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ORIGINAL RESEARCH

Classical string field mechanics with non-standard Lagrangians

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Abstract We reformulate the string classical mechanics by substituting the standard Lagrangian by a non-standard exponential Lagrangian where higher-order derivative terms occur naturally in the equations of motion. Our motivation is based on the accumulating evidence that higher-order derivatives play a leading role in string field theories. Since non-standard Lagrangians generate higher-order derivatives in a usual way, it will be of interest to explore their roles in classical string field mechanics. It was observed that replacing standard by non-standard Lagrangians gives another possibility to obtain new aspects which may have interesting physical effects.

Keywords String theory · Non-standard Lagrangians · Modified Euler–Lagrange equations

Mathematics Subject Classification Primary 70S05 · Secondary 83E30

Introduction

String theory is a quantum theory of one-dimensional objects called strings which come in two different types: open and closed. Geometrically, open strings are characterized by free endpoints, whereas closed strings are characterized by connected endpoints [19]. String theory arise in different forms depending on is associated quantum field theory. There exist many physical results which make string theory an appealing approach to describe a number of fundamental aspects in theoretical physics, mainly the particle theory and the unification problem as the main aim of string theory is to unify the standard electroweak model with a quantum theory of gravity. Some of these results include: the occurrence of graviton in closed string theories, the emergence of non-abelian gauge fields and chiral fermions in open string theories, the appearance of microscopic black holes at high energy limit, the Einstein–Hilbert action contained in the perturbative string theory and so on. From mathematical points of view, string theory joins algebraic geometry and differential geometry with theoretical physics and this is quite amazing. However, there is no clear physical reason why higher-order derivative curvature terms could not be present in string actions, yet there are some recent arguments which prove that higher-derivative terms are fundamentally important [12, 16, 17], i.e. higher-order derivative quantum corrections to supergravity. In fact, higher-derivative corrections in string theories are significantly investigated in a number of ways, e.g. nonlinear sigma model, duality, scattering amplitude and so on [1, 13, 14 and references therein]. These higher-order derivative corrections may be also added to the Einstein–Hilbert action and hence represent an additional way to describe gravitational string theories. This will help us to construct a perturbative low-energy effective action [20]. In fact, the simplest theory which describes the emergence of gravity in the string theory model is known as the bosonic string theory, which is formulated by the Polyakov action that is nothing but the action of the non-linear sigma model in two-dimensional conformal field theory [15]. The aim of this paper is to modify string actions, mainly the Nambu–Goto and the Polyakov actions by replacing the string standard Lagrangian by a string non-standard Lagrangian and to derive the corresponding

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equations of motion and study their main consequences. Our motivation to non-standardize the string Lagrangian arises from the observation that higher-order derivatives arise naturally in non-standard Lagrangian (NSL) theories and not by implementing these later by hand. In fact, NSL plays an important role in different branches of theoretical physics and applied mathematics and is in general characterized by a deformed kinetic energy term and a deformed potential function [2–11, 21, 22, 24, 25]. The follow-on modified Euler–Lagrange equation that results from the standard calculus of variations leads to equations of motion that correspond to physically interesting nonlinear dynamical systems. NSL comes in different forms, yet in this paper we choose the exponential NSL (ENSL) and prove that many interesting consequences will arise in bosonic string theory. We will deal with a bosonic string embedded in a Minkowskian flat spacetime of signature $(1, D-1)$.

The paper is organized as follows: in "The modified Nambu–Goto string action and equations of motion", we introduce basic concepts of the modified Nambu–Goto string actions and derive the corresponding equations of motion; in "The modified Polyakov string action and equations of motion", we discuss the modified Polyakov string action and the main consequences of NSL formulation; finally conclusions and perspectives are given in "Conclusions and perspectives".

The modified Nambu–Goto string action and equations of motion

In general, the action for a relativistic string must be a functional of the string trajectory. When a particle moves through a spacetime, it traces out a world line, whereas a string would trace out a surface, the worldsheet. There exist two different types of strings: the open string which traces out a flat sheet and the closed string which traces out a closed tube-like surface. When dealing with action functional, it is notable that the string action is proportional to the worldsheet proper area, in contrast to the relativistic particle where the action is proportional to the proper distance of the world line. The string action is recognized as the Nambu–Goto action. Usually, the Nambu–Goto action (NGA) which is proportional to the proper area is defined as follows [18]: we consider a scalar parameter τ and the following Lagrangian coordinates $\{X^{\mu}(\sigma), h^{ab}(\sigma)\}$ assumed to be classical fields in the curved $1+1$ worldsheet geometry $\mathcal{V}_2$ spanned by (σ, τ) and characterized by the worldsheet pure gauge metric $h_{ab}(\sigma, \tau)$. The NGA is then defined by $S = -\frac{T}{c}\int\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}\mathrm{d}\tau\mathrm{d}\sigma$, where the dot refers to the time derivative and the prime refers to the space derivative. Here, T and c are, respectively, the tension of the string and the celerity of light. In our approach, the following definition holds:

Definition 2.1 Let $X^{\mu}(\sigma)$ be string coordinates and (σ, τ) be coordinates on the worldsheet augmented by the constraint $\dot{X}_0(\text{endpoints}) \neq 0$. We define the exponentially non-standard Nambu–Goto action by

$$\mathbb{S} = -\frac{\xi T}{c}\int e^{\zeta\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}\mathrm{d}\tau\mathrm{d}\sigma, \tag{1}$$

where ξ is a parameter introduced to take into account the dimensional problem, c is the celerity of light and ζ is a constant. The NSL-density along the string is $L = -\frac{\xi T}{c}e^{\zeta\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \equiv -\frac{\xi T}{c}e^{\mathbb{L}}$.

