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Long-term dynamics of fractional stochastic delay reaction–diffusion equations on unbounded domains

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Abstract
In this paper, we investigate the long-term dynamics of fractional stochastic delay reaction-diffusion equations on unbounded domains with a polynomial drift term of arbitrary order driven by nonlinear noise. We first define a mean random dynamical system in a Hilbert space for the solutions of the equation and prove the existence and uniqueness of weak pullback mean random attractors. We then establish the existence and regularity of invariant measures of the system under further conditions on the nonlinear delay and diffusion terms. We also prove the tightness of the set of all invariant measures of the equation when the time delay varies in a bounded interval. We finally show that every limit of a sequence of invariant measures of the delay equation must be an invariant measure of the limiting system as delay approaches zero. The uniform tail-estimates and the Ascoli–Arzelà theorem are used to derive the tightness of distribution laws of solutions in order to overcome the non-compactness of Sobolev embeddings on unbounded domains.

Keywords Fractional equation · Stochastic equation · Time delay · Invariant measure · Tightness · Weak mean attractor

Mathematics Subject Classification 37L40 · 37L55 · 60H15 · 35B40

Contents

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1 Introduction

This paper is concerned with the long-term dynamics of the fractional stochastic delay reaction-diffusion equation with a polynomial drift term defined on $\mathbb{R}^n$:

$$\begin{aligned} du(t) + (-\triangle)^\alpha u(t)dt + \lambda u(t)dt + F(t, x, u(t))dt \\ = G(t, u(t-\rho))dt + \sigma(t, u(t))dW(t), \quad t > \tau, \end{aligned} \tag{1.1}$$

with initial data

$$u(\tau + s) = \varphi(s), \quad s \in [-\rho, 0], \tag{1.2}$$

where $(-\triangle)^\alpha$ with $\alpha \in (0, 1)$ is the fractional Laplace operator, λ is a positive constant, $\rho \in [0, 1]$ is a time delay parameter, F, G and σ are nonlinear functions, and W is a two-sided cylindrical Wiener process in a Hilbert space U defined on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

We will investigate mean random attractors and invariant measures of (1.1)–(1.2) under certain conditions on the nonlinear drift term F, delay term G and diffusion term σ. Indeed, in the non-autonomous case, we will prove the existence and uniqueness of weak mean random attractors for the dynamical system associated with (1.1)–(1.2) in $L^2(\Omega, \mathscr{F}; L^2(\mathbb{R}^n)) \times L^2\big(\Omega, \mathscr{F}; L^2((-\rho, 0), L^2(\mathbb{R}^n))\big)$ when F is a polynomial nonlinearity of arbitrary order, G and σ are locally Lipschitz continuous. Notice that the diffusion coefficient σ of white noise in (1.1) is nonlinear, and hence the pathwise random attractors theory does not apply to (1.1)–(1.2). That is why we study the weak mean random attractors instead of pathwise random attractors in this paper. Nevertheless, we remark that the pathwise random attractors theory is very effective for dealing with stochastic equations driven by linear white noise, see, e.g., [1–9] and the references therein.

On the other hand, because the weak mean random attractors theory is built up on reflexive Banach spaces [10–12] and $C([-\rho, 0]; L^2(\mathbb{R}^n))$ of continuous functions from $[-\rho, 0]$ to $L^2(\mathbb{R}^n)$ is not reflexive, we need to choose the Hilbert space $L^2(\Omega, \mathscr{F}; L^2(\mathbb{R}^n)) \times L^2\big(\Omega, \mathscr{F}; L^2((-\rho, 0), L^2(\mathbb{R}^n))\big)$ rather than the space $L^2\big(\Omega, \mathscr{F}; C([-\rho, 0], L^2(\mathbb{R}^n))\big)$ as a phase space for studying mean random attractors of (1.1)–(1.2), though the space $C([-\rho, 0]; L^2(\mathbb{R}^n))$ is often chosen as a phase space for pathwise random attractors.

The main goal of this paper is to investigate the existence and the limiting behavior of invariant measures of the autonomous version of (1.1)–(1.2) in the product space

$L^2(\mathbb{R}^n) \times L^2((-\rho,0), L^2(\mathbb{R}^n))$ when the delay ρ varies over a bounded interval. The concept of invariant measure is an important tool for understanding the asymptotic behavior of stochastic systems from the point of statistical dynamics view. For instance, the existence of such invariant measures has been studied in [13–16] for finite-dimensional stochastic delay systems in $\mathbb{R}^n$, and in [17–19] for infinite-dimensional stochastic delay lattice systems in l^2.

When $F \equiv 0$, G and σ are globally Lipschitz continuous, the existence of invariant measures of (1.1)–(1.2) in $C([-\rho,0], L^2(\mathbb{R}^n))$ was recently investigated in [20]. In the present paper, we will deal with the case where F has a polynomial growth rate of arbitrary order. The polynomial nonlinearity of F introduces an essential difficulty for establishing the tightness of distribution laws of a family of solutions in $L^2(\mathbb{R}^n) \times L^2((-\rho,0), L^2(\mathbb{R}^n))$. Indeed, in this case, we have to derive the uniform estimates of solutions in $L^r(\Omega, L^r(\mathbb{R}^n))$ for sufficiently large r (see Lemma 4.6). We will employ the Ito formula for the norm of solutions in the space $L^r(\mathbb{R}^n)$ as given in [21] to derive such uniform estimates. Furthermore, we need to establish the regularity of solutions in $L^2(\Omega, H^\alpha(\mathbb{R}^n))$ for initial data in $L^2(\Omega, L^2(\mathbb{R}^n)) \times L^2(\Omega, L^2(-\rho,0), L^2(\mathbb{R}^n))$ (see Lemma 4.7) as well as the regularity in $L^{r_0}(\Omega, H^\alpha(\mathbb{R}^n))$ for initial data in $L^{r_0}(\Omega, H^\alpha(\mathbb{R}^n)) \times L^{r_0}(\Omega, L^{r_0}(-\rho,0), H^\alpha(\mathbb{R}^n))$ for some appropriate $r_0 > 1$ depending on the nonlinear terms in (1.1) (see Lemma 4.9). All these uniform estimates will be used to prove the Hölder continuity of solutions in time in the space $L^{r_0}(\Omega, L^2(\mathbb{R}^n))$ (see Lemma 4.10), which will be further used to obtain the pathwise equicontinuity of solutions in time based on the Kolmogorov theorem.

Note that the stochastic equation (1.1) is defined on the unbounded domain $\mathbb{R}^n$, and hence the standard Sobolev embeddings are non-compact. This introduces another major difficulty for proving the tightness of distribution laws of a set of solutions in $L^2(\mathbb{R}^n) \times L^2((-\rho,0), L^2(\mathbb{R}^n))$. We will overcome this difficulty by the idea of uniform tail-estimates of solutions outside a sufficiently large ball in $\mathbb{R}^n$. More precisely, we will first show the uniform smallness of solutions for large space variables (see Lemma 4.4), and then apply the compactness of Sobolev embeddings in bounded domains as well as the pathwise equicontinuity of solutions to establish the tightness of distribution laws of solutions (see Lemma 4.12). The tightness of distributions of solutions immediately yields the existence of invariant measures of (1.1)–(1.2) in $L^2(\mathbb{R}^n) \times L^2((-\rho,0), L^2(\mathbb{R}^n))$ by the Krylov-Bogolyubov method (see Theorem 4.1). For existence of invariant measures of stochastic PDEs without delay in unbounded domains, we refer the reader to [22–30] for more details.

Based on the existence of invariant measures, we will further investigate the limits of a family of invariant measures of (1.1)–(1.2) in $L^2(\mathbb{R}^n) \times L^2((-\rho,0), L^2(\mathbb{R}^n))$ as the delay $\rho \to \rho_0 \in [0,1)$. To that end, we need to establish the regularity of invariant measures of (1.1)–(1.2) in $H^\alpha(\mathbb{R}^n) \times L^\infty((-\rho,0), H^\alpha(\mathbb{R}^n))$ (see Theorem 5.1), which means that every invariant measure of (1.1)–(1.2) in $L^2(\mathbb{R}^n) \times L^2((-\rho,0), L^2(\mathbb{R}^n))$ is supported by $H^\alpha(\mathbb{R}^n) \times L^\infty((-\rho,0), H^\alpha(\mathbb{R}^n))$. We then prove the tightness of the collection of all invariant measures of (1.1)–(1.2) as ρ varies on the interval $[0,1]$ by using the regularity of invariant measures as well as the uniform estimates of solutions with respect to $\rho \in [0,1]$ (see Theorem 6.1). We finally prove that every limit of a family of invariant measures of (1.1)–(1.2) as $\rho \to \rho_0 \in [0,1)$ must be an invariant measure of the corresponding limiting system (see Theorem 7.2). For the limits of invariant measures of stochastic PDEs without delay as the noise intensity approaches zero, the reader is referred to [31] and [32] for bounded and unbounded domains, respectively.

This paper is organized as follows. In Sect. 2, we prove the existence and uniqueness of solutions and define a mean random dynamical system. Section 3 is devoted to the existence and uniqueness of weak mean random attractors in $L^2(\Omega, \mathscr{F}; L^2(\mathbb{R}^n)) \times L^2\big(\Omega, \mathscr{F}; L^2((-\rho, 0), L^2(\mathbb{R}^n))\big)$. In Sect. 4, we derive all necessary uniform estimates of solutions and prove the existence of invariant measures in $L^2(\mathbb{R}^n) \times L^2((-\rho, 0), L^2(\mathbb{R}^n))$. Sections 5 and 6 are devoted to the regularity of invariant measures and the tightness of the collection of all invariant measures of (1.1)–(1.2) when ρ varies on $[0, 1]$, respectively. In the last section, we show every limit of a family of invariant measures of (1.1)–(1.2) as $\rho \to \rho_0 \in [0, 1)$ must be an invariant measure of the limiting system.

Throughout this paper, we write $H_\rho = L^2(\mathbb{R}^n) \times L^2((-\rho, 0), L^2(\mathbb{R}^n))$ if $\rho \in (0, 1]$, and $H_\rho = L^2(\mathbb{R}^n)$ if $\rho = 0$. For convenience, we also denote $L^2(\mathbb{R}^n)$ by H with inner product $(\cdot, \cdot)$ and norm and $\|\cdot\|$. If $u(t)$, $t > \tau - \rho$, is an H-valued stochastic process, then for every $t \geqslant \tau$, define $u_t : (-\rho, 0) \to L^2(\mathbb{R}^n)$ by $u_t(s) = u(t + s), \forall s \in (-\rho, 0)$. Given a Banach space Z, we use $L^2(\Omega, \mathscr{F}; Z)$ for the space of all strongly $\mathscr{F}$-measurable square-integrable Z-valued random variables. The notation $L^2(\Omega, \mathscr{F}_t; Z)$ with $t \in \mathbb{R}$ will be understood similarly. We also use $\mathcal{L}_2(U, H)$ for the space of Hilbert-Schmidt operators from a separable Hilbert space U to H with norm $\|\cdot\|_{\mathcal{L}_2(U,H)}$.

2 Mean random dynamical systems

In this section, we prove the existence and uniqueness of solutions of (1.1)–(1.2) and define a mean random dynamical system based on the solution operators. For that purpose, we first discuss the assumptions on the nonlinear functions in (1.1).

(F1). $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and $F(\cdot, \cdot, 0) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ and

$$F(t, x, u)u \geqslant \lambda_1 |u|^p - \psi_1(t, x), \tag{2.1}$$

$$|F(t, x, u)| \leqslant \psi_2(t, x)|u|^{p-1} + \psi_3(t, x), \tag{2.2}$$

$$\frac{\partial F(t, x, u)}{\partial u} \geqslant -\psi_4(t, x), \tag{2.3}$$

where $\lambda_1 > 0$ and $p > 2$ are constants, $\psi_1 \in L^1_{loc}(\mathbb{R}, L^1(\mathbb{R}^n))$, $\psi_2 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n))$, $\psi_3 \in L^q_{loc}(\mathbb{R}, L^q(\mathbb{R}^n))$ and $\psi_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)) \cap L^{\frac{2q}{2-q}}_{loc}(\mathbb{R}, L^{\frac{2q}{2-q}}(\mathbb{R}^n))$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(F2). $F(t, x, u)$ is locally Lipschitz continuous in u uniformly with respect to $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$; that is, for any bounded interval I, there exists a constant $C^F_I > 0$ such that

$$|F(t, x, u_1) - F(t, x, u_2)| \leqslant C^F_I |u_1 - u_2|, \quad \forall\, t \in \mathbb{R},\ x \in \mathbb{R}^n,\ u_1, u_2 \in I. \tag{2.4}$$

(G1). $G : \mathbb{R} \times H \to H$ is continuous such that

$$\|G(t,u)\| \leqslant \|h(t)\| + a\|u\|, \quad \forall\, t \in \mathbb{R}, \quad u \in H, \tag{2.5}$$

where $a > 0$ is a constant and $h \in L^2_{loc}(\mathbb{R}, H)$.
(G2). $G(t,u)$ is locally Lipschitz continuous in $u \in H$ uniformly with respect to $t \in \mathbb{R}$; that is, for any $r > 0$, there exists a constant $C_r^G > 0$ such that

$$\|G(t,u_1) - G(t,u_2)\| \leqslant C_r^G \|u_1 - u_2\|, \quad \forall\, t \in \mathbb{R},\ \|u_1\| \leqslant r,\ \|u_2\| \leqslant r. \tag{2.6}$$

For the diffusion coefficients of noise, we assume that $\sigma : \mathbb{R} \times H \to \mathcal{L}_2(U,H)$ is continuous and
(Σ1). $\sigma(t,u)$ is locally Lipschitz continuous in $u \in H$ uniformly with respect to $t \in \mathbb{R}$; that is, for every $r > 0$, there exists a constant $C_r^\sigma > 0$ such that

$$\|\sigma(t,u_1) - \sigma(t,u_2)\|_{\mathcal{L}_2(U,H)} \leqslant C_r^\sigma \|u_1 - u_2\|, \quad \forall\, t \in \mathbb{R},\ \|u_1\| \leqslant r,\ \|u_2\| \leqslant r. \tag{2.7}$$

(Σ2). $\sigma(t,u)$ grows linearly in $u \in H$ uniformly for $t \in \mathbb{R}$; that is, there exists a constant $L > 0$ such that for all $(t,u) \in \mathbb{R} \times H$,

$$\|\sigma(t,u)\|_{\mathcal{L}_2(U,H)} \leqslant L(1 + \|u\|). \tag{2.8}$$

Recall that for $\alpha \in (0,1)$, the Hilbert space $H^\alpha(\mathbb{R}^n)$ is defined by

$$H^\alpha(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} dx dy < \infty \right\},$$

with inner product

$$\begin{aligned}(u,v)_{H^\alpha(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} u(x)v(x)dx \\ &\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2\alpha}} dx dy, \quad \forall\, u,v \in H^\alpha(\mathbb{R}^n),\end{aligned}$$

and norm $\|u\|_{H^\alpha(\mathbb{R}^n)} = (u,u)^{\frac{1}{2}}_{H^\alpha(\mathbb{R}^n)}$ for $u \in H^\alpha(\mathbb{R}^n)$. Note that for all $u \in H^\alpha(\mathbb{R}^n)$,

$$\|u\|_{H^\alpha(\mathbb{R}^n)} = \left(\|u\|^2 + \frac{2}{C(n,\alpha)} \|(-\triangle)^{\frac{\alpha}{2}} u\|^2 \right)^{\frac{1}{2}},$$

where $C(n,\alpha) = \frac{\alpha 4^\alpha \Gamma(\frac{n+2\alpha}{2})}{\pi^{\frac{n}{2}} \Gamma(1-\alpha)}$ and $(-\triangle)^\alpha$ is the fractional Laplace operator given by (see, e.g., [33]):

$$(-\triangle)^\alpha u(x) = -\frac{1}{2} C(n,\alpha) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2\alpha}} dy, \quad x \in \mathbb{R}^n.$$

For convenience, we write $V = H^\alpha(\mathbb{R}^n)$ with inner product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$.

A solution of problem (1.1)–(1.2) will be understood in the following sense.

Definition 2.1 Suppose $u^0 \in L^2(\Omega, \mathscr{F}_\tau; H)$ and $\varphi \in L^2\big(\Omega, \mathscr{F}_\tau; L^2((-\rho, 0), H)\big)$. Then an H-valued stochastic process $u(t)$, $t \geq \tau - \rho$, is called a weak solution of problem (1.1)–(1.2) in the sense of PDEs if

(i) $u \in L^2\big(\Omega, \mathscr{F}_\tau; L^2((\tau-\rho, \tau), H)\big)$ and $u_\tau = \varphi$.
(ii) u is pathwise continuous on $[\tau, \infty)$, and $\mathscr{F}_t$-adapted for all $t \geqslant \tau$, $u(\tau) = u^0$, and $u \in L^2\big(\Omega, C([\tau, \tau+T], H)\big) \cap L^2\big(\Omega, L^2(\tau, \tau+T; V)\big) \cap L^p\big(\Omega, L^p(\tau, \tau+T; L^p(\mathbb{R}^n))\big)$ for all $T > 0$.
(iii) For all $t \geqslant \tau$ and $\xi \in V \cap L^p(\mathbb{R}^n)$,

$$
\begin{aligned}
&(u(t), \xi) + \int_\tau^t \big((-\triangle)^{\frac{\alpha}{2}} u(s),\ (-\triangle)^{\frac{\alpha}{2}} \xi\big) ds + \lambda \int_\tau^t \big(u(s),\ \xi\big) ds \\
&\quad + \int_\tau^t \int_{\mathbb{R}^n} F(s, x, u(s)) \xi \, dx ds \\
&\quad = \big(u^0,\ \xi\big) + \int_0^t \big(G(s, u(s-\rho)),\ \xi\big) ds + \int_0^t \big(\xi,\ \sigma(s, u(s)) dW(s)\big), \quad \mathbb{P}\text{-almost surely.}
\end{aligned}
$$

Next, we show the existence and uniqueness of solutions of problem (1.1)–(1.2).

Theorem 2.2 *Suppose* **(F1)-(F2)**, **(G1)-(G2)** *and* **(Σ1)-(Σ2)** *hold. Then for any* $u^0 \in L^2(\Omega, \mathscr{F}_\tau; H)$ *and* $\varphi \in L^2\big(\Omega, \mathscr{F}_\tau; L^2((-\rho, 0), H)\big)$, *problem (1.1)–(1.2) has a unique solution u in the sense of Definition 2.1. Moreover, for any* $T > 0$,

$$
\begin{aligned}
&\mathbb{E}\left(\|u\|^2_{C([\tau,\tau+T],H)}\right) + \mathbb{E}\left(\|u\|^2_{L^2(\tau,\tau+T;V)}\right) + \mathbb{E}\left(\|u\|^p_{L^p(\tau,\tau+T;L^p(\mathbb{R}^n))}\right) \\
&\quad \leqslant M\left[\mathbb{E}\left(\|u^0\|^2\right) + \int_{-\rho}^0 \mathbb{E}\left(\|\varphi(s)\|^2\right) ds + T + \int_\tau^{\tau+T} \left(\|\psi_1(s)\|_{L^1(\mathbb{R}^n)} + \|h(s)\|^2\right) ds\right] e^{MT},
\end{aligned}
\tag{2.9}
$$

where M is a positive constant independent of u^0, φ, ρ, τ and T.

Proof We first show the existence of solutions on $[\tau, \tau+\rho]$. By **(G1)** we have for any $\varphi \in L^2\big(\Omega, \mathscr{F}_\tau; L^2((-\rho, 0), H)\big)$,

$$
\begin{aligned}
\int_\tau^{\tau+\rho} \mathbb{E}\left(\|G(t, u(t-\rho))\|^2\right) dt &= \int_\tau^{\tau+\rho} \mathbb{E}\left(\|G(t, \varphi(t-\rho-\tau))\|^2\right) dt \\
&\leqslant 2\int_\tau^{\tau+\rho} \|h(t)\|^2 dt + 2a^2 \int_{-\rho}^0 \mathbb{E}\left(\|\varphi(s)\|^2\right) ds < \infty.
\end{aligned}
\tag{2.10}
$$

In terms of (2.10), (1.1)–(1.2) on $[\tau, \tau+\rho]$ is equivalent to the following system without delay:

$$
\begin{cases}
du(t) + (-\triangle)^\alpha u(t) dt + \lambda u(t) dt + F(t, \cdot, u(t)) dt \\
\quad = G(t, \varphi(t-\rho-\tau)) dt + \sigma(t, u(t)) dW(t), \ t \in (\tau, \tau+\rho], \\
u(\tau) = u^0.
\end{cases}
\tag{2.11}
$$

Then by Theorem 6.3 in [25], under conditions **(F1)-(F2)**, **(G1)-(G2)** and **(Σ1)-(Σ2)**, problem (2.11) has a unique solution u defined on $[\tau, \tau+\rho]$ such that $u \in L^2\big(\Omega, C([\tau, \tau+\rho], H)\big) \cap L^2\big(\Omega, L^2(\tau, \tau+\rho; V)\big) \cap L^p\big(\Omega, L^p(\tau, \tau+\rho; L^p(\mathbb{R}^n))\big)$. Repeating this argument, one can extend the solution u to the interval $[\tau, \infty)$ such that $u \in L^2\big(\Omega, C([\tau, \tau+T], H)\big) \cap L^2\big(\Omega, L^2(\tau, \tau+T; V)\big) \cap L^p\big(\Omega, L^p(\tau, \tau+T; L^p(\mathbb{R}^n))\big)$ for any $T > 0$.

Next, we derive the uniform estimates of solutions. Applying Ito's formula to (1.1), we obtain

$$\begin{aligned}
&\|u(t)\|^2 + 2\int_\tau^t \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2 ds + 2\lambda \int_\tau^t \|u(s)\|^2 ds + 2\int_\tau^t \int_{\mathbb{R}^n} F(s, x, u(s)) u(s) dx ds \\
&\quad = \|u(\tau)\|^2 + 2\int_\tau^t \big(G(s, u(s-\rho)),\ u(s)\big) ds + \int_\tau^t \|\sigma(s, u(s))\|^2_{\mathcal{L}_2(U,H)} ds \\
&\qquad + 2\int_\tau^t \big(u(s),\ \sigma(s, u(s)) dW(s)\big). \qquad (2.12)
\end{aligned}$$

For the fourth term on the left-hand side of (2.12), by (2.1), we have

$$2\int_\tau^t \int_{\mathbb{R}^n} F(s, x, u(s)) u(s) dx ds \geqslant 2\lambda_1 \int_\tau^t \|u(s)\|^p_{L^p(\mathbb{R}^n)} ds - 2\int_\tau^t \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds. \qquad (2.13)$$

For the second term on the right-hand side of (2.12), by **(G1)**, we obtain

$$\begin{aligned}
&2\int_\tau^t \big(G(s, u(s-\rho)),\ u(s)\big) ds \\
&\quad \leqslant \left(1 + 2a^2\right) \int_\tau^t \|u(s)\|^2 ds + 2\int_\tau^t \|h(s)\|^2 ds + 2a^2 \int_{-\rho}^0 \|\varphi(s)\|^2 ds. \qquad (2.14)
\end{aligned}$$

Then by (2.12)–(2.14), we get

$$\begin{aligned}
&\|u(t)\|^2 + 2\int_\tau^t \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2 ds + 2\lambda_1 \int_\tau^t \|u(s)\|^p_{L^p(\mathbb{R}^n)} ds \\
&\quad \leqslant \|u(\tau)\|^2 + 2a^2 \int_{-\rho}^0 \|\varphi(s)\|^2 ds + 2\int_\tau^t \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds + \big(1 + 2a^2 - 2\lambda\big) \int_\tau^t \|u(s)\|^2 ds \\
&\qquad + 2\int_\tau^t \|h(s)\|^2 ds + \int_\tau^t \|\sigma(s, u(s))\|^2_{\mathcal{L}_2(U,H)} ds + 2\int_\tau^t \big(u(s),\ \sigma(s, u(s)) dW(s)\big). \qquad (2.15)
\end{aligned}$$

By (2.15), we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{\tau \leqslant r \leqslant t} \|u(r)\|^2\right) \\
&\quad \leqslant \|u(\tau)\|^2 + 2a^2 \mathbb{E}\left(\int_{-\rho}^0 \|\varphi(s)\|^2 ds\right) + 2\int_\tau^t \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds
\end{aligned}$$

$$+2\int_\tau^t \|h(s)\|^2 ds + \left(1+2a^2\right)\int_\tau^t \mathbb{E}\left(\|u(s)\|^2\right) ds$$
$$+\mathbb{E}\Big(\int_\tau^t \|\sigma(s,u(s))\|^2_{\mathcal{L}_2(U,H)} ds\Big) + 2\mathbb{E}\Big(\sup_{s\in[\tau,t]}\Big|\int_\tau^s \big(u(s),\ \sigma(s,u(s))dW(s)\big)\Big|\Big). \tag{2.16}$$

For the last two terms on the right-hand side of (2.16), by ($\Sigma 2$) and Burkholder-Davis-Gundy's inequality, we have

$$\mathbb{E}\Big(\int_\tau^t \|\sigma(s,u(s))\|^2_{\mathcal{L}_2(U,H)} ds\Big) + 2\mathbb{E}\Big(\sup_{s\in[\tau,t]}\Big|\int_\tau^s \big(u(s),\ \sigma(s,u(s))dW(s)\big)\Big|\Big)$$
$$\leqslant \frac{1}{2}\mathbb{E}\left(\sup_{\tau\leqslant s\leqslant t}\|u(s)\|^2\right) + (1+2c^2)\mathbb{E}\Big(\int_\tau^t \|\sigma(s,u(s))\|^2_{\mathcal{L}_2(U,H)} ds\Big)$$
$$\leqslant \frac{1}{2}\mathbb{E}\left(\sup_{\tau\leqslant s\leqslant t}\|u(s)\|^2\right) + 2(1+2c^2)L^2T + 2(1+2c^2)L^2\int_\tau^t \mathbb{E}\left(\|u(s)\|^2\right) ds, \tag{2.17}$$

where c is the positive constant in Burkholder–Davis–Gundy's inequality.

