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A SIR epidemic model on a refining spatial grid II-central limit theorem

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Abstract
A stochastic SIR epidemic model taking into account the heterogeneity of the spatial environment is constructed. The deterministic model is given by a partial differential equation and the stochastic one by a space-time jump Markov process. The consistency of the two models is given by a law of large numbers. In this paper, we study the deviation of the spatial stochastic model from the deterministic model by a functional central limit theorem. The limit is a distribution-valued Ornstein–Uhlenbeck Gaussian process, which is the mild solution of a stochastic partial differential equation.

Keywords Spatial model · Deterministic · Stochastic · Stochastic partial differential equation · Central limit theorem

Mathematics Subject Classification 60F05 · 60G15 · 60G65 · 60H15 · 92D30

1 Introduction

A stochastic spatial model of epidemic has been described by N'zi et al. [10] to study the oubreak of infectious diseases in a bounded domain. Such a model takes into account heterogeneity, spatial connectivity and movement of individuals, which play an important role in the spread of the infectious diseases. It is based on the

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compartmental SIR model of Kermack and Mckendrick [6]. Let us summarize the results in N'zi et al. [10] in the case of one dimensional space.
Consider a deterministic and a stochastic SIR model on a grid D_ε of the torus $\mathbb{T}^1 = [0, 1)$ with migration between neighboring sites (two neighboring sites are at distance ε apart, $\varepsilon^{-1} \in \mathbb{N}^*$). Let $S_\varepsilon(t, x_i)$ (resp. $I_\varepsilon(t, x_i)$, resp. $R_\varepsilon(t, x_i)$) be the proportion of the total population which is both susceptible (resp. infectious, resp. removed) and located at site x_i at time t. The dynamics of susceptible, infected and removed individuals at each site can be expressed as

$$\begin{cases} \dfrac{d\,S_\varepsilon}{dt}(t,x_i) = \mu_S\,\Delta_\varepsilon S_\varepsilon(t,x_i) - \dfrac{\beta(x_i)\,S_\varepsilon(t,x_i)I_\varepsilon(t,x_i)}{S_\varepsilon(t,x_i)+I_\varepsilon(t,x_i)+R_\varepsilon(t,x_i)}, \\ \dfrac{d\,I_\varepsilon}{dt}(t,x_i) = \mu_I\,\Delta_\varepsilon I_\varepsilon(t,x_i) + \dfrac{\beta(x_i)\,S_\varepsilon(t,x_i)I_\varepsilon(t,x_i)}{S_\varepsilon(t,x_i)+I_\varepsilon(t,x_i)+R_\varepsilon(t,x_i)} - \alpha(x_i)\,I_\varepsilon(t,x_i), \\ \dfrac{d\,R_\varepsilon}{dt}(t,x_i) = \mu_R\,\Delta_\varepsilon R_\varepsilon(t,x_i) + \alpha(x_i)\,I_\varepsilon(t,x_i), \quad (t,x_i) \in (0,T)\times \mathrm{D}_\varepsilon, \\ S_\varepsilon(0,x_i), I_\varepsilon(0,x_i), R_\varepsilon(0,x_i) \geq 0,\ \ 0 < S_\varepsilon(0,x_i)+I_\varepsilon(0,x_i)+R_\varepsilon(0,x_i) \leq M, \\ \text{for some } M < \infty, \end{cases} \tag{1}$$

Δ_ε is the discrete Laplace operator defined as follows

$$\Delta_\varepsilon f(x_i) := \varepsilon^{-2}\big[f(x_i+\varepsilon) - 2f(x_i) + f(x_i-\varepsilon)\big].$$

The rates $\beta : [0, 1] \longrightarrow \mathbb{R}_+$ and $\alpha : [0, 1] \longrightarrow \mathbb{R}_+$ are continuous periodic functions, and μ_S, μ_I and μ_R are positive diffusion coefficients for the susceptible, infectious and removed subpopulations, respectively.

In what follows, we use the notations $S_\varepsilon(t) := \begin{pmatrix} S_\varepsilon(t,x_1) \\ \vdots \\ S_\varepsilon(t,x_\ell) \end{pmatrix}$, $I_\varepsilon(t) := \begin{pmatrix} I_\varepsilon(t,x_1) \\ \vdots \\ I_\varepsilon(t,x_\ell) \end{pmatrix}$,

$R_\varepsilon(t) := \begin{pmatrix} R_\varepsilon(t,x_1) \\ \vdots \\ R_\varepsilon(t,x_\ell) \end{pmatrix}$, and $\quad Z_\varepsilon(t) = \big(S_\varepsilon(t)\,,\ I_\varepsilon(t)\,,\ R_\varepsilon(t)\big)^{\mathrm{T}}$. Here $\ell = \varepsilon^{-1}$.
Note that (1) is the discrete space approximation of the following system of PDEs

$$\begin{cases} \dfrac{\partial\,\mathbf{s}}{\partial t}(t,x) = \mu_S\,\Delta\mathbf{s}(t,x) - \dfrac{\beta(x)\,\mathbf{s}(t,x)\mathbf{i}(t,x)}{\mathbf{s}(t,x)+\mathbf{i}(t,x)+\mathbf{r}(t,x)}, \\ \dfrac{\partial\,\mathbf{i}}{\partial t}(t,x) = \mu_I\,\Delta\mathbf{i}(t,x) + \dfrac{\beta(x)\,\mathbf{s}(t,x)\mathbf{i}(t,x)}{\mathbf{s}(t,x)+\mathbf{i}(t,x)+\mathbf{r}(t,x)} - \alpha(x)\,\mathbf{i}(t,x), \\ \dfrac{\partial\,\mathbf{r}}{\partial t}(t,x) = \mu_R\,\Delta\mathbf{r}(t,x) + \alpha(x)\,\mathbf{i}(t,x), \quad (t,x) \in (0,T)\times D, \\ \mathbf{s}(0,x), \mathbf{i}(0,x), \mathbf{r}(0,x) \geq 0,\ \ 0 < \mathbf{s}(0,x)+\mathbf{i}(0,x)+\mathbf{r}(0,x) \leq M, \end{cases} \tag{2}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2}$. In the sequel, we set $\mathbf{X} := (\mathbf{s}\,,\ \mathbf{i}\,,\ \mathbf{r})^{\mathrm{T}}$.
Let $\mathbf{N}$ be the total population size. The stochastic version of (1) is given by the following system

$$\begin{cases} S_{\mathbf{N},\varepsilon}(t,x_i) = S_{\mathbf{N},\varepsilon}(0,x_i) - \dfrac{1}{\mathbf{N}}\mathrm{P}^{inf}_{x_i}\left(\mathbf{N}\displaystyle\int_0^t \dfrac{\beta(x_i)S_{\mathbf{N},\varepsilon}(r,x_i)I_{\mathbf{N},\varepsilon}(r,x_i)}{S_{\mathbf{N},\varepsilon}(r,x_i)+I_{\mathbf{N},\varepsilon}(r,x_i)+R_{\mathbf{N},\varepsilon}(r,x_i)}dr\right) \\ \qquad - \displaystyle\sum_{y_i\sim x_i}\dfrac{1}{\mathbf{N}}\mathrm{P}^{mig}_{S,x_i,y_i}\left(\mathbf{N}\int_0^t \dfrac{\mu_S}{\varepsilon^2}S_{\mathbf{N},\varepsilon}(r,x_i)dr\right) \\ \qquad + \displaystyle\sum_{y_i\sim x_i}\dfrac{1}{\mathbf{N}}\mathrm{P}^{mig}_{S,y_i,x_i}\left(\mathbf{N}\int_0^t \dfrac{\mu_S}{\varepsilon^2}S_{\mathbf{N},\varepsilon}(r,y_i)dr\right), \\ I_{\mathbf{N},\varepsilon}(t,x_i) = I_{\mathbf{N},\varepsilon}(0,x_i) + \dfrac{1}{\mathbf{N}}\mathrm{P}^{inf}_{x_i}\left(\mathbf{N}\displaystyle\int_0^t \dfrac{\beta(x_i)S_{\mathbf{N},\varepsilon}(r,x_i)I_{\mathbf{N},\varepsilon}(r,x_i)}{S_{\mathbf{N},\varepsilon}(r,x_i)+I_{\mathbf{N},\varepsilon}(r,x_i)+R_{\mathbf{N},\varepsilon}(r,x_i)}dr\right) \\ \qquad - \dfrac{1}{\mathbf{N}}\mathrm{P}^{rec}_{x_i}\left(\mathbf{N}\displaystyle\int_0^t \alpha(x_i)I_{\mathbf{N},\varepsilon}(r,x_i)dr\right) - \displaystyle\sum_{y_i\sim x_i}\dfrac{1}{\mathbf{N}}\mathrm{P}^{mig}_{I,x_i,y_i}\left(\mathbf{N}\int_0^t \dfrac{\mu_I}{\varepsilon^2}I_{\mathbf{N},\varepsilon}(r,x_i)dr\right) \\ \qquad + \displaystyle\sum_{y_i\sim x_i}\dfrac{1}{\mathbf{N}}\mathrm{P}^{mig}_{I,y_i,x_i}\left(\mathbf{N}\int_0^t \dfrac{\mu_I}{\varepsilon^2}I_{\mathbf{N},\varepsilon}(r,y_i)dr\right), \\ R_{\mathbf{N},\varepsilon}(t,x_i) = R_{\mathbf{N},\varepsilon}(0,x_i) + \dfrac{1}{\mathbf{N}}\mathrm{P}^{rec}_{x_i}\left(\mathbf{N}\displaystyle\int_0^t \alpha(x_i)I_{\mathbf{N},\varepsilon}(r,x_i)dr\right) \\ \qquad - \displaystyle\sum_{y_i\sim x_i}\dfrac{1}{\mathbf{N}}\mathrm{P}^{mig}_{R,x_i,y_i}\left(\mathbf{N}\int_0^t \dfrac{\mu_R}{\varepsilon^2}R_{\mathbf{N},\varepsilon}(r,x_i)dr\right) \\ \qquad + \dfrac{1}{\mathbf{N}}\displaystyle\sum_{y_i\sim x_i}\mathrm{P}^{mig}_{R,y_i,x_i}\left(\mathbf{N}\int_0^t \dfrac{\mu_R}{\varepsilon^2}R_{\mathbf{N},\varepsilon}(r,y_i)dr\right), \\ \qquad (t,x_i)\in[0,T]\times \mathrm{D}_\varepsilon \end{cases} \tag{3}$$

where all the P_j's are standard Poisson processes, which are mutually independent. For each given site, these processes count the number of new infectious, recoveries and the migrations between sites. $y_i \sim x_i$ means that $y_i \in \{x_i+\varepsilon\,,\ x_i-\varepsilon\}$.

Let $S_{\mathbf{N},\varepsilon}(t) := \begin{pmatrix} S_{\mathbf{N},\varepsilon}(t,x_1) \\ \vdots \\ S_{\mathbf{N},\varepsilon}(t,x_\ell)\end{pmatrix}$, $I_{\mathbf{N},\varepsilon}(t) := \begin{pmatrix} I_{\mathbf{N},\varepsilon}(t,x_1) \\ \vdots \\ I_{\mathbf{N},\varepsilon}(t,x_\ell)\end{pmatrix}$,
$R_{\mathbf{N},\varepsilon}(t) := \begin{pmatrix} R_{\mathbf{N},\varepsilon}(t,x_1) \\ \vdots \\ R_{\mathbf{N},\varepsilon}(t,x_\ell)\end{pmatrix}$,

$$Z_{\mathbf{N},\varepsilon}(t) := \big(S_{\mathbf{N},\varepsilon}(t),\ I_{\mathbf{N},\varepsilon}(t),\ R_{\mathbf{N},\varepsilon}(t)\big)^{\mathrm{T}} \text{ and } b_\varepsilon\big(t, Z_{\mathbf{N},\varepsilon}(t)\big) := \sum_{j=1}^{K} h_j\beta_j(Z_{\mathbf{N},\varepsilon}(t))$$

(K being the number of Poisson's processes in the system), where the vectors $h_j \in \{-1,0,1\}^{3\ell}$ denote the respective jump directions with jump rates β_j. The SDE (3) can be rewritten as follows

$$Z_{\mathbf{N},\varepsilon}(t) = Z_{\mathbf{N},\varepsilon}(0) + \int_0^t b_\varepsilon\big(r, Z_{\mathbf{N},\varepsilon}(r)dr + \frac{1}{\mathbf{N}}\sum_{j=1}^{K} h_j \mathrm{P}_j\left(\mathbf{N}\int_0^t \beta_j\left(Z_{\mathbf{N},\varepsilon}(r)\right)dr\right). \tag{4}$$

Also the sytem (1) can be written as follows

$$\frac{dZ_\varepsilon(t)}{dt} = b_\varepsilon(t, Z_\varepsilon(t)). \tag{5}$$

The authors show the consistency of the two models by a law of large numbers. More precisely, the following two results were proved in N'zi et al. [10].

Theorem 1.1 (Law of Large Numbers: $N \to \infty$, ε being fixed)
Let $Z_{\mathbf{N},\varepsilon}$ denote the solution (4) and Z_ε the solution of (5).
Let us fix an arbitrary $T > 0$ and assume that $Z_{\mathbf{N},\varepsilon}(0) \longrightarrow Z_\varepsilon(0)$, as $\mathbf{N} \to +\infty$.
Then $\sup_{0\le t\le T}\left\| Z_{\mathbf{N},\varepsilon}(t) - Z_\varepsilon(t)\right\| \longrightarrow 0$ *a.s.* , *as* $\mathbf{N} \to +\infty$.

Moreover, for all $x_i \in D_\varepsilon$, $V_i := [x_i - \varepsilon/2, x_i + \varepsilon/2)$ denote the cell centered in the site x_i. We define

$$\begin{aligned}\mathcal{S}_\varepsilon(t,x) &:= \sum_{i=1}^{\varepsilon^{-1}} S_\varepsilon(t,x_i)\mathbb{1}_{V_i}(x),\ \ \mathcal{I}_\varepsilon(t,x)\\ &= \sum_{i=1}^{\varepsilon^{-1}} I_\varepsilon(t,x_i)\mathbb{1}_{V_i}(x),\ \ \mathcal{R}_\varepsilon(t,x) := \sum_{i=1}^{\varepsilon^{-1}} R_\varepsilon(t,x_i)\mathbb{1}_{V_i}(x),\\ \beta(x) &:= \sum_{i=1}^{\varepsilon^{-1}} \beta(x_i)\mathbb{1}_{V_i}(x),\ \alpha(x) := \sum_{i=1}^{\varepsilon^{-1}} \alpha(x_i)\mathbb{1}_{V_i}(x)\end{aligned}$$

, and we set

$$\mathbf{X}_\varepsilon := (\mathcal{S}_\varepsilon\,,\ \mathcal{I}_\varepsilon\,,\ \mathcal{R}_\varepsilon)^{\mathrm{T}}. \tag{6}$$

We introduce the canonical projection $P_\varepsilon : L^2(\mathbb{T}^1) \longrightarrow \mathrm{H}_\varepsilon$ defined by

$$f \longmapsto P_\varepsilon f(x) = \varepsilon^{-1}\int_{V_i} f(y)dy,\ \text{if}\ \ x \in V_i.$$

Throughout this paper, we assume that the initial condition satisfies

Assumption 1.1 $\mathbf{s}(0,.)$, $\mathbf{i}(0,.)$, $\mathbf{r}(0,.) \in C^1(\mathbb{T}^1)$, $\forall x \in \mathbb{T}^1$, $\mathcal{S}_\varepsilon(0,x) = P_\varepsilon\mathbf{s}(0,x)$, $\mathcal{I}_\varepsilon(0,x) = P_\varepsilon\mathbf{i}(0,x)$, $\mathcal{R}_\varepsilon(0,x) = P_\varepsilon\mathbf{r}(0,x)$, and $\int_{\mathbb{T}^1}(\mathbf{s}(0,x) + \mathbf{i}(0,x) + \mathbf{r}(0,x))\,dx = 1$.

Assumption 1.2 There exists a constant $c > 0$ such that $\inf_{x\in\mathbb{T}^1} \mathbf{s}(0, x) \geq c$.

We use the notation $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$ to denote the supremun norm of f in $[0, 1]$ and define $\left\|\left(f, g, h\right)^{\mathrm{T}}\right\|_\infty := \|f\|_\infty + \|g\|_\infty + \|h\|_\infty$.

We have the

Theorem 1.2 *For all* $T > 0$, $\sup_{0\leq t\leq T} \left\|\mathbf{X}_\varepsilon(t) - \mathbf{X}(t)\right\|_\infty \longrightarrow 0$, *as* $\varepsilon \to 0$.

Next, defining $\mathcal{S}_{\mathbf{N},\varepsilon}(t, x) := \sum_{i=1}^{\varepsilon^{-1}} S_{\mathbf{N},\varepsilon}(t, x_i)\mathbb{1}_{V_i}(x)$, $\mathcal{I}_{\mathbf{N},\varepsilon}(t, x) := \sum_{i=1}^{\varepsilon^{-1}} I_{\mathbf{N},\varepsilon}(t, x_i)$ $\mathbb{1}_{V_i}(x)$,

$\mathcal{R}_{\mathbf{N},\varepsilon}(t, x) := \sum_{i=1}^{\varepsilon^{-1}} R_{\mathbf{N},\varepsilon}(t, x_i)\mathbb{1}_{V_i}(x)$, and setting $\mathbf{X}_{\mathbf{N},\varepsilon} := \left(\mathcal{S}_{\mathbf{N},\varepsilon}\,,\, \mathcal{I}_{\mathbf{N},\varepsilon}\,,\, \mathcal{R}_{\mathbf{N},\varepsilon}\right)^{\mathrm{T}}$, the following theorem is proved in N'zi et al. [10].

