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Inviscid limit for stochastic second-grade fluid equations

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Abstract
We consider in a smooth bounded and simply connected two dimensional domain the convergence in the L^2 norm, uniformly in time, of the solution of the stochastic second-grade fluid equations with transport noise and no-slip boundary conditions to the solution of the corresponding Euler equations. We prove, that assuming proper regularity of the initial conditions of the Euler equations and a proper behavior of the parameters ν and α, then the inviscid limit holds without requiring a particular dissipation of the energy of the solutions of the second-grade fluid equations in the boundary layer.

Keywords Inviscid limit · Turbulence · Transport noise · No-slip boundary conditions · Boundary layer · Additive noise · Second-grade complex fluid

1 Introduction

The second-grade fluid equations are a model for viscoelastic fluids, with two parameters: $\alpha > 0$, corresponding to the elastic response, and $\nu > 0$, corresponding to viscosity. Considering a constant density, $\rho = 1$, their stress tensor is given by

$$T = -pI + \nu A_1 + \alpha^2 A_2 - \alpha^2 A_1^2,$$

where

$$A_1 = \frac{\nabla u + \nabla u^T}{2},$$
$$A_2 = \partial_t A_1 + A_1 \nabla u + \nabla u^T A_1,$$

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being p the pressure and u the velocity field. Given this stress tensor, the equations of motion for an incompressible homogeneous fluid of grade 2 are given by

$$\begin{cases} \partial_t v = \nu \Delta u - \mathrm{curl}(v) \times u + \nabla p + f \\ \mathrm{div}\, u = 0 \\ v = u - \alpha^2 \Delta u \\ u|_{\partial D} = 0 \\ u(0) = u_0. \end{cases} \tag{1}$$

where f describes some external forces, possibly stochastic, acting on the fluid, see [10, 36] for further details on the physics behind this system. The analysis of the deterministic system started with [7]. They proved global existence and uniqueness without restricting the problem to the two dimensional case. Setting, formally, $\alpha = 0$ in Eq. (1) we can reduce the system to the well-known Navier–Stokes one:

$$\begin{cases} \partial_t u = \nu \Delta u - u \cdot \nabla u + \nabla p + f \\ \mathrm{div}\, u = 0 \\ u|_{\partial D} = 0 \\ u(0) = u_0. \end{cases} \tag{2}$$

Thus (1) can be seen as a generalization of (2). Moreover, in [19], it has been shown that second-grade fluid equations are a good approximation of the Navier–Stokes system. Due to these good properties of the system it is a legitimate question trying to understand if the second-grade fluid equations behave better than the Navier–Stokes ones in problems related to turbulence, like the inviscid limit for domain with boundary and no-slip boundary conditions. In fact, such question is far for being solved for system (2) also in the deterministic framework. Partial results are available:

1. Unconditioned results. They are based on strong assumptions about the flows. For example flows with radial symmetry [25, 26], or flows with analytic boundary layers [29, 37].
2. Conditioned results. They are based on stating some criteria about the behavior of the solutions of the Navier–Stokes equations in the boundary layer in order to prove the inviscid limit. This line of research started with the famous work by Kato [21], see [8, 43, 44] for other results. For what concerns the Stochastic framework few results are available, see for example [28] for a generalization of the Kato's results to the additive noise case and a wider set of initial conditions and [3] for some analysis on the validity of a Large Deviation Principle for the inviscid limit of the Navier–Stokes equations in two-dimensional bounded domains perturbed by additive noise.

The analysis of the inviscid limit for the deterministic second-grade fluid equations is a partially well-understood topic. In particular, in [27], the authors show that the behavior of the system changes considering different scaling between ν and α^2.

If we set, formally, $\nu = 0$ in system (1) second-grade fluid equations reduce to the so-called Euler-α equations:

$$\begin{cases} \partial_t v = -\operatorname{curl}(v) \times u + \nabla p + f \\ \operatorname{div} u = 0 \\ v = u - \alpha^2 \Delta u \\ u|_{\partial D} = 0 \\ u(0) = u_0. \end{cases} \tag{3}$$

This system models the averaged motion of an ideal incompressible fluid when filtering over spatial scales smaller than α and its well-posedness has been treated in [30, 39]. Euler-α equations, formally, satisfies the condition of [27, Theorem 3]. Therefore we can expect that the inviscid limit holds also in this framework. Indeed, this is true as has been showed in [24].

In this work, we will consider stochastic second-grade fluid equations and stochastic Euler-α equations with transport noise which scales with respect to the elasticity. We want to understand if the good behavior proved in [27] if $\nu = O(\alpha^2)$ and in [24] if $\nu = 0$ is preserved also in this case. There are several motivations to consider transport noise, as the effect of small scales on large scales in fluid dynamics problems, see [9, 13, 14, 18] for several discussions on this topic. A first issue related to the analysis of the inviscid limit in the case of the transport noise is the well-posedness of the systems. In fact the existence of strong probabilistic solutions of such systems is outside the framework treated in [32, 34], thus we need to improve slightly these results thanks to the properties of the transport noise. In the following $\nu \geq 0$ and we will always speak of second-grade fluid equations even if $\nu = 0$.

The paper is organized as follows. In Sect. 2 we introduce the mathematical problem, we state our main theorems and we give some well-known results for the Euler equations and the analysis of the stochastic second-grade fluid equations. In Sect. 3 we prove that the stochastic second-grade fluid equations with transport noise and no-slip boundary conditions are well posed. In Sect. 4, thanks to the already proven well-posedness and Hypothesis 6 below we improve the energy estimates obtained in Sect. 3 in order to get some estimates crucial for the proof of Theorem 9. The proof of our main theorem on the inviscid limit occupies Sect. 5. Lastly in Sect. 6 we add some remarks for the analysis of the additive noise case.

2 Main results

Let us start this section introducing some general assumptions which will be always adopted under our analysis even if not recalled.

Hypothesis 1 The following hold:

- $0 < T < +\infty$.
- D is a bounded, smooth, simply connected domain.

- $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(\mathcal{F}_t)_{t\in[0,T]}$ is a right continuous filtration and $\mathcal{F}_0$ contains every $\mathbb{P}$ null subset of Ω.

For square integrable semimartingales taking value in separable Hilbert spaces $U_1,\ U_2$ we will denote by $[M, N]_t$ the quadratic covariation process. If M, N take values in the same separable Hilbert space U with orthonormal basis u_i, we will denote by $\langle\langle M, N\rangle\rangle_t = \sum_{i\in\mathbb{N}}[\langle M, u_i\rangle_U, \langle N, u_i\rangle_U]_t$. For each $k \in \mathbb{N},\ 1 \le p \le \infty$ we will denote by $L^p(D)$ and $W^{k,p}(D)$ the well-known Lebesgue and Sobolev spaces. We will denote by $C_c^\infty(D)$ the space of smooth functions with compact support and by $W_0^{k,p}(D)$ their closure with respect to the $W^{k,p}(D)$ topology. If $p = 2$, we will write $H^k(D)$ (resp. $H_0^k(D)$) instead of $W^{k,2}(D)$ (resp. $W_0^{k,2}(D)$). Let X be a separable Hilbert space, denote by $L^p(\mathcal{F}_{t_0}, X)$ the space of p integrable random variables with values in X, measurable with respect to $\mathcal{F}_{t_0}$. We will denote by $L^p(0, T; X)$ the space of measurable functions from $[0, T]$ to X such that

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p\, dt\right)^{1/p} < +\infty, \quad 1 \le p < \infty$$

and obvious generalization for $p = \infty$. For any $r,\ p \ge 1$, we will denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; L^r(0, T; X))$ the space of processes with values in X such that

1. $u(\cdot, t)$ is progressively measurable.
2. $u(\omega, t) \in X$ for almost all (ω, t) and

$$\mathbb{E}\left[\|u(\omega, \cdot)\|_{L^r(0,T;X)}^p\right] < +\infty.$$

Obvious generalizations for $p = \infty$ or $r = \infty$.

