Math Sci (2015) 9:33–38
DOI 10.1007/s40096-015-0145-x

ORIGINAL RESEARCH

Some fixed point theorems for contractive maps in N-cone metric spaces

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Received: 15 August 2014 / Accepted: 23 February 2015 / Published online: 13 March 2015

Abstract In present paper, we prove unique fixed point theorems for contractive maps in N-cone metric spaces. Our results extend and generalize some well-known results of (Banach, Fund Math 3:133–181 1992; Chatterjee, Rend Acad Bulgare Sci 25:727–730 1972; Kannan, Bull Calcutta Math Soc 60:71–76 1968; Rezapour and Hamlbarani, J Math Anal Appl 345:719–724 2008) in the setting of N-cone metric spaces.

Keywords N-cone metric space · Fixed point · Contractive map

Mathematics Subject Classification 54H25 · 47H10

Introduction and preliminaries

The notion of cone metric space was introduced in [7]. In this paper, Huang and Zhang replace the real numbers by ordering Banach space and define cone metric space. They also gave an example of a function which is a contraction in the category of cone metric but not contraction if considered over metric spaces and hence by proving fixed point theorem in cone metric spaces ensured that this map must have a unique fixed point.

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Subsequently, Rezapour and Hamlbarani [14] omitted the assumption of normality in cone metric space. After that a series of articles in cone metric space started to appear (see, [3, 10, 15, 16]).

Recently, Aage and Salunke [1] introduced a generalized D^*-metric space and Ismat Beg et al. [4] introduced G-cone metric space. Very recently, Malviya and Fisher [13] introduced the notion of N-cone metric space and proved fixed point theorems for asymptotically regular maps and sequence. This new notion generalized the notion of generalized G-cone metric space [4] and generalized D^*-metric space [1]. In [12], the authors defined expansive maps in N-cone metric spaces and proved various fixed point theorems.

In present paper, we prove the Banach contraction theorem [2] and fixed point theorems of Kannan [11], Chatterjee [5] and Rezapour et al. [14] in N-cone metric space. The examples and application in support of our results are also given.

Throughout this paper, let E be a real Banach space and P be a subset of E. P is called a cone, if and only if

(1) P is closed, nonempty and $P \neq 0$;
(2) $ax + by \in P$, for all $x, y \in P$ and non-negative real numbers a, b;
(3) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$, if and only if $y - x \in P$, $x < y$ will stand for $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P.

The cone P is called normal if there is a number $N > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$. The least positive number satisfying the above is called the normal constant of P [7].

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 1.1 [14] *Every regular cone is normal.*

Definition 1.1 [13] Let X be a non-empty set. An N-cone metric on X is a function $N : X^3 \to E$, that satisfies the following conditions: for all $x, y, z, a \in X$

(1) $N(x, y, z) \geq 0$;
(2) $N(x, y, z) = 0$ if and only if $x = y = z$;
(3) $N(x, y, z) \leq N(x, x, a) + N(y, y, a) + N(z, z, a)$.

Then, the function N is called an N-cone metric and the pair (X, N) is called an N-cone metric space.

Remark 1.1 [13] It is easy to see that every generalized-D^*-metric space is an N-cone metric space but in general, the converse is not true, see the following example.

Example 1.1 [13] Let $E = R^3$, $P = \{(x, y, z) \in E, x, y, z \geq 0\}$, $X = R$, and $N : X \times X \times X \to E$ is defined by

$$N(x,y,z) = \Big(\alpha(\,|\,y+z-2x\,|+|\,y-z\,|\,), \beta(\,|\,y+z-2x\,| + |\,y-z\,|\,), \gamma(\,|\,y+z-2x\,|+|\,y-z\,|\,)\Big)$$

where α, β, γ are positive constants. Then, (X, N) is an N-cone metric space but not a generalized-D^*-metric space because N is not symmetric.

Proposition 1.1 [13] *If (X, N) is an N-cone metric space, then for all $x, y, z \in X$, we have $N(x, x, y) = N(y, y, x)$.*

Definition 1.2 [13] Let (X, N) be an N-cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is $\mathtt{N}$ such that for all $n > \mathtt{N}$, $N(x_n, x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$ as $(n \to \infty)$.