Then by (2.16)–(2.17), we obtain

$$\mathbb{E}\left(\sup_{\tau\leqslant r\leqslant t}\|u(r)\|^2\right) \leqslant 2\mathbb{E}\left(\|u^0\|^2\right) + 4a^2\mathbb{E}\left(\int_{-\rho}^0 \|\varphi(s)\|^2 ds\right)$$
$$+4\int_\tau^t \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds$$
$$+4\int_\tau^t \|h(s)\|^2 ds + 4(1+2c^2)L^2T$$
$$+2\left[\left(1+2a^2\right) + 2(1+2c^2)L^2\right]\int_\tau^t \mathbb{E}\left[\sup_{\tau\leqslant r\leqslant s}\|u(r)\|^2\right] ds. \tag{2.18}$$

From (2.18) and Gronwall's inequality, it follows that for all $t\in[\tau,\tau+T]$ with $T>0$,

$$\mathbb{E}\left(\sup_{\tau\leqslant r\leqslant t}\|u(r)\|^2\right) \leqslant \Big\{2\mathbb{E}\left(\|u^0\|^2\right) + 4a^2\mathbb{E}\left(\int_{-\rho}^0 \|\varphi(s)\|^2 ds\right)$$
$$+4\int_\tau^{\tau+T} \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds + 4\int_\tau^{\tau+T} \|h(s)\|^2 ds$$
$$+4(1+2c^2)L^2T\Big\} e^{\left[2\left(1+2a^2\right)+4(1+2c^2)L^2\right](t-\tau)},$$

which together with (2.15) concludes the proof. □

Now, for all $\tau \in \mathbb{R}$ and $t \in \mathbb{R}^+$, let $\Phi(t, \tau)$ be a mapping from $L^2(\Omega, \mathscr{F}_\tau; H) \times L^2\left(\Omega, \mathscr{F}_\tau; L^2((-\rho, 0), H)\right)$ to $L^2(\Omega, \mathscr{F}_{t+\tau}; H) \times L^2\left(\Omega, \mathscr{F}_{t+\tau}; L^2((-\rho, 0), H)\right)$ given by

$$\Phi(t, \tau)(u^0, \varphi) = \Big(u(t + \tau; \tau, u^0, \varphi), u_{t+\tau}(\cdot; \tau, u^0, \varphi)\Big),$$

for all $(u^0, \varphi) \in L^2(\Omega, \mathscr{F}_\tau; H) \times L^2\left(\Omega, \mathscr{F}_\tau; L^2((-\rho, 0), H)\right)$, where $u(t; \tau, u^0, \varphi)$ is the solution of (1.1) with initial data u^0 and φ, and $u_{t+\tau}(\theta; \tau, u^0, \varphi) = u(t + \tau + \theta; \tau, u^0, \varphi)$ for $\theta \in (-\rho, 0)$. Then, we find that Φ is a mean random dynamical system on

$$L^2(\Omega, \mathscr{F}; H) \times L^2\Big(\Omega, \mathscr{F}; L^2((-\rho, 0), H)\Big)$$

over the filtration $\{\mathscr{F}_t\}_{t\in\mathbb{R}}$.

In what follows, we investigate the existence and uniqueness of weak mean random attractors of (1.1).

3 Weak pullback mean random attractors

In this section, we study weak pullback mean random attractors of (1.1). For simplicity, for every $\tau \in \mathbb{R}$, we set $\mathcal{H}_\tau = L^2(\Omega, \mathscr{F}_\tau; H) \times L^2\big(\Omega, \mathscr{F}_\tau; L^2((-\rho, 0), H)\big)$. Then $\mathcal{H}_\tau$ is a Hilbert space with inner product $\big((u^0, \varphi), (v^0, \psi)\big)_{\mathcal{H}_\tau} = \mathbb{E}\left(u^0, v^0\right) + \mathbb{E}\Big(\int_{-\rho}^0 (\varphi(s), \psi(s))\, ds\Big)$ and norm $\|(u^0, \varphi)\|_{\mathcal{H}_\tau} = \Big(\mathbb{E}(\|u^0\|^2) + \int_{-\rho}^0 \mathbb{E}\left(\|\varphi(s)\|^2\right) ds\Big)^{\frac{1}{2}}$ for (u^0, φ) and $(v^0, \psi) \in \mathcal{H}_\tau$.

Assume a in **(G1)** and L in **(Σ2)** are sufficiently small in the following sense:

$$\sqrt{2}a + L^2 < \lambda. \tag{3.1}$$

By (3.1), there exists a positive constant μ such that

$$\mu - 2\lambda + \sqrt{2}a(1 + e^{\mu\rho}) + 2L^2 < 0. \tag{3.2}$$

Let $B = \{B(\tau) \subseteq \mathcal{H}_\tau : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets such that

$$\lim_{\tau\to-\infty} e^{\mu\tau} \|B(\tau)\|^2_{\mathcal{H}_\tau} = 0, \tag{3.3}$$

where $\|B(\tau)\|_{\mathcal{H}_\tau} = \sup_{(u^0,\varphi)\in B(\tau)} \|(u^0, \varphi)\|_{\mathcal{H}_\tau}$. Denote by

$$\mathcal{D} = \Big\{B = \{B(\tau) \subseteq \mathcal{H}_\tau : \tau \in \mathbb{R}\} : B \text{ satisfies (3.3)}\Big\}.$$

We will show (1.1) has a unique weak $\mathcal{D}$-pullback mean random attractor for which we further assume that for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\mu(s-\tau)} \left(\|\psi_1(s)\|_{L^1(\mathbb{R}^n)} + \|h(s)\|^2 \right) ds < \infty, \tag{3.4}$$

where μ is the positive constant as in (3.2).

Lemma 3.1 *Suppose* **(F1)-+(F2)**, **(G1)–(G2)**, **(Σ1)–(Σ2)**, *(3.1) and (3.4) hold. Then for any $\tau \in \mathbb{R}$ and $B = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, there exists $T = T(\tau, B) > \rho$ such that for all $t \geqslant T$,*

$$\begin{aligned} &\mathbb{E}\left(\|u(\tau;\tau-t,u^0,\varphi)\|^2\right) + \int_{\tau-\rho}^{\tau} \mathbb{E}\left(\|u(s;\tau-t,u^0,\varphi)\|^2\right) ds \\ &\leqslant \left(1+\rho e^{\mu\rho}\right)\left[1 + \frac{2L^2}{\mu} + \int_{-\infty}^{\tau} e^{\mu(s-\tau)} \left(\|\psi_1(s)\|_{L^1(\mathbb{R}^n)} + \frac{\sqrt{2}}{a}\|h(s)\|^2 \right) ds \right], \end{aligned} \tag{3.5}$$

where μ is the same constant as in (3.2) and $(u^0, \varphi) \in B(\tau - t)$.

Proof For any $t > 0$ and $r \in (\tau - t, \tau]$, by (2.12) we get

$$\begin{aligned} &e^{\mu r}\mathbb{E}\left(\|u(r)\|^2\right) + 2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right) ds \\ &= e^{\mu(\tau-t)}\mathbb{E}\left(\|u^0\|^2\right) + (\mu - 2\lambda)\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\left(\|u(s)\|^2\right) ds \\ &\quad - 2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\Big(\int_{\mathbb{R}^n} F(s,x,u(s))u(s)dx\Big)ds + 2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\big(G(s,u(s-\rho)),u(s)\big)ds \\ &\quad + \int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\big(\|\sigma(s,u(s))\|^2_{\mathcal{L}_2(U,H)}\big)ds. \end{aligned} \tag{3.6}$$

We now estimate the right-hand side of (3.6). For the third term on the right-hand side of (3.6), by **(F1)**, we obtain

$$\begin{aligned} &2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\left(\int_{\mathbb{R}^n} F(s,x,u(s))u(s)dx\right) ds \\ &\geqslant 2\lambda_1 \int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right) ds - 2\int_{\tau-t}^{r} e^{\mu s}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)}ds. \end{aligned} \tag{3.7}$$

For the fourth term on the right-hand side of (3.6), by **(G1)** we have

$$\begin{aligned} &2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\left(G(s,u(s-\rho)),u(s)\right) ds \\ &\leqslant \sqrt{2}a(1+e^{\mu\rho})\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\left(\|u(s)\|^2\right) ds + \frac{\sqrt{2}}{a}\int_{\tau-t}^{r} e^{\mu s}\|h(s)\|^2 ds \end{aligned}$$

$$+\sqrt{2}ae^{\mu\rho}e^{\mu(\tau-t)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds. \tag{3.8}$$

For the fifth term on the right-hand side of (3.6), by **(Σ2)** we get

$$\int_{\tau-t}^{r}e^{\mu s}\mathbb{E}\left(\|\sigma(s,u(s))\|^2_{\mathcal{L}_2(U,H)}\right)ds \leqslant 2L^2\int_{\tau-t}^{r}e^{\mu s}ds+2L^2\int_{\tau-t}^{r}e^{\mu s}\mathbb{E}\left(\|u(s)\|^2\right)ds. \tag{3.9}$$

From (3.6)–(3.9), it follows that for all $r\in(\tau-t,\tau]$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(r;\tau-t,u^0,\varphi)\|^2\right)+2\int_{\tau-t}^{r}e^{\mu(s-r)}\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds\\
&\quad+2\lambda_1\int_{\tau-t}^{r}e^{\mu(s-r)}\mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right)ds\\
&\leqslant e^{\mu(\tau-t-r)}\mathbb{E}\left(\|u^0\|^2\right)+\sqrt{2}ae^{\mu\rho}e^{\mu(\tau-t-r)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds\\
&\quad+2\int_{\tau-t}^{r}e^{\mu(s-r)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)}ds+\frac{\sqrt{2}}{a}\int_{\tau-t}^{r}e^{\mu(s-r)}\|h(s)\|^2ds+\frac{2L^2}{\mu}\\
&\quad+\left[\mu-2\lambda+\sqrt{2}a(1+e^{\mu\rho})+2L^2\right]\int_{\tau-t}^{r}e^{\mu(s-r)}\mathbb{E}\left(\|u(s)\|^2\right)ds.
\end{aligned} \tag{3.10}$$

By (3.2) and (3.10) we find

$$\begin{aligned}
&\mathbb{E}\left(\|u(\tau;\tau-t,u^0,\varphi)\|^2\right)+2\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds\\
&\quad+2\lambda_1\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right)ds\\
&\leqslant e^{-\mu t}\mathbb{E}\left(\|u^0\|^2\right)+\sqrt{2}ae^{\mu(\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds+\frac{2L^2}{\mu}\\
&\quad+2\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)}ds+\frac{\sqrt{2}}{a}\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\|h(s)\|^2ds,
\end{aligned} \tag{3.11}$$

and for $t\geqslant\rho$,

$$\begin{aligned}
&\sup_{\tau-\rho\leqslant r\leqslant\tau}\mathbb{E}\left(\|u(r;\tau-t,u^0,\varphi)\|^2\right)\\
&\leqslant e^{\mu(\rho-t)}\mathbb{E}\left(\|u^0\|^2\right)+\sqrt{2}ae^{\mu(2\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds+\frac{2L^2}{\mu}\\
&\quad+2e^{\mu\rho}\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)}ds+\frac{\sqrt{2}}{a}e^{\mu\rho}\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\|h(s)\|^2ds.
\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain that for $t \geqslant \rho$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(\tau;\tau-t,u^0,\varphi)\|^2\right)+\int_{\tau-\rho}^{\tau}\mathbb{E}\left(\|u(s;\tau-t,u^0,\varphi)\|^2\right)ds \\
&\quad\leqslant\left(1+\rho e^{\mu\rho}\right)\left[e^{-\mu t}\mathbb{E}\left(\|u^0\|^2\right)+\sqrt{2}ae^{\mu(\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds\right. \\
&\quad\quad\left.+2\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)}ds+\frac{\sqrt{2}}{a}\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\|h(s)\|^2ds+\frac{2L^2}{\mu}\right].
\end{aligned} \tag{3.13}$$

For the first two terms on the right-hand side of (3.13), by $(u^0,\varphi)\in B(\tau-t)$ we obtain

$$\begin{aligned}
&e^{-\mu t}\mathbb{E}\left(\|u^0\|^2\right)+\sqrt{2}ae^{\mu(\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds \\
&\quad\leqslant(e^{-\mu\tau}+\sqrt{2}ae^{\mu(\rho-\tau)})e^{\mu(\tau-t)}\|B(\tau-t)\|^2\to 0,\quad \text{as } t\to\infty,
\end{aligned}$$

and hence there exists $T=T(\tau,B)\geqslant\rho$ such that for all $t\geqslant T$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(\tau;\tau-t,u^0,\varphi)\|^2\right)+\int_{\tau-\rho}^{\tau}\mathbb{E}\left(\|u(s;\tau-t,u^0,\varphi)\|^2\right)ds \\
&\leqslant\left(1+\rho e^{\mu\rho}\right)\left[1+\frac{2L^2}{\mu}+2\int_{-\infty}^{\tau}e^{\mu(s-\tau)}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)}ds+\frac{\sqrt{2}}{a}\int_{-\infty}^{\tau}e^{\mu(s-\tau)}\|h(s)\|^2ds\right],
\end{aligned}$$

as desired. □

In order to prove the existence and uniqueness of weak pullback mean random attractor, we need the following result (see Theorem 2.13 in [11]), which is given here for convenience.

Theorem 3.1 ([11]) *Suppose X is a reflexive Banach space and $p\in(1,\infty)$. Let $\mathcal{D}_0$ be an inclusion-closed collection of some families of nonempty bounded subsets of $L^p(\Omega,\mathscr{F};X)$ and Φ be a mean random dynamical system on $L^p(\Omega,\mathscr{F};X)$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$. If Φ has a weakly compact $\mathcal{D}_0$-pullback absorbing set $K\in\mathcal{D}_0$ on $L^p(\Omega,\mathscr{F};X)$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$, then Φ has a unique weak $\mathcal{D}_0$-pullback mean random attractor $\mathcal{A}\in\mathcal{D}_0$ on $L^p(\Omega,\mathscr{F};X)$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$, which is given by, for each $\tau\in\mathbb{R}$,*

$$\mathcal{A}(\tau)=\Omega^w(K,\tau)=\bigcap_{r\geqslant 0}\overline{\bigcup_{t\geqslant r}\Phi(t,\tau-t)(K(\tau-t))}^{w}$$

where the closure is taken with respect to the weak topology of $L^p(\Omega,\mathscr{F}_\tau;X)$.

We are now in a position to present the main result of this section.

Theorem 3.2 *Suppose* **(F1)–(F2)**, **(G1)–(G2)**, **(Σ1)–(Σ2)**, (3.1) *and* (3.4) *hold. Then the mean random dynamical system* Φ *associated with* (1.1) *has a unique weak* $\mathcal{D}$*-pullback mean random attractor* $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *in* $L^2(\Omega, \mathscr{F}; H) \times L^2\big(\Omega, \mathscr{F}; L^2((-\rho, 0), H)\big)$*, that is:*
(i) $\mathcal{A}(\tau)$ *is weakly compact in* $L^2(\Omega, \mathscr{F}_\tau; H) \times L^2\big(\Omega, \mathscr{F}_\tau; L^2((-\rho, 0), H)\big)$ *for all* $\tau \in \mathbb{R}$.
(ii) $\mathcal{A}$ *is a* $\mathcal{D}$*-pullback weakly attracting set of* Φ.
(iii) $\mathcal{A}$ *is the minimal element of* $\mathcal{D}$ *with properties* (i) *and* (ii).

Proof For each $\tau \in \mathbb{R}$, define

$$K_0(\tau) = \left\{(u, \varphi) \in \mathcal{H}_\tau : \|(u, \varphi)\|^2_{\mathcal{H}_\tau} \leqslant R_0(\tau)\right\},$$

where

$$R_0(\tau) = \big(1 + \rho e^{\mu\rho}\big)\left[1 + \frac{2L^2}{\mu} + \int_{-\infty}^{\tau} e^{\mu(s-\tau)}(\|\psi_1(s)\|_{L^1(\mathbb{R}^n)} + \frac{\sqrt{2}}{a}\|h(s)\|^2)ds\right].$$

Then $K_0(\tau)$ is a bounded closed convex subset of $\mathcal{H}_\tau$ and hence is weakly compact in $\mathcal{H}_\tau$. By (3.4) we have

$$\lim_{\tau\to-\infty} e^{\mu\tau}\|K_0(\tau)\|^2_{\mathcal{H}_\tau} = \lim_{\tau\to-\infty} e^{\mu\tau} R_0(\tau) = 0,$$

which means that $K = \{K_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$.

By Lemma 3.1, we see that for every $\tau \in \mathbb{R}$ and $B = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, there exists $T = T(\tau, B) \geq \rho$ such that for all $t \geqslant T$,

$$\Phi(t, \tau - t)(B(\tau - t)) \subseteq K_0(\tau).$$

Consequently, K_0 is a weakly compact $\mathcal{D}$-pullback absorbing set of Φ. Then by Theorem 3.1, Φ has a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A} \in \mathcal{D}$ in $L^2(\Omega, \mathscr{F}; H) \times L^2\big(\Omega, \mathscr{F}; L^2((-\rho, 0), H)\big)$. □

4 Existence of invariant measures

In this section, we investigate invariant measures of the autonomous version of (1.1) when the nonlinear functions F, G and σ are time-independent. More precisely, consider the following stochastic delay equation:

$$\begin{aligned} du(t) &+ (-\triangle)^\alpha u(t)dt + \lambda u(t)dt + F(x, u(t))dt \\ &= G(x, u(t-\rho))dt + \sum_{k=1}^{\infty}\big(\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(t))\big)\,dW_k(t), \quad t > 0, \end{aligned} \tag{4.1}$$

where $\sigma_{1,k} \in L^2(\mathbb{R}^n)$, $\kappa \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and $\{W_k\}_{k=1}^\infty$ is a sequence of real-valued mutually independent Wiener processes on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

The autonomous version of assumption **(F1)** is given below:
(F′). $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and $F(\cdot, 0) \in L^2(\mathbb{R}^n)$, and for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$,

$$F(x, u)u \geqslant \lambda_1 |u|^p - \psi_1(x), \tag{4.2}$$

$$|F(x, u)| \leqslant \psi_2(x)|u|^{p-1} + \psi_3(x), \tag{4.3}$$

$$\frac{\partial F(x, u)}{\partial u} \geqslant -\psi_4(x), \tag{4.4}$$

where $\lambda_1 > 0$ and $p > 2$ are constants, $\psi_1 \in L^1(\mathbb{R}^n)$, $\psi_2 \in L^\infty(\mathbb{R}^n)$, $\psi_3 \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and $\psi_4 \in L^\infty(\mathbb{R}^n) \cap L^{\frac{2q}{2-q}}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In addition, $F(x, u)$ is locally Lipschitz continuous in $u \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{R}^n$; that is, for any bounded interval I, there exists a constant $C_I^F > 0$ such that

$$|F(x, u_1) - F(x, u_2)| \leqslant C_I^F |u_1 - u_2|, \quad \forall\, x \in \mathbb{R}^n,\ u_1, u_2 \in I. \tag{4.5}$$

(G′). $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous such that

$$|G(x, u)| \leqslant |h(x)| + a|u|, \quad \forall\, x \in \mathbb{R}^n,\ u \in \mathbb{R}, \tag{4.6}$$

where $a > 0$ is a constant and $h \in L^2(\mathbb{R}^n)$.
In addition, $G(x, u)$ is Lipschitz continuous in $u \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{R}$; that is, there exists a constant $C^G > 0$ such that

$$|G(x, u_1) - G(x, u_2)| \leqslant C^G |u_1 - u_2|, \quad \forall\, x \in \mathbb{R}^n,\ u_1, u_2 \in \mathbb{R}. \tag{4.7}$$

For the diffusion coefficients of noise, we now assume:
(Σ′).

$$\sum_{k=1}^\infty \|\sigma_{1,k}\|^2 < \infty. \tag{4.8}$$

In addition, for each $k \in \mathbb{N}$, we assume that $\sigma_{2,k} : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous; that is, for each $k \in \mathbb{N}$, there exists a positive number α_k such that for all $s_1, s_2 \in \mathbb{R}$,

$$|\sigma_{2,k}(s_1) - \sigma_{2,k}(s_2)| \leqslant \alpha_k |s_1 - s_2|. \tag{4.9}$$

We further assume that for each $k \in \mathbb{N}$, there exist positive numbers β_k and γ_k such that

$$|\sigma_{2,k}(s)| \leqslant \beta_k + \gamma_k |s|, \quad \forall\, s \in \mathbb{R}, \tag{4.10}$$

where $\sum_{k=1}^{\infty}(\alpha_k^2+\beta_k^2)<+\infty$.

In order to prove the existence of invariant measures of (4.1), we need to assume ψ_4, a, α_k and γ_k in **(F′)**, **(G′)** and **(Σ′)** are sufficiently small in the sense that there exists a constant $\theta \geqslant 1$ such that

$$\theta\|\psi_4\|_{L^\infty(\mathbb{R}^n)} + a2^{1-\frac{1}{2\theta}}(2\theta-1)^{\frac{2\theta-1}{2\theta}} + 2\theta(2\theta-1)\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}(\alpha_k^2+\gamma_k^2) < \theta\lambda. \tag{4.11}$$

Note that (4.11) implies the following conditions:

$$\sqrt{2}a + 2\sum_{k=1}^{\infty}\gamma_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)} < \lambda \tag{4.12}$$

and

$$a2^{1-\frac{1}{2\theta}}(2\theta-1)^{\frac{2\theta-1}{2\theta}} + 2\theta(2\theta-1)\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2 < \theta\lambda. \tag{4.13}$$

These inequalities are useful for deriving uniform estimates of solutions which are needed for proving the tightness of distribution laws of a family of solutions on the space $H \times L^2((-\rho,0),H)$.

4.1 Uniform estimates of solutions

We now derive uniform estimates of solutions for proving existence of invariant measures. We start with the estimates in $L^2(\Omega,\mathscr{F}_t;H)$.

Lemma 4.1 *Suppose* **(F′)**, **(G′)**, **(Σ′)** *and (4.12) hold. Then for any* $u^0 \in L^2(\Omega,\mathscr{F}_0;H)$ *and* $\varphi \in L^2\big(\Omega,\mathscr{F}_0;L^2((-\rho,0),H)\big)$, *the solution* u *of (4.1) satisfies that for all* $t \geqslant 0$,

$$\begin{aligned}&\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^2\right) + \int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^2_{H^\alpha(\mathbb{R}^n)}\right)ds + \int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^p_{L^p}\right)ds\\ &\quad\leqslant M_1\left\{\left[\mathbb{E}\left(\|u^0\|^2\right) + \mathbb{E}\Big(\int_{-\rho}^0\|\varphi(s)\|^2ds\Big)\right]e^{-\nu t}+1\right\},\end{aligned} \tag{4.14}$$

and for $t \geqslant 1+\rho$,

$$\begin{aligned}&\int_{t-1}^t \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds + \int_{t-1}^t \mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right)ds\\ &\quad\leqslant M_1\left\{\left[\mathbb{E}\left(\|u^0\|^2\right) + \mathbb{E}\Big(\int_{-\rho}^0\|\varphi(s)\|^2ds\Big)\right]e^{-\nu t}+1\right\},\end{aligned} \tag{4.15}$$

where ν and M_1 are positive constant independent of ρ, u^0 and φ.

Proof By (2.12), we have for all $t \geqslant 0$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^2\right) + 2\int_0^t \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds + 2\lambda\int_0^t \mathbb{E}\left(\|u(s)\|^2\right)ds \\
&\quad = \mathbb{E}\left(\|u^0\|^2\right) - 2\int_0^t \mathbb{E}\left(\int_{\mathbb{R}^n} F(x,u(s))u(s)dx\right)ds + 2\int_0^t \mathbb{E}\left(G(\cdot,u(s-\rho)),u(s)\right)ds \\
&\qquad + \sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left(\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2\right)ds. \qquad (4.16)
\end{aligned}$$

By (4.16) we have for all $t > 0$,

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left(\|u(t)\|^2\right) + 2\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^2\right) + 2\lambda\mathbb{E}\left(\|u(t)\|^2\right) \\
&\quad = -2\mathbb{E}\left(\int_{\mathbb{R}^n} F(x,u(t))u(t)dx\right) + 2\mathbb{E}\left(G(\cdot,u(t-\rho)),u(t)\right) \\
&\qquad + \sum_{k=1}^{\infty}\mathbb{E}\left(\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(t))\|^2\right). \qquad (4.17)
\end{aligned}$$

We now estimate the terms on the right-hand side of (4.17). For the first term on the right-hand side of (4.17), by (4.2) we have

$$2\mathbb{E}\left(\int_{\mathbb{R}^n} F(x,u(t))u(t)dx\right) \geqslant 2\lambda_1\mathbb{E}\left(\|u(t)\|^p_{L^p(\mathbb{R}^n)}\right) - 2\|\psi_1\|_{L^1(\mathbb{R}^n)}. \qquad (4.18)$$

For the second term on the right-hand side of (3.6), by (4.6) we have

$$\begin{aligned}
2\mathbb{E}\left(G(\cdot,u(t-\rho)),u(t)\right) &\leqslant 2\mathbb{E}\left(\|G(\cdot,u(t-\rho))\|\|u(t)\|\right) \\
&\leqslant \sqrt{2}a\mathbb{E}\left(\|u(t)\|^2\right) + \frac{\sqrt{2}}{a}\|h\|^2 + \sqrt{2}a\mathbb{E}\left(\|u(t-\rho)\|^2\right). \qquad (4.19)
\end{aligned}$$

For the third term on the right-hand side of (4.17), by (4.10) we have

$$\sum_{k=1}^{\infty}\mathbb{E}\left(\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(t))\|^2\right) \leqslant 2\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa\|^2\right) + 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\mathbb{E}\left(\|u(t)\|^2\right). \qquad (4.20)$$

It follows from (4.17)–(4.20) that for $t \geq 0$,

$$\frac{d}{dt}\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^2\right) + 2\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^2\right) + 2\lambda_1\mathbb{E}\left(\|u(t)\|^p_{L^p(\mathbb{R}^n)}\right)$$

$$\leqslant -\Big(2\lambda - \sqrt{2}a - 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\Big)\mathbb{E}\left(\|u(t)\|^2\right) + \sqrt{2}a\mathbb{E}\left(\|u(t-\rho)\|^2\right)$$
$$+ 2\|\psi_1\|_{L^1(\mathbb{R}^n)} + \frac{\sqrt{2}}{a}\|h\|^2 + 2\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa\|^2\right). \tag{4.21}$$

By (4.12) we infer that there exists a positive constant ν such that

$$2\nu - 2\lambda + \sqrt{2}a + 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)} + \sqrt{2}ae^{\nu} < 0. \tag{4.22}$$

Then by (4.21), we obtain

$$\frac{d}{dt}e^{\nu t}\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^2\right) + 2e^{\nu t}\mathbb{E}\left(\|(-\Delta)^{\frac{\alpha}{2}}u(t)\|^2\right) + 2\lambda_1 e^{\nu t}\mathbb{E}\left(\|u(t)\|^p_{L^p(\mathbb{R}^n)}\right)$$
$$\leqslant -\Big(2\lambda - \nu - \sqrt{2}a - 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\Big)e^{\nu t}\mathbb{E}\left(\|u(t)\|^2\right) + \sqrt{2}ae^{\nu t}\mathbb{E}\left(\|u(t-\rho)\|^2\right)$$
$$+ e^{\nu t}\Big(2\|\psi_1\|_{L^1(\mathbb{R}^n)} + \frac{\sqrt{2}}{a}\|h\|^2 + 2\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa\|^2\right)\Big). \tag{4.23}$$

By (4.23), we get that for $t \geqslant 0$,

$$\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^2\right) + 2\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2\right)ds + 2\lambda_1\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right)ds$$
$$+ \Big(2\lambda - \nu - \sqrt{2}a - 4\sum_{k=1}^{\infty}\gamma_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\Big)\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^2\right)ds$$
$$\leqslant \mathbb{E}\left(\|u^0\|^2\right)e^{-\nu t} + \sqrt{2}ae^{\nu(\rho-t)}\int_{-\rho}^0\mathbb{E}\left(\|\varphi(s)\|^2\right)ds + \sqrt{2}ae^{\nu\rho}\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^2\right)ds$$
$$+ \frac{1}{\nu}\Big(2\|\psi_1\|_{L^1(\mathbb{R}^n)} + \frac{\sqrt{2}}{a}\|h\|^2 + 2\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa\|^2\right)\Big). \tag{4.24}$$

By (4.22) and (4.24), we have, for all $t \geqslant 0$,

$$\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^2\right) + \nu\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^2\right)ds$$
$$+ 2\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2\right)ds + 2\lambda_1\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right)ds$$
$$\leqslant (1+\sqrt{2}ae^{\nu\rho})e^{-\nu t}\left[\mathbb{E}\left(\|u^0\|^2\right) + \int_{-\rho}^0\mathbb{E}\left(\|\varphi(s)\|^2\right)ds\right]$$
$$+ \frac{1}{\nu}\Big(2\|\psi_1\|_{L^1(\mathbb{R}^n)} + \frac{\sqrt{2}}{a}\|h\|^2 + 2\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa\|^2\right)\Big), \tag{4.25}$$

which yields (4.14).