Theorem 1.3 *Let us assume that* $(\varepsilon, \mathbf{N}) \to (0, \infty)$, *in such way that*

(i) $\dfrac{\mathbf{N}}{\log(1/\varepsilon)} \longrightarrow \infty$ *as* $\mathbf{N} \to \infty$ *and* $\varepsilon \to 0$*;*

(ii) $\left\|\mathbf{X}_{\mathbf{N},\varepsilon}(0) - \mathbf{X}(0)\right\|_\infty \longrightarrow 0$ *in probability.*

Then for all $T > 0$, $\sup_{0\leq t\leq T} \left\|\mathbf{X}_{\mathbf{N},\varepsilon}(t) - \mathbf{X}(t)\right\|_\infty \longrightarrow 0$ *in probability* .

We devote this paper to study the deviation of the stochastic model from the deterministic one as the mesh size of the grid goes to zero. In this work, we focus our attention to the periodic boundary conditions on the unit interval [0, 1], which we denote by $\mathbb{T}^1$. Let us mention that Blount [2] and Kotelenez [7] described similar spatial model for chemical reactions. The resulting process has one component and is compared with the corresponding deterministic model. They proved a functional central limit theorem under some restriction on the respective speeds of convergence of the initial number of particles in each cell and the number of cells.
The rest of this paper is organized as follows. In Sect. 2, we give some notations and preliminaries which will be useful in the sequel of this paper. In Sect 3, we establish a functional central limit theorem, the main result of this paper, by letting the mesh size ε of the grid go to zero. The fluctuation limit is a distribution valued generalized Ornstein–Uhlenbeck Gaussian process and can be represented as the solution of a linear stochastic partial differential equation, whose driving terms are Gaussian martingales.

2 Notations and preliminaries

In this section, we give some notations and collect some standard facts on the Sobolev spaces $\mathrm{H}^\gamma(\mathbb{T}^1)$, $\gamma \in \mathbb{R}$. First of all, let us describe some of the properties of the (discrete)-Laplace operator. Let $\mathrm{H}_\varepsilon \subset L^2(\mathbb{T}^1)$ denote the space of real valued step functions that are constant on each cell V_i. For $f \in \mathrm{H}_\varepsilon$, let us define

$$\nabla_\varepsilon^+ f(x_i) := \frac{f(x_i+\varepsilon) - f(x_i)}{\varepsilon} \text{ and } \nabla_\varepsilon^- f(x_i) := \frac{f(x_i) - f(x_i-\varepsilon)}{\varepsilon}.$$

For $f, g \in L^2(\mathbb{T}^1)$, $\langle f, g \rangle := \int_{\mathbb{T}^1} f(x)g(x)dx$ denotes the scalar product in $\mathrm{L}^2(\mathbb{T}^1)$. It is not hard to see that

$$\langle \nabla_\varepsilon^+ f, g \rangle = -\langle f, \nabla_\varepsilon^- g \rangle \text{ and } \Delta_\varepsilon f = \nabla_\varepsilon^- \nabla_\varepsilon^+ f = \nabla_\varepsilon^+ \nabla_\varepsilon^- f.$$

For m even and $x \in \mathbb{R}$ we define

$$\varphi_m(x) := \begin{cases} 1, & \text{for } m = 0 \\ \sqrt{2}\cos(m\pi x), & \text{for } m \neq 0 \text{ and even,} \end{cases}$$

$$\psi_m(x) := \begin{cases} 0, & \text{for } m = 0 \\ \sqrt{2} sin(m\pi x), & \text{for } m \neq 0 \text{ and even.} \end{cases}$$

$\{1, \varphi_m, \psi_m, m = 2k, k \geq 1\}$ is a complete orthonormal system (CONS) of eigenvectors of Δ in $L^2(\mathbb{T}^1)$ with eigenvalues $-\lambda_m = -\pi^2 m^2$. Consequently, the semigroup $\mathsf{T}(t) := \exp(\Delta t)$ acting on $L^2(\mathbb{T}^1)$ generated by Δ can be represented as

$$\mathsf{T}(t)f = \langle f, 1 \rangle + \sum_{k\geq 1} \exp(-\lambda_{2k} t)\Big[\langle f, \varphi_{2k} \rangle \varphi_{2k} + \langle f, \psi_{2k} \rangle \psi_{2k}\Big], \quad f \in L^2(\mathbb{T}^1).$$

Assume that ε^{-1} is an odd integer. For $m \in \{0, 2, \cdots, \varepsilon^{-1} - 1\}$, we define $\varphi_m^\varepsilon(x) = \sqrt{2}\cos(\pi m j \varepsilon)$, if $x \in V_j$ and $\psi_m^\varepsilon(x) = \sqrt{2}\sin(\pi m j \varepsilon)$, if $x \in V_j$. $\{\varphi_m^\varepsilon, \psi_m^\varepsilon, m\}$ form an orthonormal basis of H_ε as a subspace of $L^2(\mathbb{T}^1)$. These vectors are eigenfunctions of Δ_ε with the associated eigenvalues $-\lambda_m^\varepsilon = -2\varepsilon^{-2}\big(1 - \cos(m\pi\varepsilon)\big)$. Note that $\lambda_m^\varepsilon \longrightarrow \lambda_m$, as $\varepsilon \to 0$. Basic computations show that there exists a constant c, such that for each m and ε, $\varepsilon^{-2}\big(1-\cos(\pi m\varepsilon)\big) > c\, m^2$. Let us set $n_\varepsilon = \frac{\varepsilon^{-1}-1}{2}$. Δ_ε generates a contraction semigroup $\mathsf{T}_\varepsilon(t) := \exp(\Delta_\varepsilon t)$ whose action on each $f \in \mathrm{H}_\varepsilon$ is given by

$$\mathsf{T}_\varepsilon(t)f = \sum_{k=0}^{n_\varepsilon} \exp(-\lambda_{2k}^\varepsilon t)\Big[\langle f, \varphi_{2k}^\varepsilon \rangle \varphi_{2k}^\varepsilon + \langle f, \psi_{2k}^\varepsilon \rangle \psi_{2k}^\varepsilon\Big]. \tag{7}$$

Note that both Δ_ε and $\mathsf{T}_\varepsilon(t)$ are self-adjoint and that $\mathsf{T}_\varepsilon(t)\Delta_\varepsilon\varphi = \Delta_\varepsilon \mathsf{T}_\varepsilon(t)\varphi$.
For any $J \in \{S, I, R\}$, the semigroup generated by $\mu_J \Delta$ is $\mathsf{T}(\mu_J t)$. In the sequel, we will use the notation $\mathsf{T}_J(t) := \mathsf{T}(\mu_J t)$ and similarly, in the discrete case, we will use the notation $\mathsf{T}_{\varepsilon,J}(t) := \mathsf{T}_\varepsilon(\mu_J t)$. Also, for any $J \in \{S, I, R\}$, we set $\lambda_{m,J} := \mu_J \lambda_m$ and $\lambda_{m,J}^\varepsilon := \mu_J \lambda_m^\varepsilon$. For $\gamma \in \mathbb{R}_+$, we define the Hilbert space $\mathrm{H}^\gamma(\mathbb{T}^1)$ as follows.

$$\mathrm{H}^\gamma(\mathbb{T}^1) := \Big\{ f \in L^2(\mathbb{T}^1), \|f\|_{\mathrm{H}^\gamma}^2 := \sum_{m \text{ even}} \big[\langle f, \varphi_m \rangle^2 + \langle f, \psi_m \rangle^2\big](1+\lambda_m)^\gamma < \infty \Big\}.$$

We shall use the notations $\mathrm{H}^\gamma := \mathrm{H}^\gamma(\mathbb{T}^1)$ and $L^2 := L^2(\mathbb{T}^1)$.
Note that $\|\varphi\|_{\mathrm{H}^\gamma} = \|(\mathbf{I}-\Delta)^{\gamma/2}\varphi\|_{L^2}$, where $\mathbf{I}$ is the identity operator on $L^2\left(\mathbb{T}^1\right)$. For any three-dimensional vector-valued function $\Phi = (\Phi_1, \Phi_2, \Phi_3)^{\mathrm{T}}$, we use the notation $\|\Phi\|_{\mathrm{H}^\gamma} := \left(\|\Phi_1\|^2_{\mathrm{H}^\gamma} + \|\Phi_2\|^2_{\mathrm{H}^\gamma} + \|\Phi_3\|^2_{\mathrm{H}^\gamma}\right)^{1/2}$.
For $\gamma \in \mathbb{R}$, we also define

$$\|f\|_{\mathrm{H}^{\gamma,\varepsilon}} := \left[\sum_{m \text{ even}} \left(\langle f, \varphi_m^\varepsilon\rangle^2 + \langle f, \psi_m^\varepsilon\rangle^2\right)(1+\lambda_m^\varepsilon)^\gamma\right]^{1/2}, \quad f \in \mathrm{H}_\varepsilon.$$

For f, $g \in \mathrm{H}_\varepsilon$, we have

$$\left|\langle f, g\rangle\right| \le \|f\|_{\mathrm{H}^{-\gamma,\varepsilon}} \|g\|_{\mathrm{H}^{\gamma,\varepsilon}}, \quad \gamma \ge 0. \tag{8}$$

Elementary calculation shows that for $f \in \mathrm{H}_\varepsilon$, and $\gamma > 0$ there exist positive constants $c_1(\gamma)$ and $c_2(\gamma)$ such that for all $\varepsilon > 0$

$$c_1(\gamma)\|f\|_{\mathrm{H}^{-\gamma,\varepsilon}} \le \|f\|_{\mathrm{H}^{-\gamma}} \le c_2(\gamma)\|f\|_{\mathrm{H}^{-\gamma,\varepsilon}}. \tag{9}$$

$f' := \dfrac{\partial f}{\partial x}$ will denote the derivative of f.

In the sequel of this paper we may use the same notation for different constants (we use the generic notation C for a positive constant). These constants can depend upon some parameters of the model, as long as these are independent of ε and $\mathbf{N}$, we will not necessarily mention this dependence explicitly. However, we use $C(\gamma, T)$ to denote a constant which depends on γ and T (and possibly on some unimportant constants). The exact value may change from line to line.

Let us now consider the deviation of the stochastic model around its determinsitc law of large numbers limit. To this end we introduce the rescaled difference between $Z_{\mathbf{N},\varepsilon}(t)$ and Z_ε, namely

$$\Psi_{\mathbf{N},\varepsilon}(t) := \begin{pmatrix} U_{\mathbf{N},\varepsilon}(t) \\ V_{\mathbf{N},\varepsilon}(t) \\ W_{\mathbf{N},\varepsilon}(t) \end{pmatrix},$$

where

$$U_{\mathbf{N},\varepsilon}(t) := \begin{pmatrix} \sqrt{\mathbf{N}}\Big(S_{\mathbf{N},\varepsilon}(t,x_1) - S_\varepsilon(t,x_1)\Big) \\ \vdots \\ \sqrt{\mathbf{N}}\Big(S_{\mathbf{N},\varepsilon}(t,x_\ell) - S_\varepsilon(t,x_\ell)\Big) \end{pmatrix},$$

$$V_{\mathbf{N},\varepsilon}(t) := \begin{pmatrix} \sqrt{\mathbf{N}}\Big(I_{\mathbf{N},\varepsilon}(t,x_1) - I_\varepsilon(t,x_1)\Big) \\ \vdots \\ \sqrt{\mathbf{N}}\Big(I_{\mathbf{N},\varepsilon}(t,x_\ell) - I_\varepsilon(t,x_\ell)\Big) \end{pmatrix}$$

and

$$W_{\mathbf{N},\varepsilon}(t) := \begin{pmatrix} \sqrt{\mathbf{N}}\Big(R_{\mathbf{N},\varepsilon}(t,x_1) - R_\varepsilon(t,x_1)\Big) \\ \vdots \\ \sqrt{\mathbf{N}}\Big(R_{\mathbf{N},\varepsilon}(t,x_\ell) - R_\varepsilon(t,x_\ell)\Big) \end{pmatrix}.$$

In the sequel, we denote by " $\Longrightarrow$ " weak convergence. By fixing the mesh size ε of the grid and letting $\mathbf{N}$ go to infinity, we obtain the following theorem.

Theorem 2.1 *(**Central Limit Theorem** : $N \to \infty$, ε **being fixed**)*
Assume that $\sqrt{\mathbf{N}}\big(Z_{\mathbf{N},\varepsilon}(0) - Z_\varepsilon(0)\big) \longrightarrow 0$*, as* $\mathbf{N} \to \infty$.
Then, as $\mathbf{N} \to +\infty$, $\big\{\Psi_{\mathbf{N},\varepsilon}(t),\ t \geq 0\big\} \Longrightarrow \big\{\Psi_\varepsilon(t),\ t \geq 0\big\}$*, for the topology of locally uniform convergence, where the limit process* $\Psi_\varepsilon(t) := \begin{pmatrix} U_\varepsilon(t) \\ V_\varepsilon(t) \\ W_\varepsilon(t) \end{pmatrix}$ *satisfies*

$$\Psi_\varepsilon(t) = \int_0^t \nabla_z b_\varepsilon\big(r, Z_\varepsilon(r)\big)\Psi_\varepsilon(r)dr + \sum_{j=1}^K \int_0^t \sqrt{\beta_j\big(r, Z_\varepsilon(r)\big)}dB_j(r), \quad t \geq 0, \tag{10}$$

and $\{B_1(t), B_2(t), \cdots, B_K(t)\}$ *are mutually independent standard Brownian motions. More precisely, by setting* $A_\varepsilon = S_\varepsilon + I_\varepsilon + R_\varepsilon$*, for any site* x_i*, the limit* $(U_\varepsilon, V_\varepsilon, W_\varepsilon)^{\mathrm{T}}$ *satisfies the following system*

$$\begin{aligned}
U_\varepsilon(t,x_i) &= \mu_S \int_0^t \Delta_\varepsilon U_\varepsilon(r,x_i)dr \\
&\quad - \int_0^t \beta(x_i)\frac{I_\varepsilon(r,x_i)\big(I_\varepsilon(r,x_i) + R_\varepsilon(r,x_i)\big)V_\varepsilon(r,x_i)}{A_\varepsilon^2(r,x_i)}dr \\
&\quad - \int_0^t \beta(x_i)\frac{S_\varepsilon(r,x_i)\big(S_\varepsilon(r,x_i) + R_\varepsilon(r,x_i)\big)U_\varepsilon(r,x_i)}{A_\varepsilon^2(r,x_i)}dr \\
&\quad + \int_0^t \sqrt{\beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}}\ dB_{x_i}^{inf}(r) \\
&\quad - \sum_{y_i \sim x_i} \int_0^t \sqrt{\frac{\mu_S}{\varepsilon^2}S_\varepsilon(r,x_i)}\ dB_{x_i y_i}^S(r) + \sum_{y_i \sim x_i} \int_0^t \sqrt{\frac{\mu_S}{\varepsilon^2}S_\varepsilon(r,y_i)}\ dB_{y_i x_i}^S(r), \\
V_\varepsilon(t,x_i) &= \mu_I \int_0^t \Delta_\varepsilon V_\varepsilon(r,x_i)dr + \int_0^t \beta(x_i)\frac{I_\varepsilon(r,x_i)\big(I_\varepsilon(r,x_i) + R_\varepsilon(r,x_i)\big)V_\varepsilon(r,x_i)}{A_\varepsilon^2(r,x_i)}dr \\
&\quad + \int_0^t \beta(x_i)\frac{S_\varepsilon(r,x_i)\big(S_\varepsilon(r,x_i) + R_\varepsilon(r,x_i)\big)U_\varepsilon(r,x_i)}{A_\varepsilon^2(r,x_i)}dr - \int_0^t \alpha(x_i)V_\varepsilon(r,x_i)dr \\
&\quad - \int_0^t \sqrt{\beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon^2(r,x_i)}}\ dB_{x_i}^{inf}(r) + \int_0^t \sqrt{\alpha(x_i)I_\varepsilon(r,x_i)}\ dB_{x_i}^{rec}(r) \\
&\quad - \sum_{y_i \sim x_i} \int_0^t \sqrt{\frac{\mu_I}{\varepsilon^2}I_\varepsilon(r,x_i)}dB_{x_i y_i}^I(r) + \sum_{y_i \sim x_i} \int_0^t \sqrt{\frac{\mu_I}{\varepsilon^2}I_\varepsilon(r,y_i)}\ dB_{y_i x_i}^I(r),
\end{aligned}$$

$$W_\varepsilon(t,x_i) = \mu_R \int_0^t \Delta_\varepsilon W_\varepsilon(r,x_i)dr + \int_0^t \alpha(x_i)V_\varepsilon(r,x_i)dr - \int_0^t \sqrt{\alpha(x_i)I_\varepsilon(r,x_i)}\ dB^{rec}_{x_i}(r)$$
$$- \sum_{y_i \sim x_i} \int_0^t \sqrt{\frac{\mu_R}{\varepsilon^2} R_\varepsilon(r,x_i)}\ dB^R_{x_i y_i}(r) + \sum_{y_i \sim x_i} \int_0^t \sqrt{\frac{\mu_R}{\varepsilon^2} R_\varepsilon(r,y_i)}\ dB^R_{y_i x_i}(r),$$

where $\{B^{inf}_{x_i} : x_i \in \mathrm{D}_\varepsilon\}$, $\{B^{rec}_{x_i} : x_i \in \mathrm{D}_\varepsilon\}$, $\{B^{S}_{x_i y_i} : y_i \sim x_i \in \mathrm{D}_\varepsilon\}$, $\{B^{I}_{x_i y_i} : y_i \sim x_i \in \mathrm{D}_\varepsilon\}$ *and* $\{B^{R}_{x_i y_i} : y_i \sim x_i \in \mathrm{D}_\varepsilon\}$ *are families of independent Brownian motions.*

Theorem 2.1 is a special case of Theorem 3.5 of Kurtz [9] (see also Theorem 2.3.2 in Britton and Pardoux [3]). Then, here, we do not give the proof and refer the reader to those references for a complete proof. □

Let $\mathbf{X} = (\mathbf{s}\,,\ \mathbf{i}\,,\ \mathbf{r})^{\mathrm{T}}$ satisfying the system (2) on [0, 1]. Thanks to Proposition 1.1 of Taylor [11] (chapter 15, section 1) we have the following lemma.