Theorem 2.1 *The modified Euler–Lagrange equations which correspond to the action function (1) are:*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}\right) + \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathbb{L}}{\partial X'^{\mu}}\right) = -\zeta\left(\ddot{X}^{\mu}\left(\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}\right)^2 + X''^{\mu}\left(\frac{\partial\mathbb{L}}{\partial X'^{\mu}}\right)^2\right). \tag{2}$$

Proof The variation of the action (1) gives the full set:

$$\begin{aligned}\delta\mathbb{S} &= -\frac{\xi T}{c}\int_{\tau_{\text{initial}}}^{\tau_{\text{final}}}\int_0^{\sigma_1}\mathrm{d}\sigma\mathrm{d}\tau e^{-\zeta\mathbb{L}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\delta X^{\mu}\frac{\partial e^{\zeta\mathbb{L}}}{\partial\dot{X}^{\mu}}\right)\right. \\ &\quad\left.+\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\delta X^{\mu}\frac{\partial e^{\zeta\mathbb{L}}}{\partial X'^{\mu}}\right) - \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial e^{\zeta\mathbb{L}}}{\partial\dot{X}^{\mu}} + \frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial e^{\zeta\mathbb{L}}}{\partial X'^{\mu}}\right)\delta X^{\mu}\right), \\ &= -\frac{\xi T}{c}\int_0^{\sigma_1}\mathrm{d}\sigma\delta X^{\mu}\left.\frac{\partial e^{\zeta\mathbb{L}}}{\partial\dot{X}^{\mu}}\right|_{\tau_{\text{initial}}}^{\tau_{\text{final}}} - \xi T\int_{\tau_{\text{initial}}}^{\tau_{\text{final}}}\mathrm{d}\sigma\delta X^{\mu}\left.\frac{\partial e^{\zeta\mathbb{L}}}{\partial X'^{\mu}}\right|_0^{\sigma_1} \\ &\quad+\frac{\xi T}{c}\int_{\tau_{\text{initial}}}^{\tau_{\text{final}}}\int_0^{\sigma_1}\mathrm{d}\sigma\mathrm{d}\tau e^{-\zeta\mathbb{L}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial e^{\zeta\mathbb{L}}}{\partial\dot{X}^{\mu}} + \frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial e^{\zeta\mathbb{L}}}{\partial X'^{\mu}}\right)\delta X^{\mu}.\end{aligned}$$

Using the boundary conditions $\delta X^{\mu}(\tau_{initial}, \sigma) = \delta X^{\mu}(\tau_{final}, \sigma) = 0$, the equation of motion for an arbitrarily parameterized string reads:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial e^{\zeta\mathbb{L}}}{\partial\dot{X}^{\mu}} + \frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial e^{\zeta\mathbb{L}}}{\partial X'^{\mu}} = 0.$$

This equation may be written explicitly as:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}\right) + \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathbb{L}}{\partial X'^{\mu}}\right) = -\zeta\left(\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}\tau}\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}} + \frac{\mathrm{d}\mathbb{L}}{\mathrm{d}\sigma}\frac{\partial\mathbb{L}}{\partial X'^{\mu}}\right).$$

Using the chain rules

$$\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}\tau} = \frac{\partial\mathbb{L}}{\partial\tau} + \dot{X}^{\mu}\frac{\partial\mathbb{L}}{\partial X^{\mu}} + \ddot{X}^{\mu}\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}} = \ddot{X}^{\mu}\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}},$$
$$\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}\sigma} = \frac{\partial\mathbb{L}}{\partial\sigma} + X'^{\mu}\frac{\partial\mathbb{L}}{\partial X^{\mu}} + X''^{\mu}\frac{\partial\mathbb{L}}{\partial X'^{\mu}} = X''^{\mu}\frac{\partial\mathbb{L}}{\partial X'^{\mu}},$$

$$\frac{d\mathbb{L}}{d\sigma}=\frac{\partial\mathbb{L}}{\partial\sigma}+X'^{\mu}\frac{\partial\mathbb{L}}{\partial X^{\mu}}+X''^{\mu}\frac{\partial\mathbb{L}}{\partial X'^{\mu}}=X''^{\mu}\frac{\partial\mathbb{L}}{\partial X'^{\mu}},$$

the equation of motion reads as:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}\right)+\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathbb{L}}{\partial X'^{\mu}}\right)=-\zeta\left(\ddot{X}^{\mu}\left(\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}\right)^{2}+X''^{\mu}\left(\frac{\partial\mathbb{L}}{\partial X'^{\mu}}\right)^{2}\right).\quad\square$$

Remark 2.1 When $\zeta=0$, Eq. (2) is reduced to the standard Nambu–Goto equations of motion. Obviously, in Eq. (2) higher-order derivative terms appear and these new terms will modify string dynamics accordingly. These higher-order derivative terms are coupled to the parameter ζ.