Set

$$H = \{f \in L^2(D; \mathbb{R}^2), \quad \operatorname{div} f = 0, \quad f \cdot n|_{\partial D} = 0\},$$
$$V = H_0^1(D; \mathbb{R}^2) \cap H, \quad D(A) = H^2(D; \mathbb{R}^2) \cap V.$$

Moreover we introduce the vector space

$$W = \{u \in V :\ \operatorname{curl}(u - \alpha^2\Delta u) \in L^2(D; \mathbb{R}^2)\}$$

with norm $\|u\|_W^2 = \|u\|^2 + \alpha^2\|\nabla u\|_{L^2(D;\mathbb{R}^2)}^2 + \|\operatorname{curl}(u - \Delta u)\|_{L^2(D)}^2$. It is well-known, see for example [7], that we can identify W with the space

$$\hat{W} = \{u \in H^3(D; \mathbb{R}^2) \cap V\}.$$

Moreover there exists a constant such that

$$\|u\|_{H^3}^2 \le C\left(\|u\|^2 + \|\nabla u\|_{L^2(D;\mathbb{R}^2)}^2 + \|\operatorname{curl}(u - \Delta u)\|_{L^2(D)}^2\right). \tag{4}$$

We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and the norm in H respectively. Other norms and scalar products will be denoted with the proper subscript. On V we introduce the norm $\|u\|_V^2 = \|u\|^2 + \alpha^2 \|\nabla u\|^2_{L^2(D;\mathbb{R}^2)}$. We will shortly denote by $\|u\|_* = \|\mathrm{curl}(u - \alpha^2 \Delta u)\|_{L^2(D)}$. Obviously the following inequality holds for $u \in V$, where C_p is the Poincarè constant associated to D,

$$\frac{\|u\|_V^2}{\alpha^2 + C_p^2} \leq \|\nabla u\|^2_{L^2(D;\mathbb{R}^2)} \leq \frac{\|u\|_V^2}{\alpha^2} \tag{5}$$

Denote by P the linear projector of $L^2\left(D;\mathbb{R}^2\right)$ on H and define the unbounded linear operator $A : D(A) \subseteq H \to H$ by the identity

$$\langle Av, w \rangle = \langle \Delta v, w \rangle_{L^2(D;\mathbb{R}^2)} \tag{6}$$

for all $v \in D(A),\ w \in H$. A will be called the Stokes operator. It is well-known (see for example [42]) that A is self-adjoint, generates an analytic semigroup of negative type on H and moreover $V = D\left((-A)^{1/2}\right)$. Denote by $\mathbb{L}^4$ the space $L^4\left(D, \mathbb{R}^2\right) \cap H$, with the usual topology of $L^4\left(D, \mathbb{R}^2\right)$. Define the trilinear, continuous form $b : \mathbb{L}^4 \times V \times \mathbb{L}^4 \to \mathbb{R}$ as

$$b\,(u, v, w) = \langle u, P(\nabla v w) \rangle. \tag{7}$$

Now we introduce some assumptions on the stochastic part of the system.

Hypothesis 2 The following hold:

- K is a (possibly countable) set of indexes.
- $\sigma_k \in W^{1,\infty}(D;\mathbb{R}^2) \cap V$ satisfying

$$\sum_{k\in K} \|\sigma_k\|^2_{W^{1,\infty}} < +\infty.$$

- $u_0 \in \cap_{p\geq 2} L^p(\mathcal{F}_0, W)$.
- $\{W_t^k\}_{k\in K}$ is a sequence of real, independent Brownian motions adapted to $\mathcal{F}_t$.

Let us consider the stochastic second-grade fluid equations below. Some physical motivations for the introduction of transport noise in fluid dynamic models can be found in [13, 18].

$$\begin{cases} dv = (\nu \Delta u - \mathrm{curl}(v) \times u + \nabla p)dt + \sum_{k\in K} (\sigma_k \cdot \nabla u + \nabla \tilde{p}_k) \circ dW_t^k \\ \mathrm{div}\, u = 0 \\ v = u - \alpha^2 \Delta u \\ u|_{\partial D} = 0 \\ u(0) = u_0. \end{cases}$$

We need to add the additional pressure term $\sum_{k\in K}\nabla\tilde{p}_k\circ dW_t^k$, the so-called turbulent pressure, in the system above in order to deal with the fact that $\sum_{k\in K}\sigma_k\cdot\nabla u\circ dW_t^k$ is not divergence free, therefore an additional martingale term orthogonal to H must be added to make the system feasible.

Introducing the Stokes operator, the previous equation can be rewritten as

$$\begin{cases} d(u-\alpha^2 Au) = (\nu Au - P(\mathrm{curl}(v)\times u))dt + \sum_{k\in K} P(\sigma_k\cdot\nabla u)\circ dW_t^k \\ v = u-\alpha^2\Delta u \\ u(0)=u_0 \end{cases} \tag{8}$$

or the corresponding Itô form

$$\begin{cases} d(u-\alpha^2 Au) = (\nu Au - P(\mathrm{curl}(v)\times u))dt + \sum_{k\in K} P(\sigma_k\cdot\nabla u)dW_t^k \\ \quad +\frac{1}{2}\sum_{k\in K} P(\sigma_k\cdot\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u)))dt \\ v = u-\alpha^2\Delta u \\ u(0)=u_0. \end{cases} \tag{9}$$

Indeed each of the Stratonovich integrals in Eq. (8) can be rewritten, thanks to the Stratonovich–Itô corrector associated to previous equation, in the following form:

$$\begin{aligned} &P(\sigma_k\cdot\nabla u)\circ dW_t^k \\ &= P(\sigma_k\cdot\nabla u)dW_t^k + \frac{1}{2}d[P(\sigma_k\cdot\nabla u), W^k]_t \\ &= P(\sigma_k\cdot\nabla u)dW_t^k + \frac{1}{2}P\left(\sigma_k\cdot\nabla d[u, W^k]_t\right) \\ &= P(\sigma_k\cdot\nabla u)dW_t^k \\ &\quad + \frac{1}{2}P\left(\sigma_k\cdot\nabla d\left[\int_0^{\cdot}(I-\alpha^2 A)^{-1}\sum_{j\in K}P(\sigma_j\cdot\nabla u)dW_s^j, W^k\right]_t\right) \\ &= P(\sigma_k\cdot\nabla u)dW_t^k \\ &\quad + \frac{1}{2}P\left(\sigma_k\cdot\nabla\left((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u)\right)\right)dt. \end{aligned} \tag{10}$$