Integrating (4.21) on $[t-1, t]$ for $t \geqslant 1+\rho$, we have

$$
\begin{aligned}
&2\int_{t-1}^{t} \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2\right) ds + 2\lambda_1 \int_{t-1}^{t} \mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right) ds \\
&\quad \leqslant \mathbb{E}\left(\|u(t-1)\|^2\right) + \sqrt{2}a \int_{t-1}^{t} \mathbb{E}\left(\|u(s-\rho)\|^2\right) ds \\
&\qquad + 2\|\psi_1\|_{L^1(\mathbb{R}^n)} + \frac{\sqrt{2}}{a}\|h\|^2 + 2\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2 + 2\beta_k^2\|\kappa\|^2\right).
\end{aligned} \tag{4.26}
$$

Then from (4.25) and (4.26), we get (4.15) immediately. □

Remark 4.1 Let $(u^0, \varphi) \in L^2(\Omega, \mathscr{F}_0; H) \times L^2\big(\Omega, \mathscr{F}_0; L^2((-\rho, 0), H)\big)$ satisfy

$$
\mathbb{E}\left(\|u^0\|^2 + \int_{-\rho}^{0} \|\varphi(s)\|^2 ds\right) \leqslant R
$$

for some $R > 0$. Then by Lemma 4.1, we find that the solution u of (4.1) satisfies, for $t \geqslant 0$,

$$
\mathbb{E}\left(\|u(t; 0, u^0, \varphi)\|^2\right) + \int_0^t e^{\nu(s-t)} \mathbb{E}\left(\|u(s)\|^2_{H^\alpha(\mathbb{R}^n)}\right) ds + \int_0^t e^{\nu(s-t)} \mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right) ds \leqslant \overline{M}_1,
$$

and for $t \geqslant 1$,

$$
\int_{t-1}^{t} \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2\right) ds + \int_{t-1}^{t} \mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right) ds \leqslant \overline{M}_1,
$$

where $\overline{M}_1 > 0$ is a constant depending only on R but not on (u^0, φ) or $\rho \in [0, 1]$. Furthermore, there exists $T \geqslant 2$ depending only on R (but not on (u^0, φ) or $\rho \in [0, 1]$) such that for all $t \geqslant T$,

$$
\mathbb{E}(\|u(t; 0, u^0, \varphi)\|^2) + \int_0^t e^{\nu(s-t)} \mathbb{E}\left(\|u(s)\|^2_{H^\alpha(\mathbb{R}^n)}\right) ds + \int_0^t e^{\nu(s-t)} \mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right) ds \leqslant \widetilde{M}_1
$$

and

$$
\int_{t-1}^{t} \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2\right) ds + \int_{t-1}^{t} \mathbb{E}\left(\|u(s)\|^p_{L^p(\mathbb{R}^n)}\right) ds \leqslant \widetilde{M}_1,
$$

where $\widetilde{M}_1 > 0$ is a constant independent of R, (u^0, φ) and $\rho \in [0, 1]$.

Lemma 4.2 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.12) hold. Then for any* $u^0 \in L^2(\Omega, \mathscr{F}_0; H)$ *and* $\varphi \in L^2\big(\Omega, \mathscr{F}_0; L^2((-\rho, 0), H)\big)$, *the solution* u *of (4.1) satisfies that for all* $t \geqslant 1+\rho$,

$$\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\|u(r;0,u^0,\varphi)\|^2\right)\leqslant M_2\left\{\left[\mathbb{E}\left(\|u^0\|^2\right)+\mathbb{E}\Big(\int_{-\rho}^0\|\varphi(s)\|^2ds\Big)\right]e^{-\nu t}+1\right\},$$

where ν and M_2 are positive constant independent of u^0, φ and $\rho\in[0,1]$.

Proof The proof is based on Lemma 4.1 and is similar to that of Lemma 3.2 in [20]. So the details are omitted here. □

In order to prove the tightness of probability distributions of solutions to (4.1), we need to derive the uniform estimates on the tails of solutions with initial data in $L^2(\Omega,\mathscr{F}_0;H)\times L^2\big(\Omega,\mathscr{F}_0;L^2((-\rho,0),H)\big)$.

Lemma 4.3 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.12) hold. Then for every $\rho\in[0,1]$ and every compact subset E of $L^2(\Omega,\mathscr{F}_0;H)\times L^2\big(\Omega,\mathscr{F}_0;L^2((-\rho,0),H)\big)$, the solution u of (4.1) satisfies*

$$\limsup_{m\to\infty}\sup_{(u^0,\varphi)\in E}\sup_{t\geqslant 0}\int_{|x|\geqslant m}\mathbb{E}\left(|u(t;0,u^0,\varphi)|^2\right)dx=0.$$

Proof Let θ be a smooth function on $\mathbb{R}$ such that

$$\theta(s)=\begin{cases}0, & |s|\leqslant 1,\\ 1, & |s|\geqslant 2,\end{cases}$$

and $0\leqslant\theta(s)\leqslant 1$ for all $s\in\mathbb{R}$.

For given $m\in\mathbb{N}$, denote by $\theta_m(x)=\theta(\frac{x}{m})$. By (4.1) and Ito's formula, we obtain

$$\begin{aligned}&\mathbb{E}\left(\|\theta_m u(t)\|^2\right)+2\lambda\int_0^t\mathbb{E}\left(\|\theta_m u(s)\|^2\right)ds\\&=\mathbb{E}\left(\|\theta_m u^0\|^2\right)-2\int_0^t\mathbb{E}\left[\left((-\triangle)^{\frac{\alpha}{2}}u(s),(-\triangle)^{\frac{\alpha}{2}}\theta_m^2u(s)\right)\right]ds\\&\quad-2\int_0^t\mathbb{E}\left(\int_{\mathbb{R}^n}\theta_m^2(x)F(x,u(s))u(s)dx\right)ds+2\int_0^t\mathbb{E}\left[(\theta_m G(\cdot,u(s-\rho)),\theta_m u(s))\right]ds\\&\quad+\sum_{k=1}^\infty\int_0^t\mathbb{E}\left(\|\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(s))\|^2\right)ds,\end{aligned}\tag{4.27}$$

and hence for $t>0$,

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}\left(\|\theta_m u(t)\|^2\right)+2\lambda\mathbb{E}\left(\|\theta_m u(t)\|^2\right)\\&\quad=-2\mathbb{E}\left[\left((-\triangle)^{\frac{\alpha}{2}}u(t),(-\triangle)^{\frac{\alpha}{2}}\theta_m^2u(t)\right)\right]-2\mathbb{E}\left(\int_{\mathbb{R}^n}\theta_m^2(x)F(x,u(t))u(t)dx\right)\\&\qquad+2\mathbb{E}\left(\theta_m G(\cdot,u(t-\rho)),\theta_m u(t)\right)+\sum_{k=1}^\infty\mathbb{E}\left(\|\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(t))\|^2\right).\end{aligned}\tag{4.28}$$

For the first term on the right-hand side of (4.28), as (8.18) in [25] we find that there exists a positive constant c_1 independent of m and ρ such that

$$-2\mathbb{E}((-\triangle)^{\frac{\alpha}{2}}u(t),(-\triangle)^{\frac{\alpha}{2}}\theta_m^2u(t)) \leqslant c_1m^{-\alpha}\mathbb{E}\left(\|u(t)\|^2+\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^2\right). \quad (4.29)$$

For the second term on the right-hand side of (4.28), by (2.1) we get

$$-2\mathbb{E}\left(\int_{\mathbb{R}^n}\theta_m^2(x)F(x,u(t))u(t)dx\right) \leqslant -2\lambda_1\mathbb{E}\left(\|\theta_m u(t)\|^p_{L^p(\mathbb{R}^n)}\right)+2\int_{|x|\geqslant m}\psi_1(x)dx. \quad (4.30)$$

For the third term on the right-hand side of (4.28), by (4.6), we deduce

$$\begin{aligned}&2\mathbb{E}\left(\theta_m G(\cdot,u(t-\rho)),\theta_m u(t)\right)\\&\quad\leqslant \sqrt{2}a\mathbb{E}\left(\|\theta_m u(t)\|^2\right)+\frac{\sqrt{2}}{a}\int_{|x|\geqslant m}h^2(x)dx+\sqrt{2}a\mathbb{E}\left(\left\|\theta_m u(t-\rho)\right\|^2\right).\end{aligned} \quad (4.31)$$

For the last term on the right-hand side of (4.28), by (4.10) we get

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{E}\left(\|\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(t))\|^2\right) &\leqslant 2\sum_{k=1}^{\infty}\int_{|x|\geqslant m}|\sigma_{1,k}(x)|^2dx+4\sum_{k=1}^{\infty}\beta_k^2\int_{|x|\geqslant m}|\kappa(x)|^2dx\\&\quad+4\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\mathbb{E}\left(\|\theta_m u(t)\|^2\right).\end{aligned} \quad (4.32)$$

From (4.28)–(4.32), it follows that for $t>0$,

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}\left(\|\theta_m u(t)\|^2\right)+2\lambda_1\mathbb{E}\left(\|\theta_m u(t)\|^p_{L^p(\mathbb{R}^n)}\right)\\&\quad\leqslant -(2\lambda-\sqrt{2}a-4\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2)\mathbb{E}\left(\|\theta_m u(t)\|^2\right)+\sqrt{2}a\mathbb{E}\left(\|\theta_m u(t-\rho)\|^2\right)\\&\qquad+c_1m^{-\alpha}\mathbb{E}\left(\|u(t)\|^2+\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^2\right)+2\int_{|x|\geqslant m}\psi_1(x)dx+\frac{\sqrt{2}}{a}\int_{|x|\geqslant m}h^2(x)dx\\&\qquad+2\int_{|x|\geqslant m}\sum_{k=1}^{\infty}|\sigma_{1,k}(x)|^2dx+4\sum_{k=1}^{\infty}\beta_k^2\int_{|x|\geqslant m}|\kappa(x)|^2dx.\end{aligned} \quad (4.33)$$

Let $\nu>0$ be a constant satisfying (4.22). Then by (4.22) and (4.33) we obtain that for all $\rho\in[0,1]$ and $t\geqslant\rho$,

$$\begin{aligned}\mathbb{E}\left(\|\theta_m u(t)\|^2\right) &\leqslant \sup_{0\leqslant s\leqslant\rho}\mathbb{E}\left(\|\theta_m u(s)\|^2\right)\mathrm{e}^{-\nu(t-\rho)}\\&\quad+c_1m^{-\alpha}\int_\rho^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^2+\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds\end{aligned}$$

$$
+\frac{1}{\nu}\bigg[2\int_{|x|\geqslant m}\psi_1(x)dx+\frac{\sqrt{2}}{a}\int_{|x|\geqslant m}h^2(x)dx
+2\int_{|x|\geqslant m}\sum_{k=1}^{\infty}|\sigma_{1,k}(x)|^2dx+4\sum_{k=1}^{\infty}\beta_k^2\int_{|x|\geqslant m}|\kappa(x)|^2dx\bigg]. \tag{4.34}
$$

Next, we estimate the first term on the right-hand side of (4.34). By (4.12), (4.33) and Theorem 2.2, we obtain that for all $\rho\in[0,1]$ and $t\in[0,\rho]$,

$$
\begin{aligned}
\mathbb{E}\left(\|\theta_m u(t)\|^2\right)\leqslant{}& \mathbb{E}\left(\|\theta_m u^0\|^2\right)+\sqrt{2}a\int_{-\rho}^{0}\mathbb{E}\left(\|\theta_m\varphi(s)\|^2\right)ds\\
&+c_2m^{-\alpha}\left(\mathbb{E}(\|u^0\|^2)+\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds+1\right)\\
&+\rho\bigg[2\int_{|x|\geqslant m}\psi_1(x)dx+\frac{\sqrt{2}}{a}\int_{|x|\geqslant m}h^2(x)dx\\
&+2\int_{|x|\geqslant m}\sum_{k=1}^{\infty}|\sigma_{1,k}(x)|^2dx+4\sum_{k=1}^{\infty}\beta_k^2\int_{|x|\geqslant m}|\kappa(x)|^2dx\bigg].
\end{aligned}\tag{4.35}
$$

For the second and third terms on the right-hand side of (4.34), by Lemma 4.1, we obtain

$$
\begin{aligned}
&c_1m^{-\alpha}\int_0^t e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^2+\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds\\
&\leqslant c_3m^{-\alpha}\left[\mathbb{E}\left(\|u^0\|^2\right)+\mathbb{E}\left(\int_{-\rho}^{0}\|\varphi(s)\|^2ds\right)+1\right].
\end{aligned}\tag{4.36}
$$

Then from (4.34), (4.35) and (4.36), it follows that for all $\rho\in[0,1]$ and $t\geqslant\rho$,

$$
\begin{aligned}
\mathbb{E}\left(\|\theta_m u(t)\|^2\right)\leqslant{}&(1+\sqrt{2}a)\left(\mathbb{E}\left(\|\theta_m u^0\|^2\right)+\int_{-\rho}^{0}\mathbb{E}\left(\|\theta_m\varphi(s)\|^2\right)ds\right)\mathrm{e}^{-\nu(t-\rho)}\\
&+(c_2+c_3)m^{-\alpha}\left(\mathbb{E}(\|u^0\|^2)+\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds+1\right)\\
&+\left(1+\frac{1}{\nu}\right)\bigg[2\int_{|x|\geqslant m}\psi_1(x)dx+\frac{\sqrt{2}}{a}\int_{|x|\geqslant m}h^2(x)dx\\
&+2\int_{|x|\geqslant m}\sum_{k=1}^{\infty}|\sigma_{1,k}(x)|^2dx+4\sum_{k=1}^{\infty}\beta_k^2\int_{|x|\geqslant m}|\kappa(x)|^2dx\bigg].
\end{aligned}\tag{4.37}
$$

For any $\varepsilon > 0$, since E is compact in $L^2(\Omega, \mathscr{F}_0; H) \times L^2\big(\Omega, \mathscr{F}_0; L^2((-\rho, 0), H)\big)$, E has a finite open cover of balls with radius $\frac{\sqrt{\varepsilon}}{2}$ which is denoted by $\big\{B\big((u^i, \varphi^i), \frac{\sqrt{\varepsilon}}{2}\big)\big\}_{i=1}^l$. Since $(u^i, \varphi^i) \in E \subseteq L^2(\Omega, \mathscr{F}_0; H) \times L^2\big(\Omega, \mathscr{F}_0; L^2((-\rho, 0), H)\big)$ for $i = 1, 2, \ldots, l$, it follows that there exists $R_1 = R_1(\varepsilon, E) \geqslant 1$ such that for all $m \geqslant R_1$, $i = 1, 2, \ldots, l$,

$$\int_{|x|\geqslant m} \left(\mathbb{E}\left(|u^i(x)|^2 \right) + \int_{-\rho}^0 \mathbb{E}\left(|\varphi^i(s, x)|^2 \right) ds \right) dx < \frac{\varepsilon}{4}.$$

Then for all $(u^0, \varphi) \in E$ and $m \geqslant R_1$,

$$\int_{|x|\geqslant m} \left(\mathbb{E}\left(|u^0(x)|^2 \right) + \int_{-\rho}^0 \mathbb{E}\big(|\varphi(s, x)|^2 \big) ds \right) dx < \varepsilon. \tag{4.38}$$

From (4.38) and the definition of θ_m, we obtain that for all $(u^0, \varphi) \in E$ and $m \geqslant R_1$,

$$\mathbb{E}\left(\|\theta_m u^0\|^2 \right) + \int_{-\rho}^0 \mathbb{E}\left(\|\theta_m \varphi(s)\|^2 \right) ds < \varepsilon. \tag{4.39}$$

By (4.37) and (4.39), we infer that there exists $R_2 = R_2(\varepsilon, E) \geqslant R_1$ such that for all $m \geqslant R_2$, $(u^0, \varphi) \in E$ and $t \geqslant \rho$,

$$\sum_{|n|\geqslant 2k} \mathbb{E}(|u_n(t)|^2) \leqslant \mathbb{E}(\|\theta_m u(t)\|^2) \leqslant \left(2 + \sqrt{2}a \right) \varepsilon. \tag{4.40}$$

On the other hand, by (4.35) and (4.39), we find that there exists $R_3 = R_3(\varepsilon, E) \geqslant R_2$ such that for all $m \geqslant R_3$, $(u^0, \varphi) \in E$ and $t \in [0, \rho]$,

$$\sum_{|n|\geqslant 2k} \mathbb{E}(|u_n(t)|^2) \leqslant \mathbb{E}(\|\theta_m u(t)\|^2) \leqslant 2\varepsilon,$$

which along with (4.40) shows that

$$\limsup_{m\to\infty} \sup_{(u^0,\varphi)\in E} \sup_{t\geqslant 0} \int_{|x|\geqslant m} \mathbb{E}\left(|u(t, x; 0, u^0, \varphi)|^2 \right) dx = 0,$$

as desired. □

Based on Lemma 4.3 we have the following uniform tail-estimates on the segments of solutions.

Lemma 4.4 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.12) hold. Then for every* $\rho \in [0, 1]$ *and every compact subset* E *in* $L^2(\Omega, \mathscr{F}_0; H) \times L^2\big(\Omega, \mathscr{F}_0; L^2((-\rho, 0), H)\big)$*, the solution* u *of (4.1) satisfies*

$$\limsup_{m\to\infty} \sup_{(u^0,\varphi)\in E} \sup_{t\geqslant\rho} \mathbb{E}\Big(\sup_{t-\rho\leqslant r\leqslant t} \int_{|x|\geqslant m} |u(r; 0, u^0, \varphi)|^2 dx \Big) = 0.$$

Proof Let θ be the smooth function as defined in Lemma 4.3 and ν be the positive number determined by (4.22). For $t \geqslant \rho$ and $t-\rho \leqslant r \leqslant t$, by Ito's formula, (4.2) and (4.22), we obtain that for all $t \geqslant \rho$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t-\rho\leqslant r\leqslant t}\|\theta_m u(r)\|^2\Big)\\
&\leqslant \mathbb{E}\left(\|\theta_m u(t-\rho)\|^2\right)+\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\left(-2\int_{t-\rho}^{r} e^{\nu(s-r)}\left((-\triangle)^{\frac{\alpha}{2}}u(s),(-\triangle)^{\frac{\alpha}{2}}\theta_m^2 u(s)\right)ds\right)\right)\\
&\quad+\frac{2}{\nu}\int_{|x|\geqslant m}\psi_1(x)dx+2\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\int_{t-\rho}^{r} e^{\nu(s-r)}\left|(\theta_m G(\cdot,u(s-\rho)),\theta_m u(s))\right|ds\right)\\
&\quad+\sum_{k=1}^{\infty}\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\int_{t-\rho}^{r} e^{\nu(s-r)}\|\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(s))\|^2ds\right)\\
&\quad+2\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\left|\sum_{k=1}^{\infty}\int_{t-\rho}^{r} e^{\nu(s-r)}\left(\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(s)),\theta_m u(s)\right)dW_k(s)\right|\right).
\end{aligned} \tag{4.41}$$

Now we estimate the terms on the right-hand of (4.41). For the second term on the right-hand of (4.41), we obtain by the arguments of (8.18) in [25] and Remark 4.1 that for all $t \geqslant \rho$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\left(-2\int_{t-\rho}^{r} e^{\nu(s-r)}\left((-\triangle)^{\frac{\alpha}{2}}u(s),(-\triangle)^{\frac{\alpha}{2}}\theta_m^2 u(s)\right)ds\right)\right)\\
&\qquad\leqslant c_1 m^{-\alpha}e^{\nu\rho}\int_{t-\rho}^{t} e^{\nu(s-t)}\mathbb{E}\left(\|u(s)\|^2_{H^\alpha(\mathbb{R}^n)}\right)ds \le c_2 m^{-\alpha},
\end{aligned} \tag{4.42}$$

where $c_2>0$ is a constant depending only on E but not on m, (u^0,φ) or $\rho\in[0,1]$.

For the fourth term on the right-hand of (4.41), by (4.6), we get that for all $t \geqslant \rho$,

$$\begin{aligned}
&2\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\int_{t-\rho}^{r} e^{\nu(s-r)}\left|(\theta_m G(\cdot,u(s-\rho)),\theta_m u(s))\right|ds\right)\\
&\leqslant 2\|\theta_m h\|^2+2a^2\int_{t-2\rho}^{t-\rho}\mathbb{E}\left(\|\theta_m u(s)\|^2\right)ds+\sup_{s\ge 0}\mathbb{E}\left(\|\theta_m u(s)\|^2\right)\\
&\leqslant 2\|\theta_m h\|^2+2a^2\int_{-\rho}^{0}\mathbb{E}\left(\|\theta_m\varphi(s)\|^2\right)ds+(1+2a^2)\sup_{s\geqslant 0}\mathbb{E}\left(\|\theta_m u(s)\|^2\right).
\end{aligned} \tag{4.43}$$

For the fifth term on the right-hand of (4.41), by (4.10), we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\int_{t-\rho}^{r} e^{\nu(s-r)}\|\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(s))\|^2ds\right)\\
&\leqslant 2\sum_{k=1}^{\infty}\|\theta_m\sigma_{1,k}\|^2+4\sum_{k=1}^{\infty}\beta_k^2\|\theta_m\kappa\|^2+4\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\geqslant 0}\mathbb{E}\left(\|\theta_m u(s)\|^2\right).
\end{aligned} \tag{4.44}$$

For the sixth term on the right-hand of (4.41), by Burkholder-Davis-Gundy's inequality and (4.44), we have

$$\begin{aligned}
&2\mathbb{E}\left(\sup_{t-\rho\leqslant r\leqslant t}\left|\sum_{k=1}^{\infty}\int_{t-\rho}^{r} e^{\nu(s-r)}\left(\theta_m\sigma_{1,k}+\theta_m\kappa\sigma_{2,k}(u(s)),\theta_m u(s)\right)dW_k(s)\right|\right)\\
&\quad\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{t-\rho\leqslant s\leqslant t}\|\theta_m u(s)\|^2\Big)\\
&\qquad+4c^2e^{2\nu\rho}\sum_{k=1}^{\infty}\|\theta_m\sigma_{1,k}\|^2+8c^2e^{2\nu\rho}\sum_{k=1}^{\infty}\beta_k^2\|\theta_m\kappa\|^2\\
&\quad+8c^2e^{2\nu\rho}\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\geqslant 0}\mathbb{E}\left(\|\theta_m u(s)\|^2\right).
\end{aligned}\tag{4.45}$$

Then from (4.41)–(4.45), it follows that for all $t\geqslant\rho$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t-\rho\leqslant r\leqslant t}\|\theta_m u(r)\|^2\Big)\\
&\quad\leqslant 2\mathbb{E}\left(\|\theta_m u(t-\rho)\|^2\right)+2c_2m^{-\alpha}+\frac{4}{\nu}\int_{|x|\geqslant m}\psi_1(x)dx+4\|\theta_m h\|^2\\
&\quad+4a^2\int_{-\rho}^{0}\mathbb{E}\left(\|\theta_m\varphi(s)\|^2\right)ds+2(1+2a^2)\sup_{s\geqslant 0}\mathbb{E}\left(\|\theta_m u(s)\|^2\right)\\
&\quad+4(1+4c^2e^{2\nu\rho})\left(\sum_{k=1}^{\infty}\|\theta_m\sigma_{1,k}\|^2+2\sum_{k=1}^{\infty}\beta_k^2\|\theta_m\kappa\|^2+2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\geqslant 0}\mathbb{E}\left(\|\theta_m u(s)\|^2\right)\right).
\end{aligned}\tag{4.46}$$

By (4.39), (4.46) and Lemma 4.3, we find

$$\limsup_{m\to\infty}\sup_{(u^0,\varphi)\in E}\sup_{t\geqslant\rho}\mathbb{E}\Big(\sup_{t-\rho\leqslant r\leqslant t}\int_{|x|\geqslant m}|u(r,0,u^0,\varphi)|^2dx\Big)=0,$$

which concludes the proof. □

Remark 4.2 From (4.34) and Remark 4.1, we see that for every $R>0$ and $\varepsilon>0$, there exist $T=T(R,\varepsilon)\geqslant 2$ and $K=K(\varepsilon)\geqslant 1$ such that for all $t\geqslant T$, $m\geqslant K$ and $\rho\in[0,1]$, the solution u of (4.1) satisfies

$$\int_{|x|\geqslant m}\mathbb{E}\left(|u(t;0,u^0,\varphi)|^2\right)dx<\varepsilon,\tag{4.47}$$

for any $(u^0,\varphi)\in L^2(\Omega,\mathscr{F}_0;H)\times L^2\big(\Omega,\mathscr{F}_0;L^2((-\rho,0),H)\big)$ such that

$$\mathbb{E}\left(\|u^0\|^2\right)+\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(s)\|^2\right)ds\leqslant R.\tag{4.48}$$

Based on (4.47), similar to Lemma 4.4, one can further show that there exist $T_1 = T_1(R,\varepsilon) \geqslant T$ and $K_1 = K_1(\varepsilon) \geqslant K$ such that for all $t \geqslant T_1, m \geqslant K_1$ and $\rho \in [0,1]$,

$$\mathbb{E}\Big(\sup_{t-\rho\leqslant r\leqslant t}\int_{|x|\geqslant m}|u(r;0,u^0,\varphi)|^2dx\Big) < \varepsilon,$$

for any (u^0,φ) satisfying (4.48).