Lemma 1.2 [13] *Let (X, N) be an N-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ also converges to y then $x = y$. That is the limit of $\{x_n\}$, if exists, is unique.*

Definition 1.3 [13] Let (X, N) be an N-cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$ there is $\mathtt{N}$ such that for all $m, n > \mathtt{N}$, $N(x_n, x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.4 [13] Let (X, N) be an N-cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete N-cone metric space.

Lemma 1.3 [13] *Let (X, N) be an N-cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Definition 1.5 [13] Let (X, N) and (X', N') be N-cone metric spaces. Then, a function $f : X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is, whenever $\{x_n\}$ is convergent to x we have $\{fx_n\}$ is convergent to $f(x)$.

Lemma 1.4 [13] *Let (X, N) be an N-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and suppose that $x_n \to x$, $y_n \to y$ as $n \to \infty$. Then $N(x_n, x_n, y_n) \to N(x, x, y)$ as $n \to \infty$.*

Remark 1.2 [13] If $x_n \to x$ in an N-cone metric space X, then every subsequence of $\{x_n\}$ converges to x in X.

Proposition 1.2 [13] *Let (X, N) be an N-cone metric space and P be a cone in a real Banach space E. If $u \leq v$, $v \ll w$ then $u \ll w$.*

Lemma 1.5 [13] *Let (X, N) be an N-cone metric space, P be an N-cone in a real Banach space E and $k_1, k_2, k_3, k_4, k > 0$. If $x_n \to x$, $y_n \to y$, $z_n \to z$ and $p_n \to p$ in X and*

$$ka \leq k_1 N(x_n, x_n, x) + k_2 N(y_n, y_n, y) + k_3 N(z_n, z_n, z) + k_4 N(p_n, p_n, p), \text{ then } a = 0.$$

The following lemmas are often used.

Lemma 1.6 [10] *Let P be a cone and $\{x_n\}$ be a sequence in E. If $c \in intP$ and $0 \leq x_n \to 0$ (as $n \to \infty$), then there exists N such that for all $n > N$, we have $x_n \ll c$.*

Lemma 1.7 [10] *Let $x, y, z \in E$, if $x \leq y$ and $y \ll z$, then $x \ll z$.*

Lemma 1.8 [9] *Let P be a cone and $0 \leq u \ll c$ for each $c \in intP$, then $u = 0$.*

Lemma 1.9 [6] *Let P be a cone. If $u \in P$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = 0$.*

Lemma 1.10 [10] *Let P be a cone and $a \leq b + c$ for each $c \in intP$, then $a \leq b$.*

Topology of N-cone metric space Let (X, N) be an N-cone metric space, each N-cone metric N on X generates a topology τ_N on X whose base is the family of open N-balls defined as

$$B_N(x, c) = \{y \in X : N(y, y, x) \ll c\},$$

for $c \in E$ with $0 \ll c$ and for all $x \in X$.

Definition 1.6 Let (X, N) be an N-cone metric space. A map $f : X \to X$ is said to be a contractive mapping if there exists a constant $0 \le k < 1$ such that

$$N(fx, fx, fy) \le kN(x, x, y) \text{ for all } x, y \in X.$$

Example 1.2 Let $E = R^3$, $P = \{(x, y, z) \in E,\ x, y, z \ge 0\}$ and $X = R$ and $N : X \times X \times X \to E$ is defined by

$$N(x, y, z) = (\alpha(|x - z| + |y - z|), \beta(|x - z| + |y - z|), \gamma(|x - z| + |y - z|)),$$

where α, β, γ are positive constants. Then (X, N) is an N-cone metric space. Define a self-map f on X as follows $fx = \frac{x}{4}$ for all x. Clearly, f is a contractive map in X.

Main results

Theorem 2.1 *Let (X, N) be a complete N-cone metric space and the mapping $T : X \to X$ satisfies the contractive condition*

$$N(Tx, Tx, Ty) \le kN(x, x, y) \tag{2.1}$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then, T has a unique fixed point in X. For each $x \in X$, the sequence of iterates $\{T^n x\}_{n \ge 1}$ converges to the fixed point.

Proof For each $x_0 \in X$ and $n \ge 1$, set $x_1 = Tx_0$ and $x_{n+1} = T^{n+1}x_0$.