We denote by $F(u)=\frac{1}{2}\sum_{k\in K}P(\sigma_k\cdot\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u)))$ and $G^k(u)=P(\sigma_k\cdot\nabla u)$. By Corollary 22 below

$$F\in\mathcal{L}(V; H\cap H^1(D;\mathbb{R}^2)),\quad G^k\in\mathcal{L}(V;H).$$

Definition 3 A stochastic process with weakly continuous trajectories with values in W is a weak solution of Eq. (9) if

$$u\in L^p(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;W)),\quad \forall p\geq 2$$

and $\mathbb{P}-a.s.$ for every $t \in [0, T]$ and $\phi \in W$ we have

$$\begin{aligned}
&\langle u(t)-u_0, \phi\rangle_V + \int_0^t \nu\langle \nabla u(s), \nabla\phi\rangle_{L^2(D;\mathbb{R}^2)} \\
&\quad + \langle \operatorname{curl}(u(s)-\alpha^2\Delta u(s)) \times u(s), \phi\rangle_{L^2(D)} ds \\
&\quad = \int_0^t \langle F(u(s)), \phi\rangle ds + \sum_{k\in K}\int_0^t \langle G^k(u(s)), \phi\rangle dW_s^k.
\end{aligned}$$

Theorem 4 *Under Hypothesis 1–2, Eq. (9) has a unique solution in the sense of Definition 3. Moreover, almost surely, the stochastic process u has V continuous paths.*

Remark 5 Actually we can weaken the integrability assumption of u_0 with respect to $\mathbb{P}$ in order to get less integrable solution, but regular enough to prove that the inviscid limit holds. Indeed $u_0 \in L^4(\mathcal{F}_0, W)$ is the minimal assumption to prove either the well-posedness, see [2] and Sect. 3.2 below, and the inviscid limit, see Sects. 4 and 5. However, we prefer to not stress this assumption in order to make our results comparable to [34].

As stated in Sect. 1, the proof of Theorem 4 will be the object of Sect. 3. Usually, in stochastic analysis, the well-posedness of a stochastic partial differential equation is obtained considering some approximating sequence, $\{u^N\}_{N\in\mathbb{N}}$, which solves an approximate equation in the original probability space and showing the tightness of their law in some spaces of functions. Then, by Prokhorov's theorem and Skorokhod's representation theorem, one can find an auxiliary probability space and a solution of the limit equation in this auxiliary probability space, u. Lastly, by a Gyongy–Krylov argument, one can recover that the limit process belongs to the original probability space and that the approximating sequence converge in probability to u. See [1, 4, 13] for some examples of this method. Here, we follow a different, perhaps, more direct approach introduced by Breckner in [2] for Navier–Stokes equations with multiplicative noise with particular regularity properties, but well-suited to treat transport noise, which a priori does not satisfy the general assumptions of [2, Section 2]. This approach uses, in particular, the properties of stopping times, some basic convergence principles from functional analysis and some properties of fluid dynamic non-linearities. Therefore, even if the results of [2] were related to Navier–Stokes equations, this approach can be applied also to other fluid dynamic models, see [5, 34] for some examples to different fluid dynamic systems. An important byproduct of this way of proceed is that the approximations converge in mean square to the solution of the second-grade fluid equations, see Theorem 33 below. This fact will be crucial in order to obtain some apriori estimates on the solution, see Lemma 35 below.

Now we move to consider the inviscid limit problem and introduce a new set of hypotheses.

Hypothesis 6 The following hold:

- $\nu = O(\alpha^2),\ \tilde{\nu} = O(\alpha^2)$.
- $\overline{u}_0 \in H^s(D;\mathbb{R}^2) \cap H$ for some $s \geq 3$.

- $$\mathbb{E}\left[\|u_0^\alpha - \bar{u}_0\|^2\right] \to 0; \tag{11}$$
$$\mathbb{E}\left[\alpha^2\|\nabla u_0^\alpha\|^2_{L^2(D;\mathbb{R}^2)}\right] = o(1); \tag{12}$$
$$\mathbb{E}\left[\alpha^6\|u_0^\alpha\|^2_{H^3(D;\mathbb{R}^2)}\right] = O(1). \tag{13}$$

Let us consider the family of equations

$$\begin{cases} d(u^\alpha - \alpha^2 A u^\alpha) = (\nu A u^\alpha - P(\mathrm{curl}(v^\alpha) \times u^\alpha))dt + \sqrt{\tilde{\nu}} \sum_{k\in K} P(\sigma_k \cdot \nabla u^\alpha) dW_t^k \\ \quad + \frac{\tilde{\nu}}{2} \sum_{k\in K} P(\sigma_k \cdot \nabla((I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^\alpha)))dt \\ v^\alpha = u^\alpha - \alpha^2 \Delta u^\alpha \\ u^\alpha(0) = u_0^\alpha, \end{cases} \tag{14}$$

where σ_k are independent from ν, $\tilde{\nu}$, α and u_0^α are random variable satisfying the assumptions of Theorem 4. Energy relations and the behavior of the H^3 norm of u^α play a crucial role in the analysis of the inviscid limit in the deterministic framework, see Eqs. (3.2) and (3.7) in [27]. If we want to have some hope of replicating the approach of [27] we need some estimates in that direction. This is exactly what happens. Indeed, under Hypothesis 6, Eqs. (3.2) and (3.7) in [27] continue to hold in the stochastic framework, see Lemma 35 below. Therefore there is some hope to generalize the results of [24, 27] to our stochastic framework. Now, let us consider the Euler equations

$$\begin{cases} \partial_t \bar{u} + \nabla \bar{u} \cdot \bar{u} + \nabla p = 0 \ (x,t) \in D \times (0,T) \\ \mathrm{div}\, \bar{u} = 0 \\ \bar{u} \cdot n|_{\partial D} = 0 \\ \bar{u}(0) = \bar{u}_0. \end{cases} \tag{15}$$

Definition 7 Given $\bar{u}_0 \in H$, we say that $\bar{u} \in C(0,T;H)$ is a weak solution of Eq. (15) if for every $\phi \in C_0^\infty([0,T] \times D) \cap C^1([0,T];H)$

$$\langle \bar{u}(t), \phi(t)\rangle = \langle \bar{u}_0, \phi(0)\rangle + \int_0^t \langle \bar{u}(s), \partial_s \phi(s)\rangle ds + \int_0^t b(\bar{u}(s), \phi(s), \bar{u}(s)) ds$$

for every $t \in [0,T]$ and the energy inequality

$$\|\bar{u}(t)\|^2 \leq \|\bar{u}_0\|^2$$

holds.

For what concerns the well posedness of the Euler equations the following results hold true, see [22, 41].

Theorem 8 *Fix $T > 0$, $s \geq 3$. Let $\bar{u}_0 \in H^s(D;\mathbb{R}^2) \cap H$. Then there exist a unique weak solution of* (15) *with initial condition $\bar{u}_0$ such that*

$$\bar{u} \in C([0,T]; H^s(D;\mathbb{R}^2)) \cap C^1([0,T]; H^{s-1}(D;\mathbb{R}^2))$$

and $\|\bar{u}(t)\| = \|\bar{u}_0\|$, $\forall t \in [0,T]$.