Lemma 2.1 *Let* $\gamma \geq 0$ *and assume that the initial data* $\mathbf{X}(0)$ *bebong to* $(\mathrm{H}^\gamma)^3$*, then the parabolic system* (2) *has a unique solution* $\mathbf{X} \in C\big([0,T];(\mathrm{H}^\gamma)^3\big)$.

The rest of this section is devoted to the proof of some estimates for the solution of the system of equations (1). We first note that $S_\varepsilon(t,x_i) \geq 0$, $I_\varepsilon(t,x_i) \geq 0$, $R_\varepsilon(t,x_i) \geq 0$ for all $t \geq 0$, $x_i \in D_\varepsilon$ and $\varepsilon > 0$. Moreover for any $T > 0$, there exists a contant C_T such that

$$\sup_{0\leq t\leq T} (\|S_\varepsilon(t)\|_\infty \vee \|I_\varepsilon(t)\|_\infty \vee \|R_\varepsilon(t)\|_\infty) \leq C_T,\ \forall \varepsilon > 0\,. \tag{11}$$

Indeed we first note that $\|S_\varepsilon(t)\|_\infty \leq M$, since S_ε is upper bounded by the solution of the ODE

$$\frac{dX_\varepsilon}{dt}(t,x_i) = \mu_S \Delta_\varepsilon X_\varepsilon(t,x_i), \quad X_\varepsilon(0,x_i) = M\,. \tag{12}$$

Next $I_\varepsilon(t,x_i)$ is upper bounded by the solution of the ODE (with $\bar\beta := \sup_x \beta(x)$)

$$\frac{dY_\varepsilon}{dt}(t,x_i) = \mu_I \Delta_\varepsilon Y_\varepsilon(t,x_i) + \bar\beta Y_\varepsilon(t,x_i), \quad Y_\varepsilon(0,x_i) = M\,.$$

The result for R_ε is now easy.

Let us set $\mathcal{A}_\varepsilon := \mathcal{S}_\varepsilon + \mathcal{I}_\varepsilon + \mathcal{R}_\varepsilon$. We have the

Lemma 2.2 *For any* $T > 0$*, there exists a positive constant* c_T *such that*

$$\mathcal{A}_\varepsilon(t,x) \geq c_T, \quad \textit{for any } \varepsilon > 0,\ 0 \leq t \leq T,\ x \in \mathbb{T}^1\,.$$

Proof We consider the ODE

$$\frac{d\mathcal{S}_\varepsilon}{dt}(t,x) = \mu_S \Delta_\varepsilon \mathcal{S}_\varepsilon(t,x) - \frac{\beta(x)\mathcal{S}_\varepsilon(t,x)\mathcal{I}_\varepsilon(t,x)}{\mathcal{S}_\varepsilon(t,x) + \mathcal{I}_\varepsilon(t,x) + \mathcal{R}_\varepsilon(t,x)}.$$

Since $\mathcal{S}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)\ge 0$ and $\mathcal{I}_\varepsilon(t,x)\ge 0$, it is plain that

$$0\le \frac{\beta(x)\mathcal{I}_\varepsilon(t,x)}{\mathcal{S}_\varepsilon(t,x)+\mathcal{I}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)}\le \overline{\beta},\quad \text{where } \overline{\beta}:=\sup_{x\in\mathbb{T}^1}|\beta(x)|.$$

Define $\overline{\mathcal{S}}_\varepsilon(t,x)=e^{\overline{\beta}t}\mathcal{S}_\varepsilon(t,x)$. We have

$$\frac{d\overline{\mathcal{S}}_\varepsilon}{dt}(t,x)=\mu_S\Delta_\varepsilon\overline{\mathcal{S}}_\varepsilon(t,x)+\left(\overline{\beta}-\frac{\beta(x)\mathcal{I}_\varepsilon(t,x)}{\mathcal{S}_\varepsilon(t,x)+\mathcal{I}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)}\right)\overline{\mathcal{S}}_\varepsilon(t,x).$$

Combining this with the last inequality, we deduce that

$$\overline{\mathcal{S}}_\varepsilon(t,x)\ge [e^{t\mu_S\Delta_\varepsilon}\overline{\mathcal{S}}_\varepsilon(0,\cdot)](x)\ge c,$$

from Assumption 1.2.

Going back to $\mathcal{S}_\varepsilon$, we note that we have proved that

$$\mathcal{S}_\varepsilon(t,x)\ge ce^{-\overline{\beta}t}.$$

In other words, for any $T>0$, there exists a constant $c_T:=ce^{-\overline{\beta}T}$ which is such that

$$\mathcal{S}_\varepsilon(t,x)\ge c_T,\qquad \text{for any } \varepsilon>0,\ 0\le t\le T,\ x\in\mathbb{T}^1.$$

And since $I_\varepsilon(t,x_i)+R_\varepsilon(t,x_i)\ge 0$, $\mathcal{A}_\varepsilon(t,x)$ satisfies the same lower bound. □

Lemma 2.3 *For any $T>0$, there exists a constant C such that for each $\varepsilon>0$*

$$\sup_{0\le t\le T}\left(\left\|\mathcal{S}_\varepsilon(t)\right\|_{L^2}^2+\left\|\mathcal{I}_\varepsilon(t)\right\|_{L^2}^2+\left\|\mathcal{R}_\varepsilon(t)\right\|_{L^2}^2\right)$$
$$+2\int_0^T\left(\mu_S\left\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(r)\right\|_{L^2}^2+\mu_I\left\|\nabla_\varepsilon^+\mathcal{I}_\varepsilon(r)\right\|_{L^2}^2+\mu_R\left\|\nabla_\varepsilon^+\mathcal{R}_\varepsilon(r)\right\|_{L^2}^2\right)dr\le C. \tag{13}$$

Proof For all $(t,x)\in[0,T]\times[0,1]$, we have

$$\frac{d\,\mathcal{S}_\varepsilon}{dt}(t,x)=\mu_S\,\Delta_\varepsilon\mathcal{S}_\varepsilon(t,x)-\frac{\beta(x)\,\mathcal{S}_\varepsilon(t,x)\mathcal{I}_\varepsilon(t,x)}{\mathcal{A}_\varepsilon(t,x)},$$

which implies

$$2\left\langle\mathcal{S}_\varepsilon(t),\,\frac{d\,\mathcal{S}_\varepsilon}{dt}(t)\right\rangle=2\mu_S\left\langle\Delta_\varepsilon\mathcal{S}_\varepsilon(t),\,\mathcal{S}_\varepsilon(t)\right\rangle-2\left\langle\frac{\beta(.)\,\mathcal{S}_\varepsilon(t)\mathcal{I}_\varepsilon(t)}{\mathcal{A}_\varepsilon(t)},\,\mathcal{S}_\varepsilon(t)\right\rangle$$
$$=-2\mu_S\left\langle\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t),\,\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t)\right\rangle-2\left\langle\frac{\beta(.)\,\mathcal{S}_\varepsilon(t)\mathcal{I}_\varepsilon(t)}{\mathcal{A}_\varepsilon(t)},\,\mathcal{S}_\varepsilon(t)\right\rangle.$$

Then, $\forall\, t \in [0, T]$,

$$\left\| \mathcal{S}_\varepsilon(t) \right\|_{L^2}^2 + 2\mu_S \int_0^t \left\| \nabla_\varepsilon^+ \mathcal{S}_\varepsilon(r) \right\|_{L^2}^2 dr = \left\| \mathcal{S}_\varepsilon(0) \right\|_{L^2}^2 - 2 \int_0^t \Big\langle \frac{\beta(.)\, \mathcal{S}_\varepsilon(r) \mathcal{I}_\varepsilon(r)}{\mathcal{A}_\varepsilon(r)}\, ,\, \mathcal{S}_\varepsilon(r) \Big\rangle dr.$$

In the same way, we obtain

$$\begin{aligned} &\left\| \mathcal{I}_\varepsilon(t) \right\|_{L^2}^2 + 2\mu_I \int_0^t \left\| \nabla_\varepsilon^+ \mathcal{I}_\varepsilon(r) \right\|_{L^2}^2 dr \\ &\quad = \left\| \mathcal{I}_\varepsilon(0) \right\|_{L^2}^2 + 2 \int_0^t \Big\langle \frac{\beta(.)\, \mathcal{S}_\varepsilon(r) \mathcal{I}_\varepsilon(r)}{\mathcal{A}_\varepsilon(r)}\, ,\, \mathcal{I}_\varepsilon(r) \Big\rangle dr \\ &\quad -2 \int_0^t \big\langle \alpha(.) \mathcal{I}_\varepsilon(r)\, ,\, \mathcal{I}_\varepsilon(r) \big\rangle dr\, , \end{aligned}$$

and

$$\left\| \mathcal{R}_\varepsilon(t) \right\|_{L^2}^2 + 2\mu_R \int_0^t \left\| \nabla_\varepsilon^+ \mathcal{R}_\varepsilon(r) \right\|_{L^2}^2 dr = \left\| \mathcal{R}_\varepsilon(0) \right\|_{L^2}^2 + 2 \int_0^t \big\langle \alpha(.) \mathcal{I}_\varepsilon(r)\, ,\, \mathcal{R}_\varepsilon(r) \big\rangle dr.$$

Then, we deduce that

$$\begin{aligned} &\left\| \mathcal{S}_\varepsilon(t) \right\|_{L^2}^2 + \left\| \mathcal{I}_\varepsilon(t) \right\|_{L^2}^2 + \left\| \mathcal{R}_\varepsilon(t) \right\|_{L^2}^2 + 2 \int_0^t \\ &\quad \left(\mu_S \left\| \nabla_\varepsilon^+ \mathcal{S}_\varepsilon(r) \right\|_{L^2}^2 + \mu_I \left\| \nabla_\varepsilon^+ \mathcal{I}_\varepsilon(r) \right\|_{L^2}^2 + \mu_R \left\| \nabla_\varepsilon^+ \mathcal{R}_\varepsilon(r) \right\|_{L^2}^2 \right) dr \\ &\quad \le \left\| \mathcal{S}_\varepsilon(0) \right\|_{L^2}^2 + \left\| \mathcal{I}_\varepsilon(0) \right\|_{L^2}^2 + \left\| \mathcal{R}_\varepsilon(0) \right\|_{L^2}^2 \\ &\qquad + \int_0^t \left((2\overline{\beta} + \overline{\alpha}) \left\| \mathcal{I}_\varepsilon(r) \right\|_{L^2}^2 + \overline{\alpha} \left\| \mathcal{R}_\varepsilon(r) \right\|_{L^2}^2 \right) dr, \end{aligned}$$

where $\overline{\alpha} = \sup\limits_{x \in \mathbb{T}^1} |\alpha(x)|$.

It then follows from Gronwall's lemma that

$$\begin{aligned} &\left\| \mathcal{S}_\varepsilon(t) \right\|_{L^2}^2 + \left\| \mathcal{I}_\varepsilon(t) \right\|_{L^2}^2 + \left\| \mathcal{R}_\varepsilon(t) \right\|_{L^2}^2 \\ &\quad +2 \int_0^t \left(\mu_S \left\| \nabla_\varepsilon^+ \mathcal{S}_\varepsilon(r) \right\|_{L^2}^2 + \mu_I \left\| \nabla_\varepsilon^+ \mathcal{I}_\varepsilon(r) \right\|_{L^2}^2 + \mu_R \left\| \nabla_\varepsilon^+ \mathcal{R}_\varepsilon(r) \right\|_{L^2}^2 \right) dr \\ &\le \left(\left\| \mathcal{S}_\varepsilon(0) \right\|_{L^2}^2 + \left\| \mathcal{I}_\varepsilon(0) \right\|_{L^2}^2 + \left\| \mathcal{R}_\varepsilon(0) \right\|_{L^2}^2 \right) e^{C(\overline{\alpha}, \overline{\beta})} \\ &\le C(\overline{\alpha}, \overline{\beta})\, . \end{aligned}$$

□

We now add the following assumption.

Assumption 2.1 The functions β, α satisfy $\alpha \in C^1(\mathbb{T}^1)$ and $\beta \in C^2(\mathbb{T}^1)$.

Let $f_\varepsilon(t,x) := \beta(x)\frac{\mathcal{S}_\varepsilon(t,x)\left[\mathcal{S}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)\right]}{\mathcal{A}_\varepsilon^2(t,x)}$,
and $g_\varepsilon(t,x) := \beta(x)\frac{\mathcal{I}_\varepsilon(t,x)\left[\mathcal{I}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)\right]}{\mathcal{A}_\varepsilon^2(t,x)}$.

Lemma 2.4 *For any $T > 0$, there exists a positive constant C such that for all $\varepsilon > 0$,*

$$\int_0^T \left(\left\|\nabla_\varepsilon^+ f_\varepsilon(t)\right\|_{L^2}^2 + \left\|\nabla_\varepsilon^+ g_\varepsilon(t)\right\|_{L^2}^2\right) dt \le C. \tag{14}$$

Proof $\forall x \in \mathbb{T}^1$, $\forall t \ge 0$ we have

$$\begin{aligned}
&\nabla_\varepsilon^+ f_\varepsilon(t,x)\\
&\quad = -\frac{\beta(x+\varepsilon)\mathcal{S}_\varepsilon(t,x+\varepsilon)\left[\mathcal{S}_\varepsilon(t,x+\varepsilon)+\mathcal{R}_\varepsilon(t,x+\varepsilon)\right]\left[\mathcal{A}_\varepsilon(t,x+\varepsilon)+\mathcal{A}_\varepsilon(t,x)\right]\nabla_\varepsilon^+\mathcal{A}_\varepsilon(t,x)}{\mathcal{A}_\varepsilon^2(t,x)\mathcal{A}_\varepsilon^2(t,x+\varepsilon)}\\
&\qquad + \frac{\beta(x+\varepsilon)\mathcal{S}_\varepsilon(t,x+\varepsilon)}{\mathcal{A}_\varepsilon^2(t,x)}\nabla_\varepsilon^+\left(\mathcal{S}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)\right)\\
&\qquad + \frac{\beta(x+\varepsilon)\left[\mathcal{S}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)\right]}{\mathcal{A}_\varepsilon^2(t,x)}\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t,x)\\
&\qquad + \frac{\mathcal{S}_\varepsilon(t,x)\left[\mathcal{S}_\varepsilon(t,x)+\mathcal{R}_\varepsilon(t,x)\right]}{\mathcal{A}_\varepsilon^2(t,x)}\nabla_\varepsilon^+\beta(x),
\end{aligned} \tag{15}$$

from which we obtain

$$\begin{aligned}
\int_0^T\int_{\mathbb{T}^1}\left|\nabla_\varepsilon^+ f_\varepsilon(t,x)\right|^2 dxdt \le C\int_0^T\int_{\mathbb{T}^1}\\
\left(\left|\nabla_\varepsilon^+\beta(x)\right|^2 + \left|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t,x)\right|^2\right.\\
\left. + \left|\nabla_\varepsilon^+\mathcal{R}_\varepsilon(t,x)\right|^2 + \left|\nabla_\varepsilon^+\mathcal{A}_\varepsilon(t,x)\right|^2\right) dxdt,
\end{aligned}$$

where we have used Assumption 2.1, inequality (11) and Lemma 2.2. The result now follows from Lemma 2.3.