Then

$$\begin{aligned} N(x_n, x_n, x_{n+1}) &= N(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\le kN(x_{n-1}, x_{n-1}, x_n) \\ &\le k^2 N(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\vdots \\ &\vdots \\ &\le k^n N(x_0, x_0, x_1) \end{aligned}$$

So for $m > n$,

$$\begin{aligned} N(x_n, x_n, x_m) \le\ & 2\ N(x_n, x_n, x_{n+1}) + 2N(x_{n+1}, x_{n+1}, x_{n+2}) \\ & + \cdots + 2N(x_{m-2}, x_{m-2}, x_{m-1}) \\ & + N(x_{m-1}, x_{m-1}, x_m) \\ \le\ & 2N(x_n, x_n, x_{n+1}) + 2N(x_{n+1}, x_{n+1}, x_{n+2}) \\ & + \cdots + 2N(x_{m-2}, x_{m-2}, x_{m-1}) \\ & + 2N(x_{m-1}, x_{m-1}, x_m) \\ \le\ & [2k^n + 2k^{n+1} + \cdots + 2k^{m+n-2} \\ & + 2k^{m+n-1}]N(x_0, x_0, x_1) \\ =\ & 2k^n[1 + k + k^2 + \cdots + k^{m+n-1}]N(x_0, x_0, x_1) \\ <\ & \frac{2k^n}{1-k} N(x_0, x_0, x_1) \end{aligned}$$

Let $0 \ll c$ be given. Choose a natural number N_1 such that $\frac{2k^n}{1-k} N(x_0, x_0, x_1) \ll c$ for all $n \ge N_1$. Thus, $N(x_n, x_n, x_m) \ll c$ for all $n > m$.

Therefore, $\{x_n\}_{n \ge 1}$ is a cauchy sequence in (X, N). Since (X, N) is a complete N-cone metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Choose a natural number N_2 such that $N(x^*, x^*, x_n) \ll \frac{c}{4k}$ and $N(x^*, x^*, x_{n+1}) \ll \frac{c}{2}$ for all $n \ge N_2$.

Hence, for all $n \ge N_2$, we have

$$\begin{aligned} N(Tx^*, Tx^*, x^*) &\le N(Tx^*, Tx^*, Tx_n) + N(Tx^*, Tx^*, Tx_n) \\ &\quad + N(x^*, x^*, Tx_n) \\ &\le 2N(Tx^*, Tx^*, Tx_n) + N(x^*, x^*, Tx_n) \\ &\le 2kN(x^*, x^*, x_n) + N(x^*, x^*, x_{n+1}) \\ &\ll 2k\frac{c}{4k} + \frac{c}{2} \\ &= c \end{aligned}$$

for all $n \ge N_2$. Thus, $N(Tx^*, Tx^*, x^*) \ll \frac{c}{m}$ for all $m \ge 1$. So $\frac{c}{m} - N(Tx^*, Tx^*, x^*) \in P$ for all $m \ge 1$. Since $\frac{c}{m} \to 0$ (as $m \to \infty$) and P is closed, $-N(Tx^*, Tx^*, x^*) \in P$, but $N(Tx^*, Tx^*, x^*) \in P$. Therefore, $N(Tx^*, Tx^*, x^*) = 0$ and so $Tx^* = x^*$.

To prove uniqueness, let y^* be another fixed point of T, then

$$\begin{aligned} N(x^*, x^*, y^*) &= N(Tx^*, Tx^*, Ty^*) \\ &\le kN(x^*, x^*, y^*), \end{aligned}$$

which implies that by Lemma (1.9) $N(x^*, x^*, y^*) = 0$. Hence the fixed point of T is unique. □

Corollary 2.1 *Let (X, N) be a complete N-cone metric space. Suppose the mapping $T : X \to X$ satisfies for some positive integer n,*

$$N(T^n x, T^n x, T^n y) \le k.N(x, x, y),$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then, T has a unique fixed point in X.

Proof From Theorem (2.1), T^n has a unique fixed point x^*. But $T^n(Tx^*) = T(T^n x^*) = Tx^*$. So Tx^* is also a fixed point of T^n. Hence $Tx^* = x^*$, x^* is a fixed point of T. Since the fixed point of T is also fixed point of T^n, then fixed point of T is unique. □

Theorem 2.2 *Let (X, N) be a complete N-cone metric space. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$N(Tx, Tx, Ty) \le k[N(Tx, Tx, x) + N(Ty, Ty, y)],$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. Then, T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n \ge 1}$ converges to the fixed point.