In what follows, we derive uniform estimates on the higher-order moments of solutions to (4.1).

Lemma 4.5 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.13) hold. If* $(u^0,\varphi) \in L^{2\theta}(\Omega,\mathcal{F}_0;H) \times L^{2\theta}(\Omega,\mathcal{F}_0;L^{2\theta}((-\rho,0),H))$, *then there exists a positive constant* μ *such that the solution* u *of (4.1) satisfies for any* $t \geqslant 0$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^{2\theta}\right) + \mathbb{E}\Big(\int_0^t e^{\mu(s-t)}\|u(s;0,u^0,\varphi)\|^{2\theta-2}\|(-\triangle)^{\frac{\alpha}{2}}u(s;0,u^0,\varphi)\|^2ds\Big)\\
&\quad+\mathbb{E}\Big(\int_0^t e^{\mu(s-t)}\|u(s;0,u^0,\varphi)\|^{2\theta-2}\|u(s;0,u^0,\varphi)\|_{L^p}^p ds\Big)\\
&\quad\leqslant M_3\Big(\mathbb{E}\left(\|u^0\|^{2\theta}\right)+\mathbb{E}\Big(\int_{-\rho}^0\|\varphi(s)\|^{2\theta}ds\Big)\Big)e^{-\mu t}+M_3,
\end{aligned}$$

where M_3 *is a positive constant independent of* u^0, φ *and* $\rho \in [0,1]$.

Proof The proof is similar to Lemma 3.6 in [20]. For the reader's convenience, we here sketch the main idea.

If $\theta = 1$, then this result is already covered by Lemma 4.1. Next, we assume $\theta > 1$. By (4.13), there exist positive constants μ and ε_1 such that

$$\begin{aligned}
&\mu + 2(\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}} + ae^{\frac{\mu}{2\theta}}2^{2-\frac{1}{2\theta}}(2\theta-1)^{\frac{2\theta-1}{2\theta}}\\
&\quad+2(\theta-1)(2\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-2}}\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2\\
&\quad+4(\theta-1)(2\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-2}}\|\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2+4\theta(2\theta-1)\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\\
&\quad< 2\theta\lambda.
\end{aligned}\tag{4.49}$$

Given $m \in \mathbb{N}$, let τ_m be a stopping time as defined by $\tau_m = \inf\{t \geqslant 0 : \|u(t)\| > m\}$. As usual, $\inf\emptyset = \infty$. Note that the pathwise continuity of u implies $\lim_{m\to\infty}\tau_m = \infty$.

Applying Ito's formula to (4.1), we obtain for $t \geqslant 0$,

$$\mathbb{E}\left(e^{\mu(t\wedge\tau_m)}\|u(t\wedge\tau_m)\|^{2\theta}\right)+2\theta\mathbb{E}\left(\int_0^{t\wedge\tau_m}e^{\mu s}\|u(s)\|^{2\theta-2}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2ds\right)$$

$$
\begin{aligned}
&= \mathbb{E}\left(\|u^0\|^{2\theta}\right) + (\mu - 2\theta\lambda)\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta} ds\right)\\
&\quad - 2\theta\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}\int_{\mathbb{R}^n} F(x,u(s))u(s)dxds\right)\\
&\quad + 2\theta\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}(u(s),\ G(\cdot,u(s-\rho)))ds\right)\\
&\quad + \theta\sum_{k=1}^{\infty}\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right)\\
&\quad + 2\theta(\theta-1)\sum_{k=1}^{\infty}\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-4}|(u(s),\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))|^2 ds\right).
\end{aligned}
\tag{4.50}
$$

For the third term on the right-hand side of (4.50), by Young's inequality and (4.2), we get

$$
\begin{aligned}
&- 2\theta\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}\int_{\mathbb{R}^n} F(x,u(s))u(s)dxds\right)\\
&\quad \leqslant -2\theta\lambda_1\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}\|u(s)\|^p_{L^p(\mathbb{R}^n)} ds\right)\\
&\qquad + 2(\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta} ds\right) + \frac{2}{\mu\varepsilon_1^{\theta}}\|\psi_1\|^{\theta}_{L^1(\mathbb{R}^n)}e^{\mu t}.
\end{aligned}
\tag{4.51}
$$

Similar to Lemma 3.6 in [20], by Young's inequality and (4.6), we get

$$
\begin{aligned}
&2\theta\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}\,(u(s),\ G(\cdot,u(s-\rho)))\,ds\right)\\
&\quad \leqslant ae^{\frac{\mu\rho}{2\theta}}2^{2-\frac{1}{2\theta}}(2\theta-1)^{\frac{2\theta-1}{2\theta}}\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta} ds\right) + \frac{1}{\mu}\left(\frac{4\theta-2}{a^{2\theta}e^{\mu\rho}}\right)^{\frac{2\theta-1}{2\theta}}\|h\|^{2\theta}e^{\mu t}\\
&\qquad + ae^{\frac{\mu\rho}{2\theta}}(4\theta-2)^{\frac{2\theta-1}{2\theta}}\mathbb{E}\left(\int_{-\rho}^0 \|\varphi(s)\|^{2\theta})ds\right).
\end{aligned}
\tag{4.52}
$$

For the fifth term on the right-hand side of (4.50), we have

$$
\begin{aligned}
&\theta\sum_{k=1}^{\infty}\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right)\\
&\quad \leqslant 2(\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-2}}\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta} ds\right) + \frac{2}{\mu\varepsilon_1^{\theta}}\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2 e^{\mu t}\\
&\qquad + 4(\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-2}}\|\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2\mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta} ds\right) + \frac{4}{\mu\varepsilon_1^{\theta}}\|\kappa\|^2\sum_{k=1}^{\infty}\beta_k^2 e^{\mu t}
\end{aligned}
$$

$$+ 4\theta \|\kappa\|^2_{L^\infty(\mathbb{R}^n)} \sum_{k=1}^{\infty} \gamma_k^2 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right). \tag{4.53}$$

For the sixth term on the right-hand side of (4.50), by (4.53), we have

$$\begin{aligned}
&2\theta(\theta-1)\sum_{k=1}^{\infty} \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta-4} |(u(s), \sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))|^2 ds \right)\\
&\leqslant 4(\theta-1)^2 \varepsilon_1^{\frac{2\theta}{2\theta-2}} \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right)\\
&\quad + 8(\theta-1)^2 \varepsilon_1^{\frac{2\theta}{2\theta-2}} \|\kappa\|^2 \sum_{k=1}^{\infty} \beta_k^2 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right) + \frac{4(\theta-1)}{\mu\varepsilon_1^{\theta}} \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2 e^{\mu t}\\
&\quad + 8\theta(\theta-1) \|\kappa\|^2_{L^\infty(\mathbb{R}^n)} \sum_{k=1}^{\infty} \gamma_k^2 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right) + \frac{8(\theta-1)}{\mu\varepsilon_1^{\theta}} \|\kappa\|^2 \sum_{k=1}^{\infty} \beta_k^2 e^{\mu t}.
\end{aligned} \tag{4.54}$$

It follows from (4.50)–(4.54) that for $t \geqslant 0$,

$$\begin{aligned}
&\mathbb{E}\left(e^{\mu(t\wedge\tau_m)} \|u(t\wedge\tau_m)\|^{2\theta} \right) + 2\theta \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta-2} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2 ds \right)\\
&\quad + 2\theta\lambda_1 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta-2} \|u(s)\|^p_{L^p(\mathbb{R}^n)} ds \right)\\
&\leqslant \mathbb{E}\left(\|u^0\|^{2\theta} \right) + (\mu - 2\theta\lambda) \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right)\\
&\quad + 2(\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}} \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right) + \frac{2}{\mu\varepsilon_1^{\theta}} \|\psi_1\|^{\theta}_{L^1} e^{\mu t}\\
&\quad + a e^{\frac{\mu\rho}{2\theta}} 2^{2-\frac{1}{2\theta}} (2\theta-1)^{\frac{2\theta-1}{2\theta}} \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right) + \frac{1}{\mu} \left(\frac{4\theta-2}{a^{2\theta} e^{\mu\rho}} \right)^{\frac{2\theta-1}{2\theta}} \|h\|^{2\theta} e^{\mu t}\\
&\quad + a e^{\frac{\mu\rho}{2\theta}} (4\theta-2)^{\frac{2\theta-1}{2\theta}} \mathbb{E}\left(\int_{-\rho}^{0} \|\varphi(s)\|^{2\theta} ds \right)\\
&\quad + 2(\theta-1)(2\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-2}} \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right) + \frac{2(2\theta-1)}{\mu\varepsilon_1^{\theta}} \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2 e^{\mu t}\\
&\quad + 4(\theta-1)(2\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-2}} \|\kappa\|^2 \sum_{k=1}^{\infty} \beta_k^2 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right) + \frac{4(2\theta-1)}{\mu\varepsilon_1^{\theta}} \|\kappa\|^2 \sum_{k=1}^{\infty} \beta_k^2 e^{\mu t}\\
&\quad + 4\theta(2\theta-1) \|\kappa\|^2_{L^\infty(\mathbb{R}^n)} \sum_{k=1}^{\infty} \gamma_k^2 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta} ds \right).
\end{aligned} \tag{4.55}$$

Then by (4.49) and (4.55) we get for $t \geqslant 0$,

$$\mathbb{E}\left(e^{\mu(t\wedge\tau_m)} \|u(t\wedge\tau_m)\|^{2\theta} \right) + 2\theta \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s} \|u(s)\|^{2\theta-2} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2 ds \right)$$

$$
\begin{aligned}
&+ 2\theta\lambda_1 \mathbb{E}\left(\int_0^{t\wedge\tau_m} e^{\mu s}\|u(s)\|^{2\theta-2}\|u(s)\|^p_{L^p(\mathbb{R}^n)}ds\right)\\
&\leqslant \mathbb{E}\left(\|u^0\|^{2\theta}\right) + ae^{\frac{\mu\rho}{2\theta}}(4\theta-2)^{\frac{2\theta-1}{2\theta}}\mathbb{E}\left(\int_{-\rho}^0\|\varphi(s)\|^{2\theta}ds\right) + \frac{1}{\mu}\left(\frac{4\theta-2}{a^{2\theta}e^{\mu\rho}}\right)^{\frac{2\theta-1}{2\theta}}\|h\|^{2\theta}e^{\mu t}\\
&+ \frac{2}{\mu\varepsilon_1^\theta}\|\psi_1\|^\theta_{L^1(\mathbb{R}^n)}e^{\mu t} + \frac{2(2\theta-1)}{\mu\varepsilon_1^\theta}\sum_{k=1}^\infty\|\sigma_{1,k}\|^2e^{\mu t} + \frac{4(2\theta-1)}{\mu\varepsilon_1^\theta}\|\kappa\|^2\sum_{k=1}^\infty\beta_k^2e^{\mu t}.
\end{aligned}
\tag{4.56}
$$

Letting $m\to\infty$ in (4.56), by Fatou's theorem we can obtain the desired estimate. This completes the proof. □

4.2 Regularity of solutions

In order to prove the existence of invariant measures of (4.1), we need to derive further regularity of solutions. To that end, we assume:

$$
|F(x,u)-F(y,u)| \leqslant |\phi(x)-\phi(y)|, \quad \forall\, x,y\in\mathbb{R}^n,\ u\in\mathbb{R}; \tag{4.57}
$$

$$
\sigma_{1,k}\in V\cap L^{3p-4}(\mathbb{R}^n),\ \forall\, k\in\mathbb{N} \quad\text{and}\quad \sum_{k=1}^\infty(\|\sigma_{1,k}\|_V^2+\|\sigma_{1,k}\|^2_{L^{3p-4}(\mathbb{R}^n)})<\infty; \tag{4.58}
$$

$$
\kappa\in V \quad\text{and}\quad |\nabla\kappa(x)|\leqslant C, \quad \forall\, x\in\mathbb{R}^n; \tag{4.59}
$$

$$
h\in L^2(\mathbb{R}^n)\cap L^{3p-4}(\mathbb{R}^n),\ \psi_1\in L^1(\mathbb{R}^n)\cap L^{\frac{3p-4}{2}}(\mathbb{R}^n),\ \psi_2\in L^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n), \tag{4.60}
$$

where $C>0$ is a constant and $\phi\in V$.

When proving the Hölder continuity of solutions in Lemma 4.10, the regularity of solutions of (1.1) is needed, for which the coefficients $\sigma_{1,k}$ and κ should belong to the space V instead of H. Since the nonlinear drift term F has a polynomial growth of arbitrary order, the assumptions (4.58) and (4.60) are further required when establishing the higher-order moment estimates of F in $L^r(\Omega, L^r(\mathbb{R}^n))$ with $r>0$.

Next, we derive uniform estimates of solutions in $L^{3p-4}\big(\Omega, L^{3p-4}(\mathbb{R}^n)\big)$.

Lemma 4.6 *Assume* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ'), *(4.12) and (4.58)–(4.60) hold. If*

$$
\|u^0\|_{L^2(\Omega,\mathcal{F}_0;H)}+\|\varphi\|_{L^2(\Omega,\mathcal{F}_0;L^2((-\rho,0),H))}\leqslant R,
$$

with $R>0$*, then the solution* u *of (4.1) satisfies, for all* $t\geqslant 6$ *and* $\rho\in[0,1]$*,*

$$
\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\right)+\mathbb{E}\left(\int_{t-1}^t\|u(s;0,u^0,\varphi)\|^{4p-6}_{L^{4p-6}(\mathbb{R}^n)}ds\right)\leqslant C_1, \tag{4.61}
$$

where C_1 is positive constant depending on R and p, but not on (u^0, φ) or ρ.

Proof The proof consists of several steps. We first derive the uniform estimates of solutions in $L^p\big(\Omega, L^p(\mathbb{R}^n)\big)$. The calculations are formal, but can be justified by a limiting procedure like the Galerkin method.

Step (i). We will show that for all $t \geqslant 2$,

$$\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^p_{L^p(\mathbb{R}^n)}\right) + \mathbb{E}\left(\int_{t-1}^{t} \|u(s;0,u^0,\varphi)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds\right) \leqslant L_1, \tag{4.62}$$

where L_1 is positive constant depending on R and p, but not on (u^0, φ) or ρ.

By Ito's formula [21], we get for $t \geqslant r \geqslant 0$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(t)\|^p_{L^p(\mathbb{R}^n)}\right) + p\mathbb{E}\left(\int_r^t \left(|u(s)|^{p-2}u(s), (-\triangle)^\alpha u(s)\right) ds\right) \\
&= \mathbb{E}\left(\|u(r)\|^p_{L^p(\mathbb{R}^n)}\right) - p\lambda\mathbb{E}\bigg(\int_r^t \|u(s)\|^p_{L^p(\mathbb{R}^n)} ds\bigg) \\
&\quad - p\mathbb{E}\bigg(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{p-2}u(s,x)F(x,u(s,x))dx\bigg) \\
&\quad + p\mathbb{E}\bigg(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{p-2}u(s,x)G(x,u(s-\rho,x))dx\bigg) \\
&\quad + \frac{p(p-1)}{2}\mathbb{E}\bigg(\sum_{k=1}^{\infty}\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{p-2}|\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s,x))|^2 dx\bigg).
\end{aligned} \tag{4.63}$$

For the second term on the left-hand side of (4.63), we have

$$\begin{aligned}
&p\mathbb{E}\left(\int_r^t \left(|u(s,x)|^{p-2}u(s), (-\triangle)^\alpha u(s)\right) ds\right) \\
&= p\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(s,x)-u(s,y))\left(|u(s,x)|^{p-2}u(s,x) - |u(s,y)|^{p-2}u(s,y)\right)}{|x-y|^{n+2\alpha}} dxdy\right) \\
&\geqslant cp\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(s,x)-u(s,y)|^p}{|x-y|^{n+2\alpha}} dxdy\right) \\
&\geqslant 0.
\end{aligned} \tag{4.64}$$

For the third term on the right-hand side of (4.63), by assumption **(F′)** and Young's inequality we get

$$\begin{aligned}
&- p\mathbb{E}\bigg(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{p-2}u(s,x)F(x,u(s,x))dx\bigg) \\
&\quad \leqslant -\lambda_1 p\mathbb{E}\left(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds\right) + p\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{p-2}\psi_1(x)dx\right)
\end{aligned}$$

$$
\leqslant -\lambda_1 p\mathbb{E}\left(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right) + (p-2)\mathbb{E}\left(\int_r^t \|u(s)\|^{p}_{L^{p}(\mathbb{R}^n)}ds\right)
+ 2\|\psi_1\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^n)}(t-r). \tag{4.65}
$$

For the fourth term on the right-hand side of (4.63), by Young's inequality and (4.6) we get

$$
\begin{aligned}
&p\mathbb{E}\left(\int_r^t ds\int_{\mathbb{R}^n} |u(s,x)|^{p-2}u(s,x)G(x,u(s-\rho,x))dx\right)\\
&\leqslant \frac{1}{2}p\lambda_1\mathbb{E}\left(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right) + \frac{p}{2\lambda_1}\mathbb{E}\left(\int_r^t \|G(\cdot,u(s-\rho))\|^2ds\right)\\
&\leqslant \frac{1}{2}p\lambda_1\mathbb{E}\left(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right) + \frac{p}{\lambda_1}\|h\|^2(t-r) + \frac{a^2p}{\lambda_1}\mathbb{E}\left(\int_r^t \|u(s-\rho)\|^2ds\right).
\end{aligned} \tag{4.66}
$$

For the last term on the right-hand side of (4.63), by (4.10) we deduce that

$$
\begin{aligned}
&\frac{p(p-1)}{2}\mathbb{E}\left(\sum_{k=1}^{\infty}\int_r^t ds\int_{\mathbb{R}^n} |u(s,x)|^{p-2}|\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s,x))|^2dx\right)\\
&\leqslant p(p-1)\mathbb{E}\left(\sum_{k=1}^{\infty}\int_r^t ds\int_{\mathbb{R}^n} |u(s,x)|^{p-2}\sigma_{1,k}^2(x)dx\right)\\
&\quad + p(p-1)\mathbb{E}\left(\sum_{k=1}^{\infty}\int_r^t ds\int_{\mathbb{R}^n} |u(s,x)|^{p-2}|\kappa(x)|^2|\sigma_{2,k}(u(s,x))|^2dx\right)\\
&\leqslant (p-1)(p-2)\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^p(\mathbb{R}^n)}\mathbb{E}\left(\int_r^t \|u(s)\|^p_{L^p(\mathbb{R}^n)}ds\right)\\
&\quad + 2(p-1)\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^p(\mathbb{R}^n)}(t-r)\\
&\quad + 2(p-1)(p-2)\sum_{k=1}^{\infty}\beta_k^2\mathbb{E}\left(\int_r^t \|u(s)\|^p_{L^p(\mathbb{R}^n)}ds\right) + 4(p-1)\|\kappa\|^p_{L^p(\mathbb{R}^n)}\sum_{k=1}^{\infty}\beta_k^2(t-r)\\
&\quad + 2p(p-1)\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\mathbb{E}\left(\int_r^t \|u(s)\|^p_{L^p(\mathbb{R}^n)}ds\right).
\end{aligned} \tag{4.67}
$$

Then from (4.63)–(4.67), it follows that for all $t \geqslant r \geqslant 0$,

$$
\begin{aligned}
&\mathbb{E}\left(\|u(t)\|^p_{L^p(\mathbb{R}^n)}\right) + \frac{1}{2}\lambda_1 p\mathbb{E}\left(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right)\\
&\leqslant c_1\left(1+\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^p(\mathbb{R}^n)} + \sum_{k=1}^{\infty}\beta_k^2 + \|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\right)\mathbb{E}\left(\int_r^t \|u(s)\|^p_{L^p(\mathbb{R}^n)}ds\right)\\
&\quad + \mathbb{E}\left(\|u(r)\|^p_{L^p(\mathbb{R}^n)}\right) + \frac{a^2p}{\lambda_1}\sup_{s\geqslant 0}\mathbb{E}\left(\|u(s)\|^2\right)(t-r) + \frac{a^2p}{\lambda_1}\mathbb{E}\left(\int_{-\rho}^0 \|\varphi(s)\|^2ds\right)
\end{aligned}
$$

$$+ 2(p-1)\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^p(\mathbb{R}^n)}(t-r) + 2\|\psi_1\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^n)}(t-r) + \frac{p}{\lambda_1}\|h\|^2(t-r)$$
$$+ 4(p-1)\|\kappa\|^p_{L^p(\mathbb{R}^n)}\sum_{k=1}^{\infty}\beta_k^2(t-r), \tag{4.68}$$

where $c_1 = c_1(p) > 0$ is a constant. Then integrating (4.68) with respect to r on $[t-1, t]$ for $t \geqslant 1$, we get

$$\begin{aligned}
&\mathbb{E}\left(\|u(t)\|^p_{L^p(\mathbb{R}^n)}\right)\\
&\leqslant c_1\left(1+\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^p(\mathbb{R}^n)} + \sum_{k=1}^{\infty}\beta_k^2 + \|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\right)\mathbb{E}\left(\int_{t-1}^{t}\|u(s)\|^p_{L^p(\mathbb{R}^n)}ds\right)\\
&\quad + \mathbb{E}\left(\int_{t-1}^{t}\|u(r)\|^p_{L^p(\mathbb{R}^n)}dr\right) + \frac{a^2p}{\lambda_1}\sup_{s\geqslant 0}\mathbb{E}\left(\|u(s)\|^2\right) + \frac{a^2p}{\lambda_1}\mathbb{E}\left(\int_{-\rho}^{0}\|\varphi(s)\|^2ds\right)\\
&\quad + 2(p-1)\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^p(\mathbb{R}^n)} + 2\|\psi_1\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^n)} + \frac{p}{\lambda_1}\|h\|^2 + 4(p-1)\|\kappa\|^p_{L^p(\mathbb{R}^n)}\sum_{k=1}^{\infty}\beta_k^2.
\end{aligned} \tag{4.69}$$

By (4.69) and Remark 4.1, we find that there exists a positive number c_2 depending only on R and p but not on (u^0, φ) or $\rho \in [0, 1]$ such that for all $t \geqslant 1$,

$$\mathbb{E}\left(\|u(t)\|^p_{L^p(\mathbb{R}^n)}\right) \leqslant c_2. \tag{4.70}$$

By (4.68) and (4.70), we obtain for $t \geqslant 2$,

$$\mathbb{E}\left(\int_{t-1}^{t}\|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right) \leqslant c_3, \tag{4.71}$$

where $c_3 > 0$ depends only on R and p but not on (u^0, φ) or $\rho \in [0, 1]$. Then (4.62) follows from (4.70)–(4.71) immediately.

Step (ii). We now show that for all $t \geqslant 4$,

$$\mathbb{E}\left(\|u(t; 0, u^0, \varphi)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}\right) + \mathbb{E}\left(\int_{t-1}^{t}\|u(s; 0, u^0, \varphi)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right) \leqslant L_2, \tag{4.72}$$

where L_2 is positive constant depending on R and p, but not on (u^0, φ) or ρ.

It follows from Ito's formula [21] that for $t \geqslant r \geqslant 0$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(t)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}\right) + (2p-2)\mathbb{E}\left(\int_r^t\left(|u(s,x)|^{2p-4}u(s), (-\triangle)^\alpha u(s)\right)ds\right)\\
&= \mathbb{E}\left(\|u(r)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}\right) - (2p-2)\lambda\mathbb{E}\left(\int_r^t\|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right)
\end{aligned}$$

$$
\begin{aligned}
&-(2p-2)\mathbb{E}\Big(\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}u(s,x)F(x,u(s,x))dx\Big)\\
&+(2p-2)\mathbb{E}\Big(\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}u(s,x)G(x,u(s-\rho,x))dx\Big)\\
&+(p-1)(2p-3)\mathbb{E}\Big(\sum_{k=1}^{\infty}\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}|\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s,x))|^2dx\Big).
\end{aligned}
\tag{4.73}
$$

For the second term on the left-hand side of (4.63),

$$
(2p-2)\mathbb{E}\left(\int_r^t\left(|u(s,x)|^{2p-4}u(s),(-\triangle)^{\alpha}u(s)\right)ds\right)\geqslant 0. \tag{4.74}
$$

For the third term on the right-hand side of (4.73), by assumption **(F′)** and Young's inequality we get

$$
\begin{aligned}
&-(2p-2)\mathbb{E}\Big(\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}u(s,x)F(x,u(s,x))dx\Big)\\
&\leqslant -(2p-2)\lambda_1\mathbb{E}\left(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right)+(2p-2)\mathbb{E}\left(\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}\psi_1(x)dx\right)\\
&\leqslant -(2p-2)\lambda_1\mathbb{E}\left(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right)+(2p-4)\mathbb{E}\left(\int_r^t\|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right)\\
&\quad+2\|\psi_1\|^{p-1}_{L^{p-1}(\mathbb{R}^n)}(t-r).
\end{aligned}
\tag{4.75}
$$

For the fourth term on the right-hand side of (4.73), by Young's inequality and (4.6) we get

$$
\begin{aligned}
&(2p-2)\mathbb{E}\Big(\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}u(s,x)G(x,u(s-\rho,x))dx\Big)\\
&\leqslant (2p-3)\mathbb{E}\left(\int_r^t\|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right)+\mathbb{E}\left(\int_r^t\|G(\cdot,u(s-\rho))\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right)\\
&\leqslant (2p-3)\mathbb{E}\left(\int_r^t\|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right)+2^{2p-3}\|h\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}(t-r)\\
&\quad+2^{2p-3}a^{2p-2}\mathbb{E}\left(\int_r^t\|u(s-\rho)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}ds\right).
\end{aligned}
\tag{4.76}
$$

For the last term on the right-hand side of (4.73), by (4.10) we deduce that

$$
\begin{aligned}
&(p-1)(2p-3)\mathbb{E}\Big(\sum_{k=1}^{\infty}\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}|\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s,x))|^2dx\Big)\\
&\leqslant 2(p-1)(2p-3)\mathbb{E}\Big(\sum_{k=1}^{\infty}\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{2p-4}\sigma_{1,k}^2(x)dx\Big)
\end{aligned}
$$

$$+ 2(p-1)(2p-3)\mathbb{E}\Big(\sum_{k=1}^{\infty} \int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{2p-4} |\kappa(x)|^2 |\sigma_{2,k}(u(s,x))|^2 dx \Big)$$
$$\leqslant (2p-3)(2p-4) \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2_{L^{2p-2}(\mathbb{R}^n)} \mathbb{E}\Big(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds \Big)$$
$$+ 2(2p-3) \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2_{L^{2p-2}(\mathbb{R}^n)} (t-r)$$
$$+ 4(2p-3)(p-2) \sum_{k=1}^{\infty} \beta_k^2 \mathbb{E}\Big(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds \Big)$$
$$+ 4(2p-3) \|\kappa\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \beta_k^2 (t-r)$$
$$+ 4(2p-3)(p-1) \|\kappa\|^2_{L^{\infty}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \gamma_k^2 \mathbb{E}\Big(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds \Big). \tag{4.77}$$

Then from (4.73)–(4.77), it follows that for all $t \geqslant r \geqslant 0$,

$$\mathbb{E}\left(\|u(t)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} \right) + (2p-2)\lambda_1 \mathbb{E}\left(\int_r^t \|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)} ds \right)$$
$$\leqslant c_4 \left(1 + \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2_{L^{2p-2}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} \beta_k^2 + \|\kappa\|^2_{L^{\infty}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \gamma_k^2 \right) \mathbb{E}\Big(\int_r^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds \Big)$$
$$+ \mathbb{E}\left(\|u(r)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} \right) + 2^{2p-3} a^{2p-2} \mathbb{E}\Big(\int_r^t \|u(s-\rho)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds \Big)$$
$$+ 2(2p-3) \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2_{L^{2p-2}(\mathbb{R}^n)} (t-r) + 2\|\psi_1\|^{p-1}_{L^{p-1}(\mathbb{R}^n)} (t-r) + 2^{2p-3} \|h\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} (t-r)$$
$$+ 4(2p-3) \|\kappa\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \beta_k^2 (t-r), \tag{4.78}$$

where $c_4 = c_4(p) > 0$ is a constant. Then integrating (4.78) with respect to r on $[t-1, t]$ for $t \geqslant 1$, we get

$$\mathbb{E}\left(\|u(t)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} \right)$$
$$\leqslant c_4 \left(1 + \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2_{L^{2p-2}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} \beta_k^2 + \|\kappa\|^2_{L^{\infty}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \gamma_k^2 \right) \mathbb{E}\Big(\int_{t-1}^t \|u(s)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds \Big)$$
$$+ \mathbb{E}\left(\int_{t-1}^t \|u(r)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} dr \right) + 2^{2p-3} a^{2p-2} \mathbb{E}\Big(\int_{t-1}^t \|u(s-\rho)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} ds \Big)$$
$$+ 2(2p-3) \sum_{k=1}^{\infty} \|\sigma_{1,k}\|^2_{L^{2p-2}(\mathbb{R}^n)} + 2\|\psi_1\|^{p-1}_{L^{p-1}(\mathbb{R}^n)} + 2^{2p-3} \|h\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}$$

$$+ 4(2p-3)\|\kappa\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \beta_k^2. \tag{4.79}$$

By (4.71) and (4.79) we find that there exists a positive number c_5 depending only on R and p but not on (u^0, φ) or $\rho \in [0, 1]$ such that for all $t \geqslant 3$,

$$\mathbb{E}\left(\|u(t)\|^{2p-2}_{L^{2p-2}(\mathbb{R}^n)}\right) \leqslant c_5. \tag{4.80}$$

By (4.78) and (4.80), we obtain for $t \geqslant 4$,

$$\mathbb{E}\left(\int_{t-1}^{t} \|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)} ds\right) \leqslant c_6, \tag{4.81}$$

where $c_6 > 0$ depends only on R and p but not on (u^0, φ) or $\rho \in [0, 1]$. By (4.80)–(4.81) we obtain (4.72).