□

Lemma 2.5 *For any $T > 0$, there exists a positive constant C such that*

$$\begin{aligned}
\sup_{0\le t\le T}\ &\left(\|\nabla_\varepsilon^+ S_\varepsilon(t)\|_\infty \vee \|\nabla_\varepsilon^+ I_\varepsilon(t)\|_\infty \vee \|\nabla_\varepsilon^+\right.\\
&\left. R_\varepsilon(t)\|_\infty \vee \|\nabla_\varepsilon^+ f_\varepsilon(t)\|_\infty \vee \|\nabla_\varepsilon^+ g_\varepsilon(t)\|_\infty\right) \le C,
\end{aligned} \tag{16}$$

$$\int_0^T \left(\left\|\Delta_\varepsilon\mathcal{S}_\varepsilon(t)\right\|_{L^2}^2 + \left\|\Delta_\varepsilon\mathcal{I}_\varepsilon(t)\right\|_{L^2}^2 + \left\|\Delta_\varepsilon\mathcal{R}_\varepsilon(t)\right\|_{L^2}^2\right) dt \le C, \tag{17}$$

$$\int_0^T \left(\left\|\Delta_\varepsilon f_\varepsilon(t)\right\|_{L^2}^2 + \left\|\Delta_\varepsilon g_\varepsilon(t)\right\|_{L^2}^2\right) dt \le C, \tag{18}$$

and

$$\sup_{0\le t\le T}\left(\left\|f_\varepsilon(t)\right\|_{\mathrm{H}^{1,\varepsilon}}\vee\left\|g_\varepsilon(t)\right\|_{\mathrm{H}^{1,\varepsilon}}\right)\le C. \tag{19}$$

Proof We first etablish (16). Applying the operator ∇_ε^+ to the first equation in (1), we get

$$\frac{d\nabla_\varepsilon^+\mathcal{S}_\varepsilon}{dt}(t,x)=\mu_S\Delta_\varepsilon\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t,x)-\nabla_\varepsilon^+\left(\frac{\beta\mathcal{S}_\varepsilon\mathcal{I}_\varepsilon}{\mathcal{A}_\varepsilon}\right)(t,x) \tag{20}$$

The last term on the above right hand side is easily explicited thanks to a computation similar to that done in (15). Combining that formula with Assumption 2.1, inequality (11) and Lemma 2.2, we deduce that

$$\left\|\nabla_\varepsilon^+\left(\frac{\beta\mathcal{S}_\varepsilon\mathcal{I}_\varepsilon}{\mathcal{A}_\varepsilon}\right)(t)\right\|_\infty\le C\left(\left\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t)\right\|_\infty+\left\|\nabla_\varepsilon^+\mathcal{I}_\varepsilon(t)\right\|_\infty+\left\|\nabla_\varepsilon^+\mathcal{R}_\varepsilon(t)\right\|_\infty\right).$$

From the Duhamel formula,

$$\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t)=e^{t\mu_S\Delta_\varepsilon}\nabla_\varepsilon^+\mathcal{S}_\varepsilon(0)+\int_0^t e^{(t-s)\mu_S\Delta_\varepsilon}\nabla_\varepsilon^+\left(\frac{\beta\mathcal{S}_\varepsilon\mathcal{I}_\varepsilon}{\mathcal{A}_\varepsilon}\right)(s)ds$$

Since the semigroup $e^{t\mu_S\Delta_\varepsilon}$ is contracting in L^∞, we deduce that

$$\begin{aligned}\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t)\|_\infty\le\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(0)\|_\infty+C\int_0^t\big(&\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(s)\|_\infty+\|\nabla_\varepsilon^+\mathcal{I}_\varepsilon(s)\|_\infty\\&+\|\nabla_\varepsilon^+\mathcal{R}_\varepsilon(s)\|_\infty\big)\,ds\,.\end{aligned}$$

Applying similar arguments to the two other equations in (1), we obtain

$$\begin{aligned}&\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t)\|_\infty+\|\nabla_\varepsilon^+\mathcal{I}_\varepsilon(t)\|_\infty+\|\nabla_\varepsilon^+\mathcal{R}_\varepsilon(t)\|_\infty\\&\quad\le\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(0)\|_\infty+\|\nabla_\varepsilon^+\mathcal{I}_\varepsilon(0)\|_\infty+\|\nabla_\varepsilon^+\mathcal{R}_\varepsilon(0)\|_\infty\\&\quad\quad+C\int_0^t\left(\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(s)\|_\infty+\|\nabla_\varepsilon^+\mathcal{I}_\varepsilon(s)\|_\infty+\|\nabla_\varepsilon^+\mathcal{R}_\varepsilon(s)\|_\infty\right)ds\,.\end{aligned}$$

(16) now follows from Gronwall's Lemma and Assumption 1.1.

We now multiply (20) by $\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t,x)$ and integrate on $[0,t]\times\mathbb{T}^1$, yielding

$$\begin{aligned}\|\nabla_\varepsilon^+\mathcal{S}_\varepsilon(t)\|_{L^2}+2\mu_S\int_0^t\|\Delta_\varepsilon\mathcal{S}_\varepsilon(s)\|_{L^2}^2ds&=2\int_0^t\left(\frac{\beta\mathcal{S}_\varepsilon\mathcal{I}_\varepsilon}{\mathcal{A}_\varepsilon}(s),\,\Delta_\varepsilon\mathcal{S}_\varepsilon(s)\right)ds\\&\le Ct+\mu_S\int_0^t\|\Delta_\varepsilon\mathcal{S}_\varepsilon(s)\|_{L^2}^2ds\,,\end{aligned}$$

which yields one third of (17). The rest of (17) is proved by similar computations applied to the equations for $\nabla_\varepsilon^+\mathcal{I}_\varepsilon$ and $\nabla_\varepsilon^+\mathcal{R}_\varepsilon$. Next (18) follows from (17), (16), Assumption 2.1, (11) and Lemma 2.2.

Since

$$\| f_\varepsilon(t)\|^2_{\mathrm{H}^{1,\varepsilon}} \leq C\left(\| f_\varepsilon(t)\|^2_{L^2} + \|\nabla_\varepsilon^+ f_\varepsilon(t)\|^2_{L^2}\right),$$

the estimate (19) follows from (16), Assumption 2.1, inequality (11), Lemma 2.2 and the fact that the norm in $L^2(\mathbb{T}^1)$ is bounded by the norm in $L^\infty(\mathbb{T}^1)$.

□

Lemma 2.6 *For any $T > 0$, as $\varepsilon \to 0$*

$$f_\varepsilon \longrightarrow f, \quad g_\varepsilon \longrightarrow g, \quad \nabla_\varepsilon^+ f_\varepsilon \longrightarrow \nabla f, \textit{ and}$$
$$\nabla_\varepsilon^+ g_\varepsilon \longrightarrow \nabla g in C\left([0,T]; L^2(\mathbb{T}^1)\right),$$

where $f(t,x) = \dfrac{\mathbf{s}(t,x)\,[\mathbf{s}(t,x)+\mathbf{r}(t,x)]}{\mathbf{a}^2(t,x)}$ *and* $g(t,x) = \dfrac{\mathbf{i}(t,x)\,[\mathbf{i}(t,x)+\mathbf{r}(t,x)]}{\mathbf{a}^2(t,x)}$, $\forall t \in [0,T],\ x \in \mathbb{T}^1$.

Moreover $f, g \in L^2\left(0,T;\mathrm{H}^1\right)$.

Proof Let d be the function such that , $\forall\, t \in [0,T]$, $x \in \mathbb{T}^1$ and $\varepsilon > 0$

$$f_\varepsilon(t,x) = d\big(\mathcal{S}_\varepsilon(t,x), \mathcal{I}_\varepsilon(t,x), \mathcal{R}_\varepsilon(t,x)\big) \quad \text{and} \quad f(t,x) = d\big(\mathbf{s}(t,x), \mathbf{i}(t,x), \mathbf{r}(t,x)\big).$$

Furthermore, we know that $\mathcal{S}_\varepsilon \longrightarrow \mathbf{s}$, $\mathcal{I}_\varepsilon \longrightarrow \mathbf{i}$ and $\mathcal{R}_\varepsilon \longrightarrow \mathbf{r}$ uniformly on $[0,T]\times\mathbb{T}^1$. Since d is continuous on $\{(s,i,r) \in (\mathbb{R}_+)^3 : s+i+r > 0\}$, then we deduce that $f_\varepsilon \longrightarrow f$ uniformly on $[0,T]\times\mathbb{T}^1$, and in particular in $C\left([0,T]; L^2(\mathbb{T}^1)\right)$.

From (20) and similar equations for $\nabla_\varepsilon^+\mathcal{I}_\varepsilon(t,x)$ and $\nabla_\varepsilon^+\mathcal{R}_\varepsilon(t,x)$, we obtain the convergence of $\nabla_\varepsilon^+ f_\varepsilon \longrightarrow \nabla f$ by an argument similar to the previous one.

The proofs of $g_\varepsilon \longrightarrow g$ and $\nabla_\varepsilon^+ g_\varepsilon \longrightarrow \nabla g$ are obtained in the same way.

□

In the sequel, we will write "$f_\varepsilon(t) \longrightarrow f(t)$ in H^1" to mean that "$f_\varepsilon(t) \longrightarrow f(t)$ in $L^2(\mathbb{T}^1)$ and $\nabla_\varepsilon^+ f_\varepsilon(t) \longrightarrow \nabla f(t)$ in $L^2(\mathbb{T}^1)$".

We have the following compactness result.

Lemma 2.7 (Theorem 1.69 of Bahouri et al. [1], page 47)

For any compact subset E of $\mathbb{R}^d$ and $s_1 < s_2$, the embedding of $\mathrm{H}^{s_2}(E)$ into $\mathrm{H}^{s_1}(E)$ is a compact linear operator.

In the next section, we study the behavior of the process $\{\Psi_\varepsilon, 0 < \varepsilon < 1\}$ as ε goes to zero.

3 Functional central limit theorem

Let us define $\mathscr{U}_\varepsilon(t,x) = \frac{1}{\varepsilon^{1/2}}\sum_{i=1}^{\varepsilon^{-1}} U_\varepsilon(t,x_i)\mathbb{1}_{V_i}(x)$, $\mathscr{V}_\varepsilon(t,x) = \frac{1}{\varepsilon^{1/2}}\sum_{i=1}^{\varepsilon^{-1}} V_\varepsilon(t,x_i)$ $\mathbb{1}_{V_i}(x)$,

$$\mathscr{W}_\varepsilon(t,x) = \frac{1}{\varepsilon^{1/2}}\sum_{i=1}^{\varepsilon^{-1}} W_\varepsilon(t,x_i)\mathbb{1}_{V_i}(x).$$

Moreover, we set

$$\begin{aligned}
\mathscr{M}_\varepsilon^S(t,x) &= \int_0^t \varepsilon^{-1/2}\sum_{i=1}^{\varepsilon^{-1}}\sqrt{\beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}}\mathbb{1}_{V_i}(x)\;\; dB_{x_i}^{inf}(r)\\
&\quad+\sqrt{\mu_S}\int_0^t \varepsilon^{-1/2}\sum_{i=1}^{\varepsilon^{-1}}\sum_{\substack{i\,,\,j\\ x_i\sim x_j}}\sqrt{S_\varepsilon(r,x_i)}\frac{\left(\mathbb{1}_{V_j}(x)-\mathbb{1}_{V_i}(x)\right)}{\varepsilon}dB_{x_ix_j}^S(r),\\
\mathscr{M}_\varepsilon^I(t,x) &= -\int_0^t \varepsilon^{-1/2}\sum_{i=1}^{\varepsilon^{-1}}\sqrt{\beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}}\mathbb{1}_{V_i}(x)\;\; dB_{x_i}^{inf}(r)\\
&\quad+\int_0^t \varepsilon^{-1/2}\sum_{i=1}^{\varepsilon^{-1}}\sqrt{\alpha(x_i)I_\varepsilon(r,x_i)}\mathbb{1}_{V_i}(x)\; dB_{x_i}^{rec}(r)\\
&\quad+\sqrt{\mu_I}\int_0^t \varepsilon^{-1/2}\sum_{i=1}^{\varepsilon^{-1}}\sum_{\substack{i\,,\,j\\ x_i\sim x_j}}\sqrt{I_\varepsilon(r,x_i)}\frac{\left(\mathbb{1}_{V_j}(x)-\mathbb{1}_{V_i}(x)\right)}{\varepsilon}dB_{x_ix_j}^S(r),\\
\mathscr{M}_\varepsilon^R(t,x) &= -\int_0^t \varepsilon^{-1/2}\sum_{i=1}^{\varepsilon^{-1}}\sqrt{\alpha(x_i)I_\varepsilon(r,x_i)}\mathbb{1}_{V_i}(x)\; dB_{x_i}^{rec}(r)\\
&\quad+\sqrt{\mu_R}\int_0^t \varepsilon^{-1/2}\sum_{i=1}^{\varepsilon^{-1}}\sum_{\substack{i\,,\,j\\ x_i\sim x_j}}\sqrt{R_\varepsilon(r,x_i)}\frac{\left(\mathbb{1}_{V_j}(x)-\mathbb{1}_{V_i}(x)\right)}{\varepsilon}dB_{x_ix_j}^S(r).
\end{aligned}$$

$(\mathscr{U}_\varepsilon, \mathscr{V}_\varepsilon, \mathscr{W}_\varepsilon)$ satisfies the following system

$$\begin{cases} \mathscr{U}_\varepsilon(t) = \displaystyle\int_0^t \mu_S \Delta_\varepsilon \mathscr{U}_\varepsilon(r)dr - \int_0^t \beta(.) \frac{\mathcal{I}_\varepsilon(r)\big(\mathcal{I}_\varepsilon(r) + \mathcal{R}_\varepsilon(r)\big)\mathscr{V}_\varepsilon(r)}{\mathcal{A}_\varepsilon^2(r)} dr \\ \qquad - \displaystyle\int_0^t \beta(.) \frac{\mathcal{S}_\varepsilon(r)\big(\mathcal{S}_\varepsilon(r) + \mathcal{R}_\varepsilon(r)\big)\mathscr{U}_\varepsilon(r)}{\mathcal{A}_\varepsilon^2(r)} dr + \mathscr{M}_\varepsilon^S(t), \\ \mathscr{V}_\varepsilon(t) = \displaystyle\int_0^t \mu_I \Delta_\varepsilon \mathscr{V}_\varepsilon(r)dr + \int_0^t \beta(.) \frac{\mathcal{I}_\varepsilon(r)\big(\mathcal{I}_\varepsilon(r) + \mathcal{R}_\varepsilon(r)\big)\mathscr{V}_\varepsilon(r)}{\mathcal{A}_\varepsilon^2(r)} dr \\ \qquad + \displaystyle\int_0^t \beta(.) \frac{\mathcal{S}_\varepsilon(r)\big(\mathcal{S}_\varepsilon(r) + \mathcal{R}_\varepsilon(r)\big)\mathscr{U}_\varepsilon(r)}{\mathcal{A}_\varepsilon^2(r)} dr - \int_0^t \alpha(.)\mathscr{V}_\varepsilon(r)dr + \mathscr{M}_\varepsilon^I(t), \\ \mathscr{W}_\varepsilon(t) = \displaystyle\int_0^t \mu_R \Delta_\varepsilon \mathscr{W}_\varepsilon(r)dr + \int_0^t \alpha(.)\mathscr{V}_\varepsilon(r)dr + \mathscr{M}_\varepsilon^R(t). \end{cases} \tag{21}$$

For $\gamma \in \mathbb{R}_+$, we denote by $C\big([0,T];\mathrm{H}^{-\gamma}\big)$ the complete separable metric space of continuous functions defined on $[0,T]$ with values in $\mathrm{H}^{-\gamma}$. For any $\varepsilon > 0$, $\mathscr{U}_\varepsilon$, $\mathscr{V}_\varepsilon$ and $\mathscr{W}_\varepsilon$ can be viewed as continuous processes taking values in some Hilbert space $\mathrm{H}^{-\gamma}$. Hence we will study the weak convergence of the process $(\mathscr{U}_\varepsilon, \mathscr{V}_\varepsilon, \mathscr{W}_\varepsilon)$ in $C\big([0,T];(\mathrm{H}^{-\gamma})^3\big)$.

In the sequel we will need to control the stochastic convolution integrals $\displaystyle\int_0^t \mathsf{T}_{\varepsilon,J}(t-r)d\mathscr{M}_\varepsilon^J(r)$, with $J \in \{S, I, R\}$. For that sake, we shall need a maximal inequality which is a special case of Theorem 2.1 of Kotelenez [8], which we first recall.

Lemma 3.1 (Kotelenez [8]) *Let* $(\mathrm{H}\,; \|.\|_\mathrm{H})$ *be a separable Hilbert space,* $\mathcal{M}$ *an* H*-valued locally square integrable càdlàg martingale and* $\mathsf{T}(t)$ *a contraction semigroup operator of* $\mathcal{L}(\mathrm{H})$*. Then, there is a finite constant c depending only on the Hilbert norm* $\|.\|_\mathrm{H}$ *such that for all* $T \geq 0$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left\|\int_0^t \mathsf{T}(t-r)d\mathcal{M}(r)\right\|_\mathrm{H}^2\right) \leq c\, e^{4\sigma T}\mathbb{E}\left(\left\|\mathcal{M}(T)\right\|_\mathrm{H}^2\right), \tag{22}$$

where σ *is a real number such that* $\|\mathsf{T}(t)\|_{\mathcal{L}(\mathrm{H})} \leq e^{\sigma t}$.

We want to take the limit as $\epsilon \to 0$ in the system of SDEs (21) satisfied by $\mathscr{Y}_\varepsilon$. To this end we will split our system into two subsystems.

