Example 4.1

and  $|\phi_n - y|$  [obtained by the existing MADM (10)] for  $n = 1, 2, 3$ . It is observed that the proposed MDMGF provides not only better numerical results but also avoids solving a sequence of transcendental equations for unknown constant.

In Fig. 1, we plot the exact solution  $y$  and the approximate solution  $\psi_1 = y_0 + y_1$ . We observe that only two-term approximations  $\psi_1 = y_0 + y_1$  coincide with the exact solution  $y$ .

**Example 4.2** Consider the following nonlinear fourth-order BVPs for Volterra IDE [3]

$$\left. \begin{aligned} y^{(iv)}(x) &= g(x) + \int_0^x e^{-t} y^2(t) dt, \quad x \in [0, 1], \\ y(0) &= y'(0) = 1, \quad y(1) = y'(1) = e, \end{aligned} \right\} \quad (31)$$

where  $g(x) = 1$ . The exact solution is  $y(x) = e^x$ .

Here,  $b = 1$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$ ,  $\alpha_3 = e$ ,  $\alpha_4 = e$ ,  $K(x, t) = e^{-t}$ , and  $f(y) = y^2(t)$ . In view of the MDMGF (20), we transform the problem (31) into the following recursive scheme

$$\left. \begin{aligned} y_0 &= 1, y_1 = x + (2e - 5)x^2 - (e - 3)x^3 \\ &+ \int_0^1 G(x, \xi) \left\{ g(\xi) + \int_0^\xi K(\xi, t) A_0 dt \right\} d\xi, \\ y_j &= \int_0^1 G(x, \xi) \left\{ \int_0^\xi K(\xi, t) A_{j-1} dt \right\} d\xi, \quad j = 2, 3, \dots \end{aligned} \right\} \quad (32)$$

where  $g(\xi) = 1$  and  $G(x, \xi)$  is given by the Eq. (30). The Adomian’s polynomial  $f(y) = y^2$  are obtained as

$$A_0 = y_0^2, \quad A_1 = 2y_0y_1, \quad A_2 = y_1^2 + 2y_0y_2, \dots \quad (33)$$

Using (32) and (33), we obtain the solution components  $y_j$  as

$$\begin{aligned} y_0 &= 1, \\ y_1 &= 1 - e^{-x} + 0.991415x^2 + 0.0114132x^3 + 0.0833333x^4, \\ y_2 &= 346.216 + 0.0625e^{-2x} - 184.271x + 43.4637x^2 \\ &\quad - 5.66685x^3 + 0.379275x^4 + \dots \\ &\vdots \end{aligned}$$

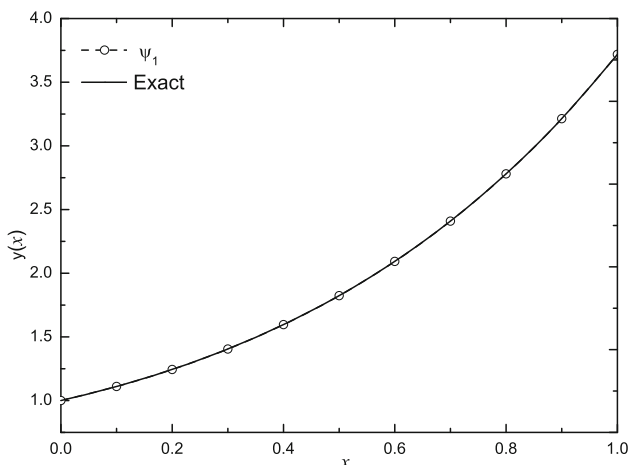
In Table 2, we present the numerical results of the absolute errors  $|\psi_n - y|$  (obtained by the proposed MDMGF) and  $|\phi_n - y|$  (obtained by MADM) for  $n = 1, 2, 3$ . It is observed that the proposed DMGF provides not only better numerical results but also avoids solving a sequence of transcendental equations for unknown constant. In Fig. 2, the exact solution  $y$  and the approximate solution  $\psi_1$  are plotted. From this figure, we observe that only two-term approximations  $\psi_1$  coincide with the exact one.