Now we can state our main Theorem. According to the analysis started in [16] and continued, recently, in [11, 12] the influence of the transport noise on the averaged solution is related to the ℓ^2 norm of its coefficients, therefore we expect that the solution of Eq. (14) converges to the solution of the Euler equations with null forcing term.

Theorem 9 *Under Hypotheses 1–6, calling u^α the solution of* (14) *and $\bar{u}$ the solution of* (15)*, it holds*

$$\lim_{\alpha\to 0} \mathbb{E}\left[\sup_{t\in[0,T]} \|u^\alpha(t) - \bar{u}(t)\|^2\right] = 0.$$

Remark 10 If $\overline{u}_0 \in H \cap H^1(D;\mathbb{R}^2)$, the existence of a family u_0^α satisfying Eqs. (11), (12), (13) is guaranteed by Proposition 1 of [24].

Remark 11 Due to the poor regularity of the coefficients F and G^k, Eq. (9) are not guaranteed to be well-posed from the results of [34]. Indeed, neither F nor G^k satisfy the assumptions of [33] or [34]. However, due to relation (47) and the good estimates of Corollary 22, we will be able to prove in Lemmas 25 and 26 the same, actually stronger, energy estimates that are available in [34]. These and Lemma 28 are the main ingredients in order to prove the well-posedness of system (9). On the contrary the well-posedness in the case of additive noise is completely solved by the results of [34], thus in Sect. 6 we will only explain some remarks about the inviscid limit and the well-posedness in the additive noise framework.

Remark 12 Both Theorems 4 and 9 continue to hold for $\nu = 0$. We will give the proof of all the statements below in full details considering the case $\nu > 0$. However if something in the proof changes considering $\nu = 0$ we will explain in a remark at the end of each proof what we need to change in order to deal with the other case.

Remark 13 The arbitrariness in the choice of the parameters ν and $\tilde{\nu}$ allows us to generalize to this stochastic framework, via Theorem 9, some results of [24, 27]. As a byproduct of its proof we obtain that under Hypotheses 1–2–6

$$\alpha^2\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}\|\nabla u^\alpha(t)\|^2_{L^2(D;\mathbb{R}^2)}\right] \to 0.$$

Moreover, considering $\tilde{\nu} = \nu > 0$ we recover the scaling introduced by Kuksin [23] which is relevant for the inviscid limit at the level of invariant measures. The scaling above has been proved of being of interest also for the evolution of the solutions of the stochastic Navier–Stokes equations in a Kato-type regime, see [17, Corollary 2.5.1].

Remark 14 The results of these notes are in a certain sense complementary to what we obtained in [28]. In [28] we required poor regularity on the initial conditions of the Euler equations and the Navier–Stokes equations but we got a conditioned result. On the contrary, in these notes we require strong regularity on the initial conditions of the two problems and a special type of convergence of the initial conditions but we arrive at a not conditioned result.

Remark 15 The assumption on $\nu = O(\alpha^2)$ is hidden in Eq. (14). For high frequencies Δu is a damping term in Eq. (14). In fact, for high frequencies $v \approx -\alpha^2 \Delta u$, thus the equation becomes, formally,

$$-\alpha^2 \partial_t \Delta u - \nu \Delta u + \cdots = 0.$$

Asking $\nu = O(\alpha^2)$, means requiring that the damping coefficient does not blow-up.

We conclude this section with few notations that will be adopted: by C we will denote several constant independent from ν, α^2 and σ_k, perhaps changing value line by line. In the case C depends by ν, α or σ_k we will add the dependence as a subscript. Sometimes we will use the notation $a \lesssim b$, if it exists a constant independent from ν and α^2 such that $a \le Cb$. In order to simplify the notation we will denote Sobolev spaces by H^s, forgetting domain and range.

3 Well-posedness

3.1 Preliminaries

Before starting with the analysis of Eq. (9), we need to recall some preliminaries results on the nonlinear term in the second-grade fluid equations, the Stokes operator A and the embedding between W and V. We will consider the Hilbert triple

$$W \hookrightarrow V \hookrightarrow W^*$$

We start recalling in a single lemma some classical facts on the nonlinear part of Eq. (1). We refer to [7], [33, Lemma 2.4], [34, Lemma 2.4] for the proof of the various statements.

Lemma 16 *For any smooth, divergence free $\phi,\ v,\ w$ the following relation holds*

$$\langle \operatorname{curl} \phi \times v, w \rangle_{L^2} = b(v, \phi, w) - b(w, \phi, v). \tag{16}$$

Moreover for $u,\ v,\ w$ the following inequalities hold

$$|\langle \operatorname{curl}(u - \alpha^2 \Delta u) \times v, w \rangle_{L^2}| \le C \|u\|_{H^3} \|v\|_V \|w\|_W \tag{17}$$

$$|\langle \operatorname{curl}(u - \alpha^2 \Delta u) \times u, w \rangle_{L^2}| \le C \|u\|_V^2 \|w\|_W \tag{18}$$

Therefore there exists a bilinear operator $\hat{B} : W \times V \to W^*$ such that

$$\langle \hat{B}(u, v), w\rangle_{W^*,W} = \langle P(\mathrm{curl}(u - \alpha^2 \Delta u) \times v), w\rangle \tag{19}$$

which satisfies for $u \in V,\ v \in W$

$$\|\hat{B}(v, u)\|_{W^*} \le C\|u\|_V \|v\|_W \tag{20}$$

$$\|\hat{B}(u, u)\|_{W^*} \le C\|u\|_V^2. \tag{21}$$

Lastly, for $u \in W,\ v \in V,\ w \in W$

$$\langle \hat{B}(u, v), w\rangle_{W^*,W} = -\langle \hat{B}(u, w), v\rangle_{W^*,W}. \tag{22}$$

We need a basis orthonormal either in W and in V in order to deal with the Galerkin approximation of Eq. (9). The existence of such basis is guaranteed by the lemma below. The first part is a consequence of the spectral theorem for self-adjoint compact operators stated in [35], we refer to [6, Lemma 4.1] for the proof of the second part.

Lemma 17 *The injection of W into V is compact. Let I be the isomorphism of W^* onto W, then the restriction of I to V is a continuous compact operator into itself. Thus, there exists a sequence e_i of elements of W which forms an orthonormal basis in W, and an orthogonal basis in V. This sequence verifies:*

$$\text{for any } v \in W \quad \langle v, e_i\rangle_W = \lambda_i \langle v, e_i\rangle_V \tag{23}$$

where $\lambda_{i+1} > \lambda_i > 0,\ i = 1, 2, \ldots$. Thus $\sqrt{\lambda_i} e_i$ is an orthonormal basis of V. Moreover e_i belong to $H^4(D; \mathbb{R}^2)$.

We will use also some properties of the projection operator P and the solution map of the Stokes operator. We refer to [42] for the proof of the lemmas below.