Step (iii). We now prove (4.61). Again, by Ito's formula [21], for $t \geqslant r \geqslant 0$,

$$\begin{aligned}
&\mathbb{E}\left(\|u(t)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\right) + (3p-4)\mathbb{E}\left(\int_r^t \left(|u(s,x)|^{3p-6}u(s), (-\triangle)^{\alpha}u(s)\right) ds\right)\\
&\quad = \mathbb{E}\left(\|u(r)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\right) - (3p-4)\lambda\mathbb{E}\left(\int_r^t \|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)} ds\right)\\
&\qquad - (3p-4)\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{3p-6}u(s,x)F(x,u(s,x))dx\right)\\
&\qquad + (3p-4)\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{3p-6}u(s,x)G(x,u(s-\rho,x))dx\right)\\
&\qquad + \frac{1}{2}(3p-4)(3p-5)\mathbb{E}\left(\sum_{k=1}^{\infty}\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{3p-6}|\sigma_{1,k}(x) + \kappa(x)\sigma_{2,k}(u(s,x))|^2 dx\right).
\end{aligned} \tag{4.82}$$

For the second term on the left-hand side of (4.82),

$$(3p-4)\mathbb{E}\left(\int_r^t \left(|u(s,x)|^{3p-6}u(s), (-\triangle)^{\alpha}u(s)\right) ds\right) \geqslant 0. \tag{4.83}$$

For the third term on the right-hand side of (4.82), by assumption **(F′)** and Young's inequality we get

$$\begin{aligned}
&- (3p-4)\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{3p-6}u(s,x)F(x,u(s,x))dx\right)\\
&\quad \leqslant -(3p-4)\lambda_1\mathbb{E}\left(\int_r^t \|u(s)\|^{4p-6}_{L^{4p-6}(\mathbb{R}^n)} ds\right) + (3p-4)\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n} |u(s,x)|^{3p-6}\psi_1(x)dx\right)\\
&\quad \leqslant -(3p-4)\lambda_1\mathbb{E}\left(\int_r^t \|u(s)\|^{4p-6}_{L^{4p-6}(\mathbb{R}^n)} ds\right) + (3p-6)\mathbb{E}\left(\int_r^t \|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)} ds\right)\\
&\qquad + 2\|\psi_1\|^{\frac{3p-4}{2}}_{L^{\frac{3p-4}{2}}(\mathbb{R}^n)}(t-r).
\end{aligned} \tag{4.84}$$

For the fourth term on the right-hand side of (4.82), by Young's inequality and (4.6) we get

$$
\begin{aligned}
&(3p-4)\mathbb{E}\Big(\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{3p-6}u(s,x)G(x,u(s-\rho,x))dx\Big)\\
&\quad\leqslant(3p-5)\mathbb{E}\left(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right)+\mathbb{E}\left(\int_r^t\|G(\cdot,u(s-\rho))\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right)\\
&\quad\leqslant(3p-5)\mathbb{E}\left(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right)\\
&\qquad+2^{3p-5}\|h\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}(t-r)+2^{3p-5}a^{3p-4}\mathbb{E}\left(\int_r^t\|u(s-\rho)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right).
\end{aligned}
\tag{4.85}
$$

For the last term on the right-hand side of (4.82), by (4.10) we deduce that

$$
\begin{aligned}
&\frac{1}{2}(3p-4)(3p-5)\mathbb{E}\Big(\sum_{k=1}^\infty\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{3p-6}|\sigma_{1,k}(x)+\kappa(x)\sigma_{2,k}(u(s,x))|^2dx\Big)\\
&\quad\leqslant(3p-4)(3p-5)\mathbb{E}\Big(\sum_{k=1}^\infty\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{3p-6}\sigma_{1,k}^2(x)dx\Big)\\
&\qquad+(3p-4)(3p-5)\mathbb{E}\Big(\sum_{k=1}^\infty\int_r^t ds\int_{\mathbb{R}^n}|u(s,x)|^{3p-6}|\kappa(x)|^2|\sigma_{2,k}(u(s,x))|^2dx\Big)\\
&\quad\leqslant(3p-5)(3p-6)\sum_{k=1}^\infty\|\sigma_{1,k}\|^2_{L^{3p-4}(\mathbb{R}^n)}\mathbb{E}\Big(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\Big)\\
&\qquad+2(3p-5)\sum_{k=1}^\infty\|\sigma_{1,k}\|^2_{L^{3p-4}(\mathbb{R}^n)}(t-r)\\
&\qquad+2(3p-5)(3p-6)\sum_{k=1}^\infty\beta_k^2\mathbb{E}\Big(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\Big)\\
&\qquad+4(3p-5)\|\kappa\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\sum_{k=1}^\infty\beta_k^2(t-r)\\
&\qquad+2(3p-4)(3p-5)\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^\infty\gamma_k^2\mathbb{E}\Big(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\Big).
\end{aligned}
\tag{4.86}
$$

Then from (4.82)–(4.86), it follows that for all $t\geqslant r\geqslant 0$,

$$
\begin{aligned}
&\mathbb{E}\left(\|u(t)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\right)+(3p-4)\lambda_1\mathbb{E}\left(\int_r^t\|u(s)\|^{4p-6}_{L^{4p-6}(\mathbb{R}^n)}ds\right)\\
&\quad\leqslant c_7\left(1+\sum_{k=1}^\infty\|\sigma_{1,k}\|^2_{L^{3p-4}(\mathbb{R}^n)}+\sum_{k=1}^\infty\beta_k^2+\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\sum_{k=1}^\infty\gamma_k^2\right)\mathbb{E}\Big(\int_r^t\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\Big)\\
&\qquad+\mathbb{E}\left(\|u(r)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\right)+2^{3p-5}a^{3p-4}\mathbb{E}\Big(\int_r^t\|u(s-\rho)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\Big)
\end{aligned}
$$

$$+2(3p-5)\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^{3p-4}(\mathbb{R}^n)}(t-r)+2\|\psi_1\|^{\frac{3p-4}{2}}_{L^{\frac{3p-4}{2}}(\mathbb{R}^n)}(t-r)+2^{3p-5}\|h\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}(t-r)$$
$$+4(3p-5)\|\kappa\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\beta_k^2(t-r), \tag{4.87}$$

where $c_7 = c_7(p) > 0$ is a constant. Then integrating (4.87) with respect to r on $[t-1, t]$ for $t \geqslant 1$, we get

$$\begin{aligned}
&\mathbb{E}\left(\|u(t)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\right)\\
&\leqslant c_7\left(1+\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^{3p-4}(\mathbb{R}^n)}+\sum_{k=1}^{\infty}\beta_k^2+\|\kappa\|^2_{L^{\infty}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\gamma_k^2\right)\mathbb{E}\left(\int_{t-1}^{t}\|u(s)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right)\\
&\quad+\mathbb{E}\left(\int_{t-1}^{t}\|u(r)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\right)dr+2^{3p-5}a^{3p-4}\mathbb{E}\left(\int_{t-1}^{t}\|u(s-\rho)\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}ds\right)\\
&\quad+2(3p-5)\sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^{3p-4}(\mathbb{R}^n)}+2\|\psi_1\|^{\frac{3p-4}{2}}_{L^{\frac{3p-4}{2}}(\mathbb{R}^n)}+2^{3p-5}\|h\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\\
&\quad+4(3p-5)\|\kappa\|^{3p-4}_{L^{3p-4}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\beta_k^2.
\end{aligned}\tag{4.88}$$

By (4.81) and (4.88), we find that there exists a positive number c_8 depending only on R and p but not on (u^0, φ) or $\rho \in [0, 1]$ such that for all $t \geqslant 5$,

$$\mathbb{E}\left(\|u(t)\|^{3p-4}_{L^{3p-4}}\right) \leqslant c_8. \tag{4.89}$$

By (4.87) and (4.89), we obtain for $t \geqslant 6$,

$$\mathbb{E}\left(\int_{t-1}^{t}\|u(s)\|^{4p-6}_{L^{4p-6}(\mathbb{R}^n)}ds\right) \leqslant c_9, \tag{4.90}$$

where $c_9 > 0$ depends only on R and p but not on (u^0, φ) or $\rho \in [0, 1]$. This concludes the proof. □

Remark 4.3 The uniform estimates given by Lemma 4.6 can be further extended under additional assumptions. Suppose $\psi_1 \in L^r(\mathbb{R}^n)$ for all $r \in [1, \infty)$ and

$$h \in L^r(\mathbb{R}^n) \quad \text{and} \quad \sum_{k=1}^{\infty}\|\sigma_{1,k}\|^2_{L^r(\mathbb{R}^n)} < \infty, \ \forall\, r \in [2, \infty).$$

Then by the argument of Lemma 4.6, one can show that for every integer $k \geqslant 0$, the solution u of (4.1) satisfies, for all $t \geqslant 2(k+1)$ and $\rho \in [0, 1]$,

$$\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|^{p+(p-2)k}_{L^{p+(p-2)k}(\mathbb{R}^n)}\right)+\mathbb{E}\left(\int_{t-1}^{t}\|u(s;0,u^0,\varphi)\|^{p+(p-2)(k+1)}_{L^{p+(p-2)(k+1)}(\mathbb{R}^n)}ds\right) \leqslant L_k,$$

where L_k is positive constant depending on k, p and R but not on (u^0, φ) or ρ when

$$\|u^0\|_{L^2(\Omega, L^2(\mathbb{R}^n))} + \|\varphi\|_{L^2(\Omega, L^2((-\rho,0), L^2(\mathbb{R}^n)))} \leqslant R.$$

In addition, by Remark 4.1 and the proof of Lemma 4.6, we know that there exists $T > 2(k+1)$ depending on R and k but not on u^0, φ or $\rho \in [0, 1]$ such that for $t \geqslant T$,

$$\mathbb{E}\left(\|u(t; 0, u^0, \varphi)\|^{p+(p-2)k}_{L^{p+(p-2)k}(\mathbb{R}^n)}\right) + \mathbb{E}\left(\int_{t-1}^t \|u(s; 0, u^0, \varphi)\|^{p+(p-2)(k+1)}_{L^{p+(p-2)(k+1)}(\mathbb{R}^n)} ds\right) \leqslant \tilde{L}_k,$$

where $\tilde{L}_k$ is positive constant depending on k and p but not on R or $\rho \in [0, 1]$

Lemma 4.7 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ'), *(4.12) and (4.57)–(4.60) hold. Then for every* $R > 0$ *and initial data* $(u^0, \varphi) \in L^2(\Omega, \mathcal{F}_0; H) \times L^2\big(\Omega, \mathcal{F}_0; L^2((-\rho, 0), H)\big)$ *with*

$$\mathbb{E}\left(\|u^0\|^2 + \int_{-\rho}^0 \|\varphi(s)\|^2 ds\right) \leqslant R,$$

the solution u *of (4.1) satisfies, for all* $t \geqslant 3$,

$$\mathbb{E}\left(\|u(t; 0, u^0, \varphi)\|_V^2\right) + \mathbb{E}\left(\int_{t-1}^t \|(-\triangle)^\alpha u(s; 0, u^0, \varphi)\|^2 ds\right) \leqslant C_2, \tag{4.91}$$

where $C_2 > 0$ *depends on* R *but not on* u^0, φ, *or* $\rho \in [0, 1]$.

Proof We formally derive (4.91). By (4.1) and Ito's formula, we obtain for $t \geqslant r \geqslant 0$,

$$\begin{aligned}
&\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(t)\|^2\right) + 2\mathbb{E}\left(\int_r^t \|(-\triangle)^\alpha u(s)\|^2 ds\right) + 2\lambda\mathbb{E}\left(\int_r^t \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2 ds\right)\\
&= \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(r)\|^2\right) - 2\mathbb{E}\left(\int_r^t \big((-\triangle)^\alpha u(s), F(\cdot, u(s))\big) ds\right)\\
&\quad + 2\mathbb{E}\left(\int_r^t \big((-\triangle)^\alpha u(s), G(\cdot, u(s-\rho))\big) ds\right)\\
&\quad + \mathbb{E}\left(\sum_{k=1}^\infty \int_r^t \|(-\triangle)^{\frac{\alpha}{2}} (\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\|^2 ds\right).
\end{aligned} \tag{4.92}$$

For the second term on the right-hand side of (4.92), by (4.4) and (4.57), we get

$$\begin{aligned}
&-2\mathbb{E}\left(\int_r^t \big((-\triangle)^\alpha u(s),\ F(\cdot, u(s))\big) ds\right)\\
&= -2\mathbb{E}\left(\int_r^t \big((-\triangle)^{\frac{\alpha}{2}} u(s),\ (-\triangle)^{\frac{\alpha}{2}} F(\cdot, u(s))\big) ds\right)\\
&\leqslant C(n, \alpha)\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(s,x) - u(s,y)|\,|\phi(x) - \phi(y)|}{|x-y|^{n+2\alpha}} dxdy\right)
\end{aligned}$$

$$
\begin{aligned}
&+ C(n,\alpha)\mathbb{E}\left(\int_r^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_4(x)\frac{(u(s,x)-u(s,y))^2}{|x-y|^{n+2\alpha}}dxdy\right)\\
&\leqslant 2\mathbb{E}\left(\int_r^t \|(-\triangle)^{\frac{\alpha}{2}}u(s)\|\cdot\|(-\triangle)^{\frac{\alpha}{2}}\phi\|ds\right) + 2\|\psi_4\|_{L^\infty(\mathbb{R}^n)}\mathbb{E}\left(\int_r^t \|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2 ds\right)\\
&\leqslant \left(1+2\|\psi_4\|_{L^\infty(\mathbb{R}^n)}\right)\int_r^t \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds + \|(-\triangle)^{\frac{\alpha}{2}}\phi\|^2(t-r).
\end{aligned}
\tag{4.93}
$$

For the third term on the right-hand side of (4.92), by (4.6) and Remark 4.1, we obtain for $t \geqslant r \geqslant \rho$

$$
\begin{aligned}
&2\mathbb{E}\left(\int_r^t \left((-\triangle)^{\alpha}u(s),\ G(\cdot,u(s-\rho))\right)ds\right)\\
&\quad\leqslant \frac{1}{2}\mathbb{E}\left(\int_r^t \|(-\triangle)^{\alpha}u(s)\|^2 ds\right) + 4\left(\|h\|^2 + a^2 \sup_{r-\rho\leqslant s\leqslant t}\mathbb{E}\left(\|u(s)\|^2\right)\right)(t-r).
\end{aligned}
\tag{4.94}
$$

For the fourth term on the right-hand side of (4.92), by the inequality (4.13) in [20], we have

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{k=1}^{\infty}\int_r^t \|(-\triangle)^{\frac{\alpha}{2}}(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2 ds\right)\\
&\quad\leqslant 2\sum_{k=1}^{\infty}\left(\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^2 + 4\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2 + 2C(n,\alpha)c_1\gamma_k^2 \sup_{r\leqslant s\leqslant t}\mathbb{E}\left(\|u(s)\|^2\right)\right)(t-r)\\
&\qquad + 4\sum_{k=1}^{\infty}\alpha_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\int_r^t \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds,
\end{aligned}
\tag{4.95}
$$

where the constant $c_1 > 0$ depends only on n, α and κ.

By (4.92)–(4.95), we have for $t \geqslant r \geqslant \rho$,

$$
\begin{aligned}
&\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^2\right) + \mathbb{E}\left(\int_r^t \|(-\triangle)^{\alpha}u(s)\|^2 ds\right)\\
&\quad\leqslant \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(r)\|^2\right) + \left(1+2\|\psi_4\|_{L^\infty(\mathbb{R}^n)}\right)\int_r^t \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds\\
&\qquad + \|(-\triangle)^{\frac{\alpha}{2}}\phi\|^2(t-r) + 4\left(\|h\|^2 + a^2\sup_{r-\rho\leqslant s\leqslant t}\mathbb{E}\left(\|u(s)\|^2\right)\right)(t-r)\\
&\qquad + 2\sum_{k=1}^{\infty}\left(\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^2 + 4\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2 + 2C(n,\alpha)c_1\gamma_k^2\sup_{r\leqslant s\leqslant t}\mathbb{E}\left(\|u(s)\|^2\right)\right)(t-r)\\
&\qquad + 4\sum_{k=1}^{\infty}\alpha_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)}\int_r^t \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^2\right)ds.
\end{aligned}
\tag{4.96}
$$

For $t \geqslant 1+\rho$, integrating (4.96) on $[t-1, t]$ with respect to r, we have

$$
\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^2\right)
$$

$$\begin{aligned}
&\leqslant \int_{t-1}^{t} \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(r)\|^2\right) dr + \left(1+2\|\psi_4\|_{L^\infty(\mathbb{R}^n)}\right) \int_{t-1}^{t} \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2\right) ds \\
&\quad + \|(-\triangle)^{\frac{\alpha}{2}} \phi\|^2 + 4\left(\|h\|^2 + a^2 \sup_{t-1-\rho\leqslant s\leqslant t} \mathbb{E}\left(\|u(s)\|^2\right)\right) \\
&\quad + 2\sum_{k=1}^{\infty}\left(\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^2 + 4\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2 + 2C(n,\alpha)c_1\gamma_k^2 \sup_{t-1\leqslant s\leqslant t}\mathbb{E}\left(\|u(s)\|^2\right)\right) \\
&\quad + 4\sum_{k=1}^{\infty}\alpha_k^2\|\kappa\|^2_{L^\infty(\mathbb{R}^n)} \int_{t-1}^{t}\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^2\right) ds. \qquad (4.97)
\end{aligned}$$

By Remark 4.1 and (4.97), we see that there exists $c_2 > 0$ depending on R but not on u^0, φ or $\rho \in [0, 1]$ such that

$$\mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(t)\|^2\right) \leqslant c_2, \quad \forall\, t \geqslant 2. \qquad (4.98)$$

Then by (4.96) and (4.98), we have for $t \geqslant 3$,

$$\mathbb{E}\left(\int_{t-1}^{t} \|(-\triangle)^{\alpha} u(s)\|^2 ds\right) \leqslant c_3, \qquad (4.99)$$

where $c_3 > 0$ depends on R but not on u^0, φ or $\rho \in [0, 1]$. Then (4.98)–(4.99) and Lemma 4.1 conclude the proof. □

Lemma 4.8 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ'), *(4.12) and (4.57)–(4.60) hold. Then for any* $R > 0$ *and the initial data* $(u^0, \varphi) \in L^2(\Omega, \mathcal{F}_0; H) \times L^2\big(\Omega, \mathcal{F}_0; L^2((-\rho, 0), H)\big)$ *with* $\mathbb{E}\Big(\|u^0\|^2 + \int_{-\rho}^{0} \|\varphi(s)\|^2 ds\Big) \leqslant R$, *the solution* u *of (4.1) satisfies, for all* $t \geqslant 4$,

$$\mathbb{E}\left(\sup_{t-1\leqslant r\leqslant t} \|u(r; 0, u^0, \varphi)\|_V^2\right) \leqslant C_3,$$

where $C_3 > 0$ *depends on* R *but not on* u^0, φ *or* $\rho \in [0, 1]$.

Proof The proof is based on Lemma 4.7 and is similar to that of Lemma 4.2 in [20]. So the details are omitted here. □

Remark 4.4 Suppose the assumptions of Lemma 4.7 hold and $\mathbb{E}\Big(\|u^0\|^2 + \int_{-\rho}^{0} \|\varphi(s)\|^2 ds\Big) \leqslant R$ for some $R > 0$. Then by Remark 4.1 and the proof of Lemma 4.7, we find that there exists $T \geqslant 4$ depending only on R (but not on u^0, φ or $\rho \in [0, 1]$) such that for all $t \geqslant T$, the solution u of (4.1) satisfies $\mathbb{E}\Big(\sup_{t-1\leqslant r\leqslant t} \|u(r)\|_V^2\Big) \leqslant \widetilde{C}_3$, where $\widetilde{C}_3 > 0$ is a constant independent of R, u^0, φ and $\rho \in [0, 1]$.

Lemma 4.9 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ'), *(4.11) and (4.57)–(4.60) hold. If* $R > 0$ *and* $(u^0, \varphi) \in L^{2\theta}(\Omega, \mathcal{F}_0; V)) \times L^{2\theta}(\Omega, \mathcal{F}_0; L^{2\theta}(-\rho, 0]; V))$ *such that* $\mathbb{E}\Big(\|u^0\|_V^{2\theta} +$

$\int_{-\rho}^{0} \|\varphi(s)\|_V^{2\theta} ds \Big) \leqslant R$, *then the solution* u *of (4.1) satisfies*

$$\sup_{t \geqslant 0} \mathbb{E}\left(\|(-\triangle)^{\frac{\alpha}{2}} u(t; 0, u^0, \varphi)\|^{2\theta} \right) \leqslant C_4,$$

where C_4 *is a positive constant depending on* R *but not on* u^0, φ *or* $\rho \in [0, 1]$.