First, we consider the following linear system

$$\begin{cases} du_\varepsilon(t) = \mu_S\, \Delta_\varepsilon u_\varepsilon(t)dt + d\mathscr{M}_\varepsilon^S(t), \\ dv_\varepsilon(t) = \mu_I\, \Delta_\varepsilon v_\varepsilon dt + d\mathscr{M}_\varepsilon^I(t), \\ dw_\varepsilon(t) = \mu_R\, \Delta_\varepsilon w_\varepsilon(t)dt + d\mathscr{M}_\varepsilon^R(t), \\ u_\varepsilon(0) = v_\varepsilon(0) = w_\varepsilon(0) = 0. \end{cases} \tag{23}$$

Next, we shall consider the second system

$$
\begin{cases}
\dfrac{d\overline{u}_\varepsilon}{dt}(t) = \mu_S\, \Delta_\varepsilon \overline{u}_\varepsilon(t) - f_\varepsilon(t)\overline{u}_\varepsilon(t) - g_\varepsilon(t)\overline{v}_\varepsilon(t) - f_\varepsilon(t)u_\varepsilon(t) - g_\varepsilon(t)v_\varepsilon(t), \\
\dfrac{d\overline{v}_\varepsilon}{dt}(t) = \mu_I\, \Delta_\varepsilon \overline{v}_\varepsilon(t) + f_\varepsilon(t)\overline{u}_\varepsilon(t) + (g_\varepsilon(t)-\alpha)\overline{v}_\varepsilon(t) + f_\varepsilon(t)u_\varepsilon(t) \\
\qquad\qquad + (g_\varepsilon(t)-\alpha)v_\varepsilon(t), \\
\dfrac{d\overline{w}_\varepsilon}{dt}(t) = \mu_R\, \Delta_\varepsilon \overline{w}_\varepsilon + \alpha\left(v_\varepsilon + \overline{v}_\varepsilon\right), \\
\overline{u}_\varepsilon(0) = \overline{v}_\varepsilon(0) = \overline{w}_\varepsilon(0) = 0,
\end{cases}
\tag{24}
$$

and finally, we note that

$$
\mathscr{U}_\varepsilon = u_\varepsilon + \overline{u}_\varepsilon, \quad \mathscr{V}_\varepsilon = u_\varepsilon + \overline{v}_\varepsilon, \quad \mathscr{W}_\varepsilon = w_\varepsilon + \overline{w}_\varepsilon.
$$

Then the convergence of $\mathscr{Y}_\varepsilon := (\mathscr{U}_\varepsilon, \mathscr{V}_\varepsilon, \mathscr{W}_\varepsilon)$ will follow from both the convergence of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ and of $(\overline{u}_\varepsilon, \overline{v}_\varepsilon, \overline{w}_\varepsilon)$.

Let us first look at the convergence of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$.

Let $\mathscr{M}_\varepsilon = \left(\mathscr{M}_\varepsilon^S, \mathscr{M}_\varepsilon^I, \mathscr{M}_\varepsilon^R\right)^{\mathrm{T}}$.

Recall that we denote by " $\Longrightarrow$ " the weak convergence.

Proposition 3.1 *For any $\gamma > 3/2$, the Gaussian martingale $\mathscr{M}_\varepsilon \Longrightarrow \mathscr{M} := \left(\mathscr{M}^S, \mathscr{M}^I, \mathscr{M}^R\right)^{\mathrm{T}}$ in $C\big([0,T]; (\mathrm{H}^{-\gamma})^3\big)$ as $\varepsilon \to 0$, where for all $\varphi \in \mathrm{H}^\gamma$*

$$
\begin{aligned}
\langle \mathscr{M}^S(t), \varphi \rangle &= -\int_0^t \int_{\mathbb{T}^1} \varphi(x) \sqrt{\frac{\beta(x)\mathbf{s}(r,x)\mathbf{i}(r,x)}{\mathbf{a}(r,x)}}\ \dot{W}_1(dr,dx) \\
&\quad -\sqrt{2\mu_S} \int_0^t \int_{\mathbb{T}^1} \varphi'(x)\sqrt{\mathbf{s}(r,x)}\ \dot{W}_2(dr,dx), \\
\langle \mathscr{M}^I(t), \varphi \rangle &= \int_0^t \int_{\mathbb{T}^1} \varphi(x) \sqrt{\frac{\beta(x)\mathbf{s}(r,x)\mathbf{i}(r,x)}{\mathbf{a}(r,x)}}\ \dot{W}_1(dr,dx) \\
&\quad + \int_0^t \int_{\mathbb{T}^1} \varphi(x)\sqrt{\alpha(x)\mathbf{i}(r,x)}\,\dot{W}_3(dr,dx) \\
&\quad -\sqrt{2\mu_I} \int_0^t \int_{\mathbb{T}^1} \varphi'(x)\sqrt{\mathbf{i}(r,x)}\ \dot{W}_4(dr,dx), \\
\langle \mathscr{M}^R(t), \varphi \rangle &= -\int_0^t \int_{\mathbb{T}^1} \varphi(x)\sqrt{\alpha(x)\mathbf{i}(r,x)}\,\dot{W}_3(dr,dx) \\
&\quad -\sqrt{2\mu_R} \int_0^t \int_{\mathbb{T}^1} \varphi'(x)\sqrt{\mathbf{r}(r,x)}\ \dot{W}_5(dr,dx),
\end{aligned}
$$

and $\dot{W}_1$, $\dot{W}_2$, $\dot{W}_3$, $\dot{W}_4$ and $\dot{W}_5$ are standard space-time white noises which are mutually independent.

Proof First, we are going to show that there exists a positive constant C independent of ε such

$$\sup_{0<\varepsilon<1} \mathbb{E}\left(\sup_{0\le t\le T} \left\|\mathscr{M}_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}}\right) \le C. \tag{25}$$

Recall that $\left\|\mathscr{M}_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}} := \left\|\mathscr{M}^S_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}} + \left\|\mathscr{M}^I_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}} + \left\|\mathscr{M}^R_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}}$. Applying Doob's inequality to the martingale $\mathscr{M}^S_\varepsilon$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T} \left\|\mathscr{M}^S_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}}\right) \le 4\mathbb{E}\left(\left\|\mathscr{M}^S_\varepsilon(T)\right\|^2_{\mathrm{H}^{-\gamma}}\right)\\
&\quad = 4\sum_{m \text{ even}} \mathbb{E}\left(\langle \mathscr{M}^S_\varepsilon(T), \mathbf{f}_m\rangle^2\right)(1+\lambda_m)^{-\gamma}, \quad \text{with } \mathbf{f}_m \in \{\varphi_m, \psi_m\}\\
&\quad = \frac{4}{\varepsilon}\int_0^T \sum_{m \text{ even}} \sum_{i=1}^{\varepsilon^{-1}} \frac{\beta(x_i)S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\left(\int_{V_i} \mathbf{f}_m(x)\,dx\right)^2 (1+\lambda_m)^{-\gamma}\,dr\\
&\qquad + \frac{4\mu_S}{\varepsilon}\int_0^T \sum_{m \text{ even}} \sum_{i=1}^{\varepsilon^{-1}} S_\varepsilon(r,x_i)\left[\left(\int_{V_i} \nabla^+_\varepsilon \mathbf{f}_m(x)\,dx\right)^2\right.\\
&\qquad \left.+\left(\int_{V_i} \nabla^-_\varepsilon \mathbf{f}_m(x)\,dx\right)^2\right](1+\lambda_m)^{-\gamma}\,dr.
\end{aligned}$$

But since $\dfrac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)} \le M$ (indeed $\dfrac{I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)} \le 1$ and $S_\varepsilon(r,x_i) \le M$, see (11) and the line which follows) and $\left|\nabla^\pm_\varepsilon \mathbf{f}_m(x)\right|^2 \le 2\pi^2 m^2$, then we obtain

$$\mathbb{E}\left(\sup_{0\le t\le T} \left\|\mathscr{M}^S_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}}\right) \le C(\overline{\beta}, \mu_S, T)\left(\sum_{m \text{ even}} \frac{1}{m^{2\gamma}} + \sum_{m \text{ even}} \frac{1}{m^{2(\gamma-1)}}\right).$$

Since $\displaystyle\sum_{m \text{ even}} \frac{1}{m^{2(\gamma-1)}} < \infty$ iff $\gamma > 3/2$, we then have

$$\sup_{0<\varepsilon<1} \mathbb{E}\left(\sup_{0\le t\le T} \left\|\mathscr{M}^S_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}}\right) \le C(\overline{\beta}, \mu_S, T), \quad \text{for all } \gamma > 3/2. \tag{26}$$

Similar inequalities hold for the martingales $\mathscr{M}^I_\varepsilon$ and $\mathscr{M}^R_\varepsilon$. Hence we obtain

$$\sup_{0<\varepsilon<1} \mathbb{E}\left(\sup_{0\le t\le T} \left\|\mathscr{M}_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma}}\right) \le C. \tag{27}$$

Inequality (27) and standard tightness criteria for martingales (see e.g. the proof of Theorem 3.1) implies that the martingale $\mathscr{M}_\varepsilon$ is tight in $C\big([0,T];(\mathrm{H}^{-\gamma})^3\big)$, with $\gamma > 3/2$.

In what follows $<\!<\mathscr{M}_\varepsilon^{S,\gamma_0}>\!>_t$ denotes the operator–valued increasing process associated to the $L^2(\mathbb{T}^1)$–valued martingale $\mathscr{M}_\varepsilon^{S,\gamma_0}(t)$, whose trace is the increasing process associated to the real valued submartingale $\|\mathscr{M}_\varepsilon^{S,\gamma_0}(t)\|^2_{L^2(\mathbb{T}^1)}$. Let $\varphi \in \mathrm{H}^\gamma$. We set $\mathscr{M}_\varepsilon^{S,\varphi} = \langle \mathscr{M}_\varepsilon^{S}, \varphi\rangle$. $\mathscr{M}_\varepsilon^{I,\varphi}$ and $\mathscr{M}_\varepsilon^{R,\varphi}$ are defined in the same way. $\forall\, t \in [0,T]$, we have

$$<\!<\mathscr{M}_\varepsilon^{S,\varphi}>\!>_t = \frac{1}{\varepsilon}\int_0^t \sum_{i=1}^{\varepsilon^{-1}} \beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\Big(\int_{V_i}\varphi(x)\,dx\Big)^2 dr$$
$$+\frac{\mu_S}{\varepsilon}\int_0^t \sum_{i=1}^{\varepsilon^{-1}} S_\varepsilon(r,x_i)\Big[\Big(\int_{V_i}\nabla_\varepsilon^{+}\varphi(x)\,dx\Big)^2 + \Big(\int_{V_i}\nabla_\varepsilon^{-}\varphi(x)\,dx\Big)^2\Big]dr.$$

We have

$$\frac{1}{\varepsilon}\int_0^t \sum_{i=1}^{\varepsilon^{-1}} \beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\Big(\int_{V_i}\varphi(x)\,dx\Big)^2 dr$$
$$= \frac{1}{\varepsilon}\int_0^t \sum_{i=1}^{\varepsilon^{-1}} \beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\Big(\int_{V_i}\varphi(x)\,dx\Big)\Big[\int_{V_i}\Big(\varphi(x)-\varphi(x_i)\Big)\,dx\Big]dr$$
$$+\int_0^t\int_{\mathbb{T}^1}\sum_{i=1}^{\varepsilon^{-1}} \beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\varphi(x)\varphi(x_i)\mathbb{1}_{Vx_i}(x)dxdr.$$

On the one hand we have

$$\Big|\frac{1}{\varepsilon}\sum_{i=1}^{\varepsilon^{-1}} \beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\Big(\int_{V_i}\varphi(x)\,dx\Big)\Big[\int_{V_i}\Big(\varphi(x)-\varphi(x_i)\Big)\,dx\Big]\Big|$$
$$\le C\varepsilon\big\|\varphi\big\|_{\mathrm{H}^\gamma}\int_{\mathbb{T}^1}\frac{\beta_\varepsilon(x)\mathcal{S}_\varepsilon(r,x)\mathcal{I}_\varepsilon(r,x)|\varphi(x)|}{\mathcal{A}_\varepsilon(r,x)}dx \longrightarrow 0,$$

because the quantity $\displaystyle\int_{\mathbb{T}^1}\frac{\beta_\varepsilon(x)\mathcal{S}_\varepsilon(r,x)\mathcal{I}_\varepsilon(r,x)|\varphi(x)|}{\mathcal{A}_\varepsilon(r,x)}dx$ is bounded uniformly in ε.

Hence $\displaystyle\frac{1}{\varepsilon}\int_0^t \sum_{i=1}^{\varepsilon^{-1}} \beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\Big(\int_{V_i}\varphi(x)\,dx\Big)\Big[\int_{V_i}\Big(\varphi(x)-\varphi(x_i)\Big)\,dx\Big] \longrightarrow$ 0, as $\varepsilon \to 0$.

On the other hand, the fact that $\sup_{0\le t\le T}\big\|\mathbf{X}_\varepsilon(t)-\mathbf{X}(t)\big\|_\infty \longrightarrow 0$, as $\varepsilon\to 0$, leads to

$$\Big|\int_{\mathbb{T}^1}\sum_{i=1}^{\varepsilon^{-1}}\beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\varphi(x)\varphi(x_i)\mathbb{1}_{Vx_i}(x)dx \\ -\int_{\mathbb{T}^1}\beta(x)\frac{\mathbf{s}(r,x)\mathbf{i}(r,x)}{\mathbf{a}(r,x)}\varphi^2(x)dx\Big| \longrightarrow 0.$$

This shows that

$$\frac{1}{\varepsilon}\int_0^t\sum_{i=1}^{\varepsilon^{-1}}\beta(x_i)\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\Big(\int_{V_i}\varphi(x)\,dx\Big)^2dr \\ \longrightarrow \int_0^t\int_{\mathbb{T}^1}\beta(x)\frac{\mathbf{s}(r,x)\mathbf{i}(r,x)}{\mathbf{a}(r,x)}\varphi^2(x)dxdr,$$

as $\varepsilon\to 0$. Similar computation shows that

$$\frac{\mu_S}{\varepsilon}\int_0^t\sum_{i=1}^{\varepsilon^{-1}}S_\varepsilon(r,x_i)\Big[\Big(\int_{V_i}\nabla_\varepsilon^+\varphi(x)\,dx\Big)^2+\Big(\int_{V_i}\nabla_\varepsilon^-\varphi(x)\,dx\Big)^2\Big]dr \\ \longrightarrow 2\,\mu_S\int_0^t\int_{\mathbb{T}^1}\mathbf{s}(r,x)\big(\varphi'(x)\big)^2dxdr,$$

from which we deduce that

$$<<\mathscr{M}_\varepsilon^{S,\varphi}>>_t \overset{\varepsilon\to 0}{\longrightarrow} \\ \int_0^t\int_{\mathbb{T}^1}\beta(x)\frac{\mathbf{s}(r,x)\mathbf{i}(r,x)}{\mathbf{a}(r,x)}\varphi^2(x)dxdr \\ +2\,\mu_S\int_0^t\int_{\mathbb{T}^1}\mathbf{s}(r,x)\big(\varphi'(x)\big)^2dxdr.$$

Hence, if $\dot W_1$, $\dot W_2$ and $\dot W_3$ are space-time white noises which are mutually independent, so the limit of the centered Gaussian martingale $\mathscr{M}_\varepsilon^{S,\varphi}(t)$ can be identified with

$$-\int_0^t\int_{\mathbb{T}^1}\varphi(x)\sqrt{\frac{\beta(x)\mathbf{s}(r,x)\mathbf{i}(r,x)}{\mathbf{a}(r,x)}}\;\dot W_1(dr,dx) \\ -\sqrt{2\mu_S}\int_0^t\int_{\mathbb{T}^1}\varphi'(x)\sqrt{\mathbf{s}(r,x)}\;\dot W_2(dr,dx).$$

In the same way

$$\mathscr{M}_\varepsilon^{I,\varphi}(t) \Longrightarrow \int_0^t \int_{\mathbb{T}^1} \varphi(x)\sqrt{\frac{\beta(x)\mathbf{s}(r,x)\mathbf{i}(r,x)}{\mathbf{a}(r,x)}}\ \dot{W}_1(dr,dx) + \int_0^t \int_{\mathbb{T}^1} \varphi(x)\sqrt{\alpha(x)\mathbf{i}(r,x)}\dot{W}_3(dr,dx) - \sqrt{2\mu_I}\int_0^t \int_{\mathbb{T}^1} \varphi'(x)\sqrt{\mathbf{i}(r,x)}\ \dot{W}_4(dr,dx)$$

and

$$\mathscr{M}_\varepsilon^{R,\varphi}(t) \Longrightarrow -\int_0^t \int_{\mathbb{T}^1} \varphi(x)\sqrt{\alpha(x)\mathbf{i}(r,x)}\dot{W}_3(dr,dx) - \sqrt{2\mu_R}\int_0^t \int_{\mathbb{T}^1} \varphi'(x)\sqrt{\mathbf{r}(r,x)}\ \dot{W}_5(dr,dx),$$

where $\dot{W}_3$, $\dot{W}_4$ and $\dot{W}_5$ are also space-time white noises which are mutually independent, and independent from $\dot{W}_1$, $\dot{W}_2$. ◻

Let set $\Im_\varepsilon = \big(u_\varepsilon\ ,\ v_\varepsilon\ ,\ w_\varepsilon\big)^{\mathrm{T}}$.
We need to check tightness of the sequence of process $\{\Im_\varepsilon(t)\ ,\ t\in[0,T]\ , 0<\varepsilon<1\}$.