To simplify the equations of motion, we follow the standard arguments by using temporal and σ-parameterizations in Minkowski spacetime $(c\mathrm{d}t,\mathrm{d}x,\mathrm{d}y,\mathrm{d}z)$ [1]. For the case of temporal parameterization, we set $\tau=t$ which is the coordinate time, as it was for the point particle.

In the temporal gauge, we can choose $X'^{\mu}=\begin{pmatrix}0\\X'\end{pmatrix}$ and $\dot{X}^{\mu}=\begin{pmatrix}c\\\dot{X}\end{pmatrix}$. We can write Eq. (1) as:

$$\mathbb{S}=-\frac{\xi T}{c}\int e^{\zeta\sqrt{(\dot{X}\cdot X')^{2}-(-c^{2}+\dot{X}\cdot\dot{X})^{2}(X'\cdot\dot{X})}}\mathrm{d}t\mathrm{d}\sigma.$$

In the σ-parameterization gauge, we can choose $s=\sigma$, i.e. arc-length parameterization which gives $|X'|=1$. We can use this to define the transverse component of the string velocity $\mathbf{v}_{\perp}=\dot{X}-(\dot{X}\cdot X')X'$, where $\dot{X}=\mathrm{d}X/\mathrm{d}t$ and $X'=\mathrm{d}X/\mathrm{d}s$. Then we can write $\mathbf{v}_{\perp}^{2}=\dot{X}\cdot\dot{X}-(\dot{X}\cdot X')^{2}$, i.e.

$$\mathbb{S}=-\frac{\xi T}{c}\int e^{\zeta\sqrt{1-v_{\perp}^{2}}}\mathrm{d}t\mathrm{d}s.$$

Corollary 2.1 *Consider the boundary condition in this parameterization, together with our pre-gauge fixed expression for $\partial\mathbb{L}/\partial X^{\sigma}$, then the equations of motion are given by*:

$$\frac{1}{c^{2}}\left(1+\zeta\frac{v^{2}}{c^{2}}\right)\ddot{X}-\left(1-\zeta\left(1-\frac{v^{2}}{c^{2}}\right)\right)X''=0.\tag{3}$$

Proof In fact, we have:

$$\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}=-\frac{\xi T}{c}\frac{(\dot{X}\cdot X')X'_{\mu}-(X'\cdot X')\dot{X}_{\mu}}{\sqrt{(\dot{X}_{\mu}X'^{\mu}_{\mu})^{2}-(\dot{X}_{\mu}\dot{X}^{\mu})^{2}(X'\cdot X')}},$$

$$\frac{\partial\mathbb{L}}{\partial X'^{\mu}}=-\frac{\xi T}{c}\frac{(\dot{X}\cdot X')\dot{X}_{\mu}-(\dot{X}\cdot\dot{X})X'_{\mu}}{\sqrt{(\dot{X}_{\mu}X'^{\mu}_{\mu})^{2}-(\dot{X}_{\mu}\dot{X}^{\mu})^{2}(X'\cdot X')}}.$$

Assuming $\tau=t$, i.e. static gauge and choosing a σ parameter in a way that $\dot{X}\cdot X'=0$, an identification made on a two-dimensional grid [26], then we can simplify these later equations respectively to:

$$\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}=\frac{\xi T}{c}\frac{(X'\cdot X')\dot{X}_{\mu}}{\sqrt{(c^{2}-v^{2})(X'\cdot X')}},$$

$$\frac{\partial\mathbb{L}}{\partial X'^{\mu}}=-\frac{\xi T}{c}\frac{(c^{2}-v^{2})X'_{\mu}}{\sqrt{(c^{2}-v^{2})(X'\cdot X')}}.$$

As we have $X'_{0}=0$, $\dot{X}_{0}=-1$ and $\ddot{X}_{0}=0$, the equation of motion is $\frac{\mathrm{d}}{\mathrm{d}\tau}(\xi T(X'\cdot X')\Big/\sqrt{(c^{2}-v^{2})(X'\cdot X')}=0$, and for the case of an arc length, we found $\sqrt{X'\cdot X'}=f(\sigma)\sqrt{c^{2}-v^{2}}\Big/\xi T$, where $f(\sigma)$ is an arbitrary function of σ.

With the choice $f(\sigma)=f_{0}=\frac{\xi T}{c}$(a constant), $X'\cdot X'=1-\frac{1}{c^{2}}\dot{X}\cdot\dot{X}=1-\frac{v^{2}}{c^{2}}$, $\frac{\partial\mathbb{L}}{\partial\dot{X}^{\mu}}=\frac{\xi T}{c^{2}}\dot{X}$ and $\frac{\partial\mathbb{L}}{\partial X'^{\mu}}=-\xi TX'$, we find:

$$\frac{1}{c^{2}}\left(1+\zeta\frac{v^{2}}{c^{2}}\right)\ddot{X}-\left(1-\zeta\left(1-\frac{v^{2}}{c^{2}}\right)\right)X''=0.\quad\square$$

Remark 2.2 This is a modified wave equation which is reduced to its standard form when $\zeta=0$.