**Example 4.3** Consider the following nonlinear fourth-order BVPs for Volterra IDE

$$\left. \begin{aligned} y^{(iv)}(x) &= g(x) + \int_0^x e^{y(t)} dt, \quad x \in [0, 1], \\ y(0) &= \ln(4), \quad y'(0) = \frac{1}{4}, \quad y(1) = \ln(5), \quad y'(1) = \frac{1}{5} \end{aligned} \right\} \quad (34)$$

**Table 2** The absolute error  $|\psi_n - y|$  and  $|\phi_n - y|$  for  $n = 1, 2, 3$  of Example 4.2

$x$	MDMGF			MADM [27]		
	$ \psi_1 - y $	$ \psi_2 - y $	$ \psi_3 - y $	$ \phi_1 - y $	$ \phi_2 - y $	$ \phi_3 - y $
0.1	7.44E-05	1.43E-05	4.32E-08	6.44E-04	8.48E-05	2.32E-06
0.3	4.67E-04	9.17E-05	2.75E-07	4.67E-03	9.16E-05	7.72E-05
0.5	7.63E-04	1.56E-04	4.54E-07	6.63E-03	3.66E-04	7.52E-05
0.7	6.22E-04	1.34E-04	3.72E-07	5.22E-03	4.54E-04	4.72E-05
0.9	1.32E-04	3.00E-05	7.61E-08	3.32E-03	3.00E-05	7.61E-06



**Fig. 2** Plots of the exact and the approximate solution of Example 4.2

where  $g(x) = -\frac{x^2}{2} - 4x - \frac{6}{(x+4)^4}$ . The exact solution is  $y(x) = \ln(4 + x)$ .

Here, we have  $b = 1$ ,  $\alpha_1 = \ln(4)$ ,  $\alpha_2 = \frac{1}{4}$ ,  $\alpha_3 = \ln(5)$ ,  $\alpha_4 = \frac{1}{5}$ ,  $K(x, t) = 1$ , and  $f(y) = e^{y(t)}$ . According to the MDMGF (20), the problem (34) is transformed into the following recursive scheme

$$\left. \begin{aligned} & y_0 = \ln(4), \\ & y_1 = 0.25x - 0.030569x^2 + 0.0037128x^3 \\ & \quad + \int_0^1 G(x, \xi) \left\{ g(\xi) + \int_0^\xi K(\xi, t) A_0 dt \right\} d\xi, \\ & y_j = \int_0^1 G(x, \xi) \left\{ \int_0^\xi K(\xi, t) A_{j-1} dt \right\} d\xi \quad j = 2, 3, \dots \end{aligned} \right\} \quad (35)$$

**Table 3** The absolute error  $|\psi_n - y|$  and  $|\phi_n - y|$  for  $n = 1, 2, 3$  of Example 4.3

$x$	MDMGF			MADM [27]		
	$ \psi_1 - y $	$ \psi_2 - y $	$ \psi_3 - y $	$ \phi_1 - y $	$ \phi_2 - y $	$ \phi_3 - y $
0.1	3.61E-05	1.71E-06	7.13E-08	5.61E-03	5.71E-04	5.23E-06
0.3	2.26E-04	1.09E-05	4.64E-07	6.56E-03	5.19E-04	3.44E-06
0.5	3.69E-04	1.85E-05	8.02E-07	7.69E-03	3.95E-04	9.41E-06
0.7	3.00E-04	1.57E-05	7.02E-07	5.40E-03	4.07E-04	9.42E-06
0.9	6.31E-05	3.49E-06	1.61E-07	4.31E-04	6.49E-04	5.71E-06

where  $g(\xi) = -\frac{\xi^2}{2} - 4\xi - \frac{6}{(\xi+4)^4}$  and  $G(x, \xi)$  is given by the Eq. (30). The Adomian’s polynomials for  $f(y) = e^{y(x)}$  are calculated as