Proof For each $x_0 \in X$ and $n \geq 1$, set $x_1 = Tx_0$ and $x_{n+1} = T^{n+1}x_0$.

Then, we have

$$\begin{aligned} N(x_n, x_n, x_{n+1}) &= N(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq k[N(Tx_{n-1}, Tx_{n-1}, x_{n-1}) + N(Tx_n, Tx_n, x_n)] \\ &= k[N(x_n, x_n, x_{n-1}) + N(x_{n+1}, x_{n+1}, x_n)] \\ &= k[N(x_n, x_n, x_{n-1}) + N(x_n, x_n, x_{n+1})] \\ &\quad [\text{by Proposition 1.1.}] \end{aligned}$$

So

$$\begin{aligned} N(x_n, x_n, x_{n+1}) &\leq \frac{k}{1-k} N(x_n, x_n, x_{n-1}) \\ &= hN(x_n, x_n, x_{n-1}) \quad \text{where } h = \frac{k}{1-k} \\ &= hN(x_{n-1}, x_{n-1}, x_n) \quad [\text{by Proposition 1.1.}] \\ &\vdots \\ &\leq h^n N(x_0, x_0, x_1), \end{aligned} \tag{2.2}$$

Now using (2.2), we can prove $\{x_n\}_{n \geq 1}$ is a Cauchy sequence as proved in Theorem (2.1).

Since (X, N) is a complete N-cone metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Choose a natural number N_2 such that $N(x_{n+1}, x_{n+1}, x_n) \ll \frac{(1-2k)c}{4k}$ and $N(x^*, x^*, x_{n+1}) \ll \frac{(1-2k)c}{2}$, for all $n \geq N_2$.

Hence, for $n \geq N_2$, we have

$$\begin{aligned} N(Tx^*, Tx^*, x^*) &\leq N(Tx^*, Tx^*, Tx_n) + N(Tx^*, Tx^*, Tx_n) \\ &\quad + N(x^*, x^*, Tx_n) \\ &= 2N(Tx^*, Tx^*, Tx_n) + N(x^*, x^*, Tx_n) \\ &\leq 2k[N(Tx^*, Tx^*, x^*) + N(Tx_n, Tx_n, x_n)] \\ &\quad + N(x^*, x^*, Tx_n). \end{aligned}$$

Thus,

$$\begin{aligned} N(Tx^*, Tx^*, x^*) &\leq \frac{1}{1-2k}[2kN(x_{n+1}, x_{n+1}, x_n) + N(x^*, x^*, x_{n+1})] \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Thus, $N(Tx^*, Tx^*, x^*) \ll \frac{c}{m}$ for all $m \geq 1$.

So $\frac{c}{m} - N(Tx^*, Tx^*, x^*) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$ and P is closed, $-N(Tx^*, Tx^*, x^*) \in P$. But $N(Tx^*, Tx^*, x^*) \in P$. Therefore, $N(Tx^*, Tx^*, x^*) = 0$ and so $Tx^* = x^*$.

Now, if y^* is another fixed point of T, then

$$\begin{aligned} N(x^*, x^*, y^*) &= N(Tx^*, Tx^*, Ty^*) \\ &\leq k[N(Tx^*, Tx^*, x^*) + N(Ty^*, Ty^*, y^*)] \\ &= k[N(x^*, x^*, x^*) + N(y^*, y^*, y^*)] \\ &= 0 \quad [\text{by Definition 1.1. and by Lemma 1.5}] \end{aligned}$$

Hence $x^* = y^*$. Therefore, the fixed point of T is unique. □

Theorem 2.3 *Let (X, N) be a complete N-cone metric space. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$N(Tx, Tx, Ty) \leq k[N(Tx, Tx, y) + N(x, x, Ty)],$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. Then, T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

Proof For each $x_0 \in X$ and $n \geq 1$, set $x_1 = Tx_0$ and $x_{n+1} = T^{n+1}x_0$.