Lemma 18 *The restriction of the projection operator $P : L^2(D; \mathbb{R}^2) \to H$ to $H^r(D; \mathbb{R}^2)$ is a continuous and linear map between $H^r(D; \mathbb{R}^2)$ and itself.*

Lemma 19 *Let $f \in H^m(D; \mathbb{R}^2)$. Then, there exists a unique couple (u, p), with p defined up to an additive constant, solution of*

$$\begin{aligned} u - \alpha^2 \Delta u + \nabla p &= f \\ \mathrm{div}\, u &= 0 \\ u|_{\partial D} &= 0. \end{aligned}$$

Moreover $u = (I - \alpha^2 A)^{-1} f \in H^{m+2}(D; \mathbb{R}^2),\ p \in H^{m+1}(D)$,

$$\|u\|_{H^{m+2}} + \|p\|_{H^{m+1}} \le C\|f\|_{H^m}.$$

Lemma 20 *The injection of V in H is compact. Thus there exists a sequence $\tilde{e}_i$ of elements of H which forms an orthonormal basis in H and an orthogonal basis in V. This sequence verifies*

$$-A\tilde{e}_i = \tilde{\lambda}_i \tilde{e}_i$$

where $\tilde{\lambda}_{i+1} > \tilde{\lambda}_i > 0,\ i = 1, 2, \ldots$. Moreover $\tilde{\lambda}_i \to +\infty$. Lastly $\tilde{e}_i \in C^\infty(\overline{D}; \mathbb{R}^2)$ under our assumptions on D

Combining Lemmas 17 and 19 above, it follows that for each $f \in H^1$, $i \in \mathbb{N}$

$$\langle (I - \alpha^2 A)^{-1} f, e_i \rangle_W = \lambda_i \langle (I - \alpha^2 A)^{-1} f, e_i \rangle_V = \lambda_i \langle f, e_i \rangle. \tag{24}$$

Moreover, Lemmas 18, 19, 20 above allow us to prove some useful estimates that will be exploited along the paper. We will need Corollary 22 in order to evaluate the regularity of the linear operators appearing in Eq. (9). Instead we will need Lemma 21 in order to quantify explicitly the dependence from α in several embeddings and operators. This will be crucial in Sects. 4 and 5.

We recall first that by Poincaré inequality, Eq. (4), triangle inequality and Eq. (5) the following relations hold:

$$\begin{aligned} \|u\|_{H^3}^2 &\le C(\|\nabla u\|_{L^2}^2 + \|\mathrm{curl}(u - \Delta u)\|_{L^2}^2) \\ &\le C\left(\frac{\alpha^2+1}{\alpha^2}\right)^2 \|\nabla u\|_{L^2}^2 + \frac{C}{\alpha^4}\|\mathrm{curl}(u - \alpha^2 \Delta u)\|_{L^2}^2 \end{aligned} \tag{25}$$

$$\|\nabla u\|_{L^2}^2 \le \frac{\|u\|_V^2}{\alpha^2}. \tag{26}$$

Before going on with the statements of Lemma 21 and Corollary 22 we recall the definitions of the linear operators, F, $\{G^k\}_{k\in K}$, appearing in Eq. (9):

$$F(u) = \frac{1}{2}\sum_{k\in K} P(\sigma_k \cdot \nabla((I - \alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u))), \quad G^k(u) = P(\sigma_k \cdot \nabla u).$$

Lemma 21 *Let $h \in H,\ u \in V,\ w \in V \cap H^2$, then*

$$\|G^k(u)\| \le \|\sigma_k\|_{L^\infty} \|\nabla u\|_{L^2} \le \|\sigma\|_{L^\infty} \frac{\|u\|_V}{\alpha}, \tag{27}$$

$$\|\nabla G^k(w)\|_{L^2} \le C\|\sigma_k\|_{W^{1,\infty}} \|w\|_{H^2}, \tag{28}$$

$$\|(I - \alpha^2 A)^{-1} h\| \le \|h\|, \tag{29}$$

$$\|(-A)^{1/2}(I - \alpha^2 A)^{-1} h\| \le \frac{1}{2\alpha}\|h\|, \tag{30}$$

$$\|-A(I - \alpha^2 A)^{-1} h\| \le \frac{1}{\alpha^2}\|h\|, \tag{31}$$

$$\|(I - \alpha^2 A)^{-1}(P(\sigma_k \cdot \nabla w))\|_W \le C\|\sigma_k\|_{W^{1,\infty}} \|w\|_{H^2}. \tag{32}$$

Therefore, if $u \in V$ the following inequalities hold true

$$\|(I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u)\| \leq \|\sigma_k\|_{L^\infty} \|\nabla u\|_{L^2} \leq \|\sigma_k\|_{L^\infty} \frac{\|u\|_V}{\alpha}, \tag{33}$$

$$\|\nabla (I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u)\|_{L^2} \leq \frac{\|\sigma_k\|_{L^\infty}}{2\alpha} \|\nabla u\|_{L^2}, \tag{34}$$

$$\|P(\sigma_k \cdot \nabla((I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u)))\| \leq \frac{\|\sigma_k\|_{L^\infty}^2}{2\alpha} \|\nabla u\|_{L^2}. \tag{35}$$

$$\|P(\sigma_k \cdot \nabla((I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u)))\|_{H^1} \leq \frac{C\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\|\nabla u\|_{L^2}}{\alpha^2}. \tag{36}$$

Proof Inequalities (27), (28) are trivial. Indeed, by Lemma 18 it holds

$$\|G^k(u)\| = \|P(\sigma_k \cdot \nabla u)\| \leq \|\sigma_k \cdot \nabla u\|_{L^2} \leq \|\sigma_k\|_{L^\infty} \|\nabla u\|_{L^2} \leq \|\sigma_k\|_{L^\infty} \frac{\|u\|_V}{\alpha}.$$

$$\begin{aligned}\|\nabla G^k(w)\|_{L^2} &= \|\nabla P(\sigma_k \cdot \nabla w)\|_{L^2} \\ &\leq C\|\sigma_k \cdot \nabla w\|_{H^1} \\ &\leq C\|\sigma_k\|_{W^{1,\infty}} \|\nabla w\|_{H^1} \\ &\leq C\|\sigma_k\|_{W^{1,\infty}} \|w\|_{H^2}.\end{aligned}$$

In order to prove inequalities (29), (30), (31) we exploit the Fourier decomposition $h = \sum_{i\in\mathbb{N}} \langle h, \tilde{e}_i\rangle \tilde{e}_i$. Therefore it holds

$$\|(I-\alpha^2 A)^{-1} h\|^2 = \sum_{i\in\mathbb{N}} \frac{\langle h, \tilde{e}_i\rangle^2}{(1+\alpha^2\tilde{\lambda}_i)^2} \leq \|h\|^2,$$

$$\|(-A)^{1/2}(I-\alpha^2 A)^{-1} h\|^2 = \sum_{i\in\mathbb{N}} \frac{\tilde{\lambda}_i}{(1+\alpha^2\tilde{\lambda}_i)^2} \langle h, \tilde{e}_i\rangle^2 \leq \frac{1}{4\alpha^2}\|h\|^2,$$

$$\|-A(I-\alpha^2 A)^{-1} h\|^2 = \sum_{i\in\mathbb{N}} \frac{\tilde{\lambda}_i^2}{(1+\alpha^2\tilde{\lambda}_i)^2} \langle h, \tilde{e}_i\rangle^2 \leq \frac{1}{\alpha^4}\|h\|^2.$$