Proof The proof for $\theta = 1$ is easier. So we assume $\theta > 1$ in the sequel. Let μ and ε_1 be positive constants to be specified later. By (4.1) and Ito's formula, we get for $t \geqslant 0$,

$$\begin{aligned}
& e^{\mu t}\mathbb{E}\left[\|(-\triangle)^{\frac{\alpha}{2}} u(t)\|^{2\theta} \right] + 2\theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} \|(-\triangle)^{\alpha} u(s)\|^2 ds \right] \\
&= \mathbb{E}\left[\|(-\triangle)^{\frac{\alpha}{2}} u^0\|^{2\theta} \right] + (\mu - 2\theta\lambda)\mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta} ds \right] \\
&\quad - 2\theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} ((-\triangle)^{\alpha} u(s),\ F(\cdot, u(s))) ds \right] \\
&\quad + 2\theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} ((-\triangle)^{\alpha} u(s),\ G(\cdot, u(s-\rho))) ds \right] \\
&\quad + \theta \sum_{k=1}^{\infty} \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} \|(-\triangle)^{\frac{\alpha}{2}} (\sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))\|^2 ds \right] \\
&\quad + 2\theta(\theta-1) \sum_{k=1}^{\infty} \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-4} |((-\triangle)^{\alpha} u(s), \sigma_{1,k} + \kappa\sigma_{2,k}(u(s)))|^2 ds \right].
\end{aligned} \tag{4.100}$$

For the third term on the right-hand side of (4.100), similar to (4.93), we obtain

$$\begin{aligned}
& 2\theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} ((-\triangle)^{\alpha} u(s),\ F(\cdot, u(s))) ds \right] \\
&\leqslant 2\theta \|\psi_4\|_{L^\infty(\mathbb{R}^n)} \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta} ds \right] \\
&\quad + 2\theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-1} \|(-\triangle)^{\frac{\alpha}{2}} \psi_5\| ds \right] \\
&\leqslant \left(2\theta \|\psi_4\|_{L^\infty(\mathbb{R}^n)} + (2\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-1}} \right) \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta} ds \right] + \varepsilon_1^{-2\theta} \mu^{-1} e^{\mu t} \|\phi\|_V^{2\theta}.
\end{aligned} \tag{4.101}$$

For the fourth term on the right-hand side of (4.100), by (4.6) we obtain

$$\begin{aligned}
& 2\theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} ((-\triangle)^{\alpha} u(s),\ G(\cdot, u(s-\rho))) ds \right] \\
&\leqslant \theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} \|(-\triangle)^{\alpha} u(s)\|^2 ds \right] \\
&\quad + \theta \mathbb{E}\left[\int_0^t e^{\mu s} \|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta-2} \|G(\cdot, u(s-\rho))\|^2 ds \right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \theta \mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta-2}\|(-\triangle)^{\alpha}u(s)\|^2 ds\right] \\
&\quad + 4(\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta} ds\right] \\
&\quad + \frac{2}{\mu\varepsilon_1^{\theta}}\|h\|^{2\theta}e^{\mu t} + \frac{2a^{2\theta}}{\mu\varepsilon_1^{\theta}}\sup_{s\geqslant 0}\mathbb{E}[\|u(s)\|^{2\theta}]e^{\mu t} + \frac{2a^{2\theta}}{\varepsilon_1^{\theta}}e^{\mu\rho}\mathbb{E}\left[\int_{-\rho}^0 \|\varphi(s)\|^{2\theta} ds\right].
\end{aligned} \tag{4.102}
$$

For the fifth term on the right-hand side of (4.100), by (4.31) in [20], we have

$$
\begin{aligned}
&\theta\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta-2}\|(-\triangle)^{\frac{\alpha}{2}}(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2 ds\right] \\
&\quad \leqslant 2\sum_{k=1}^{\infty}\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^2\left((\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta} ds\right] + \frac{1}{\varepsilon_1^{\theta}}\int_0^t e^{\mu s} ds\right) \\
&\qquad + 4\theta\sum_{k=1}^{\infty}\alpha_k^2\|\kappa\|_{L^\infty}^2\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta} ds\right] \\
&\qquad + 8\sum_{k=1}^{\infty}\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2\left((\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta} ds\right] + \frac{1}{\varepsilon_1^{\theta}}\int_0^t e^{\mu s} ds\right) \\
&\qquad + 4C(n,\alpha)c_1\sum_{k=1}^{\infty}\gamma_k^2\Bigg((\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta} ds\right] \\
&\qquad + \frac{1}{\varepsilon_1^{\theta}}\mathbb{E}\left[\int_0^t e^{\mu s}\|u(s)\|^{2\theta} ds\right]\Bigg).
\end{aligned} \tag{4.103}
$$

For the sixth term on the right-hand side of (4.100), we have

$$
\begin{aligned}
&2\theta(\theta-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta-4}|((-\triangle)^{\alpha}u(s),\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))|^2 ds\right] \\
&\quad \leqslant 2\theta(\theta-1)\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta-2}\|(-\triangle)^{\frac{\alpha}{2}}(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s)))\|^2 ds\right].
\end{aligned} \tag{4.104}
$$

Then by (4.100)–(4.104), we get for all $t \geqslant 0$,

$$
\begin{aligned}
&e^{\mu t}\mathbb{E}\left[\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^{2\theta}\right] \\
&\quad \leqslant \mathbb{E}\left[\|(-\triangle)^{\frac{\alpha}{2}}u^0\|^{2\theta}\right] \\
&\qquad + \Bigg(\mu - 2\theta\lambda + 2\theta\|\psi_4\|_{L^\infty(\mathbb{R}^n)} + (2\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-1}} + 2(2\theta+1)(\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}} \\
&\qquad + 4\theta(2\theta-1)\sum_{k=1}^{\infty}\alpha_k^2\|\kappa\|_{L^\infty}^2 + 8(\theta-1)(2\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\sum_{k=1}^{\infty}\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2(\theta-1)(2\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\sum_{k=1}^{\infty}\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^2\\
&+4(\theta-1)(2\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}C(n,\alpha)c_1\sum_{k=1}^{\infty}\gamma_k^2\Big)\mathbb{E}\left[\int_0^t e^{\mu s}\|(-\triangle)^{\frac{\alpha}{2}}u(s)\|^{2\theta}ds\right]\\
&+\varepsilon_1^{-2\theta}\mu^{-1}e^{\mu t}\|\phi\|_V^{2\theta}+\frac{2}{\mu\varepsilon_1^{\theta}}\|h\|^{2\theta}e^{\mu t}+\frac{2a^{2\theta}}{\mu\varepsilon_1^{\theta}}\sup_{s\geqslant 0}\mathbb{E}[\|u(s)\|^{2\theta}]e^{\mu t}\\
&+\frac{2a^{2\theta}}{\varepsilon_1^{\theta}}e^{\mu\rho}\mathbb{E}\left[\int_{-\rho}^0\|\varphi(s)\|^{2\theta}]ds\right]+\frac{2(2\theta-1)}{\mu\varepsilon_1^{\theta}}\sum_{k=1}^{\infty}\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^2e^{\mu t}\\
&+8(2\theta-1)\frac{1}{\mu\varepsilon_1^{\theta}}\sum_{k=1}^{\infty}\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2e^{\mu t}\\
&+4(2\theta-1)\frac{1}{\varepsilon_1^{\theta}}C(n,\alpha)c_1\sum_{k=1}^{\infty}\gamma_k^2\mathbb{E}\left[\int_0^t e^{\mu s}\|u(s)\|^{2\theta}ds\right].
\end{aligned}
\tag{4.105}
$$

By (4.11), there exist positive constants μ and ε_1 such that

$$
\begin{aligned}
&\mu-2\theta\lambda+2\theta\|\psi_4\|_{L^\infty(\mathbb{R}^n)}+(2\theta-1)\varepsilon_1^{\frac{2\theta}{2\theta-1}}+2(2\theta+1)(\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\\
&+4\theta(2\theta-1)\sum_{k=1}^{\infty}\alpha_k^2\|\kappa\|_{L^\infty}^2+8(\theta-1)(2\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\sum_{k=1}^{\infty}\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2\\
&+2(\theta-1)(2\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}\sum_{k=1}^{\infty}\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^2\\
&+4(\theta-1)(2\theta-1)\varepsilon_1^{\frac{\theta}{\theta-1}}C(n,\alpha)c_1\sum_{k=1}^{\infty}\gamma_k^2<0.
\end{aligned}
\tag{4.106}
$$

By (4.105) and (4.106) we get for all $t\geqslant 0$,

$$
\begin{aligned}
&\mathbb{E}\left[\|(-\triangle)^{\frac{\alpha}{2}}u(t)\|^{2\theta}\right]\\
&\leqslant\mathbb{E}\left[\|(-\triangle)^{\frac{\alpha}{2}}u^0\|^{2\theta}\right]e^{-\mu t}+\varepsilon_1^{-2\theta}\mu^{-1}\|\phi\|_V^{2\theta}\\
&\quad+\frac{2}{\mu\varepsilon_1^{\theta}}\|h\|^{2\theta}+\frac{2a^{2\theta}}{\mu\varepsilon_1^{\theta}}\sup_{s\geqslant 0}\mathbb{E}[\|u(s)\|^{2\theta}]+\frac{2a^{2\theta}}{\varepsilon_1^{\theta}}e^{\mu\rho}\mathbb{E}\left[\int_{-\rho}^0\|\varphi(s)\|^{2\theta}ds\right]e^{-\mu t}\\
&\quad+\frac{2(2\theta-1)}{\mu\varepsilon_1^{\theta}}\sum_{k=1}^{\infty}\|(-\triangle)^{\frac{\alpha}{2}}\sigma_{1,k}\|^{2\theta}+8(2\theta-1)\frac{1}{\mu\varepsilon_1^{\theta}}\sum_{k=1}^{\infty}\beta_k^2\|(-\triangle)^{\frac{\alpha}{2}}\kappa\|^2\\
&\quad+4(2\theta-1)\frac{1}{\mu\varepsilon_1^{\theta}}C(n,\alpha)c_1\sum_{k=1}^{\infty}\gamma_k^2\sup_{s\geqslant 0}\mathbb{E}\left[\|u(s)\|^{2\theta}\right],
\end{aligned}
\tag{4.107}
$$

which together with Lemma 4.5 concludes the proof. □

The next lemma is concerned with the pathwise Hölder continuity of solutions.

Lemma 4.10 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.57)–(4.60) hold. Let (4.11) be fulfilled with* $\theta = \frac{3p-4}{2p-2}$. *If* $R > 0$ *and* $(u^0, \varphi) \in L^{2\theta}(\Omega, \mathcal{F}_0; V) \times L^{2\theta}\big(\Omega, \mathcal{F}_0; L^{2\theta}((-\rho, 0), V)\big)$ *such that*

$$\mathbb{E}\left(\|u^0\|_V^{2\theta} + \int_{-\rho}^0 \|\varphi(s)\|_V^{2\theta} ds\right) \leqslant R,$$

then the solution u of (4.1) satisfies

$$\mathbb{E}\big[\|u(t;0,u^0,\varphi) - u(r;0,u^0,\varphi)\|^{2\theta}\big] \leqslant C_5\left(|t-r|^\theta + |t-r|^{2\theta}\right)$$

for all $\rho \in [0,1]$ *and* $t \geqslant r \geqslant 6$, *where* $C_5 > 0$ *is a constant depending on R but not on* u^0, φ *or* ρ.

Proof Let $A = (-\triangle)^\alpha + \lambda I$, where $\alpha \in (0,1)$ and λ is the positive constant in (1.1). Then, by (4.1) we find that for $t > r \geqslant 6$,

$$\begin{aligned} u(t) =& e^{-A(t-r)}u(r) + \int_r^t e^{-A(t-s)}F(\cdot, u(s))ds + \int_r^t e^{-A(t-s)}G(\cdot, u(s-\rho))ds \\ &+ \sum_{k=1}^\infty \int_r^t e^{-A(t-s)}\left(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s))\right)dW_k(s), \end{aligned}$$

which implies that

$$\begin{aligned} &\mathbb{E}\left[\|u(t)-u(r)\|^{2\theta}\right] \\ &\leqslant 4^{2\theta-1}\mathbb{E}\left[\|(e^{-A(t-r)} - I)u(r)\|^{2\theta}\right] + 4^{2\theta-1}\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}F(\cdot,u(s))ds\|^{2\theta}\right] \\ &\quad + 4^{2\theta-1}\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}G(\cdot,u(s-\rho))ds\|^{2\theta}\right] \\ &\quad + 4^{2\theta-1}\mathbb{E}\left[\|\sum_{k=1}^\infty \int_r^t e^{-A(t-s)}\left(\sigma_{1,k} + \kappa\sigma_{2,k}(u(s))\right)dW_k(s)\|^{2\theta}\right]. \end{aligned} \tag{4.108}$$

For the first term on the right-hand side of (4.108), by Theorem 1.4.3 in [35] and Lemmas 4.5 and 4.9, there exists a positive number c_1 depending on θ such that for all $t > r \geqslant 0$,

$$4^{2\theta-1}\mathbb{E}\left[\|(e^{-A(t-r)} - I)u(r)\|^{2\theta}\right] \leqslant c_1(t-r)^\theta\mathbb{E}[\|u(r)\|_V^{2\theta}] \leqslant c_2(t-r)^\theta. \tag{4.109}$$

For the second term on the right-hand side of (4.108), by the contraction property of e^{-At}, (4.3) and Lemma 4.6, we obtain for all $t > r \geqslant 6$,

$$\begin{aligned}
&4^{2\theta-1}\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}F(\cdot,u(s))ds\|^{2\theta}\right]\\
&\leqslant 4^{2\theta-1}\mathbb{E}\left[\int_r^t \|F(\cdot,u(s))\|^{2\theta}ds\right](t-r)^{2\theta-1}\\
&\leqslant 8^{2\theta-1}\mathbb{E}\left[\left(\int_r^t \|\psi_2|u(s)|^{p-1}\|^{2\theta}ds+\|\psi_3\|^{2\theta}(t-r)\right)\right](t-r)^{2\theta-1}\\
&\leqslant 8^{2\theta-1}\|\psi_2\|^{\frac{3p-4}{p-1}}_{L^{\frac{6p-8}{p-2}}}\mathbb{E}\left[\int_r^t \|u(s)\|^{3p-4}_{L^{3p-4}}ds\right](t-r)^{2\theta-1}+8^{2\theta-1}\|\psi_3\|^{2\theta}(t-r)^{2\theta}\\
&\leqslant 8^{2\theta-1}\left(c_3\|\psi_2\|^{\frac{3p-4}{p-1}}_{L^{\frac{6p-8}{p-2}}}+\|\psi_3\|^{2\theta}\right)(t-r)^{2\theta}. \qquad (4.110)
\end{aligned}$$

For the third term on the right-hand side of (4.108), by (4.6) and Lemma 4.5, we obtain for all $t > r \geqslant 1$,

$$\begin{aligned}
&4^{2\theta-1}\mathbb{E}\left[\|\int_r^t e^{-A(t-s)}G(\cdot,u(s-\rho))ds\|^{2\theta}\right]\\
&\leqslant 8^{2\theta-1}\left(\|h\|^{2\theta}+a^{2\theta}\sup_{s\geqslant 0}\mathbb{E}\left[\|u(s)\|^{2\theta}\right]\right)(t-r)^{2\theta}\\
&\leqslant 8^{2\theta-1}\left(\|h\|^{2\theta}+a^{2\theta}c_4\right)(t-r)^{2\theta}. \qquad (4.111)
\end{aligned}$$

For the fourth term on the right-hand side of (4.108), by the BDG inequality, (4.10) and Lemma 4.5, we obtain for all $t > r \geqslant 0$,

$$\begin{aligned}
&4^{2\theta-1}\mathbb{E}\left[\|\sum_{k=1}^{\infty}\int_r^t e^{-A(t-s)}\left(\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\right)dW_k(s)\|^{2\theta}\right]\\
&\leqslant 4^{2\theta-1}c_5\mathbb{E}\left[\left(\int_r^t\sum_{k=1}^{\infty}\|\sigma_{1,k}+\kappa\sigma_{2,k}(u(s))\|^2 ds\right)^{\theta}\right]\\
&\leqslant 8^{2\theta-1}c_5\left\{\left(\sum_{k=1}^{\infty}\left(\|\sigma_{1,k}\|^2+2\|\kappa\|^2\beta_k^2\right)\right)^{\theta}+c_6\left(2\sum_{k=1}^{\infty}\|\kappa\|^2_{L^\infty}\gamma_k^2\right)^{\theta}\right\}(t-r)^{\theta}. \qquad (4.112)
\end{aligned}$$

Then from (4.108)–(4.112), it follows that there exists $c_7 > 0$ depending on R but not on u^0, φ, ρ, t or r, such that for all $t > r \geqslant 6$,

$$\mathbb{E}[\|u(t;0,u^0,\varphi)-u(r;0,u^0,\varphi)\|^{2\theta}]\leqslant c_7(|t-r|^{\theta}+|t-r|^{2\theta}).$$

This completes the proof. □

Remark 4.5 Suppose $R > 0$ and $(u^0, \varphi) \in L^{2\theta}(\Omega, \mathcal{F}_0; V) \times L^{2\theta}\big(\Omega, \mathcal{F}_0; L^{2\theta}((-\rho, 0), V)\big)$ such that

$$\mathbb{E}\left(\|u^0\|_V^{2\theta} + \int_{-\rho}^{0} \|\varphi(s)\|_V^{2\theta} ds\right) \leqslant R.$$

Then by (4.107) and Lemma 4.5 we see that there exists $T > 0$ depending only on R not on $\rho \in [0, 1]$ such that for all $t \geqslant T$

$$\mathbb{E}[\|(-\triangle)^{\frac{\alpha}{2}} u(s)\|^{2\theta}] \leqslant \widetilde{C}_4, \tag{4.113}$$

where $\widetilde{C}_4$ is a positive constant independent of R, (u^0, φ) and $\rho \in [0, 1]$.

Moreover, by Lemma 4.5, Remark 4.3 and (4.113), we find from the proof of Lemma 4.10 that there exists $T \geqslant 6$ depending only on R but not on $\rho \in [0, 1]$ such that for all $t, r \geqslant T$,

$$\mathbb{E}[\|u(t) - u(r)\|^{2\theta}] \leqslant \tilde{C}_5(|t - r|^{\theta} + |t - r|^{2\theta}),$$

where $\tilde{C}_5$ is a positive constant independent of R, (u^0, φ) and $\rho \in [0, 1]$, and θ is the same as that in Lemma 4.10.

4.3 Existence of invariant measures

We now prove the existence of invariant measures of (4.1) on $H \times L^2((-\rho, 0); H)$ for which we need to show the tightness of distributions of solutions.

By Theorem 2.2, we see that for any initial time $t_0 \geqslant 0$ and any $(u^0, \varphi) \in L^2(\Omega, \mathscr{F}_{t_0}; H) \times L^2(\Omega, \mathscr{F}_{t_0}; L^2((-\rho, 0), H))$, problem (4.1) has a unique solution $u(t; t_0, u^0, \varphi)$ defined for $t \in [t_0 - \rho, \infty)$. The segment of $u(t; t_0, u^0, \varphi)$ on $(t - \rho, t)$ is written as

$$u_t(t_0, u^0, \varphi)(s) = u(t + s; t_0, u^0, \varphi) \quad \text{for all } \ s \in (-\rho, 0).$$

If $\psi : H \times L^2((-\rho, 0); H) \to \mathbb{R}$ is a bounded Borel function, then for $0 \leqslant r \leqslant t$ and $(u^0, \varphi) \in H \times L^2((-\rho, 0), H)$, we set

$$(p_{r,t}\psi)(u^0, \varphi) = \mathbb{E}\left(\psi\left(u(t; r, u^0, \varphi), u_t(r, u^0, \varphi)\right)\right).$$

In particular, for $\Gamma \in \mathcal{B}\big(H \times L^2((-\rho, 0), H)\big)$, $0 \leqslant r \leqslant t$ and $(u^0, \varphi) \in H \times L^2((-\rho, 0), H)$, we set

$$p(r, u^0, \varphi; t, \Gamma) = (p_{r,t} 1_\Gamma)(u^0, \varphi) = \mathbb{P}\left\{\omega \in \Omega \mid \left(u(t; r, u^0, \varphi), u_t(r, u^0, \varphi)\right) \in \Gamma\right\},$$

where 1_Γ is the characteristic function of Γ. We often write $p_{0,t}$ as p_t.

Let $\mathscr{P}$ be the space of all probability measures on $H \times L^2((-\rho, 0), H)$. Recall that a probability measure $\nu \in \mathscr{P}$ is called an invariant measure of (4.1) if for all $t \geqslant 0$,

$$\int_{H\times L^2((-\rho,0),H)} (p_t\psi)(u^0, \varphi)\, d\nu(u^0, \varphi) = \int_{H\times L^2((-\rho,0),H)} \psi(u^0, \varphi)\, d\nu(u^0, \varphi)$$

for every bounded Borel function $\psi : H \times L^2((-\rho, 0), H) \to \mathbb{R}$.

The following properties of $\{p_{r,t}\}_{0\leqslant r\leqslant t}$ can be proved by standard arguments as in [34].

Lemma 4.11 *If* $(\mathbf{F}')$, $(\mathbf{G}')$ *and* (Σ') *hold, then:*

(i) *The family* $\{p_{r,t}\}_{0\leqslant r\leqslant t}$ *is Feller, and is homogeneous in time.*

(ii) *For any* $(u^0, \varphi) \in H \times L^2((-\rho, 0), H)$, *the process* $\{(u(t; 0, u^0, \varphi), u_t(0, u^0, \varphi))\}_{t\geqslant 0}$ *is an* $H \times L^2((-\rho, 0), H)$*-valued Markov process with transition operators* $\{p_{r,t}\}_{0\leqslant r\leqslant t}$. *In particular, if* $\psi : H \times L^2((-\rho, 0), H) \to \mathbb{R}$ *is a bounded Borel function, then for any* $0 \leqslant s \leqslant r \leq t$, $\mathbb{P}$*-a.s.,*

$$(p_{s,t}\psi)(u^0, \varphi) = (p_{s,r}(p_{r,t}\psi))(u^0, \varphi), \quad \forall\, (u^0, \varphi) \in H \times L^2((-\rho, 0), H),$$

and the Chapman-Kolmogorov equation is valid:

$$p(s, u^0, \varphi; t, \Gamma) = \int_{H\times L^2((-\rho,0),H)} p(s, u^0, \varphi; r, dy)p(r, y; t, \Gamma)$$

for any $\Gamma \in \mathcal{B}(H \times L^2((-\rho, 0), H))$.

We will employ Krylov-Bogolyubov's method to show the existence of invariant measures of (4.1). To that end, for every $k \in \mathbb{N}$, we set

$$\mu_k = \frac{1}{k}\int_7^{k+7} p(0, 0, 0; t, \cdot)dt, \tag{4.114}$$

where $p(0, 0, 0; t, \cdot)$ is the distribution law of $(u(t; 0, 0, 0), u_t(0, 0, 0))$ corresponding to the solution $u(t; 0, 0, 0)$ of (4.1) with initial condition $(0, 0)$ at initial time 0. We first prove $\{\mu_k\}_{k\in\mathbb{N}}$ is tight on $H \times L^2((-\rho, 0), H)$.

Lemma 4.12 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.57)–(4.60) hold. Let (4.11) be fulfilled with* $\theta = \frac{3p-4}{2p-2}$. *Then* $\{\mu_k\}_{k\in\mathbb{N}}$ *is tight on* $H \times L^2((-\rho, 0), H)$.

Proof The proof is based on the uniform estimates given by Lemma 4.4, Lemma 4.8 and Lemma 4.10, and is similar to [20] regarding the tightness of distributions of solutions on $C([-\rho, 0]; H)$. We here sketch the main idea of the proof. For convenience, during the proof, we write the solution $u(t; 0, 0, 0)$ as $u(t)$.

By Lemma 4.8, for any given $\epsilon > 0$, there exists $R_1 = R_1(\epsilon) > 0$ such that for all $t \geqslant 4$ and $\rho \in [0, 1]$,

$$\mathbb{P}\left(\left\{ \sup_{s\in[-\rho,0]} \|u_t(s)\|_V \geqslant R_1 \right\}\right) \leqslant \frac{1}{3}\epsilon. \tag{4.115}$$

By Lemma 4.10, for all $t \geqslant 7$ and $r, s \in [-\rho, 0]$,

$$\mathbb{E}\Big(\|u_t(r) - u_t(s)\|^{\frac{3p-4}{p-1}} \Big) \leqslant c_1 (1 + \rho^{\frac{3p-4}{2p-2}}) |r - s|^{\frac{3p-4}{2p-2}} \leqslant 2c_1 |r - s|^{\frac{3p-4}{2p-2}}, \quad (4.116)$$

where $c_1 > 0$ is independent of ρ. By (4.116) and the technique of diadic division, we infer that for every $\epsilon > 0$, there exists $R_2 = R_2(\epsilon) > 0$ such that for all $t \geqslant 7$ and $\rho \in [0, 1]$,

$$\mathbb{P}\left(\left\{ \sup_{-\rho \leqslant s < r \leqslant 0} \frac{\|u_t(r) - u_t(s)\|}{|r - s|^{\frac{p-2}{4(3p-4)}}} \leqslant R_2 \right\} \right) > 1 - \frac{1}{3}\epsilon. \quad (4.117)$$

From Lemma 4.4, it follows that for every $\epsilon > 0$ and $m \in \mathbb{N}$, there exists $n_m = n(\epsilon, m) \geqslant 1$ such that $\mathbb{E}\Big(\sup_{t-\rho \leqslant s \leqslant t} \int_{|x| \geqslant n_m} |u(s, x)|^2 dx \Big) \leqslant \frac{\epsilon}{2^{2m+2}}$ for all $t \geqslant 1$ and $\rho \in [0, 1]$, and hence for all $t \geqslant 1$,

$$\mathbb{P}\left(\left\{ \sup_{t-\rho \leqslant s \leqslant t} \int_{|x| \geqslant n_m} |u(s, x)|^2 dx \leqslant \frac{1}{2^m}, \quad \forall m \in \mathbb{N} \right\} \right) > 1 - \frac{1}{3}\epsilon. \quad (4.118)$$

For every $\epsilon > 0$, denote by

$$\mathcal{Z}_{1,\epsilon} = \left\{ \xi : [-\rho, 0] \to V \mid \sup_{s \in [-\rho, 0]} \|\xi(s)\|_V \leqslant R_1(\epsilon) \right\}, \quad (4.119)$$

$$\mathcal{Z}_{2,\epsilon} = \left\{ \xi \in C([-\rho, 0]; H) \mid \sup_{-\rho \leqslant s < r \leqslant 0} \frac{\|\xi(r) - \xi(s)\|}{|r - s|^{\frac{p-2}{4(3p-4)}}} \leqslant R_2(\epsilon) \right\}, \quad (4.120)$$

$$\mathcal{Z}_{3,\epsilon} = \left\{ \xi \in C([-\rho, 0]; H) \mid \sup_{-\rho \leqslant s \leqslant 0} \int_{|x| \geqslant n_m} |\xi(s, x)|^2 dx \leqslant \frac{1}{2^m}, \quad \forall m \in \mathbb{N} \right\}, \quad (4.121)$$

and

$$\mathcal{Z}_\epsilon = \mathcal{Z}_{1,\epsilon} \cap \mathcal{Z}_{2,\epsilon} \cap \mathcal{Z}_{3,\epsilon}. \quad (4.122)$$

By (4.115), (4.117) and (4.118)–(4.122) wee find that for all $t \geqslant 7$ and every $\rho \in [0, 1]$,

$$\mathbb{P}\big(\{ u_t \in \mathcal{Z}_\epsilon \} \big) > 1 - \epsilon. \quad (4.123)$$

Moreover, by (4.119)–(4.122) and the Ascoli-Arzelà theorem, one may verify that the set $\{\xi(0) \mid \xi \in \mathcal{Z}_\epsilon\}$ is compact in H and $\mathcal{Z}_\epsilon$ is compact in $C([-\rho, 0]; H)$. Since the embedding $C([-\rho, 0]; H) \hookrightarrow L^2((-\rho, 0), H)$ is continuous, we find that $\mathcal{Z}_\epsilon$ is also compact in $L^2((-\rho,0),H)$. Consequently, the set $\widetilde{\mathcal{Z}}_\epsilon = \{(\xi(0),\xi) \mid \xi \in \mathcal{Z}_\epsilon\}$ is compact in $H \times L^2((-\rho,0),H)$.

Furthermore, by (4.123), we have that for all $t \geqslant 7$ and $\rho \in [0,1]$,

$$\mathbb{P}\big(\{(u(t),u_t) \in \widetilde{\mathcal{Z}}_\epsilon\}\big) = \mathbb{P}\big(\{u_t \in \mathcal{Z}_\epsilon\}\big) > 1-\epsilon,$$

which along with (4.114) implies that for every $\rho \in [0,1]$,

$$\mu_k\big(\widetilde{\mathcal{Z}}_\epsilon\big) > 1-\epsilon, \quad \forall\, k \in \mathbb{N},$$

Thus $\{\mu_k\}_{k\in\mathbb{N}}$ is tight on $H \times L^2((-\rho,0),H)$, which completes the proof. □

Theorem 4.1 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.57)–(4.60) hold. Let (4.11) be fulfilled with* $\theta = \frac{3p-4}{2p-2}$. *Then for any* $\rho \in [0,1]$, *the stochastic equation (4.1) has an invariant measure on* $H \times L^2((-\rho,0),H)$.