Then, we have

$$\begin{aligned} N(x_n, x_n, x_{n+1}) &= N(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq k[N(Tx_{n-1}, Tx_{n-1}, x_n) + N(x_{n-1}, x_{n-1}, Tx_n)] \\ &= k[N(x_n, x_n, x_n) + N(x_{n-1}, x_{n-1}, x_{n+1})] \\ &= kN(x_{n-1}, x_{n-1}, x_{n+1}) \\ &\leq k[2N(x_{n-1}, x_{n-1}, x_n) + N(x_n, x_n, x_{n+1})] \\ &\quad [\text{by Definition 1.1}] \\ &\leq \frac{2k}{1-k} N(x_{n-1}, x_{n-1}, x_n) \\ &= hN(x_{n-1}, x_{n-1}, x_n) \quad \text{where, } h = \frac{2k}{1-k} \\ &\leq h^n N(x_0, x_0, x_1). \end{aligned} \tag{2.3}$$

Now using (2.3), we can prove $\{x_n\}_{n \geq 1}$ is a Cauchy sequence as proved in Theorem (2.1).

Since (X, N) is a complete N-cone metric space, there exists $x^* \in X$ such that $x_n \to x^*$.

Now, we have

$$\begin{aligned} N(Tx^*, Tx^*, x^*) &\leq N(Tx^*, Tx^*, Tx_n) + N(Tx^*, Tx^*, Tx_n) \\ &\quad + N(x^*, x^*, Tx_n) \\ &\leq 2N(Tx^*, Tx^*, Tx_n) + N(x^*, x^*, Tx_n) \\ &\leq 2k[N(Tx^*, Tx^*, x_{n+1}) + N(x^*, x^*, Tx_n)] \\ &\quad + N(x^*, x^*, Tx_n) \\ &= 2kN(Tx^*, Tx^*, x_n) + (2k+1)N(x^*, x^*, x_{n+1}) \\ &= 2kN(Tx^*, Tx^*, x^*) + (2k+1)N(x^*, x^*, x^*) \\ &\quad \text{as } n \to \infty \\ &\leq 2kN(Tx^*, Tx^*, x^*), \quad \left[\text{As } 0 \leq k < \frac{1}{2}\right] \end{aligned}$$

which implies that, by Lemma 1.9, $N(Tx^*, Tx^*, x^*) = 0$. Hence $Tx^* = x^*$.

Now, if y^* is another fixed point of T, then

$$\begin{aligned} N(x^*, x^*, y^*) &= N(Tx^*, Tx^*, Ty^*) \\ &\leq k[N(Tx^*, Tx^*, y^*) + N(x^*, x^*, Ty^*)] \\ &= k[N(x^*, x^*, y^*) + N(x^*, x^*, y^*)] \\ &= 2kN(x^*, x^*, y^*) \quad [\text{by Lemma (1.9)}]. \end{aligned}$$

Hence, $N(x^*, x^*, y^*) = 0$ and so $x^* = y^*$. Therefore, the fixed point of T is unique. □

Theorem 2.4 *Let (X, N) be a complete N-cone metric space. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$N(Tx, Tx, Ty) \le kN(x, x, y) + lN(x, x, Ty),$$

for all $x, y \in X$, where $k, l \in [0, 1)$ is a constant. Then, T has a fixed point in X. Also the fixed point of T is unique whenever $k + 3l < 1$.

Proof For each $x_0 \in X$ and $n \ge 1$, set $x_1 = Tx_0$ and $x_{n+1} = T^{n+1}x_0$.

$$\begin{aligned}N(x_n, x_n, x_{n+1}) &= N(Tx_{n-1}, Tx_{n-1}, Tx_n)\\ &\le kN(x_{n-1}, x_{n-1}, x_n) + lN(x_{n-1}, x_{n-1}, Tx_n)\\ &= kN(x_{n-1}, x_{n-1}, x_n) + lN(x_{n-1}, x_{n-1}, x_{n+1})\\ &\le kN(x_{n-1}, x_{n-1}, x_n) + l[N(x_{n-1}, x_{n-1}, x_n)\\ &\quad + N(x_{n-1}, x_{n-1}, x_n)\\ &\quad + N(x_{n+1}, x_{n+1}, x_n)] \quad [\text{byDefinition1.1}]\\ &\le kN(x_{n-1}, x_{n-1}, x_n) + l[2N(x_{n-1}, x_{n-1}, x_n)\\ &\quad + N(x_{n+1}, x_{n+1}, x_n)]\end{aligned}$$

So

$$\begin{aligned}N(x_n, x_n, x_{n+1}) &\le \frac{k + 2l}{1 - l} N(x_{n-1}, x_{n-1}, x_n)\\ &\quad [\text{by Proposition}(1.1)]\\ &\le hN(x_{n-1}, x_{n-1}, x_n)\\ &\le h^n N(x_0, x_0, x_1),\end{aligned} \tag{2.4}$$

where $h = \frac{k+2l}{1-l}$.