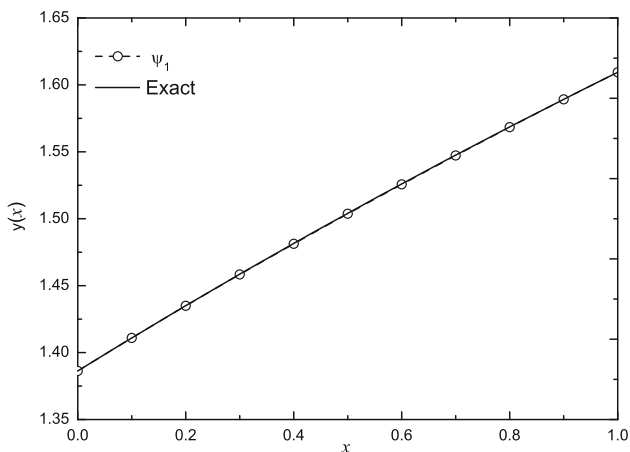
$$A_0 = e^{y_0}, \quad A_1 = e^{y_0} y_1, \quad A_2 = \frac{1}{2} e^{y_0} y_1^2 + e^{y_0} y_2(x), \dots \quad (36)$$

Using (35) and (36), we obtain the components  $y_j$  as

$$\begin{aligned} y_0 &= \ln(4), \\ y_1 &= 0.25x - 0.0354167x^2 + 0.010763x^3 - 0.000976x^4 \\ &\quad + 0.0001953x^5 + \dots, \\ y_2 &= -8.526512 \times 10^{-14} + 7.680826 \times 10^{-13}x \\ &\quad + 0.003971x^2 - 0.005310x^3 + \dots, \\ &\vdots \end{aligned}$$

In Table 3, we present the numerical results of the absolute errors  $|\psi_n - y|$  (obtained by the proposed MDMGF) and  $|\phi_n - y|$  (obtained by MADM) for  $n = 1, 2, 3$ . In this case, we also observe the same trend as was observed in last two examples that the proposed MDMGF gives better numerical results. Moreover, the curves of the exact solution  $y$  and the approximate solution  $\psi_1$  are plotted in Fig. 3. We observe that only two-term approximations  $\psi_1$  and the exact solution overlap each other.

*Example 4.4* Consider the following nonlinear fourth-order BVPs for IDEs



**Fig. 3** Plots of the exact and the approximate solution of Example 4.3

$$\left. \begin{aligned} y^{(iv)}(x) &= g(x) + \int_0^x e^{y(t)} dt, \quad x \in [0, 1], \\ y(0) &= 0, \quad y'(0) = 1, \quad y(1) = \ln(2), \quad y'(1) = \frac{1}{2}, \end{aligned} \right\} \quad (37)$$

where  $g(x) = -\frac{x^2}{2} - x - \frac{6}{(x+1)^4}$ . The exact solution is  $y(x) = \ln(1+x)$ .

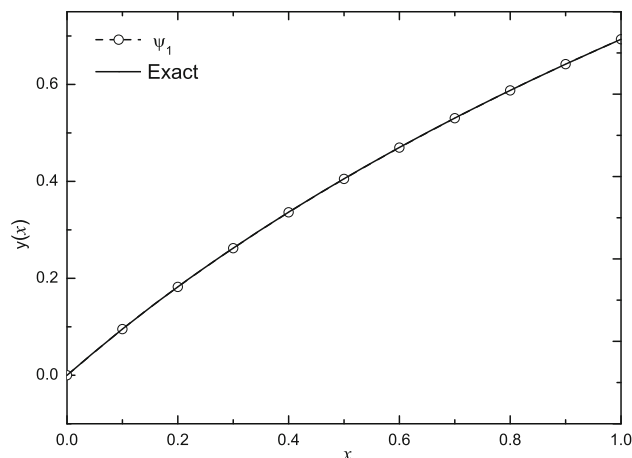
Here,  $b = 1$ ,  $\alpha_1 = 0$ ,  $\alpha_2 = 1$ ,  $\alpha_3 = \ln(2)$ ,  $\alpha_4 = \frac{1}{2}$ ,  $K(x, t) = 1$ , and  $f(y) = e^{y(t)}$ . According to MDMGF (20), we transform the problem (37) into the following recursive scheme as