Corollary 2.2 *In the exponentially non-standard Nambu–Goto framework, the effective celerity of light* c *is*:

$$\frac{1}{\mathsf{c}^{2}}=\frac{1}{c^{2}}\frac{1+\zeta\frac{\mathsf{v}^{2}}{c^{2}}}{1-\zeta\left(1-\frac{\mathsf{v}^{2}}{c^{2}}\right)}.\tag{4}$$

For $\zeta=1$, we find $\mathsf{c}^{2}=\frac{c^{2}\mathsf{v}^{2}}{c^{2}+\mathsf{V}^{2}}$, which means that for $\mathsf{v}=\alpha c$ where α is a positive constant, $\mathsf{c}=\frac{\alpha c}{\sqrt{1+\alpha^{2}}}<c\,\forall\alpha$.

However, for $\zeta=-1$, we find $\mathsf{c}^{2}=c^{2}\frac{2c^{2}-\mathsf{v}^{2}}{c^{2}-\mathsf{V}^{2}}$, and for $\mathsf{v}=\alpha c$, we find $\mathsf{c}=c\sqrt{\frac{2-\alpha^{2}}{1-\alpha^{2}}}$, and hence for $0<\alpha<1$, we find $\mathsf{c}>c$, i.e. the effective celerity of light is greater than the celerity of light. For $\alpha\geq\sqrt{2}$, $\mathsf{c}<c$, but $\mathsf{v}>c$.

Corollary 2.3 *For the case of a rotating string, the solution of Eq.* (3) *is given by:*

$$\begin{aligned}\mathrm{X}(\tau,\sigma)&=\frac{\sigma_{\mathrm{final}}}{\pi}\cos\left(\frac{\pi\sigma}{\sigma_{\mathrm{final}}}\right)\left(\cos\left(\frac{\pi\mathrm{ct}}{\sigma_{\mathrm{final}}}\right)\hat{x}+\sin\left(\frac{\pi\mathrm{ct}}{\sigma_{\mathrm{final}}}\right)\hat{y}\right),\\&\equiv\frac{\sigma_{\mathrm{final}}}{\pi}\cos\left(\frac{\pi\sigma}{\sigma_{\mathrm{final}}}\right)\left(\cos\left(\frac{\pi ct}{\sigma_{\mathrm{final}}}\sqrt{\frac{1-\zeta\left(1-\frac{\mathrm{v}^{2}}{c^{2}}\right)}{1+\zeta\frac{\mathrm{v}^{2}}{c^{2}}}}\right)\hat{x}\right.\\&\left.\quad+\sin\left(\frac{\pi ct}{\sigma_{\mathrm{final}}}\sqrt{\frac{1-\zeta\left(1-\frac{\mathrm{v}^{2}}{c^{2}}\right)}{1+\zeta\frac{\mathrm{v}^{2}}{c^{2}}}}\right)\hat{y}\right),\\&\equiv\mathrm{X}_{x}(\tau,\sigma)\hat{x}+\mathrm{X}_{y}(\tau,\sigma)\hat{y},\end{aligned}\tag{5}$$

and the perpendicular velocity which corresponds to the transverse motion is

$$\mathsf{v}_\perp = \sqrt{\dot{X}\cdot\dot{X}} = \mathsf{c}\cos\left(\frac{\pi\sigma}{\sigma_{\text{final}}}\right) \equiv c\sqrt{\frac{1-\zeta\left(1-\frac{\mathsf{v}^2}{c^2}\right)}{1+\zeta\frac{\mathsf{v}^2}{c^2}}}\cos\left(\frac{\pi\sigma}{\sigma_{\text{final}}}\right). \quad (6)$$

Proof The proof follows directly from Eq. (3) after considering a rotating string with constant angular velocity and assuming a general solution of the motion describing left and right motion of the form [26]: $X(\tau,\sigma) = \beta(A\cos\chi(\gamma-ct) + B\cos\chi(\gamma+ct))\hat{x} + \beta(C\sin\chi(\gamma-ct) + D\sin\chi(\gamma+ct))\hat{y}$, $\beta,\gamma,\chi\in\mathbb{R}$, where (A,B,C,D) are constants to be determined from boundary conditions and $(\hat{x},\hat{y})$ are unit vectors. Using the boundary condition $X'(\tau,\sigma=0) = X'(\tau,\sigma=\sigma_{\text{final}}) = 0$ gives, respectively, $A=B, D=-C$ and $\chi = n\pi/\sigma_{\text{final}}$. For $n=1$ and using the fact that $\dot{X}\cdot X' = 0$ gives $C=-A$, finally the condition $X'\cdot X' = 1 - \dot{X}\cdot\dot{X}/c^2$ gives $A = \pm\sigma_{\text{final}}/2\beta\pi$. Consequently, Eq. (5) is obtained after using Eq. (4). Equation (6) is a straightforward derivation. □

For $\zeta=-1$ and $\mathsf{v}=\alpha c$, we have $\mathsf{c} = c\sqrt{\frac{2-\alpha^2}{1-\alpha^2}}$ and then Eq. (6) takes the form:

$$\mathsf{v}_\perp = c\sqrt{\frac{2-\alpha^2}{1-\alpha^2}}\cos\left(\frac{\pi\sigma}{\sigma_{\text{final}}}\right).$$

Hence for $0<\alpha<1$, the end of the string travels at a speed larger than the velocity of light in contrast to the case $\zeta=1$ where the end of the string moves at a speed lower than the velocity of light. These results show the main differences between the standard Lagrangian and the NSL approach in string classical mechanics. We plot in Figs. 1 and 2, respectively, the variations of $X_x(\tau,\sigma)$ and $X_y(\tau,\sigma)$ for $\zeta=1, \sigma_{\text{final}}=\pi$ and $\mathsf{v}=c=1$(for illustration purpose) and in Figs. 3 and 4, respectively, the variations of $X_x(\tau,\sigma)$ and $X_y(\tau,\sigma)$ for $\zeta=-1, \sigma_{\text{final}}=\pi$ and $\mathsf{v}=\alpha c$ for $\alpha=\frac{1}{2}$.