Theorem 3.1 *For any $\gamma > 3/2$, the process $\{\Im_\varepsilon(t)\ ,\ t\in[0,T]\ , 0<\varepsilon<1\}$ is tight in $C\big([0,T]; (\mathrm{H}^{-\gamma})^3\big)$.*

Proof : We denote by $\mathcal{G}_\varepsilon^T$ the collection of $\mathcal{F}_t^\varepsilon$-stopping times $\overline{\tau}$ such that $\overline{\tau} \le T$. Following Aldous' tightness criterion (see Joffe and Metivier [5]), in oder to show that the process $\{\Im_\varepsilon(t)\ ,\ t\in[0,T]\ , 0<\varepsilon<1\}$ is tight in $C\big([0,T]; (\mathrm{H}^{-\gamma})^3\big)$, it suffices to establish the two following conditions:

[T] for $\frac{3}{2} < \gamma_0 < \gamma$, and $M > 0$ there exists C such that $\mathbb{P}\Big(\big\|\Im_\varepsilon(t)\big\|_{\mathrm{H}^{-\gamma_0}} \ge M\Big) \le C$, for all $t\in[0,T]$,

[A] $\lim\limits_{\theta\to 0}\lim\limits_{\varepsilon\to 0}\sup\limits_{\overline{\tau}\in\mathcal{G}_\varepsilon^{T-\theta}} \mathbb{E}\Bigg(\Big\|\Im_\varepsilon(\overline{\tau}+\theta) - \Im_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}}\Bigg) = 0.$

Let $\frac{3}{2} < \gamma_0 < \gamma$. Let us set $u_\varepsilon^{\gamma_0}(t,x) = (\mathbf{I}-\Delta_\varepsilon)^{-\gamma_0/2}u_\varepsilon(t,x)$. $\forall\, t\in[0,T]$, we have

$$\big\|u_\varepsilon(t)\big\|^2_{\mathrm{H}^{-\gamma_0}} = \big\langle u_\varepsilon^{\gamma_0}(t), u_\varepsilon^{\gamma_0}(t)\big\rangle.$$

If we define $\mathscr{M}_\varepsilon^{S,\gamma_0}(t) := (\mathbf{I}-\Delta_\varepsilon)^{-\gamma_0/2}\mathscr{M}_\varepsilon^{S}(t)$, since $\gamma_0 > 3/2$, it follows from (3.1) that $\mathscr{M}_\varepsilon^{S,\gamma_0}(t)$ is bounded as $\varepsilon\to 0$, as an $L^2(\mathbb{T}^1)$–valued martingale. Applying the Itô formula to $|u_\varepsilon^{\gamma_0}(t,x)|^2$ and integrating over $\mathbb{T}^1$ leads to

$$\left\|u_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma_0}} = -2\int_0^t \left\langle \nabla_\varepsilon^+ u_\varepsilon^{\gamma_0}(r), \nabla_\varepsilon^+ u_\varepsilon^{\gamma_0}(r)\right\rangle dr + 2\int_0^t \left\langle u_\varepsilon^{\gamma_0}(r), d\mathscr{M}_\varepsilon^{S,\gamma_0}(r)\right\rangle$$
$$+ \int_{\mathbb{T}^1} \langle\langle \mathscr{M}_\varepsilon^{S,\gamma_0}(., x)\rangle\rangle_t dx.$$

Letting $t = T$ and taking the expectation, we deduce that

$$\mathbb{E}(\left\|u_\varepsilon(T)\right\|^2_{\mathrm{H}^{-\gamma_0}}) + 2\mu_S\mathbb{E}\int_0^T \left\|\nabla_\varepsilon^+ u_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma_0}} dt = \mathbb{E}\left(\|\mathscr{M}_\varepsilon^{S,\gamma_0}(T)\|^2_{L^2}\right).$$

Next we want to take the supremum on $[0, T]$ in the previous identity. For that sake, we use the Burkholder Davis Gundy inequality, which implies that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \left\langle u_\varepsilon^{\gamma_0}(r), d\mathscr{M}_\varepsilon^{S,\gamma_0}(r)\right\rangle\right|\right] \le 3\mathbb{E}\sqrt{\langle\langle \int_0^\cdot \left\langle u_\varepsilon^{\gamma_0}(r), d\mathscr{M}_\varepsilon^{S,\gamma_0}(r)\right\rangle\rangle\rangle_T}$$
$$\le 3\mathbb{E}\left(\sup_{0\le t\le T}\|u_\varepsilon^{\gamma_0}(t)\|_{L^2}\sqrt{Tr\langle\langle \mathscr{M}_\varepsilon^{S,\gamma_0}\rangle\rangle_T}\right)$$
$$\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le t\le T}\|u_\varepsilon^{\gamma_0}(t)\|^2_{L^2}\right) + \frac{9}{2}\mathbb{E}(\|\mathscr{M}_\varepsilon^{S,\gamma_0}(T)\|^2_{L^2}).$$

We then obtain, thanks to (26),

$$\mathbb{E}\left(\sup_{0\le t\le T}\left\|u_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma_0}}\right) = 11\,\mathbb{E}\left(\sup_{0\le t\le T}\left\|\mathscr{M}_\varepsilon^{S,\gamma_0}(t)\right\|^2_{L^2}\right)$$
$$\le 44\, C(\overline{\beta}, \mu_S, T).$$

We also obtain similar inequalities for v_ε and w_ε. Hence there exists a constant C such that for all $\varepsilon > 0$,

$$\mathbb{E}\left(\sup_{0\le t\le T}\left\|u_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma_0}} + \sup_{0\le t\le T}\left\|v_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma_0}} + \sup_{0\le t\le T}\left\|w_\varepsilon(t)\right\|^2_{\mathrm{H}^{-\gamma_0}}\right)$$
$$+ 2\mathbb{E}\int_0^T \left[\mu_S\left\|\nabla_\varepsilon^+ u_\varepsilon(r)\right\|^2_{\mathrm{H}^{-\gamma_0}} + \mu_I\left\|\nabla_\varepsilon^+ v_\varepsilon(r)\right\|^2_{\mathrm{H}^{-\gamma_0}} + \mu_R\left\|\nabla_\varepsilon^+ w_\varepsilon(r)\right\|^2_{\mathrm{H}^{-\gamma_0}}\right] dr \le C. \tag{28}$$

Then **[T]** follows by using Markov's inequality.

Let $\theta > 0$ and $\overline{\tau} \in \mathcal{G}_\varepsilon^{T-\theta}$. We have

$$u_\varepsilon(\overline{\tau}+\theta) - u_\varepsilon(\overline{\tau}) = \left[\mathsf{T}_{\varepsilon,S}(\theta) - \mathbf{I}\right]u_\varepsilon(\overline{\tau}) + \int_{\overline{\tau}}^{\overline{\tau}+\theta} \mathsf{T}_{\varepsilon,S}(\overline{\tau}+\theta-r)d\mathscr{M}_\varepsilon^S(r).$$

So,

$$\mathbb{E}\Big(\Big\|u_\varepsilon(\overline{\tau}+\theta)-u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}}\Big) \le 2\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}}\Big) + 2\mathbb{E}\Big(\Big\|\int_{\overline{\tau}}^{\overline{\tau}+\theta}\mathsf{T}_{\varepsilon,S}(\overline{\tau}+\theta-r)d\mathscr{M}^S_\varepsilon(r)\Big\|^2_{\mathrm{H}^{-\gamma}}\Big).$$

Let us deal with each term separately. First using the inequality (9), there is a constant $C(\gamma)$ such that

$$\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}}\Big) \le C(\gamma)\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}\Big).$$

Let $3/2<\gamma'<\gamma$, and let c a positive constant. We have

$$\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}} = \sum_{\lambda^\varepsilon_m\ge c}\big\langle\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau}),\mathbf{f}^\varepsilon_m\big\rangle^2\big(1+\lambda^\varepsilon_m\big)^{-\gamma} + \sum_{\lambda^\varepsilon_m< c}\big\langle\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau}),\mathbf{f}^\varepsilon_m\big\rangle^2\big(1+\lambda^\varepsilon_m\big)^{-\gamma},$$

and

$$\begin{aligned}&\sum_{\lambda^\varepsilon_m\ge c}\big\langle\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau}),\mathbf{f}^\varepsilon_m\big\rangle^2\big(1+\lambda^\varepsilon_m\big)^{-\gamma}\\ &\quad\le(1+c)^{\gamma'-\gamma}\sum_{\lambda^\varepsilon_m\ge c}\big\langle\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau}),\mathbf{f}^\varepsilon_m\big\rangle^2\big(1+\lambda^\varepsilon_m\big)^{-\gamma'}\\ &\quad\le(1+c)^{\gamma'-\gamma}\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma'}}.\end{aligned}$$

Then

$$\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}}\Big) \le C(\gamma)(1+c)^{\gamma'-\gamma}\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma'}}\Big) + C(\gamma)\sum_{\lambda^\varepsilon_m<c}\big(e^{-\lambda^\varepsilon_m\theta}-1\big)^2\mathbb{E}\Big(\langle\, u_\varepsilon(\overline{\tau}),\mathbf{f}^\varepsilon_m\,\rangle^2\Big)\big(1+\lambda^\varepsilon_m\big)^{-\gamma}$$

On the one hand, since $\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma'}}\Big)\le C$, we can choose c large enough such that $C(\gamma)(1+c)^{\gamma'-\gamma}\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma'}}\Big)\le\varepsilon/2$. On the other hand, we have

$$\begin{aligned}
&\sum_{\lambda_m^\varepsilon<c}\big(e^{-\lambda_m^\varepsilon\theta}-1\big)^2\mathbb{E}\Big(\langle\, u_\varepsilon(\overline{\tau}),\mathbf{f}_m^\varepsilon\,\rangle^2\Big)\big(1+\lambda_m^\varepsilon\big)^{-\gamma}\\
&\qquad\le \sup_{\lambda_m^\varepsilon<c}\big(1-e^{-\lambda_n^\varepsilon\theta}\big)^2\sum_{\lambda_m^\varepsilon<c}\mathbb{E}\Big(\langle\, u_\varepsilon(\overline{\tau}),\mathbf{f}_m^\varepsilon\,\rangle^2\Big)\big(1+\lambda_m^\varepsilon\big)^{-\gamma}\\
&\qquad\le \sup_{\lambda_m^\varepsilon<c}\big(1-e^{-\lambda_m^\varepsilon\theta}\big)^2\mathbb{E}\Big(\big\|u_\varepsilon(\overline{\tau})\big\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}\Big)\\
&\qquad\le C(\gamma)\sup_{\lambda_m^\varepsilon<c}\big(1-e^{-\lambda_m^\varepsilon\theta}\big)^2\mathbb{E}\Big(\big\|u_\varepsilon(\overline{\tau})\big\|^2_{\mathrm{H}^{-\gamma}}\Big).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}\Big(\big\|u_\varepsilon(\overline{\tau})\big\|^2_{-\gamma}\Big)&\le\mathbb{E}\Big(\sup_{0\le t\le T}\big\|u_\varepsilon(t)\big\|^2_{-\gamma}\Big)\\
&\le C,
\end{aligned}$$

then for the previous choice of c, we can choose θ small enough such that $C(\gamma)\sup_{\lambda_m^\varepsilon<c}\big(1-e^{-\lambda_m^\varepsilon\theta}\big)^2\mathbb{E}\Big(\big\|u_\varepsilon(\overline{\tau})\big\|^2_{\mathrm{H}^{-\gamma}}\Big)\le\varepsilon/2$. Hence

$$\lim_{\theta\to0}\lim_{\varepsilon\to0}\sup_{\overline{\tau}\in\mathcal{G}_\varepsilon^{T-\theta}}\mathbb{E}\Big(\Big\|\big[\mathsf{T}_{\varepsilon,S}(\theta)-\mathbf{I}\big]u_\varepsilon(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}}\Big)=0\,.$$

Secondly, using the equivalence of the norms $\|.\|_{\mathrm{H}^{-\gamma}}$ and $\|.\|_{\mathrm{H}^{-\gamma,\varepsilon}}$, and the fact that $\mathsf{T}_{\varepsilon,S}$ is a contraction semigroup on $\mathtt{H}_\varepsilon$ we have

$$\begin{aligned}
&\mathbb{E}\Big(\Big\|\int_{\overline{\tau}}^{\overline{\tau}+\theta}\mathsf{T}_{\varepsilon,S}(\overline{\tau}+\theta-r)\mathscr{M}_\varepsilon^S(r)\Big\|^2_{\mathrm{H}^{-\gamma}}\Big)\\
&\quad=\mathbb{E}\Big(\Big\|\int_0^\theta\mathsf{T}_{\varepsilon,S}(\theta-r)d\mathscr{M}_\varepsilon^S(r+\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma}}\Big)\\
&\quad\le C(\gamma)\mathbb{E}\Big(\Big\|\mathscr{M}_\varepsilon^S(\overline{\tau}+\theta)-\mathscr{M}_\varepsilon^S(\overline{\tau})\Big\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}\Big)\\
&\quad\le 2C(\gamma)\mathbb{E}\Big(\Big\|\int_{\overline{\tau}}^{\overline{\tau}+\theta}\sum_{i=1}^{\varepsilon^{-1}}\frac{\sqrt{\beta(x_i)}}{\varepsilon}\sqrt{\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}}\mathbb{1}_{V_i}(.)dB_{x_i}(r)\Big\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}\Big)\\
&\qquad+2C(\gamma)\mathbb{E}\Big(\Big\|\int_{\overline{\tau}}^{\overline{\tau}+\theta}\sum_{\substack{i\,,\,j\\x_i\sim x_j}}\frac{\sqrt{\mu_S S_\varepsilon(r,x_i)}}{\varepsilon}\Big(\mathbb{1}_{V_j}(.)-\mathbb{1}_{V_i}(.)\Big)dB^S_{x_ix_j}(r)\Big\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}\Big)\\
&\quad\le\frac{2C(\gamma)\overline{\beta}}{\varepsilon}\mathbb{E}\Big(\int_{\overline{\tau}}^{\overline{\tau}+\theta}\sum_m\sum_{i=1}^{\varepsilon^{-1}}\frac{S_\varepsilon(r,x_i)I_\varepsilon(r,x_i)}{A_\varepsilon(r,x_i)}\Big(\int_{V_i}\mathbf{f}_m^\varepsilon(x)\,dx\Big)^2(1+\lambda_m^\varepsilon)^{-\gamma}dr\Big)
\end{aligned}$$

$$+\frac{2C(\gamma)\mu_S}{\varepsilon}\mathbb{E}\Big(\int_{\overline{\tau}}^{\overline{\tau}+\theta}\sum_m\sum_{i=1}^{\varepsilon^{-1}} S_\varepsilon(r,x_i)\Big(\int_{V_i}\nabla_\varepsilon^{\pm}\mathbf{f}_m^\varepsilon(x)\,dx\Big)^2(1+\lambda_m^\varepsilon)^{-\gamma}dr\Big)$$
$$\leq C(\overline{\beta},\mu_S)\,\theta \longrightarrow 0, \qquad \text{as } \theta\to 0.$$

Hence the condition **[A]** is proved.

In the way, we prove similar estimates for v_ε and w_ε. Then the process $\big\{\Im_\varepsilon(t),\ t\in[0,T], 0<\varepsilon<1\big\}$ is tight in $C\big([0,T];(\mathrm{H}^{-\gamma})^3\big)$, $\gamma>3/2$. □

Lemma 3.2 *For* $3/2<\gamma<2$, *the process* $\{\Im_\varepsilon(t),\ t\in[0,T],\ 0<\varepsilon<1\}$ *converges in law in* $C\big([0,T];\ (\mathrm{H}^{-\gamma})^3\big)\cap L^2\big(0,T;(\mathrm{H}^{-1})^3\big)$.

Proof On the one hand, from Theorem 3.1, the process $\{\Im_\varepsilon(t),\ t\in[0,T],\ 0<\varepsilon<1\}$ is tight in $C\big([0,T];\ (\mathrm{H}^{-\gamma})^3\big)$, then along a subsequence, it converges in $C\big([0,T];\ (\mathrm{H}^{-\gamma})^3\big)$. On the other hand the sequence $\{\Im_\varepsilon(t),\ t\in[0,T],\ 0<\varepsilon<1\}$ is bounded in $L^2\big(0,T\ ;\ (\mathrm{H}^{1-\gamma})^3\big)$. Indeed for all ε , we have

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^T\|u_\varepsilon(t)\|^2_{\mathrm{H}^{1-\gamma}}dt\Big)\\
&\quad\leq C(\gamma)\mathbb{E}\Big(\int_0^T\|u_\varepsilon(t)\|^2_{\mathrm{H}^{1-\gamma,\varepsilon}}dt\Big)\qquad\text{(by using the inequality (9))}\\
&\quad=C(\gamma)\sum_m\mathbb{E}\Big(\int_0^T\langle u_\varepsilon(t),\mathbf{f}_m^\varepsilon\rangle^2dt\Big)(1+\lambda_{m,S}^\varepsilon)^{1-\gamma}\\
&\quad=C(\gamma)\sum_m\mathbb{E}\Big(\int_0^T\langle u_\varepsilon(t),\mathbf{f}_m^\varepsilon\rangle^2dt\Big)(1+\lambda_{m,S}^\varepsilon)^{-\gamma}\\
&\qquad+C(\gamma)\sum_m\mathbb{E}\Big(\int_0^T\langle u_\varepsilon(t),\mathbf{f}_m^\varepsilon\rangle^2dt\Big)\lambda_{m,S}^\varepsilon(1+\lambda_{m,S}^\varepsilon)^{-\gamma}\\
&\quad=C(\gamma)\Big\{\mathbb{E}\Big(\int_0^T\|u_\varepsilon(t)\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}dt\Big)+\mathbb{E}\Big(\int_0^T\|\nabla_\varepsilon^+u_\varepsilon(t)\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}dt\Big)\Big\}\\
&\quad\leq C(\gamma)\Big\{\mathbb{E}\Big(\int_0^T\|u_\varepsilon(t)\|^2_{\mathrm{H}^{-\gamma}}dt\Big)+\mathbb{E}\Big(\int_0^T\|\nabla_\varepsilon^+u_\varepsilon(t)\|^2_{\mathrm{H}^{-\gamma}}dt\Big)\Big\},
\end{aligned}$$

where the third equality follows from the fact that

$$\|\nabla_\varepsilon^+u_\varepsilon(t)\|^2_{\mathrm{H}^{-\gamma,\varepsilon}}=\sum_m\langle u_\varepsilon(t),\mathbf{f}_m^\varepsilon\rangle^2\lambda_{m,S}^\varepsilon(1+\lambda_{m,S}^\varepsilon)^{-\gamma}$$
$$\times(\text{see }\ Lemma A.2(i)\ \text{ in the Appendix below}).$$

The inequality (28) ensures that $\mathbb{E}\int_0^T \Big[\big\|u_\varepsilon(t)\big\|^2_{\mathrm{H}^{-\gamma}}dt + \big\|\nabla_\varepsilon^+ u_\varepsilon(t)\big\|^2_{\mathrm{H}^{-\gamma}}\Big]dt$ is bounded by a constant independent of ε. It then follows that

$$\sup_{0<\varepsilon<1} \mathbb{E}\Big(\int_0^T \big\|u_\varepsilon(t)\big\|^2_{\mathrm{H}^{1-\gamma}} dt\Big) \le C(\gamma).$$

We have similar estimates for v_ε and w_ε. Thus

$$\sup_{0<\varepsilon<1} \mathbb{E}\Big(\int_0^T \big\|\Im_\varepsilon(t)\big\|^2_{\mathrm{H}^{1-\gamma}} dt\Big) \le C.$$

This implies that, from the sequence $\{\,\Im_\varepsilon(t),\ t \in [0, T],\ 0 < \varepsilon < 1\,\}$, we can extract a subsequence which converges in law in $L^2\big(0, T\ ;\ (\mathrm{H}^{1-\gamma})^3\big)$ endowed with the weak topology. Furthermore, since the imbedding of $\mathrm{H}^{1-\gamma}$ into H^{-1} is compact and we have the convergence in $C\left([0, T]\,; (\mathrm{H}^{-\gamma})^3\right)$, then the extracted sequence converges in fact in $L^2\big(0, T\ ;\ (\mathrm{H}^{-1})^3\big)$. Hence, we deduce that there exists a subsequence which converges in law in $C\big([0, T]\,;\ (\mathrm{H}^{-\gamma})^3\big) \cap L^2\big(0, T\ ;\ (\mathrm{H}^{-1})^3\big)$.