Now using (2.4), we can prove $\{x_n\}_{n \ge 1}$ is a Cauchy sequence as proved in Theorem (2.1).

Since (X, N) is a complete N-cone metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Choose a natural number N_2 such that $N(x^*, x^*, x_n) \ll \frac{c}{4k}$ and $N(x^*, x^*, x_{n+1}) \ll \frac{c}{2(2l+1)}$, for all $n \ge N_2$.

Hence, for $n \ge N_2$, we have

$$\begin{aligned}N(Tx^*, Tx^*, x^*) &\le N(Tx^*, Tx^*, Tx_n) + N(Tx^*, Tx^*, Tx_n)\\ &\quad [\text{by Definition 1.1}]\\ &\quad + N(x^*, x^*, Tx_n)\\ &\le 2N(Tx^*, Tx^*, Tx_n) + N(x^*, x^*, Tx_n)\\ &= 2kN(x^*, x^*, x_n) + 2lN(x^*, x^*, Tx_n)\\ &\quad + N(x^*, x^*, Tx_n)\\ &= 2kN(x^*, x^*, x_n) + (2l + 1)N(x^*, x^*, x_{n+1})\\ &\ll \frac{c}{2} + \frac{c}{2} = c\end{aligned}$$

Thus, $N(Tx^*, Tx^*, x^*) \ll \frac{c}{m}$ for all $m \ge 1$.

Hence, $\frac{c}{m} - N(Tx^*, Tx^*, x^*) \in P$ for all $m \ge 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$ and P is closed, $-N(Tx^*, Tx^*, x^*) \in P$, but $N(Tx^*, Tx^*, x^*) \in P$. Therefore, $N(Tx^*, Tx^*, x^*) = 0$ and so $Tx^* = x^*$.

Now, if y^* is another fixed point of T, then

$$\begin{aligned}N(x^*, x^*, y^*) &= N(Tx^*, Tx^*, Ty^*)\\ &\le kN(x^*, x^*, y^*) + lN(x^*, x^*, Ty^*)\\ &= (k + l)N(x^*, x^*, y^*)\\ &\quad [\text{by Lemma 1.9 and since } k + 3l < 1]\end{aligned}$$

Hence, $N(x^*, x^*, y^*) = 0$ and so $x^* = y^*$. Therefore, the fixed point of T is unique. □

Example 2.1 Let $E = R^3$, $P = \{(x, y, z) \in E,\ x, y, z \ge 0\}$ and $X = R$ and $N : X \times X \times X \to E$ is defined by

$$\begin{aligned}N(x, y, z) = (&\alpha(|x - z| + |y - z|), \beta(|x - z|\\ &+ |y - z|), \gamma(|x - z| + |y - z|)),\end{aligned}$$

where α, β, γ are positive constants. Then, (X, N) is an N-cone metric space. Define a self-map T on X as follows $Tx = \frac{x}{3}$ for all x. If we take $\alpha = \frac{1}{3}$, then the contractive condition (2.1) holds trivially good and 0 is the unique fixed point of the map T.

Application

In this section, we shall apply Theorem 2.1 to the following first-order periodic boundary value problem:

$$\frac{dx}{dt} = F(t, x(t)), \quad \text{with } x(0) = \xi, \tag{3.5}$$

where $F : [-h, h] \times [\xi - \delta, \xi + \delta]$ is a continuous function.

Example 3.1 Consider the boundary value problem (3.5) with the continuous function F and suppose $F(x, y)$ satisfies the local Lipschitz condition, i.e. if $|x| \le h$, $y_1, y_2 \in [\xi - \delta, \xi + \delta]$ it induces

$$|F(x, y_1) - F(x, y_2)| \le L|y_1 - y_2|$$

Set $M = \max_{[-h,h] \times [\xi-\delta, \xi+\delta]} |F(x, y)|$ such that $2h < \min[\frac{\delta}{M}, \frac{1}{L}]$, then there exists a unique solution of (3.5).