Proof By Lemma 4.12 we see that $\{\mu_k\}_{k\in\mathbb{N}}$ is tight on $H \times L^2((-\rho,0),H)$, and hence there exists a probability measure μ on $H \times L^2((-\rho,0),H)$ such that, up to a subsequence, $\mu_k \to \mu$. Then by Lemma 4.11, one can prove μ is invariant, which completes the proof. □

Given $\rho \in [0,1]$, let $\mathcal{S}^\rho$ be the collection of all invariant measures of (4.1) with delay parameter ρ. Then from Theorem 4.1 we see that $\mathcal{S}^\rho$ is nonempty. In the next section, we will prove the set $\bigcup\limits_{\rho\in[0,1]} \mathcal{S}^\rho$ is tight.

5 Regularity of invariant measures

In this section, we establish the regularity of invariant measures of (4.1), which will be useful for proving the tightness of the set of all invariant measures of (4.1) when ρ varies on the interval [0, 1] in the next section.

Theorem 5.1 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ'), *(4.12) and (4.57)–(4.60) hold. Then for every* $\rho \in [0,1]$ *and* $\mu^\rho \in \mathcal{S}^\rho$, *we have* $\mu^\rho\big(V \times L^\infty((-\rho,0),V)\big) = 1$.

Proof By Remark 4.4, we find that for every $(u^0,\varphi) \in H \times L^2((-\rho,0),H)$, there exists $T = T(u^0,\varphi) \geqslant 4$ (independent of ρ) such that for all $t \geqslant T$ and $\rho \in [0,1]$,

$$\mathbb{E}\left(\|u(t;0,u^0,\varphi)\|_V^2\right) + \mathbb{E}\Big(\sup_{r\in[-\rho,0]} \|u_t(0,u^0,\varphi)(r)\|_V^2\Big) \leqslant c_1, \tag{5.1}$$

where $c_1 > 0$ is independent of u^0, φ and ρ.

Given $R > 0$, denote by

$$\tilde{B}_R = \Big\{(u^0,\varphi) \in V \times L^\infty((-\rho,0),V) \mid \ \|(u^0,\varphi)\|_{V\times L^\infty((-\rho,0),V)} \leqslant R\Big\}.$$

Then $\tilde{B}_R$ is a closed subset of $H \times L^2((-\rho, 0), H)$.

By (5.1) we get for all $t \geqslant T$ and $\rho \in [0, 1]$,

$$\mathbb{P}\left(\left\{\left\|\left(u(t; 0, u^0, \varphi), u_t(0, u^0, \varphi)\right)\right\|_{V \times L^\infty((-\rho,0),V)} > R\right\}\right) \leqslant \frac{c_1}{R^2},$$

which implies that for all $t \geqslant T$ and $\rho \in [0, 1]$,

$$\mathbb{P}\left(\left\{\left(u(t; 0, u^0, \varphi), u_t(0, u^0, \varphi)\right) \in \tilde{B}_R\right\}\right) \geqslant 1 - \frac{c_1}{R^2}. \tag{5.2}$$

If $\mu^\rho \in \mathcal{S}^\rho$, then from the invariance of μ^ρ, it follows that for any $s \geqslant 0$,

$$\int_{H \times L^2((-\rho,0),H)} \mathbb{P}\left(\left\{\left(u(t; 0, u^0, \varphi), u_t(0, u^0, \varphi)\right) \in \tilde{B}_R\right\}\right) d\mu^\rho = \mu^\rho\big(\tilde{B}_R\big). \tag{5.3}$$

By (5.2), (5.3) and Fatou's theorem we get, for all $\rho \in [0, 1]$,

$$\mu^\rho\big(\tilde{B}_R\big) \geqslant 1 - \frac{c_1}{R^2}. \tag{5.4}$$

Letting $R \to +\infty$ in (5.4), since $\lim\limits_{R\to\infty} \mu^\rho\big(\tilde{B}_R\big) = \mu^\rho\big(V \times L^\infty((-\rho, 0), V)\big)$ we obtain for all $\rho \in [0, 1]$,

$$\mu^\rho\big(V \times L^\infty((-\rho, 0), V)\big) \geqslant 1,$$

which concludes the proof. □

6 Tightness of the set of invariant measures

In this section, we prove the set of all invariant measures of (4.1) is tight when ρ varies on $[0, 1]$. To that end, for every $\rho \in (0, 1]$, define a restriction operator $\mathcal{T}_\rho : H \times L^2((-1, 0), H) \to H \times L^2((-\rho, 0), H)$ by

$$\mathcal{T}_\rho(u^0, \varphi) = (u^0, \varphi|_{(-\rho,0)}), \quad \forall\, (u^0, \varphi) \in H \times L^2((-1, 0), H),$$

where $\varphi|_{(-\rho,0)}$ is the restriction of φ to the interval $(-\rho, 0)$.

We now prove the tightness of the set of all invariant measures of (4.1) for all $\rho \in [0, 1]$.

Theorem 6.1 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.57)–(4.60) hold. Let (4.11) be fulfilled with* $\theta = \frac{3p-4}{2p-2}$. *Then the set* $\bigcup\limits_{\rho\in[0,1]} \mathcal{S}^\rho$ *is tight in the sense that for every* $\varepsilon > 0$, *there exists a compact subset* $\mathcal{K}$ *in* $H \times L^2((-1, 0), H)$ *such that* $\mu^\rho\big(\mathcal{T}_\rho(\mathcal{K})\big) > 1 - \varepsilon$ *for all* $\mu^\rho \in \mathcal{S}^\rho$ *and* $\rho \in [0, 1]$.

Proof Given $\rho \in [0,1]$ and $(u^0, \varphi) \in V \times L^\infty((-\rho, 0), V) \subseteq V \times L^{\frac{3p-4}{p-1}}((-\rho, 0), V)$, by Remark 4.2, we find that for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exist $T_m = T(\varepsilon, m, u^0, \varphi) \geqslant 2$ and $k_m = k(\varepsilon, m) \geqslant 1$ such that for all $\rho \in [0,1]$, $t \geqslant T_m$ and $k \geqslant k_m$,

$$\mathbb{E}\Big(\sup_{t-1 \leqslant r \leqslant t} \int_{|x| \geqslant k} |u^\rho(r; 0, u^0, \varphi)|^2 dx \Big) < \frac{\varepsilon}{2^{2m+2}}. \tag{6.1}$$

On the other hand, by Remark 4.4, we see that there exists $\widetilde{T}_1 = \widetilde{T}_1(u^0, \varphi) \geqslant 1$ such that for all $t \geqslant \widetilde{T}_1$,

$$\mathbb{E}\Big(\sup_{s \in [-1,0]} \|u_t^\rho(0, u^0, \varphi)(s)\|_V^2 \Big) \leqslant c_1,$$

where $c_1 > 0$ is a constant independent of (u^0, φ) and ρ, which implies that for every $\varepsilon > 0$, there exists $R_1 = R_1(\varepsilon) > 0$ (independent of (u^0, φ) and ρ) such that for all $t \geqslant \widetilde{T}_1$,

$$\mathbb{P}\Big(\Big\{ \sup_{s \in [-1,0]} \|u_t^\rho(0, u^0, \varphi)(s)\|_V > R_1 \Big\} \Big) < \frac{1}{3}\varepsilon. \tag{6.2}$$

By Remark 4.5, there exist $\widetilde{T}_2 = \widetilde{T}_2(u^0, \varphi) \geqslant 1$ and $R_2 = R_2(\varepsilon) > 0$ (independent of (u^0, φ) and ρ) such that for all $t \geqslant \widetilde{T}_2$,

$$\mathbb{P}\Big(\Big\{ \sup_{-1 \leqslant s < r \leqslant 0} \frac{\|u^\rho\left(t + r; 0, u^0, \varphi\right) - u^\rho\left(t + s; 0, u^0, \varphi\right)\|}{|r - s|^{\frac{p-2}{4(3p-4)}}} \leqslant R_2 \Big\} \Big) > 1 - \frac{1}{3}\varepsilon. \tag{6.3}$$

For every $\varepsilon > 0$, denote by

$$\mathscr{K}_{1,\varepsilon} = \Big\{ \xi : [-1, 0] \to V \mid \sup_{s \in [-1,0]} \|\xi(s)\|_V \leqslant R_1(\epsilon) \Big\}, \tag{6.4}$$

$$\mathscr{K}_{2,\varepsilon} = \Big\{ \xi \in C([-1, 0]; H) \mid \sup_{-1 \leqslant s < r \leqslant 0} \frac{\|\xi(r) - \xi(s)\|}{|r - s|^{\frac{p-2}{4(3p-4)}}} \leqslant R_2(\epsilon) \Big\}, \tag{6.5}$$

$$\mathscr{K}_{3,\varepsilon} = \Big\{ \xi \in C([-1, 0]; H) \mid \sup_{-1 \leqslant s \leqslant 0} \int_{|x| \geqslant k_m} |\xi(s, x)|^2 dx \leqslant \frac{1}{2^m}, \ \ \forall\, m \in \mathbb{N} \Big\}, \tag{6.6}$$

It follows from (6.4)–(6.6) and the proof of Lemma 4.12 that the set

$$\mathscr{K}_\varepsilon = \big\{ (\xi(0), \xi) \mid \ \xi \in \mathscr{K}_{1,\varepsilon} \cap \mathscr{K}_{2,\varepsilon} \cap \mathscr{K}_{3,\varepsilon} \big\}$$

is compact in $H \times L^2((-1,0),H)$.

In what follows, we will prove for any $\rho \in [0,1]$ and $\mu^\rho \in \mathcal{S}^\rho$,

$$\mu^\rho\big(\mathcal{T}_\rho(\mathcal{K}_\varepsilon)\big) > 1-\varepsilon. \tag{6.7}$$

For any $m \in \mathbb{N}$, define

$$\mathcal{K}_{3,\varepsilon,m} =: \Big\{\xi \in C([-1,0];H) \mid \sup_{-1\leqslant s\leqslant 0}\int_{|x|\geqslant k_m}|\xi(s,x)|^2dx \leqslant \frac{1}{2^m}\Big\},$$

and

$$\mathcal{A}_i^\varepsilon = \Big\{(\xi(0),\xi) \mid \xi \in \mathcal{K}_{1,\varepsilon}\cap\mathcal{K}_{2,\varepsilon}\cap\Big(\bigcap_{m=1}^{i}\mathcal{K}_{3,\varepsilon,m}\Big)\Big\}, \quad \forall\, i \in \mathbb{N}.$$

Then $\mathcal{K}_\varepsilon = \bigcap_{i=1}^{\infty}\mathcal{A}_i^\varepsilon$, $\mathcal{A}_i^\varepsilon$ is a closed subset of $H \times L^2((-1,0),H)$ and $\mathcal{A}_i^\varepsilon \supseteq \mathcal{A}_{i+1}^\varepsilon$. Similarly, one can verify that $\mathcal{T}_\rho\mathcal{A}_i^\varepsilon$ is a closed subset of $H \times L^2((-\rho,0),H)$ and $\mathcal{T}_\rho(\mathcal{A}_i^\varepsilon) \supseteq \mathcal{T}_\rho(\mathcal{A}_{i+1}^\varepsilon)$ for $i \in \mathbb{N}$.

We claim:

$$\bigcap_{i=1}^{\infty}\mathcal{T}_\rho(\mathcal{A}_i^\varepsilon) = \mathcal{T}_\rho(\mathcal{K}_\varepsilon), \quad \forall\, \rho \in [0,1]. \tag{6.8}$$

It is evident that $\bigcap_{i=1}^{\infty}\mathcal{T}_\rho(\mathcal{A}_i^\varepsilon) \supseteq \mathcal{T}_\rho(\mathcal{K}_\varepsilon)$. So it is enough to prove

$$\bigcap_{i=1}^{\infty}\mathcal{T}_\rho(\mathcal{A}_i^\varepsilon) \subseteq \mathcal{T}_\rho(\mathcal{K}_\varepsilon). \tag{6.9}$$

Let $z_0 \in C([-\rho,0],H)$ such that $(z_0(0),z_0) \in \bigcap_{i=1}^{\infty}\mathcal{T}_\rho(\mathcal{A}_i^\varepsilon)$. Then for every $i \in \mathbb{N}$, we have $(z_0(0),z_0) \in \mathcal{T}_\rho(\mathcal{A}_i^\varepsilon)$, which implies that there exists $\widetilde{z}_i \in C([-1,0];H)$ such that

$$(\widetilde{z}_i(0),\widetilde{z}_i) \in \mathcal{A}_i^\varepsilon \quad \text{and} \quad z_0(s) = \widetilde{z}_i(s), \quad \forall\, s \in [-\rho,0]. \tag{6.10}$$

Consequently, we have $\widetilde{z}_i \in \mathcal{K}_{1,\varepsilon}\cap\mathcal{K}_{2,\varepsilon}\cap(\bigcap_{m=1}^{i}\mathcal{K}_{3,\varepsilon,m})$, which together with (6.10) implies

$$\sup_{s\in[-\rho,0]}\|z_0(s)\|_V \leqslant R_1(\varepsilon), \qquad \sup_{-\rho\leqslant s<r\leqslant 0}\frac{\|z_0(r)-z_0(s)\|}{|r-s|^{\frac{p-2}{4(3p-4)}}} \leqslant R_2(\varepsilon), \tag{6.11}$$

and

$$\sup_{-\rho\leqslant r\leqslant 0}\int_{|x|\geqslant k_m}|z_0(s,x)|^2dx\leqslant\frac{1}{2^m},\ \forall m\in\mathbb{N}. \tag{6.12}$$

Define a continuous function $z:[-1,0]\to H$ by

$$z(s)=z_0(s)\ \text{ if }\ s\in[-\rho,0];\quad z(s)=z_0(-\rho)\ \text{ if }\ s\in[-1,-\rho). \tag{6.13}$$

Then $(z_0(0),z_0)=\mathcal{T}_\rho(z(0),z)$. Moreover, it follows from (6.11)–(6.13) that $z\in\mathscr{K}_{1,\varepsilon}\cap\mathscr{K}_{2,\varepsilon}\cap\mathscr{K}_{3,\varepsilon}$, and hence $(z_0(0),z_0)\in\mathcal{T}_\rho(\mathscr{K}_\varepsilon)$, which gives (6.9).

By (6.8) we infer that for every $\rho\in[0,1]$ and $\mu^\rho\in\mathcal{S}^\rho$,

$$\lim_{i\to\infty}\mu^\rho\big(\mathcal{T}_\rho(\mathcal{A}_i^\varepsilon)\big)=\mu^\rho\big(\mathcal{T}_\rho(\mathscr{K}_\varepsilon)\big),$$

which implies that there exists $N_0=N_0(\varepsilon,\rho,\mu^\rho)\geqslant 1$ such that for any $i\geqslant N_0$,

$$0\leqslant\mu^\rho\big(\mathcal{T}_\rho(\mathcal{A}_i^\varepsilon)\big)-\mu^\rho\big(\mathcal{T}_\rho(\mathscr{K}_\varepsilon)\big)<\frac{1}{12}\varepsilon. \tag{6.14}$$

Next, we will prove $\mu^\rho\left(\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\right)>1-\frac{11}{12}\varepsilon$.

Since μ^ρ is an invariant measure of (4.1) with $\rho\in[0,1]$, we get

$$\int_{H\times L^2((-\rho,0),H)}\mathbb{P}\left(\left\{\left(u^\rho(t;0,u^0,\psi),u_t^\rho(0,u^0,\psi)\right)\in\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\right\}\right)d\mu^\rho=\mu^\rho\big(\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\big). \tag{6.15}$$

Then by (6.15) and Theorem 5.1, we have

$$\mu^\rho\big(\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\big)=\int_{V\times L^\infty((-\rho,0),V)}\mathbb{P}\left(\left\{\left(u^\rho(t;0,u^0,\psi),u_t^\rho(0,u^0,\psi)\right)\in\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\right\}\right)d\mu^\rho. \tag{6.16}$$

By (6.16) and Fatou's theorem, we get

$$\begin{aligned}\mu^\rho\big(\mathcal{T}_\rho(\mathcal{A}_{N_0}^\epsilon)\big)&=\liminf_{t\to\infty}\int_{V\times L^\infty((-\rho,0),V)}\mathbb{P}\left(\left\{\left(u^\rho(t;0,u^0,\psi),u_t^\rho(0,u^0,\psi)\right)\in\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\right\}\right)d\mu^\rho\\&\geqslant 1-\int_{V\times L^\infty((-\rho,0),V)}\limsup_{t\to\infty}\mathbb{P}\left(\left\{\left(u^\rho(t;0,u^0,\psi),u_t^\rho(0,u^0,\psi)\right)\notin\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\right\}\right)d\mu^\rho.\end{aligned} \tag{6.17}$$

Next, we estimate the term on the right-hand side of (6.17). For any $(u^0,\psi)\in V\times L^\infty((-\rho,0),V)$, note that $u_t^\rho(0,u^0,\psi)$ is the segment of the solution $u^\rho(t;0,u^0,\psi)$ of (4.1) on the interval $[t-\rho,t]$; that is,

$$u_t^\rho(0,u^0,\psi)(s)=u^\rho(t+s;0,u^0,\psi),\quad\forall s\in[-\rho,0].$$

We now consider the segment of $u^\rho(t; 0, u^0, \psi)$ on the interval $[t-1, t]$ with $t \geqslant 1$, which is denoted by $v_t^\rho(0, u^0, \psi)$; that is,

$$v_t^\rho(0, u^0, \psi)(s) = u^\rho(t+s; 0, u^0, \psi), \quad \forall s \in [-1, 0].$$

Then for all $t \geqslant 1$, we have $v_t^\rho(0, u^0, \psi) \in C([-1, 0]; H)$ and

$$\big(u^\rho(t; 0, u^0, \psi), u_t^\rho(0, u^0, \psi)\big) = \mathcal{T}_\rho\big(v_t^\rho(0, u^0, \psi)(0), v_t^\rho(0, u^0, \psi)\big). \quad (6.18)$$

By (6.18) we see that if $\big(u^\rho(t; 0, u^0, \psi), u_t^\rho(0, u^0, \psi)\big) \notin \mathcal{T}_\rho(\mathcal{A}^\varepsilon_{N_0})$ with $t \geq 1$, then we must have $\big(v_t^\rho(0, u^0, \psi)(0), v_t^\rho(0, u^0, \psi)\big) \notin \mathcal{A}^\varepsilon_{N_0}$, which shows that for $t \geqslant 1$,

$$\begin{aligned}
&\mathbb{P}\left(\left\{\big(u^\rho(t; 0, u^0, \psi), u_t^\rho(0, u^0, \psi)\big) \notin \mathcal{T}_\rho(\mathcal{A}^\varepsilon_{N_0})\right\}\right)\\
&\leqslant \mathbb{P}\left(\left\{v_t^\rho(0, u^0, \psi) \notin \mathscr{K}_{1,\varepsilon}\right\}\right) + \mathbb{P}\left(\left\{v_t^\rho(0, u^0, \psi) \notin \mathscr{K}_{2,\varepsilon}\right\}\right)\\
&\quad + \sum_{m=1}^{N_0} \mathbb{P}\left(\left\{v_t^\rho(0, u^0, \psi) \notin \mathscr{K}_{3,\varepsilon,m}\right\}\right)\\
&= \mathbb{P}\left(\left\{\sup_{s\in[-1,0]} \|u_t^\rho(0, u^0, \psi)(s)\|_V > R_1(\varepsilon)\right\}\right)\\
&\quad + \mathbb{P}\left(\left\{\sup_{-1\leqslant s<r\leqslant 0} \frac{\|u^\rho(t+r; 0, u^0, \psi) - u^\rho(t+s; 0, u^0, \psi)\|}{|r-s|^{\frac{p-2}{4(3p-4)}}} > R_2(\varepsilon)\right\}\right)\\
&\quad + \sum_{m=1}^{N_0} \mathbb{P}\left(\left\{\sup_{t-1\leqslant r\leqslant t} \int_{|x|\geqslant k_m} |u^\rho(r, x; 0, u^0, \psi)|^2 dx > \frac{1}{2^m}\right\}\right).
\end{aligned} \quad (6.19)$$

By (6.2), we obtain

$$\limsup_{t\to\infty} \mathbb{P}\left(\left\{\sup_{s\in[-1,0]} \|u_t^\rho(0, u^0, \psi)(s)\|_V > R_1(\varepsilon)\right\}\right) \leqslant \frac{\varepsilon}{3}. \quad (6.20)$$

By (6.3), we have

$$\limsup_{t\to\infty} \mathbb{P}\left(\left\{\sup_{-1\leqslant s<r\leqslant 0} \frac{\|u^\rho(t+r; 0, u^0, \psi) - u^\rho(t+s; 0, u^0, \psi)\|}{|r-s|^{\frac{p-2}{4(3p-4)}}} > R_2(\varepsilon)\right\}\right) \leqslant \frac{\varepsilon}{3}. \quad (6.21)$$

By (6.1), we have

$$\limsup_{t\to\infty} \sum_{m=1}^{N_0} \mathbb{P}\left(\left\{\sup_{t-1\leqslant r\leqslant t} \int_{|x|\geqslant k_m} |u^\rho(r, x; 0, u^0, \psi)|^2 dx > \frac{1}{2^m}\right\}\right)$$

$$\leqslant \sum_{m=1}^{N_0} 2^m \limsup_{t\to\infty} \mathbb{E}\left(\sup_{t-1\leqslant r\leqslant t} \int_{|x|\geqslant k_m} |u^\rho(r,x;0,u^0,\psi)|^2 dx \right)$$
$$\leqslant \sum_{m=1}^{N_0} \frac{\varepsilon}{2^{m+2}} < \frac{1}{4}\varepsilon. \tag{6.22}$$

It follows from (6.19)–(6.22) that

$$\limsup_{t\to\infty} \mathbb{P}\left(\left\{\left(u^\rho(t;0,u^0,\psi), u_t^\rho(\cdot;0,u^0,\psi)\right) \notin \mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\right\}\right) < \frac{11}{12}\varepsilon,$$

which along with (6.17) yields

$$\mu^\rho\left(\mathcal{T}_\rho(\mathcal{A}_{N_0}^\varepsilon)\right) > 1 - \frac{11}{12}\varepsilon. \tag{6.23}$$

Then (6.7) follows from (6.14) and (6.23) immediately, which completes the proof. □

7 Limits of invariant measures with respect to delay parameter

In this section, we investigate the limiting behavior of invariant measures of (4.1) as $\rho \to \rho_0$. We will show any limit of a sequence of invariant measures of (4.1) must be an invariant measure of the limiting system. We start with an abstract result regarding the limits of invariant measures.

Let Z be a separable Hilbert space with norm $\|\cdot\|_Z$. Assume that for every $\rho \in (0,1]$, $z \in Z$ and $\varphi \in L^2((-\rho,0),Z)$, $\{X^\rho(t;0,(z,\varphi)) : t \geqslant 0\}$ is a stochastic process in $Z \times L^2((-\rho,0),Z)$ with initial data (z,φ) at $t=0$. We also assume that for every $z \in Z$, $\{X^0(t;0,z) : t \geqslant 0\}$ is a stochastic process in Z with initial condition$z$ when $t=0$. Suppose the probability transition operators of X^ρ are Feller.

For each $\rho \in (0,1]$ and $\rho_1 \in [0,\rho)$, for any $(z,\varphi) \in Z \times L^2((-\rho,0),Z)$, let

$$\mathcal{T}_{\rho\to\rho_1}(z,\varphi) = \begin{cases} (z, \varphi|_{(-\rho_1,0)}), & \text{as } \rho_1 > 0; \\ z, & \text{as } \rho_1 = 0, \end{cases}$$

where $\varphi|_{(-\rho_1,0)}$ is the restriction of φ to the interval $(-\rho_1,0)$. Let $Z_\rho = Z \times L^2((-\rho,0),Z)$ if $\rho \in (0,1]$, and $Z_\rho = Z$ if $\rho = 0$.

Similar to [18, 31], we assume that X^{ρ_n} converges to X^ρ as $\rho_n \to \rho^+$ in the following sense: for every compact subset E in $Z \times L^2((-1,0),Z)$, $t \geqslant 0$ and $\zeta > 0$,

$$\lim_{\rho_n\to\rho^+} \sup_{(z,\varphi)\in E} \mathbb{P}\left(\left\{\|\mathcal{T}_{\rho_n\to\rho}\left(X^{\rho_n}(t;0,\mathcal{T}_{1\to\rho_n}(z,\varphi))\right) - X^\rho(t;0,\mathcal{T}_{1\to\rho}(z,\varphi))\|_{Z_\rho} \geqslant \zeta\right\}\right) = 0. \tag{7.1}$$

Then we have the following result as Lemma 7.1 in [18] whose proof is omitted.

Theorem 7.1 *Let $0 \leqslant \rho < \rho_n \leqslant 1$ and (7.1) hold. Suppose μ^{ρ_n} is an invariant measure of X^{ρ_n} in $Z \times L^2((-\rho_n, 0), Z)$, and for any $\epsilon > 0$, there exists a compact subset $K \subseteq Z \times L^2((-1, 0), Z)$ such that $\mu^{\rho_n}\big(\mathcal{T}_{1\to\rho_n}(K)\big) > 1-\epsilon$, $\forall\, n = 1, 2, \ldots$. Then:*

(i) The sequence $\{\mu^{\rho_n} \circ \mathcal{T}^{-1}_{\rho_n\to\rho}\}_{n=1}^{+\infty}$ is tight in Z_ρ.

(ii) If $\rho_n \to \rho$ and μ is a probability measure on Z_ρ such that $\mu^{\rho_n} \circ \mathcal{T}^{-1}_{\rho_n\to\rho}$ converges weakly to μ as $n \to \infty$, then μ must be an invariant measure of X^ρ.

Next, we will apply Theorem 7.1 to the stochastic system (4.1) with $Z = H = L^2(\mathbb{R}^n)$. Recall that $H_\rho = H \times L^2((-\rho, 0), H)$ if $\rho \in (0, 1]$, and $H_\rho = H$ if $\rho = 0$. We now write the solution of (4.1) with $\rho \in (0, 1]$ as u^ρ, and reserve u for the the solution of (4.1) with $\rho = 0$.

Lemma 7.1 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$ *and* (Σ') *hold. Then for every $\rho_1 \in [0, 1)$ and every compact subset E in $H \times L^2((-1, 0), H)$, $T \geqslant 1$ and $\eta > 0$,*

$$\lim_{\rho\to\rho_1^+} \sup_{(u^0,\varphi)\in E} \mathbb{P}\left(\left\{ \sup_{0\leqslant t\leqslant T} \|\mathcal{T}_{\rho\to\rho_1}(u^\rho(t), u^\rho_t) - (u^{\rho_1}(t), u^{\rho_1}_t)\|_{H_{\rho_1}} \geqslant \eta \right\}\right) = 0,$$

where $u^\rho(t) = u^\rho(t; 0, \mathcal{T}_{1\to\rho}(u^0, \varphi))$, and $u^\rho_t(s) = u^\rho(t+s; 0, \mathcal{T}_{1\to\rho}(u^0, \varphi))$ for $s \in (-\rho, 0)$.