It is obvious that the dynamics between cases $\zeta=1$ and $\zeta=-1$ differs. The rotating string oscillates more rapidly for the case $\zeta=-1$ than $\zeta=1$, which is due to the fact that the end of the string moves at a velocity larger than the velocity of light. For $\zeta\ll1$, we find the standard result and Figs. 5 and 6 illustrate the variations of $X_x(\tau,\sigma)$ and $X_y(\tau,\sigma)$.

The modified Polyakov string action and equations of motion

In the standard approach, the Nambu–Goto action functional is somewhat difficult due to the occurrence of the square root in the Lagrangian density. However, one way

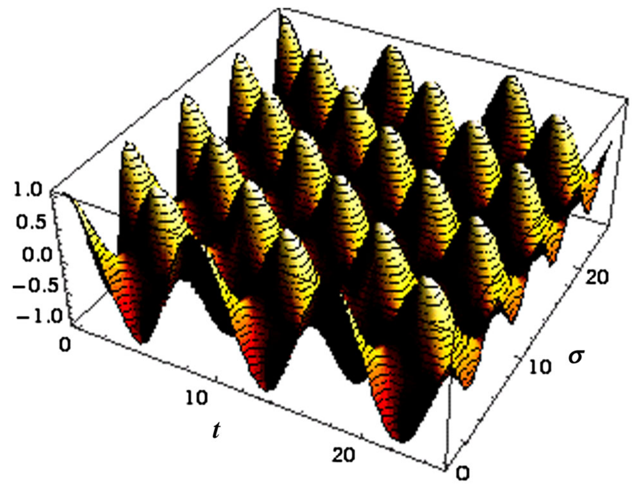

Fig. 1 Plot of $X_x(\tau,\sigma)$ for $\zeta=1$

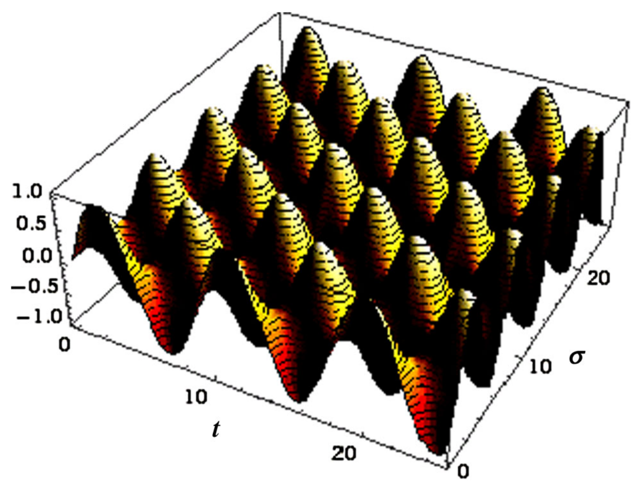

Fig. 2 Plot of $X_y(\tau,\sigma)$ for $\zeta=1$

to remove the square root is to introduce an auxiliary field in the curved 1 + 1 worldsheet geometry $\mathcal{V}_2$ known as the pure gauge metric $h_{ab}(\sigma,\tau)$ [18]. The spacetime is assumed to be flat with metric $\eta=(+,-,-,-)$ and its relation with the worldsheet geometry is through the constraints: $\eta_{\mu\nu}\partial_\tau X^\mu\partial_\tau X^\nu<0$ and $\eta_{\mu\nu}\partial_\sigma X^\mu\partial_\sigma X^\nu>0$. These conditions which correspond, respectively, for space-like and time-like tangent vectors are accompanied by the initial conditions $X(\sigma,\tau_0)=X_0(\sigma)$ and $\partial_\tau X(\sigma,\tau_0)=Y_0(\sigma)$. This will lead to the standard-Polyakov action being defined by $S = -\frac{T}{2c}\eta_{\mu\nu}\int d\sigma\sqrt{h}h^{ab}\partial_a X^\mu\partial_b X^\nu$, $(a,b=0,1),(\mu,\nu=0,1,\ldots,D)$, which is the starting point for the path integral quantization. Here, $h=\det h_{ab}$ and $(h^{-1})^{ab}=h_{ab}$. In our approach, the following definition holds.