$$\left. \begin{aligned} y_0 &= 0, \\ y_1 &= x - 0.420558x^2 + 0.113706x^3 \\ &\quad + \int_0^1 G(x, \xi) \left\{ g(\xi) + \int_0^1 K(\xi, t) A_0 dt \right\} d\xi, \\ y_j &= \int_0^1 G(x, \xi) \left\{ \int_0^1 K(\xi, t) A_{j-1} dt \right\} d\xi, \quad j = 2, 3, \dots \end{aligned} \right\} \quad (38)$$

where  $g(\xi) = -\frac{\xi^2}{2} - \xi - \frac{6}{(\xi+1)^4}$ . Using (36) and (38), we obtain the solution components  $y_j$  as

**Table 4** The absolute error  $|\psi_n - y|$  and  $|\phi_n - y|$  for  $n = 1, 2, 3$  of Example 4.4

x	MDMGF			MADM [27]		
	$ \psi_1 - y $	$ \psi_2 - y $	$ \psi_3 - y $	$ \phi_1 - y $	$ \phi_2 - y $	$ \phi_3 - y $
0.1	3.61E-05	5.39E-06	6.48E-07	5.61E-04	8.39E-05	8.18E-06
0.3	2.26E-04	3.45E-05	4.22E-06	4.96E-03	2.45E-04	7.25E-05
0.5	3.69E-04	5.83E-05	7.29E-06	3.69E-03	3.83E-04	8.20E-05
0.7	3.00E-04	4.93E-05	6.39E-06	5.70E-03	6.93E-04	7.39E-05
0.9	6.31E-05	1.09E-05	1.47E-06	5.31E-04	3.09E-04	4.47E-05



**Fig. 4** Plots of the exact and the approximate solution of Example 4.4

$$\begin{aligned} y_0 &= 0, \\ y_1 &= -0.0041666x^2 + 0.005555x^3 - 0.001388x^6 + \ln(1+x), \\ y_2 &= 2.5052108 \times 10^{-8} (-x(332640 + 1.355178 \\ &\quad \times 10^{-6}x + 2.7963706 \times 10^{-6}x^2 \\ &\quad + 2.13444 \times 10^{-6}x^3 + 759528x^4 + 66x^6 - 33x^7 + x^{10}) \\ &\quad + 332640(1+x)^5 \ln(1+x)), \\ &\vdots \end{aligned}$$

In Table 4, we present the numerical results of the absolute errors  $|\psi_n - y|$  (obtained by the proposed DMGF) and  $|\phi_n - y|$  (obtained by MADM) for  $n = 1, 2, 3$ . We also plot the curves of the exact  $y$  and the approximate solution  $\psi_1$  for  $n = 1$  in Fig. 4. Like previous examples, it is observed that only two-term approximations  $\psi_1$  coincide with the exact solution  $y$ .

### Conclusions

In this paper, we studied a reliable technique based on the decomposition method and Green’s function for the numerical solution of the fourth-order BVPs for Volterra



IDEs. The technique depends on constructing Green's function before establishing the recursive scheme for the solution components. The proposed technique provides a direct recursive scheme for obtaining the approximations to the solutions of BVPs. Unlike the existing ADM or the MADM, the proposed method DMGF or MDMGF avoids unnecessary evaluation of unknown constants and provides better numerical solutions. Convergence and error analysis of the proposed technique have also been discussed. The performance of the proposed recursive scheme have been examined by solving four numerical examples. It has been shown that only two-term series solution is enough to obtain an accurate approximation to the solution.

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