For what concerns inequality (32), by definition of the norm in the space W it holds

$$\begin{aligned}\|(I-\alpha^2 A)^{-1}(P(\sigma_k \cdot \nabla w))\|_W^2 &= \|(I-\alpha^2 A)^{-1}(P(\sigma_k \cdot \nabla w))\|_V^2 \\ &\quad + \|\operatorname{curl}((I-\alpha^2\Delta)(I-\alpha^2 A)^{-1} \\ &\quad (P(\sigma_k \cdot \nabla w)))\|_{L^2}^2.\end{aligned}$$

From Lemma 19, we know that

$$\begin{aligned}\|(I-\alpha^2 A)^{-1}(P(\sigma_k \cdot \nabla w))\|_V^2 &= \langle P(\sigma_k \cdot \nabla w), (I-\alpha^2 A)^{-1}(P(\sigma_k \cdot \nabla w))\rangle \\ &\leq \|\sigma_k\|_{L^\infty}\|\nabla w\|_{L^2}\|(I-\alpha^2 A)^{-1}(P(\sigma_k \cdot \nabla w))\|_V\end{aligned}$$

$$
\begin{aligned}
&\leq \|\sigma_k\|_{L^\infty}^2 \|\nabla w\|_{L^2}^2, &(37)\\
&\|\mathrm{curl}\left((I-\alpha^2\Delta)(I-\alpha^2 A)^{-1}(P(\sigma_k\cdot\nabla w))\right)\|_{L^2}\\
&= \|\mathrm{curl}(P(\sigma_k\cdot\nabla w))\|_{L^2} \leq C\|\sigma_k\|_{W^{1,\infty}}\|w\|_{H^2} &(38)
\end{aligned}
$$

Combining (37) and (38), inequality (32) follows.

Combining relation (27) with relations (29) and (30), inequalities (33) and (34) follow immediately. Let us now prove Eq. (35). By Hölder's inequality and relation (28) we have

$$
\begin{aligned}
\|P(\sigma_k\cdot\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u)))\| &\leq \|\sigma_k\cdot\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u))\|_{L^2}\\
&\leq \|\sigma_k\|_{L^\infty}\|\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u))\|_{L^2}\\
&\leq \frac{\|\sigma_k\|_{L^\infty}^2}{2\alpha}\|\nabla u\|_{L^2}.
\end{aligned}
$$

For what concerns the last one, by Lemma 18, 19 and relations (31) it holds

$$
\begin{aligned}
&\|P(\sigma_k\cdot\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u)))\|_{H^1}\\
&\quad\leq C\|\sigma_k\cdot\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u))\|_{H^1}\\
&\quad\leq C\|\sigma_k\|_{W^{1,\infty}}\|\nabla((I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u))\|_{H^1}\\
&\quad\leq C\|\sigma_k\|_{W^{1,\infty}}\|(I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u)\|_{H^2}\\
&\quad\leq C\|\sigma_k\|_{W^{1,\infty}}\|A(I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u)\|\\
&\quad\leq \frac{C}{\alpha^2}\|\sigma_k\|_{W^{1,\infty}}\|P(\sigma_k\cdot\nabla u)\|\\
&\quad\leq \frac{C\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\|\nabla u\|_{L^2}}{\alpha^2}.
\end{aligned}
$$

Corollary 22 *It holds*

$$
G^k \in \mathcal{L}(V;H), \quad F\in\mathcal{L}(V;H\cap H^1(D;\mathbb{R}^2)).
$$

In particular

$$
\|G^k(u)\| \leq \|\sigma_k\|_{L^\infty}\frac{\|u\|_V}{\alpha}, \tag{39}
$$

$$
\|F(u)\| \leq \frac{1}{2\alpha}\sum_{k\in K}\|\sigma_k\|_{L^\infty}^2\|\nabla u\|_{L^2} \tag{40}
$$

$$
\|F(u)\|_{H^1} \leq \frac{C}{\alpha^2}\sum_{k\in K}\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\|\nabla u\|_{L^2}. \tag{41}
$$

Lastly we recall two technical tools used in the proof of Theorem 9. We refer to [15] for the proof of the interpolation inequality and to [38] for the proof of the stochastic Grönwall's Lemma.

Theorem 23 *Each function $f \in H^2$ satisfies the following inequality:*

$$\|f\|_{H^1} \leq C\|f\|_{L^2}^{1/2}\|f\|_{H^2}^{1/2}. \tag{42}$$

Theorem 24 *Let $Z(t)$ and $H(t)$ be continuous, nonnegative, adapted processes, $\psi(t)$ a nonnegative deterministic function and $M(t)$ a continuous local martingale such that*

$$Z(t) \leq \int_0^t \psi(s)Z(s)ds + M(t) + H(t) \quad \forall t \in [0, T].$$

Then $Z(t)$ satisfies the following inequality

$$\mathbb{E}[Z(t)] \leq exp\left(\int_0^t \psi(s)ds\right)\mathbb{E}\left[\sup_{r\in[0,s]} H(s)\right]. \tag{43}$$

3.2 Galerkin approximation and limit equations

Let $W^N = \text{span}\{e_1, \ldots, e_N\} \subseteq W$ and $P^N : W \to W^N$ the orthogonal projector. We start looking for a finite dimension approximation of the solution of Eq. (9). We define

$$u^N(t) = \sum_{i=1}^N c_{i,N}(t)e_i.$$

The $c_{i,N}$ have been chosen in order to satisfy $\forall e_i, \quad 1 \leq i \leq N$

$$\begin{aligned}\langle u^N(t), e_i\rangle_V - \langle u_0^N, e_i\rangle_V = \nu \int_0^t \langle \nabla u^N(s), \nabla e_i\rangle_{L^2} ds \\ - \int_0^t b(u^N(s), u^N(s) - \alpha^2\Delta u^N(s), e_i)ds \\ - \alpha^2 \int_0^t b(e_i, \Delta u^N(s), u^N(s))ds + \int_0^t \langle F^N(s), e_i\rangle ds \\ + \sum_{k\in K}\int_0^t \langle G^{k,N}(s), e_i\rangle dW_s^k \quad \mathbb{P} - a.s.\end{aligned}$$

where $u_0^N = \sum_{i=1}^N \langle u_0, e_i\rangle_W e_i$, $F^N(s) = F(u^N(s))$ and $G^{k,N}(s) = G^k(u^N(s))$. The local well-posedness of this equation follows from classical results about stochastic differential equations with locally Lipshitz coefficients, see for example [20, 40]. The global well-posedness follows from the a priori estimates in Lemmas 25, 26.