Definition 3.1 Let $X^\mu(\sigma)$ be string coordinates, (σ,τ) be coordinates on the worldsheet geometry $\mathcal{V}_2$ with pure

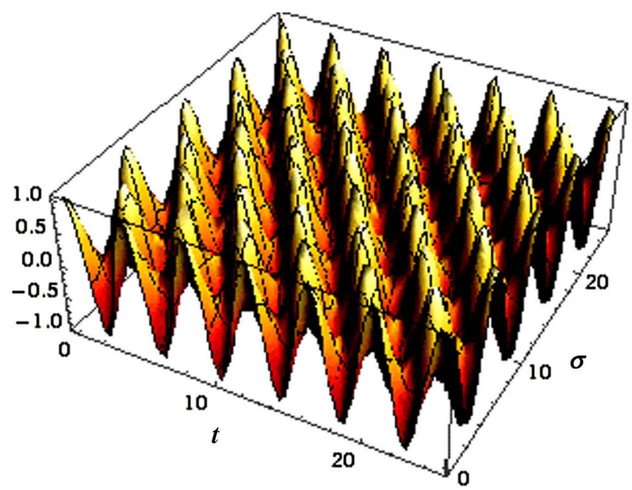

Fig. 3 Plot of $X_x(\tau,\sigma)$ for $\zeta = -1$

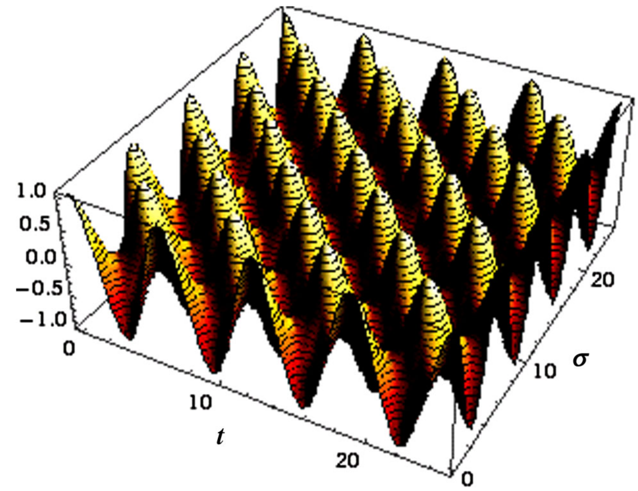

Fig. 5 Plot of $X_x(\tau,\sigma)$ for $\zeta << 1$

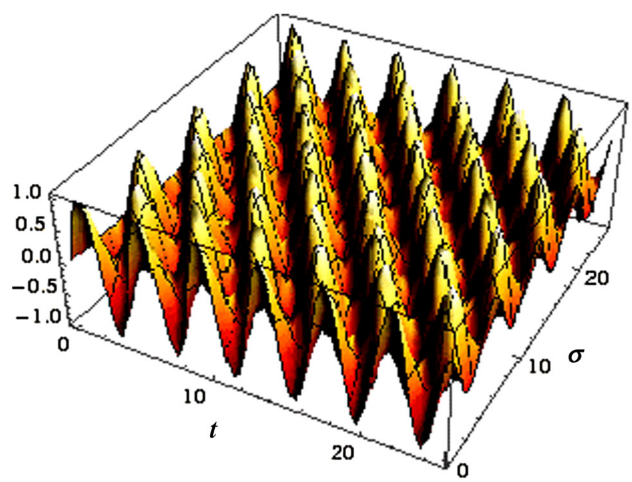

Fig. 4 Plot of $X_y(\tau,\sigma)$ for $\zeta = -1$

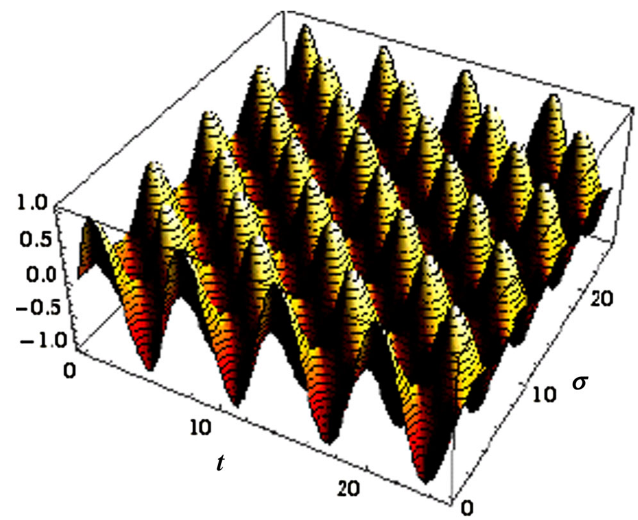

Fig. 6 Plot of $X_y(\tau,\sigma)$ for $\zeta << 1$

gauge metric $h_{ab}(\sigma,\tau)$ augmented by the constraints $\eta_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu < 0$ and $\eta_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu > 0$. We define the exponentially non-standard-Polyakov action by

$$S = -\frac{\xi T}{2c}\eta_{\mu\nu}\int e^{\zeta\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu}\mathrm{d}\sigma, \tag{7}$$

with the Lagrangian density along the string given by $L = -\frac{\xi T}{2c}\eta_{\mu\nu}e^{\zeta\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu} \equiv -\frac{\xi T}{2c}\eta_{\mu\nu}e^{\mathbb{L}}$.