We note that the limit $\Im := (u, v, w)^{\mathrm{T}}$ of any convergent subsequence satisfies the following system of stochastic PDEs

$$\begin{cases} du(t) = \mu_S\, \Delta u(t)dt + d\mathscr{M}^S(t), \\ dv(t) = \mu_I\, \Delta v dt + d\mathscr{M}^I(t), \\ dw(t) = \mu_R\, \Delta w(t)dt + d\mathscr{M}^R(t), \end{cases} \tag{29}$$

and the solution of that system is unique. Then the whole process $\{\,\Im_\varepsilon(t),\ t \in [0, T],\ 0 < \varepsilon < 1\,\}$ converges in $C\big([0, T]\,;\ (\mathrm{H}^{-\gamma})^3\big) \cap L^2\big(0, T\ ;\ (\mathrm{H}^{-1})^3\big)$. □

Lemma 3.3 *As $\varepsilon \to 0$, $f_\varepsilon u_\varepsilon \Longrightarrow fu$, and $g_\varepsilon v_\varepsilon \Longrightarrow gv$ in $L^2\Big(0, T; \mathrm{H}^{-1}\Big)$.*

Proof The convergence $f_\varepsilon u_\varepsilon \Longrightarrow fu$ follows to the fact that $u_\varepsilon \Longrightarrow u$ in $L^2\big(0, T\ ;\ \mathrm{H}^{-1}\big)$ and $f_\varepsilon \longrightarrow f$ in $C\big([0, T]\ ;\ \mathrm{H}^1\big)$. The proof of the convergence $g_\varepsilon v_\varepsilon \Longrightarrow gv$ is similar. □

We are now interested in the convergence of the process $\overline{\Im}_\varepsilon := (\overline{u}_\varepsilon, \overline{v}_\varepsilon, \overline{w}_\varepsilon)$.

Lemma 3.4 *For any $T > 0$, there exists a positive constant C such that*

$$\sup_{0\le t\le T}\Big(\big\|\overline{u}_\varepsilon(t)\big\|^2_{L^2} + \big\|\overline{v}_\varepsilon(t)\big\|^2_{L^2} + \big\|\overline{w}_\varepsilon(t)\big\|^2_{L^2}\Big) + C\int_0^T \Big(\big\|\nabla_\varepsilon^+\overline{u}_\varepsilon(s)\big\|^2_{L^2} + \big\|\nabla_\varepsilon^+\overline{v}_\varepsilon(s)\big\|^2_{L^2} + \big\|\nabla_\varepsilon^+\overline{w}_\varepsilon(s)\big\|^2_{L^2}\Big)ds \le C\eta_T e^{CT}, \tag{30}$$

where $\eta_T := \int_0^T \Big(\big\|u_\varepsilon(s)\big\|^2_{\mathrm{H}^{-1}} + \big\|v_\varepsilon(s)\big\|^2_{\mathrm{H}^{-1}}\Big)ds.$

Proof For all $t \in [0, T]$, we have

$$\int_0^t \langle \frac{d\overline{u}_\varepsilon}{ds}(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds = \mu_S \int_0^t \langle \Delta_\varepsilon \overline{u}_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds - \int_0^t \langle f_\varepsilon(s)\overline{u}_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds$$
$$- \int_0^t \langle g_\varepsilon(s)\overline{v}_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds - \int_0^t \langle f_\varepsilon(s)u_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds$$
$$- \int_0^t \langle g_\varepsilon(s)v_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds.$$

Then

$$\begin{aligned}
&\left\|\overline{u}_\varepsilon(t)\right\|_{L^2}^2 + 2\mu_S \int_0^t \left\|\nabla_\varepsilon^+ \overline{u}_\varepsilon(s)\right\|_{L^2}^2 ds \\
&= -2\int_0^t \langle f_\varepsilon(s)\overline{u}_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds - 2\int_0^t \langle g_\varepsilon(s)\overline{v}_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds \\
&\quad - 2\int_0^t \langle f_\varepsilon(s)u_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds - 2\int_0^t \langle g_\varepsilon(s)v_\varepsilon(s)\ ,\ \overline{u}_\varepsilon(s)\rangle ds.
\end{aligned}$$

Since $f_\varepsilon(t)u_\varepsilon(t) \in \mathrm{H}^{-1}$ and $g_\varepsilon(t)v_\varepsilon(t) \in \mathrm{H}^{-1}$, then

$$\begin{aligned}
&\left\|\overline{u}_\varepsilon(t)\right\|_{L^2}^2 + 2\mu_S \int_0^t \left\|\nabla_\varepsilon^+ \overline{u}_\varepsilon(s)\right\|_{L^2}^2 ds \\
&\le 2 \sup_{0\le s\le T} \|f_\varepsilon(s)\|_\infty \int_0^t \left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2 ds \\
&\quad + \int_0^t \Big(\sup_{0\le s\le T} \|g_\varepsilon(s)\|_\infty^2 \left\|\overline{v}_\varepsilon(s)\right\|_{L^2}^2 + \left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2 \Big) ds \\
&\quad +2 \sup_{0\le s\le T} \left\|f_\varepsilon(s)\right\|_{\mathrm{H}^{1,\varepsilon}} \int_0^t \left[\left\|u_\varepsilon(s)\right\|_{\mathrm{H}^{-1}} \left(\left\|\overline{u}_\varepsilon(s)\right\|_{L^2} + \left\|\nabla_\varepsilon^+ \overline{u}_\varepsilon(s)\right\|_{L^2} \right) \right] ds \\
&\quad +2 \sup_{0\le s\le T} \left\|g_\varepsilon(s)\right\|_{\mathrm{H}^{1,\varepsilon}} \int_0^t \left[\left\|v_\varepsilon(s)\right\|_{\mathrm{H}^{-1}} \left(\left\|\overline{u}_\varepsilon(s)\right\|_{L^2} + \left\|\nabla_\varepsilon^+ \overline{u}_\varepsilon(s)\right\|_{L^2} \right) \right] ds.
\end{aligned}$$

Let δ be some constant such that $0 < \delta < \dfrac{\mu_S}{C}$. We have

$$\begin{aligned}
&\left\|\overline{u}_\varepsilon(t)\right\|_{L^2}^2 + 2\mu_S \int_0^t \left\|\nabla_\varepsilon^+ \overline{u}_\varepsilon(s)\right\|_{L^2}^2 ds \\
&\le C \int_0^t \left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2 ds + \int_0^t \Big(C \left\|\overline{v}_\varepsilon(s)\right\|_{L^2}^2 + \left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2 \Big) ds \\
&\quad +C \int_0^t \left[2\delta \left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2 + 2\delta \left\|\nabla_\varepsilon^+ \overline{u}_\varepsilon(s)\right\|_{L^2}^2 + \frac{2}{\delta} \left\|u_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2 + \frac{2}{\delta} \left\|v_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2 \right] ds.
\end{aligned}$$

Then

$$\begin{aligned}\left\|\overline{u}_\varepsilon(t)\right\|_{L^2}^2+2(\mu_S-C\delta)\int_0^t\left\|\nabla_\varepsilon^+\overline{u}_\varepsilon(s)\right\|_{L^2}^2ds\\ \le(C+2C\delta)\int_0^t\left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2ds+C\int_0^t\left\|\overline{v}_\varepsilon(s)\right\|_{L^2}^2\\ +\frac{2C}{\delta}\int_0^t\left(\left\|u_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2+\left\|v_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2\right)ds.\end{aligned}\tag{31}$$

In the same way, we prove that

$$\begin{aligned}\left\|\overline{v}_\varepsilon(t)\right\|_{L^2}^2+2(\mu_I-C\delta)\int_0^t\left\|\nabla_\varepsilon^+\overline{v}_\varepsilon(s)\right\|_{L^2}^2ds\\ \le(C+2C\delta)\int_0^t\left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2ds+C\int_0^t\left\|\overline{v}_\varepsilon(s)\right\|_{L^2}^2\\ +\frac{2C}{\delta}\int_0^t\left(\left\|u_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2+\left\|v_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2\right)ds,\end{aligned}\tag{32}$$

and

$$\begin{aligned}\left\|\overline{w}_\varepsilon(t)\right\|_{L^2}^2+2\mu_R\int_0^t\left\|\nabla_\varepsilon^+\overline{w}_\varepsilon(s)\right\|_{L^2}^2ds\\ \le\frac{C}{\delta}\int_0^t\left\|v_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2ds+C\int_0^t\left\|\overline{w}_\varepsilon(s)\right\|_{L^2}^2+C\int_0^t\left\|\overline{v}_\varepsilon(s)\right\|_{L^2}^2ds.\end{aligned}\tag{33}$$

By adding the inequalities (31) , (32) and (33), we obtain

$$\begin{aligned}&\left\|\overline{u}_\varepsilon(t)\right\|_{L^2}^2+\left\|\overline{v}_\varepsilon(t)\right\|_{L^2}^2+\left\|\overline{w}_\varepsilon(t)\right\|_{L^2}^2+C\int_0^t\left(\left\|\nabla_\varepsilon^+\overline{u}_\varepsilon(s)\right\|_{L^2}^2+\left\|\nabla_\varepsilon^+\overline{v}_\varepsilon(s)\right\|_{L^2}^2\right.\\ &\quad\left.+\left\|\nabla_\varepsilon^+\overline{w}_\varepsilon(s)\right\|_{L^2}^2\right)ds\\ &\le C\int_0^t\left(\left\|\overline{u}_\varepsilon(s)\right\|_{L^2}^2+\left\|\overline{v}_\varepsilon(s)\right\|_{L^2}^2+\left\|\overline{w}_\varepsilon(s)\right\|_{L^2}^2\right)ds\\ &\quad+C\int_0^t\left(\left\|u_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2+\left\|v_\varepsilon(s)\right\|_{\mathrm{H}^{-1}}^2\right)ds.\end{aligned}$$

Hence applying Gronwall's Lemma we obtain

$$\begin{aligned}&\sup_{0\le t\le T}\left(\left\|\overline{u}_\varepsilon(t)\right\|_{L^2}^2+\left\|\overline{v}_\varepsilon(t)\right\|_{L^2}^2+\left\|\overline{w}_\varepsilon(t)\right\|_{L^2}^2\right)\\ &\quad+C\int_0^T\left(\left\|\nabla_\varepsilon^+\overline{u}_\varepsilon(s)\right\|_{L^2}^2+\left\|\nabla_\varepsilon^+\overline{v}_\varepsilon(s)\right\|_{L^2}^2+\left\|\nabla_\varepsilon^+\overline{w}_\varepsilon(s)\right\|_{L^2}^2\right)ds\le C\eta_Te^{CT}.\end{aligned}$$

□

We want to deduce from the fact that the pair $(u_\varepsilon, v_\varepsilon)$ converges in law towards (u, v) in $L^2(0, T; (\mathrm{H}^{-1})^2)$, the convergence in law of $(\overline{u}_\varepsilon, \overline{v}_\varepsilon, \overline{w}_\varepsilon)$.

Lemma 3.5 *The process* $\{(\overline{u}_\varepsilon(t), \overline{v}_\varepsilon(t), \overline{w}_\varepsilon(t)), 0 \le t \le T, \quad 0 < \varepsilon < 1\} \Rightarrow \{(\overline{u}(t), \overline{v}(t), \overline{w}(t)), 0 \le t \le T\}$ *in* $L^2\big(0, T; (L^2)^3\big) \cap C([0, T]; (\mathrm{H}^{-1})^3)$, *where the limit* $\{(\overline{u}(t), \overline{v}(t), \overline{w}(t)), 0 \le t \le T\}$ *is the unique solution of the following system of parabolic PDEs*

$$\begin{cases} \dfrac{d\overline{u}}{dt}(t) = \mu_S\, \Delta\overline{u}(t) - f(t)\overline{u}(t) - g(t)\overline{v}(t) - f(t)u(t) - g(t)v(t), \\ \dfrac{d\overline{v}}{dt}(t) = \mu_I\, \Delta\overline{v}(t) + f(t)\overline{u}(t) + g(t)\overline{v}(t) + f(t)u(t) \\ \qquad\qquad + g(t)v(t) - \alpha\,(v(t) + \overline{v}(t)\,, \\ \dfrac{d\overline{w}}{dt}(t) = \mu_R\, \Delta\overline{w}(t) + \alpha\,(v(t) + \overline{v}(t))\,, \\ \overline{u}(0) = \overline{v}(0) = \overline{w}(0) = 0. \end{cases} \tag{34}$$

Proof Let

$$\overline{\Im}_\varepsilon(t) = \begin{pmatrix} \overline{u}_\varepsilon(t) \\ \overline{v}_\varepsilon(t) \\ \overline{w}_\varepsilon(t) \end{pmatrix}, \quad F_\varepsilon(t) = \begin{pmatrix} -f_\varepsilon(t)u_\varepsilon(t) - g_\varepsilon(t)v_\varepsilon(t) \\ f_\varepsilon(t)u_\varepsilon(t) + (g_\varepsilon(t) - \alpha)v_\varepsilon(t) \\ \alpha v_\varepsilon(t) \end{pmatrix},$$

$$\Lambda_\varepsilon(t) = \begin{pmatrix} \mu_S\Delta_\varepsilon - f_\varepsilon(t) & -g_\varepsilon(t) & 0 \\ f_\varepsilon(t) & \mu_I\Delta_\varepsilon + g_\varepsilon(t) - \alpha & 0 \\ 0 & 0 & \mu_R\Delta_\varepsilon + \alpha \end{pmatrix}.$$

Note that both $\overline{\Im}_\varepsilon$ and F_ε belong to $L^2(0, T; (\mathrm{H}_\varepsilon)^3)$. We have the following system of ODEs

$$\frac{d\overline{\Im}_\varepsilon}{dt}(t) = \Lambda_\varepsilon(t)\overline{\Im}_\varepsilon(t) + F_\varepsilon(t), \quad \overline{\Im}_\varepsilon(0) = 0\,. \tag{35}$$

Lemma 3.3 tells us that whenever, as $\varepsilon \to 0$,

$$F_\varepsilon \Longrightarrow F \ \text{ in } L^2(0, T; (\mathrm{H}^{-1})^3),$$

where

$$F(t) = \begin{pmatrix} -f(t)u(t) - g(t)v(t) \\ f(t)u(t) + g(t)v(t) - \alpha v(t) \\ \alpha v(t) \end{pmatrix}. \tag{36}$$

We apply the well–known theorem due to Skorohod, which asserts that redefining the probability space, we can assume that $F_\varepsilon \to F$ a.s. strongly in $L^2((0, T); (\mathrm{H}^{-1})^3)$. Our assumptions and the hypotheses imply that both $\overline{\Im}_\varepsilon$ and $\nabla_\varepsilon^+\overline{\Im}_\varepsilon$ are bounded in $(L^2((0, T) \times \mathbb{T}^1))^3$. Hence along a subsequence $\overline{\Im}_\varepsilon \to \overline{\Im}$ and $\nabla_\varepsilon^+\overline{\Im}_\varepsilon \to \overline{G}$ in

$(L^2((0,T)\times\mathbb{T}^1))^3$ weakly. However, it follows from a duality argument that $\overline{G}=\nabla\overline{\Im}$, and taking the weak limit in (35), we deduce that $\overline{\Im}$ is the unique solution of the system of parabolic PDEs

$$\frac{d\overline{\Im}}{dt}(t)=\Lambda(t)\overline{\Im}(t)+F(t),\quad \overline{\Im}(0)=0\,,$$

with

$$\Lambda(t)=\begin{pmatrix}\mu_S\Delta-f(t) & -g(t) & 0\\ f(t) & \mu_I\Delta+g(t)-\alpha & 0\\ 0 & 0 & \mu_R\Delta+\alpha\end{pmatrix}. \tag{37}$$

Hence all converging subsequences have the same limit, and the whole sequence converges.