Lemma 25 *Assuming Hypothesis 2, the following relations hold:*

- *The Itô's formula*

$$
\begin{aligned}
d\|u^N\|_V^2 &= -2\nu\|\nabla u^N\|_{L^2}^2 dt \\
&\quad - \sum_{k\in K} b(\sigma_k, u^N, (I-P^N)(I-\alpha^2 A)^{-1} P(\sigma_k\cdot\nabla u^N))dt. \qquad (44)
\end{aligned}
$$

- *The inequality below holds uniformly in N*

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\|u^N(t)\|_V^p\right] \le C_{p,\alpha,u_0,\{\sigma_k\}_{k\in K}}, \quad \forall p\ge 1. \qquad (45)
$$

Proof If we apply the Itô's formula to $\sum_{i=1}^N \lambda_i \langle u^N(t), e_i\rangle_V^2$, we get

$$
\begin{aligned}
\|u^N(t)\|_V^2 + 2\nu\int_0^t \|\nabla u^N(s)\|_{L^2}^2 ds &= \|u_0^N\|_V^2 + 2\int_0^t \langle F^N(s), u^N(s)\rangle ds \\
&\quad + \sum_{i=1}^N\sum_{k\in K}\lambda_i\int_0^t \langle G^{k,N}(s), e_i\rangle^2 ds \\
&\quad + 2\sum_{k=1}^N\int_0^t \langle G^{k,N}(s), u^N(s)\rangle dW_s^k. \qquad (46)
\end{aligned}
$$

In the last relation we exploited the fact that $b(u^N(s), u^N(s), u^N(s)) = b(u^N(s), \Delta u^N(s), u^N(s)) = 0$. Now we observe that for each k, $\langle G^{k,N}(s), u^N(s)\rangle = 0$. In fact

$$
\langle G^{k,N}(s), u^N(s)\rangle = b(\sigma_k, u^N(s), u^N(s)) = 0. \qquad (47)
$$

Moreover we have

$$
\begin{aligned}
&2\int_0^t \langle F^N(s), u^N(s)\rangle ds + \sum_{i=1}^N\sum_{k\in K}\lambda_i\int_0^t \langle G^{k,N}(s), e_i\rangle^2 ds \\
&\quad = -\sum_{k\in K} b(\sigma_k, u^N(s), (I-P^N)(I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u^N(s))). \qquad (48)
\end{aligned}
$$

In fact,

$$
\sum_{i=1}^N \lambda_i\langle G^{k,N}(s), e_i\rangle^2 = b\left(\sigma_k, u^N(s), \sum_{i=1}^N \lambda_i e_i b(\sigma_k, u^N(s), e_i)\right).
$$

It remains to show that

$$b\left(\sigma_k, u^N(s), \sum_{i=1}^N \lambda_i e_i b(\sigma_k, u^N(s), e_i)\right) + b\left(\sigma_k, (I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^N(s)), u^N\right)$$
$$= -b(\sigma_k, u^N(s), (I-P^N)(I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^N(s))).$$

Thus it is enough to show that $\langle \sum_{i=1}^N \lambda_i e_i b(\sigma_k, u^N(s), e_i), v\rangle_V = \langle (I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^N(s))), v\rangle_V$ for all $v \in V_N$, where $V_N = \text{span}\{e_i\}_{i=1}^N$. The last claim is true, in fact

$$\langle (I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^N(s))), v\rangle_V = b(\sigma_k, u^N(s), v)$$
$$b\left(\sigma_k, u^N(s), \sum_{i=1}^N \lambda_i e_i \langle e_i, v\rangle_v\right) = \left\langle \sum_{i=1}^N \lambda_i e_i b(\sigma_k, u^N(s), e_i), v\right\rangle_V .$$

Therefore, combining Eqs. (46), (47) and (48) we obtain

$$\begin{aligned}
&\|u^N(t)\|_V^2 + 2\nu \int_0^t \|\nabla u^N(s)\|_{L^2}^2 ds \\
&\quad = \|u_0^N\|_V^2 - \sum_{k\in K} \int_0^t b(\sigma_k, u^N(s), (I-P^N)(I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^N(s)))ds \\
&\quad \le \|u_0\|_V^2 \\
&\qquad + \sum_{k\in K} \|\sigma_k\|_{L^\infty} \int_0^t \|\nabla u^N(s)\|_{L^2} \|(I-P^N)(I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^N(s))\| ds \\
&\quad \le \|u_0\|_V^2 + \sum_{k\in K} \|\sigma_k\|_{L^\infty} \int_0^t \|u^N(s)\|_V \|(I-\alpha^2 A)^{-1} P(\sigma_k \cdot \nabla u^N(s))\|_V ds \\
&\quad \le \|u_0\|_V^2 + \sum_{k\in K} \|\sigma_k\|_{L^\infty} \int_0^t \|u^N(s)\|_V \|P(\sigma_k \cdot \nabla u^N(s))\| ds \\
&\quad \le \|u_0\|_V^2 + \frac{1}{\alpha} \sum_{k\in K} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|u^N(s)\|_V^2 ds.
\end{aligned}$$

Thus, by Grönwall,

$$\sup_{t\in[0,T]} \|u^N(t)\|_V^2 \le C_{\alpha,\{\sigma_k\}_{k\in K}} \|u_0\|_V^2. \tag{49}$$

Taking the expected value of Eq. (49) we get the thesis for $p \le 2$. If $p > 2$, raising to the power $p/2$ both sides of Eq. (49) the thesis follows easily.

Lemma 26 *Assuming Hypothesis 2, the following relation holds:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u^N(t)\|_W^p\right] \le C_{p,\nu,\alpha,u_0,\{\sigma_k\}_{k\in K}}, \quad \forall p \ge 1 \tag{50}$$

where $C_{p,\nu,\alpha,\{\sigma_k\}_{k\in K}}$ is a constant independent from N.

Proof This proof is similar to Lemmas 2.4–2.5 of [33]. We will need some changes due to the poor regularity of the coefficients F and G^k. In the part where we will not need any changes, we will refer to the equations in [33]. Let

$$\tau_M^N = \inf\{t :\ \|u^N(t)\|_V + \|u^N(t)\|_* \geq M\} \wedge T$$

and $\tilde{G}^{k,N} = (I - \alpha^2 A)^{-1} G^{k,N}$ the solution of Stokes problem defined in Lemma 19. From the regularity of the eigenvectors e_i, $G^{k,N} \in H^1$, thus $\tilde{G}^{k,N} \in W$ and by Eqs. (24) and (32) the following relations hold true

$$\langle \tilde{G}^{k,N}, e_i\rangle_W = \lambda_i \langle G^{k,N}, e_i\rangle \tag{51}$$

$$\|\tilde{G}^{k,N}\|_W \leq C\|\sigma_k\|_{W^{1,\infty}}\|u^N\|_{H^2}. \tag{52}$$

Let us call

$$\phi^N = -\nu\Delta u^N + \mathrm{curl}(u^N - \alpha^2\Delta u^N) \times u^N - F^N.$$

From the regularity of the e_i, we have that $\phi^N \in H^1$. Thus we can find a $v^N \in W$ such that $v^N = (I - \alpha^2 A)^{-1}\phi^N$. We rewrite shortly the weak formulation satisfied by u^N

$$d\langle u^N, e_i\rangle_V + \langle \phi^N, e_i\rangle dt = d\langle u^N, e_i\rangle_V + \langle v^N, e_i\rangle_V dt = \sum_{k\in K}\langle G^{k,N}, e_i\rangle dW_t^k.$$