Corollary 3.1 *The Lagrangian density for the exponentially non-standard-Polyakov action is written as*:

$$\begin{aligned} L &= -\frac{\xi T}{2c}\eta_{\mu\nu}e^{\zeta\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu} \\ &= -\frac{\xi T}{2c}\eta_{\mu\nu}e^{\zeta\sqrt{h}\left(h^{\tau\tau}\dot{X}^\mu X^\nu + 2h^{\tau\sigma}\dot{X}^\mu X'^\nu + h^{\sigma\sigma}X'^\mu X'^\nu\right)}, \end{aligned} \tag{8}$$

where the dot refers to the time derivative and the prime refers to the space derivative.

Usually, for the case of ENSL $S = \int e^{\zeta\mathbb{L}(\dot{x},x,t)}\mathrm{d}t$, the Euler–Lagrange equation is modified and $\frac{\partial\mathbb{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbb{L}}{\partial\dot{x}}\right) = \zeta\frac{\partial\mathbb{L}}{\partial\dot{x}}\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}t}$, where $\frac{\mathrm{d}\mathbb{L}}{\mathrm{d}t} = \frac{\partial\mathbb{L}}{\partial t} + \dot{x}\frac{\partial\mathbb{L}}{\partial x} + \ddot{x}\frac{\partial\mathbb{L}}{\partial\dot{x}}$ is the total derivative operator [5].

Theorem 3.1 *For the case of an explicitly time-independent Lagrangian density, the string coordinates $X^\mu(\sigma)$ and the gauge metric $h_{ab}(\sigma,\tau)$, respectively, obey the following modified Euler–Lagrange equations*:

$$\frac{\partial\mathbb{L}}{\partial X^\mu} - \partial_a\frac{\partial\mathbb{L}}{\partial(\partial_a X^\mu)} = \zeta\frac{\partial\mathbb{L}}{\partial(\partial_a X^\mu)}\left(\partial_a X^\mu\frac{\partial\mathbb{L}}{\partial X^\mu} + \partial_a\partial_a X^\mu\frac{\partial\mathbb{L}}{\partial(\partial_a X^\mu)}\right), \tag{9}$$

$$\frac{\partial\mathbb{L}}{\partial h^{bc}} - \partial_a\frac{\partial\mathbb{L}}{\partial(\partial_a h^{bc})} = \zeta\frac{\partial\mathbb{L}}{\partial(\partial_a h^{bc})}\left(\partial_a h^{bc}\frac{\partial\mathbb{L}}{\partial h^{bc}} + \partial_a\partial_a h^{bc}\frac{\partial\mathbb{L}}{\partial(\partial_a h^{bc})}\right). \tag{10}$$

Corollary 3.2 *The equations of motion for the action functional (7) are:*

$$\partial_a\partial^a X_\mu - \frac{1}{2}h_{cd}\partial_a h^{cd}\partial_a X_\mu = \frac{\zeta\xi T}{c}\sqrt{h}(\partial^a X_\mu)^2\partial_a\partial_a X^\mu, \quad (11)$$

$$\partial_b X^\mu \partial_c X_\mu - \frac{h_{bc}}{2}h^{ef}\partial_e X^\mu \partial_f X_\mu = 0. \quad (12)$$

Proof From Eq. (8), we find that $\frac{\partial \mathbb{L}}{\partial X^\mu} = 0$ and $\frac{\partial \mathbb{L}}{\partial(\partial_a X^\mu)} = -\frac{\xi T}{c}\sqrt{h}\partial^a X_\mu$. Then from Eq. (9), we find:

$$\partial_a\left(\sqrt{h}\partial^a X_\mu\right) = \frac{\zeta\xi T}{c}\sqrt{h}\partial^a X_\mu\left(\sqrt{h}\partial^a X_\mu\partial_a\partial_a X^\mu\right).$$

Using the fact that $\partial_a(\sqrt{h}\partial^a X_\mu) = \sqrt{h}\partial_a\partial^a X_\mu + \partial_a\sqrt{h}\partial^a X_\mu = \sqrt{h}\partial_a\partial^a X_\mu - \frac{1}{2}\sqrt{h}h_{cd}\partial_a h^{cd}\partial_a X_\mu$, we find:

$$\partial_a\partial^a X_\mu - \frac{1}{2}h_{cd}\partial_a h^{cd}\partial_a X_\mu = \frac{\zeta\xi T}{c}\sqrt{h}(\partial^a X_\mu)^2\partial_a\partial_a X^\mu.$$

Again from Eq. (8), $\frac{\partial \mathbb{L}}{\partial h^{bc}} = -\frac{\xi T}{2c}(\partial_b X^\mu \partial_c X_\mu - \frac{h_{bc}}{2}h^{ef}\partial_e X^\mu \partial_f X_\mu)$ and $\frac{\partial \mathbb{L}}{\partial(\partial_a h^{bc})} = 0$. Then Eq. (10) gives:

$$\partial_b X^\mu \partial_c X_\mu - \frac{h_{bc}}{2}h^{ef}\partial_e X^\mu \partial_f X_\mu = 0. \qquad \square$$

Remark 3.1 In the NSL approach, only Eq. (11) which corresponds to string coordinates is modified. This equation holds higher-order derivative terms. Equation (12) is similar to the one derived in the standard approach.

Lemma 3.1 *Let* $G_{bc} = \partial_b X^\mu \partial_c X_\mu$, *then* $S = -\frac{\xi T}{2c}\int \mathrm{d}\tau \int \mathrm{d}\sigma e^{\zeta\sqrt{G}}$.