We now show that the pair $\langle\overline{\Im}_\varepsilon,\nabla_\varepsilon^+\overline{\Im}_\varepsilon\rangle$ converges strongly in $(L^2((0,T)\times\mathbb{T}^1))^6$. We first note that both $\overline{\Im}_\varepsilon$ and $\nabla_\varepsilon^+\overline{\Im}_\varepsilon$ are bounded in $(L^2((0,T)\times\mathbb{T}^1))^3$, but also $\frac{d}{dt}\overline{\Im}_\varepsilon$ is bounded in $L^2((0,T);(\mathrm{H}^{-1}(\mathbb{T}^1))^3)$. From these estimates, we deduce with the help of Theorem 5.4 in Droniou et al. [4] that $\overline{\Im}_\varepsilon\to\overline{\Im}$ strongly in $(L^2((0,T)\times\mathbb{T}^1))^3$. Next we deduce from (35) that

$$\frac{1}{2}\frac{d\left\|\overline{\Im}_\varepsilon(t)\right\|_{L^2}^2}{dt}=\langle\Lambda_\varepsilon\overline{\Im}_\varepsilon(t),\overline{\Im}_\varepsilon(t)\rangle+\langle F_\varepsilon(t),\overline{\Im}_\varepsilon(t)\rangle,$$

hence

$$\begin{aligned}&\frac{1}{2}\left\|\overline{\Im}_\varepsilon(T)\right\|_{L^2}^2+\int_0^T\left[\mu_S\left\|\nabla_\varepsilon^+\overline{u}_\varepsilon(t)\right\|_{L^2}^2+\mu_I\left\|\nabla_\varepsilon^+\overline{v}_\varepsilon(t)\right\|_{L^2}^2+\mu_R\left\|\nabla_\varepsilon^+\overline{w}_\varepsilon(t)\right\|_{L^2}^2\right]dt\\&=\int_0^T\Big[\langle f_\varepsilon(t)\overline{u}_\varepsilon(t)+g_\varepsilon(t)\overline{v}_\varepsilon(t),\overline{v}_\varepsilon(t)\\&\quad-\overline{u}_\varepsilon(t)\rangle+\left\|\sqrt{\alpha}\,\overline{w}_\varepsilon(t)\right\|_{L^2}^2-\left\|\sqrt{\alpha}\,\overline{v}_\varepsilon(t)\right\|_{L^2}^2+\langle F_\varepsilon(t),\overline{\Im}_\varepsilon(t)\rangle\Big]dt\,.\end{aligned}\tag{38}$$

We have an analogous identity for the limiting quantities, namely:

$$\begin{aligned}&\frac{1}{2}\left\|\overline{\Im}(T)\right\|_{L^2}^2+\int_0^T\left[\mu_S\left\|\nabla\overline{u}(t)\right\|_{L^2}^2+\mu_I\left\|\nabla\overline{v}(t)\right\|_{L^2}^2+\mu_R\left\|\nabla\overline{w}(t)\right\|_{L^2}^2\right]dt\\&=\int_0^T\Big[\langle f(t)\overline{u}(t)+g(t)\overline{v}(t),\overline{v}(t)-\overline{u}(t)\rangle\\&\quad+\left\|\sqrt{\alpha}\,\overline{w}(t)\right\|_{L^2}^2-\left\|\sqrt{\alpha}\,\overline{v}(t)\right\|_{L^2}^2+\langle F(t),\overline{\Im}(t)\rangle\Big]dt\,.\end{aligned}\tag{39}$$

It follows from the strong convergence of F_ε to F in $L^2(0,T;(\mathrm{H}^{-1})^3)$, the strong convergence of $\overline{\Im}_\varepsilon\to\overline{\Im}$ in $(L^2((0,T)\times\mathbb{T}^1))^3$ and the weak convergence of $\nabla_\varepsilon^+\overline{\Im}_\varepsilon$ to $\nabla\overline{\Im}$ in $(L^2((0,T)\times\mathbb{T}^1))^3$ that the right hand side of (38) converges to the right hand side of (39). Hence the left hand side of (38) converges to the left hand side of (39). Consequently

$$\frac{1}{2}\left\|\overline{\Im}_\varepsilon(T)-\overline{\Im}(T)\right\|_{L^2}^2+\int_0^T\Big[\mu_S\left\|\nabla_\varepsilon^+\overline{u}_\varepsilon(t)-\nabla\overline{u}(t)\right\|_{L^2}^2+\mu_I\left\|\nabla_\varepsilon^+\overline{v}_\varepsilon(t)-\nabla\overline{v}(t)\right\|_{L^2}^2 \\ +\mu_R\left\|\nabla_\varepsilon^+\overline{w}_\varepsilon(t)-\nabla\overline{w}(t)\right\|_{L^2}^2\Big]dt\to 0\,. \tag{40}$$

This last result follows from the convergence of the left hand side of (38) to that of (39), and the facts that

$$\langle\overline{\Im}_\varepsilon(T),\overline{\Im}(T)\rangle\to\left\|\overline{\Im}(T)\right\|_{L^2}^2,$$

and

$$\int_0^T\left[\mu_S\langle\nabla_\varepsilon^+\overline{u}_\varepsilon(t),\nabla\overline{u}(t)\rangle+\mu_I\langle\nabla_\varepsilon^+\overline{v}_\varepsilon(t),\nabla\overline{v}(t)\rangle+\mu_R\langle\nabla_\varepsilon^+\overline{w}_\varepsilon(t),\nabla\overline{w}(t)\rangle\right]dt \\ \to\int_0^T\Big[\mu_S\left\|\nabla\overline{u}(t)\right\|_{L^2}^2+\mu_I\left\|\nabla\overline{v}(t)\right\|_{L^2}^2+\mu_R\left\|\nabla\overline{w}(t)\right\|_{L^2}^2\Big]dt\,.$$

The second convergence follows from the fact that $\nabla_\varepsilon^+\overline{\Im}_\varepsilon\to\nabla\overline{\Im}$ in $(L^2((0,T)\times\mathbb{T}^1))^3$ weakly. Concerning the first one, we deduce from the equations and the above statements that $\overline{\Im}_\varepsilon(T)\to\overline{\Im}(T)$ weakly in $(\mathrm{H}^{-1})^3$. But since that sequence is bounded in $(L^2(\mathbb{T}^1))^3$, it also converges weakly in $(L^2(\mathbb{T}^1))^3$.

The fact that $\nabla_\varepsilon^+\overline{\Im}_\varepsilon\to\nabla\overline{\Im}$ strongly in $(L^2((0,T)\times\mathbb{T}^1))^3$ clearly follows from (40).

The above arguments imply that a.s.

$$\langle\overline{\Im}_\varepsilon,\nabla_\varepsilon^+\overline{\Im}_\varepsilon\rangle\to\langle\overline{\Im},\nabla\overline{\Im}\rangle\ \text{ strongly in } (L^2((0,T)\times\mathbb{T}^1))^6\,.$$

Now the convergence $\overline{\Im}_\varepsilon\to\overline{\Im}$ in $C([0,T];(H^{-1})^3)$ follows readily from the equation. □

Lemma 3.2 says that $\Im_\varepsilon\Rightarrow\Im$ in $C\big([0,T];\,(\mathrm{H}^{-\gamma})^3\big)\cap L^2\big(0,T;(\mathrm{H}^{-1})^3\big)$, we have used in Lemma 3.5 the Skorohod theorem to deduce that $\overline{\Im}_\varepsilon\Rightarrow\overline{\Im}$ in $L^2(0,T(L^2)^3)\cap C([0,T];(\mathrm{H}^{-\gamma})^{-1})$. Hence the same Skorohod theorem allows us to take the limit in the sum $\Im_\varepsilon+\overline{\Im}_\varepsilon$, which yields the following result.

Theorem 3.2 (Functional central limit theorem) *For* $3/2<\gamma<2$, *as* $\varepsilon\to 0$, $\{\mathscr{Y}_\varepsilon(t)\ ,\ 0\le t\le T\}_{0<\varepsilon<1}\Longrightarrow\{\mathscr{Y}(t)\ ,\ 0\le t\le T\}$ *in* $C\big([0,T];\,(\mathrm{H}^{-\gamma})^3\big)\cap$

$L^2\big(0, T; (\mathrm{H}^{-1})^3\big)$*, where the limit* $\mathscr{Y}$ *is solution of the following system of SPDEs : for all* $\varphi \in \mathrm{H}^1$

$$\begin{cases} \big\langle \mathscr{U}(t), \varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} \\ \quad = \mu_S \int_0^t \big\langle \mathscr{U}(r)\,,\ \Delta\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr + \int_0^t \big\langle \mathscr{V}(r),\ \beta(.)\frac{\mathbf{i}(r)\big(\mathbf{i}(r)+\mathbf{r}(r)\big)}{\mathbf{a}^2(r)}\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr \\ \qquad + \int_0^t \big\langle \mathscr{U}(r),\ \beta(.)\frac{\mathbf{s}(r)\big(\mathbf{s}(r)+\mathbf{r}(r)\big)}{\mathbf{a}^2(r)}\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr + \big\langle \mathscr{M}^S(t)\,,\ \varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1}, \\ \big\langle \mathscr{V}(t), \varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} \\ \quad = \mu_I \int_0^t \big\langle \mathscr{V}(r)\,,\ \Delta\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr - \int_0^t \big\langle \mathscr{V}(r),\ \beta(.)\frac{\mathbf{i}(r)\big(\mathbf{i}(r)+\mathbf{r}(r)\big)}{\mathbf{a}^2(r)}\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr \\ \qquad - \int_0^t \big\langle \mathscr{U}(r),\ \beta(.)\frac{\mathbf{s}(r)\big(\mathbf{s}(r)+\mathbf{r}(r)\big)}{\mathbf{a}^2(r)}\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr + \int_0^t \big\langle \mathscr{V}(r), \alpha(.)\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr \\ \qquad + \big\langle \mathscr{M}^I(t)\,,\ \varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1}, \\ \big\langle \mathscr{W}(t), \varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} \\ \quad = \mu_R \int_0^t \big\langle \mathscr{W}(r)\,,\ \Delta\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr - \int_0^t \big\langle \mathscr{V}(r), \alpha(.)\varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} dr + \big\langle \mathscr{M}^R(t)\,,\ \varphi \big\rangle_{\mathrm{H}^{-1},\mathrm{H}^1}. \end{cases} \tag{41}$$

Final remarks: • Our functional central limit theorem is established in dimension 1. The difficulty in higher dimension is the following. $\gamma > 3/2$ has to be replaced by $\gamma > 1 + d/2$. Then in Lemma 3.2 we have convergence in $L^2(0, T; (H^{1-\gamma})^3) \cap C([0, T]; (H^{-\gamma})^3)$. Note that $1 - \gamma < -d/2$. Already in dimension 2, we have $1 - \gamma < -1$, and there is a serious difficulty with the analog of Lemma 3.5.

• In this work, we have first let $\mathbf{N} \to \infty$, while $\varepsilon > 0$ is fixed, and then let $\varepsilon \to 0$. The case where $\mathbf{N} \to +\infty$ and $\varepsilon \to 0$ together, with some constraint on the relative speeds of convergence (which does not allow $\mathbf{N}$ to converge too slowly to ∞ while $\varepsilon \to 0$) will be the subject of future work.

Appendix A

Lemma A.1 *Let* $(h_\varepsilon)_{0<\varepsilon<1}$ *be a sequence of* H_ε*. If* $(h_\varepsilon)_{0<\varepsilon<1}$ *is bounded in* $\mathrm{H}^{1,\varepsilon}$*, then it is relatively compact in* L^2*, and the limit of any convergent subsequence belongs to* H^1.

Proof By using the fact that the sequence (h_ε) is bounded in L^2 and $\big\|\nabla_\varepsilon^+ h_\varepsilon\big\|_{L^2} \le C\big\|h_\varepsilon\big\|_{\mathrm{H}^1}$, then the result of the compactness follows from the compactness theorem of Kolmogorov in L^2.

The fact the limit of any convergent subsequence belong to H^1, follows from the discrete integrating by part

$$\int_{\mathbb{T}^1} \nabla_\varepsilon^+ h_\varepsilon(x)\varphi(x)dx = -\langle \int_0^{\cdot} h_\varepsilon(y)dy\ ,\ \nabla_\varepsilon^+\varphi\ \rangle,$$

and letting ε go to zero in this equation. □

Lemma A.2

For all $u_\varepsilon \in \mathrm{H}_\varepsilon$

$$\left\| \nabla_\varepsilon^- u_\varepsilon \right\|^2_{\mathrm{H}^{-\gamma,\varepsilon}} = \left\| \nabla_\varepsilon^+ u_\varepsilon \right\|^2_{\mathrm{H}^{-\gamma,\varepsilon}} = \sum_m \left(\langle u_\varepsilon, \varphi_m^\varepsilon \rangle^2 + \langle u_\varepsilon, \psi_m^\varepsilon \rangle^2 \right) \lambda_m^\varepsilon (1 + \lambda_m^\varepsilon)^{-\gamma}.$$

Proof We have

$$\nabla_\varepsilon^- \varphi_m^\varepsilon = -b_{m,\varepsilon}\varphi_m^\varepsilon - a_{m,\varepsilon}\psi_m^\varepsilon \quad \text{and} \quad \nabla_\varepsilon^- \psi_m^\varepsilon = a_{m,\varepsilon}\varphi_m^\varepsilon - b_{m,\varepsilon}\psi_m^\varepsilon,$$

where $a_{m,\varepsilon} = \varepsilon^{-1}\sin(\pi m \varepsilon)$ and $b_{m,\varepsilon} = \varepsilon^{-1}(\cos(\pi m\varepsilon) - 1)$.

We have

$$a_{m,\varepsilon}^2 + b_{m,\varepsilon}^2 = \lambda_m^\varepsilon.$$

Let $u_\varepsilon \in \mathrm{H}_\varepsilon$. We have

$$\begin{aligned}
&\left\| \nabla_\varepsilon^+ u_\varepsilon \right\|^2_{\mathrm{H}^{-\gamma,\varepsilon}} \\
&= \sum_m \left(\langle u_\varepsilon, \nabla_\varepsilon^- \varphi_m \rangle^2 + \langle u_\varepsilon, \nabla_\varepsilon^- \psi_m \rangle^2 \right) (1 + \lambda_m^\varepsilon)^{-\gamma} \\
&= \sum_m \left(\langle u_\varepsilon, -b_{m,\varepsilon}\varphi_m^\varepsilon - a_{m,\varepsilon}\psi_m^\varepsilon \rangle^2 + \langle u_\varepsilon, a_{m,\varepsilon}\varphi_m^\varepsilon - b_{m,\varepsilon}\psi_m^\varepsilon \rangle^2 \right) (1 + \lambda_m^\varepsilon)^{-\gamma} \\
&= \sum_m \left([-b_{m,\varepsilon}\langle u_\varepsilon, \varphi_m^\varepsilon \rangle - a_{m,\varepsilon}\langle u_\varepsilon, \psi_m^\varepsilon \rangle]^2 + [a_{m,\varepsilon}\langle u_\varepsilon, \varphi_m^\varepsilon \rangle - b_{m,\varepsilon}\langle u_\varepsilon, \psi_m^\varepsilon \rangle]^2 \right) \\
&\quad \times (1 + \lambda_m^\varepsilon)^{-\gamma} \\
&= \sum_m \left([a_{m,\varepsilon}^2 + b_{m,\varepsilon}^2]\{\langle u_\varepsilon, \varphi_m^\varepsilon \rangle^2 + \langle u_\varepsilon, \psi_m^\varepsilon \rangle^2\} \right) (1 + \lambda_m^\varepsilon)^{-\gamma} \\
&= \sum_m \left(\langle u_\varepsilon, \varphi_m^\varepsilon \rangle^2 + \langle u_\varepsilon, \psi_m^\varepsilon \rangle^2 \right) \lambda_m^\varepsilon (1 + \lambda_m^\varepsilon)^{-\gamma}.
\end{aligned}$$

The proof of $\left\| \nabla_\varepsilon^- u_\varepsilon \right\|^2_{\mathrm{H}^{-\gamma,\varepsilon}} = \sum_m \left(\langle u_\varepsilon, \varphi_m^\varepsilon \rangle^2 + \langle u_\varepsilon, \psi_m^\varepsilon \rangle^2 \right) \lambda_m^\varepsilon (1+\lambda_m^\varepsilon)^{-\gamma}$ is similar by noting that

$$\nabla_\varepsilon^+ \varphi_m^\varepsilon = b_{m,\varepsilon}\varphi_m^\varepsilon - a_{m,\varepsilon}\psi_m^\varepsilon \quad \text{and} \quad \nabla_\varepsilon^+ \psi_m^\varepsilon = a_{m,\varepsilon}\varphi_m^\varepsilon + b_{m,\varepsilon}\psi_m^\varepsilon.$$

□

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Conflict of interest The authors declare that they have no conflict of interest and no competing interests. The authors have no competing interests to declare that are relevant to the content of this article.

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