Proof Let $X = E = C([-h, h])$ and $P = \{u \in E : u \ge 0\}$. Put $N : X \times X \times X \to E$ as

$$\begin{aligned}N(x, x, y) &= f(t) \max_{-h \le t \le h} (|x(t) - y(t)| + |x(t) - y(t)|)\\ &= f(t) \max_{-h \le t \le h} 2|x(t) - y(t)|\end{aligned}$$

with $f : [-h, h] \to R$ such that $f(t) = e^t$.

It is clear that (X, N) is a complete N-cone metric space.

Note that (3.5) is equivalent to the integral equation

$$x(t) = \xi + \int_0^t F(\tau, x(\tau))d\tau.$$

Define a mapping $T : C([-h, h]) \to R$ by

$$Tx(t) = \xi + \int_0^t F(\tau, x(\tau))d\tau.$$

If $x(t), y(t) \in B(\xi, \delta f) \triangleq \{\psi(t) \in C([-h, h]) : N(\xi, \xi, \psi) \leq \delta f\}$, then we have

$$\begin{aligned} N(Tx, Tx, Ty) &= f(t) \max_{-h \leq t \leq h} 2 \left| \int_0^t F(\tau, x(\tau))d\tau - \int_0^t F(\tau, y(\tau))d\tau \right| \\ &= 2f(t) \max_{-h \leq t \leq h} \left| \int_0^t [F(\tau, x(\tau)) - F(\tau, y(\tau))]d\tau \right| \\ &\leq 2hf(t) \max_{-h \leq \tau \leq h} |F(\tau, x(\tau)) - F(\tau, y(\tau))| \\ &= 2hf(t) \max_{-h \leq \tau \leq h} L|x(\tau) - y(\tau)| \\ &\leq 2hLf(t) \max_{-h \leq \tau \leq h} |x(\tau) - y(\tau)| \\ &= hLN(x, x, y) \end{aligned}$$

and

$$\begin{aligned} N(Tx, Tx, \xi) &= 2f(t) \max_{-h \leq t \leq h} \left| \int_0^t F(\tau, x(\tau))d\tau \right| \\ &\leq 2hf \max_{-h \leq \tau \leq h} |F(\tau, x(\tau))| \\ &\leq 2hMf \\ &\leq \delta f \end{aligned}$$

We speculate $T : B(\xi, \delta f) \to B(\xi, \delta f)$ is a contractive mapping.

Finally, we prove that $(B(\xi, \delta f), N)$ is complete. In fact, suppose $\{x_n\}$ is a Cauchy sequence in $B(\xi, \delta f)$. Then $\{x_n\}$ is also a Cauchy sequence in X. Since (X, N) is complete, there is $x \in X$ such that $x_n \to x$ as $n \to \infty$. So for each $c \in intP$, there exists N_1, whenever $n > N_1$ we obtain $N(x, x, x_n) \ll \frac{c}{2}$. Thus, it follows that

$$\begin{aligned} N(x, x, \xi) &\leq N(x, x, x_n) + N(x, x, x_n) + N(\xi, \xi, x_n) \\ &\quad \text{[by Definition1.1]} \\ &\leq 2N(x, x, x_n) + N(Tx_{n-1}, Tx_{n-1}, \xi) \\ &\quad \text{[by Proposition 1.1 and definition of T in Theorem 2.1]} \\ &= c + \delta f \end{aligned}$$

and by Lemma (1.10), $N(x, x, \xi) \leq \delta f$ which means $x \in B(\xi, \delta f)$, that is, $(B(\xi, \delta f), N)$ is complete.

Owing to the above statement, all the conditions of Theorem 2.1 are satisfied. Hence, T has a unique fixed point $x(t) \in B(\xi, \delta f)$. That is to say, there exists a unique solution of Example (3.1).

We notice that the above-mentioned application of fixed point theorem in b-cone metric space was given by [8]. □

Conclusion

In this paper, we define topology in N-cone metric space and extend various famous results such as Banach contraction theorem and Chatterjee's theorem in this newly defined space with applications in integral equations.

Acknowledgments The first author is thankful to Prof. Pankaj Kumar Jhade, Department of Mathematics, NRI Institute of Information Science & Technology, Bhopal (MP), India and Dr. Kalpana Saxena, Department of Mathematics, Government MVM College, Bhopal (MP), India for his/her suggestions and constant encouragement. The authors are also thankful to the referee for their valuable comments and suggestions which improved greatly the quality of this paper.

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