Proof By Theorem 2.2, we find that for every $T \geqslant 1$ and every compact subset E in $\mathcal{H}_1$, there exists a positive number $c_1 = c_1(E, T)$ independent of $\rho \in [0, 1]$ such that for all $(u^0, \varphi) \in E$ and $\rho \in [0, 1]$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|u^\rho(t; 0, \mathcal{T}_{1\to\rho}(u^0, \varphi))\|^2 \Big) \leqslant c_1. \tag{7.2}$$

Applying Ito's formula to (4.1), we obtain for $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant r\leqslant t} \|u^\rho(r) - u^{\rho_1}(r)\|^2 \right) \\
&\quad\leqslant \mathbb{E}\Big(\sup_{0\leqslant r\leqslant t} \int_0^r ds \int_{\mathbb{R}^n} -2\big(F(x, u^\rho(s)) - F(x, u^{\rho_1}(s))\big) \cdot \big(u^\rho(s) - u^{\rho_1}(s)\big) dx \Big) \\
&\qquad + 2\mathbb{E}\Big(\sup_{0\leqslant r\leqslant t} \int_0^r ds \int_{\mathbb{R}^n} \big(G(x, u^\rho(s-\rho)) - G(x, u^{\rho_1}(s-\rho_1))\big) \cdot \big(u^\rho(s) - u^{\rho_1}(s)\big) dx \Big) \\
&\qquad + \mathbb{E}\Big(\sum_{k=1}^\infty \int_0^t \|\kappa\sigma_{2,k}(u^\rho(s)) - \kappa\sigma_{2,k}(u^{\rho_1}(s))\|^2 ds \Big) \\
&\qquad + 2\mathbb{E}\Big(\sup_{0\leqslant r\leqslant t} \Big| \sum_{k=1}^\infty \int_0^r \big(\kappa\sigma_{2,k}(u^\rho(s)) - \kappa\sigma_{2,k}(u^{\rho_1}(s)), u^\rho(s) - u^{\rho_1}(s)\big) dW_k(s) \Big| \Big).
\end{aligned} \tag{7.3}$$

We now estimate all terms on the right-hand side of (7.3). For the first term on the right-hand side of (7.3), it follows from (4.4) that

$$\mathbb{E}\Big(\sup_{0\leqslant r\leqslant t}\int_0^r ds\int_{\mathbb{R}^n} -2\big(F(x,u^\rho(s))-F(x,u^{\rho_1}(s))\big)\cdot\big(u^\rho(s)-u^{\rho_1}(s)\big)dx\Big)$$
$$\leqslant 2\|\psi_4\|_{L^\infty(\mathbb{R}^n)}\mathbb{E}\Big(\int_0^t \|u^\rho(s)-u^{\rho_1}(s)\|^2 ds\Big). \tag{7.4}$$

For the second term on the right-hand side of (7.3), we have

$$\begin{aligned}
&2\mathbb{E}\Big(\sup_{0\leqslant r\leqslant t}\int_0^r ds\int_{\mathbb{R}^n}\big(G(x,u^\rho(s-\rho))-G(x,u^{\rho_1}(s-\rho_1))\big)\cdot\big(u^\rho(s)-u^{\rho_1}(s)\big)dx\Big)\\
&\leqslant \mathbb{E}\Big(\int_{-\rho}^{t-\rho}\|G(\cdot,u^\rho(s))-G(\cdot,u^{\rho_1}(s-\rho_1+\rho))\|^2 ds\Big)\\
&\quad+\mathbb{E}\Big(\int_0^t\|u^\rho(s)-u^{\rho_1}(s)\|^2 ds\Big)\\
&=\mathbb{E}\Big(\int_{-\rho}^{\rho_1-\rho}\|G(\cdot,u^\rho(s))-G(\cdot,u^{\rho_1}(s-\rho_1+\rho))\|^2 ds\Big)\\
&\quad+\mathbb{E}\Big(\int_{\rho_1-\rho}^{0}\|G(\cdot,u^\rho(s))-G(\cdot,u^{\rho_1}(s-\rho_1+\rho))\|^2 ds\Big)\\
&\quad+\mathbb{E}\Big(\int_{0}^{t-\rho}\|G(\cdot,u^\rho(s))-G(\cdot,u^{\rho_1}(s-\rho_1+\rho))\|^2 ds\Big)\\
&\quad+\mathbb{E}\Big(\int_0^t\|u^\rho(s)-u^{\rho_1}(s)\|^2 ds\Big).
\end{aligned} \tag{7.5}$$

For the first term on the right-hand side of (7.5), by (4.7) we have

$$\mathbb{E}\Big(\int_{-\rho}^{\rho_1-\rho}\|G(\cdot,u^\rho(s))-G(\cdot,u^{\rho_1}(s-\rho_1+\rho))\|^2 ds\Big)$$
$$\leqslant (C^G)^2\int_{-\rho_1}^0\|\varphi(s+\rho_1-\rho)-\varphi(s)\|^2 ds. \tag{7.6}$$

Since $C([-1,0];H)$ is dense in $L^2((-1,0),H)$, we find that for each $\varphi\in L^2((-1,0),H)$, there exists $\delta_\varphi\in(0,1-\rho_1)$ such that $\int_{-\rho_1}^0\|\varphi(s-h)-\varphi(s)\|^2ds<\varepsilon$ for any $h\in(0,\delta_\varphi)$. Since E is compact in $H\times L^2((-1,0),H)$, we infer that there exists $\delta=\delta(E)\in(0,1-\rho_1)$ such that for all $h\in(0,\delta)$ and for all $(u^0,\varphi)\in E$,

$$\int_{-\rho_1}^0\|\varphi(s-h)-\varphi(s)\|^2 ds<\varepsilon. \tag{7.7}$$

By (7.6) and (7.7), we obtain that for $0 < \rho - \rho_1 < \delta$,

$$\mathbb{E}\bigg(\int_{-\rho}^{\rho_1-\rho} \|G(\cdot, u^\rho(s)) - G(\cdot, u^{\rho_1}(s - \rho_1 + \rho))\|^2 ds \bigg) \leqslant (C^G)^2 \varepsilon. \tag{7.8}$$

For the second term on the right-hand side of (7.5), it follows from (4.6) and (7.2) that for $0 < \rho - \rho_1 < \delta$,

$$\begin{aligned} &\mathbb{E}\bigg(\int_{\rho_1-\rho}^{0} \|G(\cdot, u^\rho(s)) - G(\cdot, u^{\rho_1}(s - \rho_1 + \rho))\|^2 ds \bigg) \\ &\leqslant 4\|h\|^2(\rho - \rho_1) + 2a^2 \int_{\rho_1-\rho}^{0} \|\varphi(s)\|^2 ds + 2a^2 \int_{0}^{\rho-\rho_1} \mathbb{E}\bigg(\sup_{s\in[0,T]} \|u^{\rho_1}(s)\|^2 ds \bigg) \\ &\leqslant 2(2\|h\|^2 + a^2 c_1)(\rho - \rho_1) + 2a^2 \int_{\rho_1-\rho}^{0} \|\varphi(s)\|^2 ds. \end{aligned} \tag{7.9}$$

For the third term on the right-hand side of (7.5), by (4.7) we have

$$\begin{aligned} &\mathbb{E}\bigg(\int_{0}^{t-\rho} \|G(\cdot, u^\rho(s)) - G(\cdot, u^{\rho_1}(s - \rho_1 + \rho))\|^2 ds \bigg) \\ &\quad \leqslant \mathbb{E}\bigg(\int_{0}^{t-\rho} 1_{(\rho,+\infty)}(t) \|G(\cdot, u^\rho(s)) - G(\cdot, u^{\rho_1}(s - \rho_1 + \rho))\|^2 ds \bigg) \\ &\quad \leqslant 2(C^G)^2 \mathbb{E}\bigg(\int_{0}^{t-\rho} 1_{(\rho,+\infty)}(t) \|u^\rho(s) - u^{\rho_1}(s)\|^2 ds \bigg) \\ &\qquad + 2(C^G)^2 \mathbb{E}\bigg(\int_{0}^{t-\rho} 1_{(\rho,+\infty)}(t) \|u^{\rho_1}(s) - u^{\rho_1}(s - \rho_1 + \rho)\|^2 ds \bigg). \end{aligned} \tag{7.10}$$

Next, we consider the second term on the right-hand side of (7.10). Since E is compact in $H \times L^2((-1, 0), H)$, we see that for every $\varepsilon > 0$, E has a finite open cover of balls with radius ε in $H \times L^2((-1, 0), H)$, which is denoted by $\{B\big((u_i, \varphi_i), \varepsilon\big)\}_{i=1}^m$. Then for each $(u^0, \varphi) \in E$, there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $(u^0, \varphi) \in B\big((u_{i_0}, \varphi_{i_0}), \varepsilon\big)$; that is,

$$\|u^0 - u_{i_0}\|^2 + \int_{-1}^{0} \|\varphi(s) - \varphi_{i_0}(s)\|^2 ds < \varepsilon^2. \tag{7.11}$$

Note that

$$\begin{aligned} &2(C^G)^2 \mathbb{E}\bigg(\int_{0}^{t-\rho} 1_{(\rho,+\infty)}(t) \|u^{\rho_1}(s) - u^{\rho_1}(s - \rho_1 + \rho)\|^2 ds \bigg) \\ &\quad \leqslant 6(C^G)^2 \mathbb{E}\bigg(\int_{0}^{t-\rho} 1_{(\rho,+\infty)}(t) \|u^{\rho_1}(s; 0, \mathcal{T}_{1\to\rho_1}(u^0, \varphi)) - u^{\rho_1}(s; 0, \mathcal{T}_{1\to\rho_1}(u_{i_0}, \varphi_{i_0}))\|^2 ds \bigg) \\ &\qquad + 6(C^G)^2 \mathbb{E}\bigg(\int_{0}^{t-\rho} 1_{(\rho,+\infty)}(t) \|u^{\rho_1}(s; 0, \mathcal{T}_{1\to\rho_1}(u_{i_0}, \varphi_{i_0}) \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - u^{\rho_1}(s-\rho_1+\rho;0,\mathcal{T}_{1\to\rho_1}(u_{i_0},\varphi_{i_0}))\|^2 ds\Big)\\
&+6(C^G)^2\mathbb{E}\Big(\int_0^{t-\rho} 1_{(\rho,+\infty)}(t)\|u^{\rho_1}(s-\rho_1+\rho;0,\mathcal{T}_{1\to\rho_1}(u_{i_0},\varphi_{i_0}))\\
&\qquad\qquad - u^{\rho_1}(s-\rho_1+\rho;0,\mathcal{T}_{1\to\rho_1}(u^0,\varphi))\|^2 ds\Big)\\
&\doteq I_1+I_2+I_3.
\end{aligned}
\tag{7.12}
$$

Since for all $i=1,2,\ldots,m, u^{\rho_1}(s;0,\mathcal{T}_{1\to\rho_1}(u_i,\varphi_i))\in C([0,T],L^2(\Omega,\mathcal{F}_0;H))$, which implies that $u^{\rho_1}(s;0,\mathcal{T}_{1\to\rho_1}(u_i,\varphi_i)) : [0,T]\to L^2(\Omega,\mathcal{F}_0;H)$ is uniformly continuous, and thus there exists $\delta_i=\delta_i(\varepsilon,T,u_i,\varphi_i)>0$ such that for all $t_1,t_2\in[0,T]$ with $|t_1-t_2|<\delta_i$,

$$
\mathbb{E}\Big(\|u^{\rho_1}(t_1;0,\mathcal{T}_{1\to\rho_1}(u_i,\varphi_i))-u^{\rho_1}(t_2;0,\mathcal{T}_{1\to\rho_1}(u_i,\varphi_i))\|^2\Big)<\frac{\varepsilon}{6T((C_R^G)^2+1)}.
$$

Let $\tilde\delta=\min\{\delta_i\mid i=1,2,\ldots,m\}$. Then for all $0<\rho-\rho_1<\tilde\delta$,

$$
\mathbb{E}\left(\|u^{\rho_1}(s;0,\mathcal{T}_{1\to\rho_1}(u_i,\varphi_i))-u^{\rho_1}(s-\rho_1+\rho;0,\mathcal{T}_{1\to\rho_1}(u_i,\varphi_i))\|^2\right)<\frac{\varepsilon}{6T((C_R^G)^2+1)}
\tag{7.13}
$$

for all $s\in[0,T-\rho],\ i=1,2,\ldots,m$. Then by (7.13) we obtain

$$
I_2<\varepsilon,\quad \text{for all } t\in[0,T] \text{ and } 0<\rho-\rho_1<\tilde\delta.
\tag{7.14}
$$

On the other hand, by Ito's formula and together with (4.4), (4.7), and (4.9), we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\|u^{\rho_1}(t;0,\mathcal{T}_{1\to\rho_1}(u^0,\varphi))-u^{\rho_1}(t;0,\mathcal{T}_{1\to\rho_1}(u_{i_0},\varphi_{i_0}))\|^2\right)\\
&\quad\leqslant \|u^0-u_{i_0}\|^2+\int_{-\rho_1}^0\|\varphi(s)-\varphi_{i_0}(s)\|^2ds+\big(\|\psi_4\|_{L^\infty}+(C^G)^2+1+\|\kappa\|_{L^\infty}^2\sum_{k=1}^\infty\alpha_k^2\big)\\
&\quad\times\mathbb{E}\Big(\int_0^t\|u^{\rho_1}(s;0,\mathcal{T}_{1\to\rho_1}(u^0,\varphi))-u^{\rho_1}(s;0,\mathcal{T}_{1\to\rho_1}(u_{i_0},\varphi_{i_0}))\|^2ds\Big).
\end{aligned}
\tag{7.15}
$$

Applying Gronwall's inequality to (7.15), by (7.11) we have

$$
\begin{aligned}
&\mathbb{E}\left(\|u^{\rho_1}(t;0,\mathcal{T}_{1\to\rho_1}(u^0,\varphi))-u^{\rho_1}(t;0,\mathcal{T}_{1\to\rho_1}(u_{i_0},\varphi_{i_0}))\|^2\right)\\
&\quad\leqslant\left(\|u^0-u_{i_0}\|^2+\int_{-\rho_1}^0\|\varphi(s)-\varphi_{i_0}(s)\|^2ds\right)e^{\left(\|\psi_4\|_{L^\infty}+(C^G)^2+1+\|\kappa\|_{L^\infty}^2\sum_{k=1}^\infty\alpha_k^2\right)t}\\
&\quad<e^{\left(\|\psi_4\|_{L^\infty}+C^G+1+\|\kappa\|_{L^\infty}^2\sum_{k=1}^\infty\alpha_k^2\right)t}\varepsilon^2,\quad\forall\, t\in[0,T].
\end{aligned}
\tag{7.16}
$$

By (7.16), we have for $t \in [0, T]$

$$I_1 \leqslant 6(C^G)^2\mathbb{E}\bigg(\int_0^T \|u^{\rho_1}(s; 0, \mathcal{T}_{1\to\rho_1}(u^0, \varphi)) - u^{\rho_1}(s; 0, \mathcal{T}_{1\to\rho_1}(u_{i_0}, \varphi_{i_0}))\|^2 ds \bigg) \\ \leqslant 6T(C^G)^2 e^{\left(\|\psi_4\|_{L^\infty} + (C^G)^2 + 1 + \|\kappa\|^2_{L^\infty} \sum_{k=1}^\infty \alpha_k^2\right)T} \varepsilon^2. \tag{7.17}$$

In addition, by (7.16) we have

$$I_3 \leqslant 6(C^G)^2\mathbb{E}\bigg(\int_{\rho-\rho_1}^{t-\rho_1} 1_{(\rho,+\infty)}(t)\|u^{\rho_1}(s; 0, \mathcal{T}_{1\to\rho_1}(u_{i_0}, \varphi_{i_0})) - u^{\rho_1}(s; 0, \mathcal{T}_{1\to\rho_1}(u^0, \varphi))\|^2 ds \bigg) \\ \leqslant 6T(C^G)^2 e^{\left(\|\psi_4\|_{L^\infty} + (C^G)^2 + 1 + \|\kappa\|^2_{L^\infty} \sum_{k=1}^\infty \alpha_k^2\right)T} \varepsilon^2. \tag{7.18}$$

So by (7.12), (7.14), (7.17) and (7.18), we obtain for $0 < \rho - \rho_1 < \bar{\delta}$,

$$2(C^G)^2\mathbb{E}\bigg(\int_0^{t-\rho} 1_{(\rho,+\infty)}(t)\|u^{\rho_1}(s) - u^{\rho_1}(s - \rho_1 + \rho)\|^2 ds \bigg) \\ \leqslant \varepsilon + 12T(C^G)^2 e^{\left(\|\psi_4\|_{L^\infty} + (C^G)^2 + 1 + \|\kappa\|^2_{L^\infty} \sum_{k=1}^\infty \alpha_k^2\right)T} \varepsilon^2, \tag{7.19}$$

which along with (7.10) yields that for $0 < \rho - \rho_1 < \bar{\delta}$,

$$\mathbb{E}\bigg(\int_0^{t-\rho} \|G(\cdot, u^\rho(s)) - G(\cdot, u^{\rho_1}(s - \rho_1 + \rho))\|^2 ds \bigg) \\ \leqslant 2(C^G)^2\mathbb{E}\bigg(\int_0^{t-\rho} 1_{(\rho,+\infty)}(t)\|u^\rho(s) - u^{\rho_1}(s)\|^2 ds \bigg) \\ + \varepsilon + 12T(C^G)^2 e^{\left(\|\psi_4\|_{L^\infty} + (C^G)^2 + 1 + \|\kappa\|^2_{L^\infty} \sum_{k=1}^\infty \alpha_k^2\right)T} \varepsilon^2. \tag{7.20}$$

Let $\hat{\delta} = \min\{\delta, \bar{\delta}\}$. Then for $0 < \rho - \rho_1 < \hat{\delta}$, it follows from (7.5), (7.8), (7.9) and (7.20) that

$$2\mathbb{E}\bigg(\sup_{0\leqslant r\leqslant t} \int_0^r ds \int_{\mathbb{R}^n} \big(G(x, u^\rho(s - \rho)) - G(x, u^{\rho_1}(s - \rho_1))\big) \cdot \big(u^\rho(s) - u^{\rho_1}(s)\big) dx \bigg) \\ \leqslant (C^G)^2\varepsilon + 2(2\|h\|^2 + a^2c_1)(\rho - \rho_1) + 2a^2 \int_{\rho_1-\rho}^0 \|\varphi(s)\|^2 ds \\ + \varepsilon + 12T(C^G)^2 e^{\left(\|\psi_4\|_{L^\infty} + (C^G)^2 + 1 + \|\kappa\|^2_{L^\infty} \sum_{k=1}^\infty \alpha_k^2\right)T} \varepsilon^2 \\ + [2(C^G)^2 + 1]\mathbb{E}\bigg(\int_0^t \|u^\rho(s) - u^{\rho_1}(s)\|^2 ds \bigg). \tag{7.21}$$

For the third term on the right-hand side of (7.3), by (4.9), we have

$$\mathbb{E}\Big(\sum_{k=1}^{\infty}\int_0^t \|\kappa\sigma_{2,k}(u^{\rho}(s))-\kappa\sigma_{2,k}(u^{\rho_1}(s))\|^2ds\Big) \leqslant \|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\alpha_k^2\mathbb{E}\Big(\int_0^t \|u^{\rho}(s)-u^{\rho_1}(s)\|^2ds\Big). \tag{7.22}$$

For the fourth term on the right-hand side of (7.3), by $(\boldsymbol{\Sigma 1'})$ and the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}&2\mathbb{E}\Big(\sup_{0\leqslant r\leqslant t}\Big|\sum_{k=1}^{\infty}\int_0^r \big(\kappa\sigma_{2,k}(u^{\rho}(s))-\kappa\sigma_{2,k}(u^{\rho_1}(s)), u^{\rho}(s)-u^{\rho_1}(s)\big)dW_k(s)\Big|\Big)\\ &\leqslant \frac{1}{2}\mathbb{E}\left(\sup_{0\leqslant r\leqslant t}\|u^{\rho}(r)-u^{\rho_1}(r)\|^2\right)+2c^2\|\kappa\|_{L^\infty}^2\sum_{k=1}^{\infty}\alpha_k^2\mathbb{E}\left(\int_0^t \|u^{\rho}(s)-u^{\rho_1}(s)\|^2ds\right).\end{aligned} \tag{7.23}$$

Then by (7.3), (7.4), (7.21)–(7.23), we obtain, for $\varepsilon\in(0,1)$ and $0<\rho-\rho_1<\hat{\delta}$,

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leqslant r\leqslant t}\|u^{\rho}(r)-u^{\rho_1}(r)\|^2\right)\\ &\leqslant c_2\varepsilon+c_3(\rho-\rho_1)+c_4\int_{\rho_1-\rho}^0\|\varphi(s)\|^2ds+c_5\mathbb{E}\Big(\int_0^t\|u^{\rho}(s)-u^{\rho_1}(s)\|^2ds\Big),\end{aligned} \tag{7.24}$$

where c_2, c_3, c_4 and c_5 are positive numbers depending only on E and T but not on $u^0, \varphi, \varepsilon$ or ρ. By (7.24) and Gronwall's inequality, we obtain that for all $t\in[0,T]$, $(u^0,\varphi)\in E$ and $0<\rho-\rho_1<\hat{\delta}$,

$$\mathbb{E}\left(\sup_{0\leqslant r\leqslant t}\|u^{\rho}(r)-u^{\rho_1}(r)\|^2\right)\leqslant\left(c_2\varepsilon+c_3(\rho-\rho_1)+c_4\int_{\rho_1-\rho}^0\|\varphi(s)\|^2ds\right)e^{c_5T}. \tag{7.25}$$

Furthermore, by (7.25) we obtain for all $t\in[0,T]$, $(u^0,\varphi)\in E$ and $0<\rho-\rho_1<\hat{\delta}$,

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant r\leqslant t}\int_{-\rho_1}^0\|u_r^{\rho}(s)-u_r^{\rho_1}(s)\|^2ds\right)&\leqslant\int_{-\rho_1}^0\mathbb{E}\left(\sup_{0\leqslant r\leqslant t}\|u^{\rho}(r+s)-u^{\rho_1}(r+s)\|^2\right)ds\\ &\leqslant\rho_1\mathbb{E}\left(\sup_{0\leqslant r\leqslant t}\|u^{\rho}(r)-u^{\rho_1}(r)\|^2\right)\\ &\leqslant\rho_1\left(c_2\varepsilon+c_3(\rho-\rho_1)+c_4\int_{\rho_1-\rho}^0\|\varphi(s)\|^2ds\right)e^{c_5T}.\end{aligned} \tag{7.26}$$

Since E is compact in $H \times L^2((-1, 0), H)$, there exists $\delta_0 = \delta_0(\varepsilon, E) > 0$ such that for all $h \in (0, \delta_0)$,

$$\int_{-h}^{0} \|\varphi(s)\|^2 ds < \varepsilon, \quad \forall\, (u^0, \varphi) \in E. \tag{7.27}$$

Let $\hat{\delta}_0 = \min\{\delta_0, \hat{\delta}\}$. By (7.25), (7.26) and (7.27) we get for all $(u^0, \varphi) \in E$ and $0 < \rho - \rho_1 < \hat{\delta}_0$,

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leqslant r\leqslant T} \left(\|u^\rho(r) - u^{\rho_1}(r)\|^2 + \int_{-\rho_1}^{0} \|u_r^\rho(s) - u_r^{\rho_1}(s)\|^2 ds \right)\right) \\ &\leqslant (1 + \rho_1)\,(c_2\varepsilon + c_3(\rho - \rho_1) + c_4\varepsilon)\, e^{c_5 T}. \end{aligned} \tag{7.28}$$

It follows from (7.28) that for all $0 < \rho - \rho_1 < \hat{\delta}_0$,

$$\begin{aligned} &\sup_{(u^0,\varphi)\in E} \mathbb{P}\Big(\Big\{ \sup_{0\leqslant t\leqslant T} \|\mathcal{T}_{\rho\to\rho_1}\big(u^\rho(t), u_t^\rho\big) - \big(u^{\rho_1}(t), u_t^{\rho_1}\big)\|_{H_{\rho_1}} \geqslant \eta \Big\}\Big) \\ &\leqslant (1 + \rho_1)\eta^{-2}\,(c_2\varepsilon + c_3(\rho - \rho_1) + c_4\varepsilon)\, e^{c_5 T}. \end{aligned} \tag{7.29}$$

By (7.29), we obtain

$$\lim_{\rho\to\rho_1} \sup_{(u^0,\varphi)\in E} \mathbb{P}\Big(\Big\{ \sup_{0\leqslant t\leqslant T} \|\mathcal{T}_{\rho\to\rho_1}\big(u^\rho(t), u_t^\rho\big) - \big(u^{\rho_1}(t), u_t^{\rho_1}\big)\|_{H_{\rho_1}} \geqslant \eta \Big\}\Big) = 0,$$

as desired. □

We are now ready to present the main result of this section.

Theorem 7.2 *Suppose* $(\mathbf{F}')$, $(\mathbf{G}')$, (Σ') *and (4.58)–(4.60) hold. Let (4.11) be fulfilled with* $\theta = \frac{3p-4}{2p-2}$. *Take* $\rho_0 \in [0, 1)$ *and* $\rho_n \in (\rho_0, 1]$. *If* $\rho_n \to \rho_0$ *and* $\mu^{\rho_n} \in \mathcal{S}^{\rho_n}$, *then there exist a subsequence* $\{\rho_{n_k}\}_{k=1}^{\infty}$ *and an invariant measure* $\mu^{\rho_0} \in \mathcal{S}^{\rho_0}$ *such that* $\mu^{\rho_{n_k}} \circ \mathcal{T}^{-1}_{\rho_{n_k}\to\rho_0} \to \mu^{\rho_0}$ *weakly.*

Proof Note that $\{\mu^{\rho_n}\}_{n=1}^{\infty}$ is tight by Theorem 6.1. Therefore, there exist a subsequence $\{\rho_{n_k}\}_{k=1}^{\infty}$ and probability measure μ^* such that $\mu^{\rho_{n_k}} \circ \mathcal{T}^{-1}_{\rho_{n_k}\to\rho_0} \to \mu^*$ weakly. Since $\rho_{n_k} \to \rho_0$, by Lemma 7.1 and Theorem 7.1 we infer that μ^* must be an invariant probability measure of (4.1) with $\rho = \rho_0$, which concludes the proof. □

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Declarations

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