Equation (12) is equivalent to $G_{bc} = \frac{h_{bc}}{2}tr\|G\|$ for $G_{bc} = \partial_b X^\mu \partial_c X_\mu$ which is written after some algebra as $\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu = 2G$ [18], and hence we find the require results. $\square$

In fact, Lemma 3.1 states that "the exponentially non-standard Nambu–Goto action is obtained from the exponentially non-standard-Polyakov action using Euler–Lagrange equations for the worldsheet metric". However, in our approach, the equation of motion (11) for the string coordinates $X^\mu(\sigma)$ is different.

Remark 3.2 Using diffeomorphisms and Weyl transformation [23], one can write in the conformal gauge Eq. (7) as

$$S = -\frac{\xi T}{2c}\int \mathrm{d}^2\sigma e^{\zeta\left(X'^2 - \frac{1}{c^2}\dot{X}^2\right)},$$

and in particular in the Minkowski spacetime after making the choice $h_{ab} = \eta_{ab}$, where η_{ab} is the Minkowski signature, i.e. conformal gauge and c is the effective celerity of light defined by Eq. (4).

The corresponding equation of motion for X is then given by Eq. (2) which is rewritten in that case as:

$$\frac{1}{c^2}\ddot{X}(1 - 2\zeta^2\dot{X}^2) - X''(1 + 2\zeta^2 X'^2) = 0.$$

This equation has to be consistent with the equation of motion (12) for h^{ab}, which is $T_{ab} = \partial_a X \partial_b X - \frac{1}{2}\eta_{ab}\eta^{ef}\partial_e X \partial_f X = 0$ and gives $T_{01} = X' \cdot \dot{X} = 0$ and $T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0$, known as the Virasoro constraints. Accordingly, we can write the equation of motion as $(1 - 2\zeta^2\dot{X}^2)(\ddot{X} - c^2 X'') = 0$ and then $\ddot{X} - c^2 X'' = 0$ or $1 - 2\zeta^2\dot{X}^2 = 0$. So the equation of motion implies a harmonic wave equation $\ddot{X} - c^2 X'' = 0$ or a solution of the form $\dot{X}^2 = -X'^2 = 1/2\zeta^2$. This last solution gives $X(\sigma) = \pm\sqrt{-\frac{1}{2\zeta}}\sigma + X_{\sigma 0}$ and $X(\tau) = \pm\sqrt{\frac{1}{2\zeta}}\tau + X_{\tau 0}$, where $X_{\sigma 0}$ and $X_{\tau 0}$ are initial solutions. For $\zeta < 0 (> 0)$, we find a complexified (real) evolution of $X(\tau)$ and a real (complexified) evolution of $X(\sigma)$. In other words, these solutions show that we have a complexified linear evolution in time, which is not realistic unless we perform a Wick rotation $\tau \to i\tau, i = \sqrt{-1} \in \mathbb{C}$ for $\zeta < 0$, i.e. complexified action. For $X_{\sigma 0} = X_{\tau 0} = 0$, we have $X^2(\sigma) - X^2(\tau) = -\frac{1}{2\zeta}(\tau^2 + \sigma^2)$ which means that for $\zeta < 0 (> 0)$, we have $X^2(\sigma) - X^2(\tau) > (<) 0$ that corresponds, respectively, to stable solutions and unstable solutions.

Conclusions and perspectives

The objective of this work was to discuss the impacts of exponentially non-standard Lagrangians in string classical mechanics. In reality, NSL naturally generates higher-order derivatives in the equations of motion; therefore, our basic motivation was to explore the main consequences of these higher-order terms in string theory as there exist many arguments which proved that these terms played an important role in mainly all string theories. To do this, we gave in this work the basic setups where two exponentially non-standard-string Lagrangian densities were discussed: the Nambu–Goto and the Polyakov actions. It was observed that the non-standard formulation of both fundamental actions gave some new insights into classical string theory.

For the case of an exponentially non-standard Nambu–Goto action, the wave equation is characterized by an effective celerity of light which depends on the sign of the parameter ζ. For positive value of ζ, the effective celerity of light is lower than c, whereas for the case of a negative value of ζ, the effective celerity of light is greater than c which means that the end of the string moves at a velocity larger than the celerity of light, in particular when the velocity of motion is less than the celerity of light, $v = \alpha c$ with $0 < \alpha < 1$. Moreover, for the case of a rotating string, it was observed that the dynamics depends also on the sign of the parameter ζ. Oscillations are faster for the negative case than for the positive case.

For the case of an exponentially non-standard-Polyakov action, the equation of motion for the string coordinates differs from the standard case where higher-order derivative terms occur, whereas it is identical to the standard case for the gauge metric. However, the exponentially non-standard Numbu–Goto action may be derived from the exponentially non-standard-Polyakov action using Euler–Lagrange equations for the worldsheet metric. Using diffeomorphisms and Weyl transformation in the Minkowski spacetime, we have introduced a new exponentially non-standard-Polyakov action where the celerity of light is replaced by an effective one. In that way, the equation of motion yields the harmonic wave equation characterized as in the exponentially non-standard Nambu–Goto action by an effective celerity of light, which depends on the sign of the parameter ζ.

These results must be used to explore their main consequences in different string theoretical aspects, mainly their quantization aspects. Work in this direction is under progress.

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