Multiplying each equation by λ_i we get

$$d\langle u^N, e_i\rangle_W + \langle v^N, e_i\rangle_W dt = \sum_{k\in K}\langle \tilde{G}^{k,N}, e_i\rangle_W dW_t^k.$$

Now we apply the Itô's formula to $\sum_{i=1}^N \langle u^N, e_i\rangle_W^2$ and we obtain

$$\begin{aligned} d(\|u^N\|_V^2 + \|u^N\|_*^2) \\ &+ 2\left(\langle v^N, u^N\rangle_V + \langle \mathrm{curl}(u^N - \alpha^2\Delta u^N), \mathrm{curl}(v^N - \alpha^2\Delta v^N)\rangle_{L^2}\right) dt \\ &= 2\sum_{k\in K}\langle \mathrm{curl}(\tilde{G}^{k,N} - \alpha^2\Delta\tilde{G}^{k,N}), \mathrm{curl}(u^N - \alpha^2\Delta u^N)\rangle_{L^2} dW_t^k \\ &+ \sum_{k\in K}\sum_{i=1}^N \lambda_i^2\langle \tilde{G}^{k,N}, e_i\rangle_V^2\, dt + 2\sum_{k\in K}\langle \tilde{G}^{k,N}, u^N\rangle_V dW_t^k. \end{aligned}$$

Exploiting the definition of v^N, $\tilde{G}^{k,N}$, Eq. (47) and the classical fact that $\mathrm{curl}\,\nabla = 0$ we get

$$
\begin{aligned}
&d(\|u^N\|_V^2 + \|u^N\|_*^2) + 2\left(\langle \phi^N, u^N\rangle + \langle \mathrm{curl}(\phi^N), \mathrm{curl}(u^N - \alpha^2\Delta u^N)\rangle_{L^2}\right) dt \\
&\quad = 2\sum_{k\in K}\langle \mathrm{curl}(G^{k,N}), \mathrm{curl}(u^N - \alpha^2\Delta u^N)\rangle_{L^2} dW_t^k + \sum_{k\in K}\sum_{i=1}^{N}\lambda_i^2\langle G^{k,N}, e_i\rangle^2\, dt.
\end{aligned} \tag{53}
$$

From Lemma 25 we already know that

$$
\begin{aligned}
d\|u^N\|_V^2 &= -2\nu\|\nabla u^N\|_{L^2}^2 dt \\
&\quad - \sum_{k\in K} b(\sigma_k, u^N, (I-P^N)(I-\alpha^2 A)^{-1}P(\sigma_k\cdot\nabla u^N))dt.
\end{aligned}
$$

Substituting this relation in the Itô's formula (53) we get

$$
\begin{aligned}
&d(\|u^N\|_*^2) + 2\left(\langle \mathrm{curl}(\phi^N), \mathrm{curl}(u^N - \alpha^2\Delta u^N)\rangle_{L^2}\right) dt \\
&\quad = 2\sum_{k\in K}\langle \mathrm{curl}(G^{k,N}), \mathrm{curl}(u^N - \alpha^2\Delta u^N)\rangle_{L^2} dW_t^k \\
&\qquad + \sum_{k\in K}\sum_{i=1}^{N}(\lambda_i + \lambda_i^2)\langle G^{k,N}, e_i\rangle^2\, dt.
\end{aligned} \tag{54}
$$

Analogously to Eq. (4.48) in [33], the relation below holds true

$$
\begin{aligned}
&\langle \mathrm{curl}\,\phi^N, \mathrm{curl}(u^N - \alpha^2\Delta u^N)\rangle_{L^2} \\
&\quad = \frac{\nu}{\alpha^2}\|u^N\|_*^2 - \frac{\nu}{\alpha^2}\left\langle \mathrm{curl}\,u^N + \frac{\alpha^2}{\nu}\mathrm{curl}\,F^N, \mathrm{curl}(u^N - \alpha^2\Delta u^N)\right\rangle_{L^2}.
\end{aligned}
$$

Using this relation in the Itô's formula (54) and integrating between 0 and $t \le \tau_M^N$ we get

$$
\begin{aligned}
&\|u^N(t)\|_*^2 + \frac{2\nu}{\alpha^2}\int_0^t \|u^N(s)\|_*^2 - \sum_{k\in K}\sum_{i=1}^{N}(\lambda_i + \lambda_i^2)\langle G^{k,N}(s), e_i\rangle^2 ds \\
&\quad = \|u_0^N\|_*^2 + \int_0^t \frac{2\nu}{\alpha^2}\langle \mathrm{curl}\,u^N(s) + \frac{\alpha^2}{\nu}\mathrm{curl}\,F^N(s), \mathrm{curl}(u^N(s) - \alpha^2\Delta u^N(s))\rangle_{L^2} ds \\
&\qquad + 2\sum_{k\in K}\int_0^t \langle \mathrm{curl}(G^{k,N}(s)), \mathrm{curl}(u^N(s) - \alpha^2\Delta u^N(s))\rangle_{L^2} dW_s^k \\
&\quad \le \|u_0^N\|_*^2 + \int_0^t \frac{2\nu}{\alpha^2}\|\mathrm{curl}\,u^N(s)\|_{L^2}\|u^N(s)\|_* ds + 2\int_0^t \|\mathrm{curl}\,F^N(s)\|_{L^2}\|u^N(s)\|_* ds \\
&\qquad + 2\left|\sum_{k\in K}\int_0^t \langle \mathrm{curl}(G^{k,N}(s)), \mathrm{curl}(u^N(s) - \alpha^2\Delta u^N(s))\rangle_{L^2} dW_s^k\right|.
\end{aligned} \tag{55}
$$

Taking the supremum between 0 and $r \wedge \tau_M^N$ in relation (55) and, then, the expected value we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]+\frac{2\nu}{\alpha^2}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2ds\right]\\
&\quad\le 2\mathbb{E}\left[\|u_0^N\|_*^2\right]+\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\frac{4\nu}{\alpha^2}\|\mathrm{curl}\,u^N(s)\|_{L^2}\|u^N(s)\|_*ds\right]\\
&\qquad+4\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\mathrm{curl}\,F^N(s)\|_{L^2}\|u^N(s)\|_*ds\right]\\
&\qquad+4\mathbb{E}\left[\left|\sum_{k\in K}\int_0^{r\wedge\tau_M^N}\langle\mathrm{curl}(G^{k,N}(s)),\mathrm{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}dW_s^k\right|\right]\\
&\quad\le 2\mathbb{E}\left[\|u_0^N\|_*^2\right]\\
&\qquad+4\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\left|\sum_{k\in K}\int_0^t\langle\mathrm{curl}(G^{k,N}(s)),\mathrm{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}dW_s^k\right|\right]\\
&\qquad+\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\left(\frac{2\nu\epsilon_1}{\alpha^2}+2\epsilon_2\right)\|u^N(s)\|_*^2ds\right]+\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\frac{2\nu}{\alpha^2\epsilon_1}\|\mathrm{curl}\,u^N(s)\|_{L^2}^2ds\right]\\
&\qquad+\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\frac{2}{\epsilon_2}\|\mathrm{curl}\,F^N(s)\|_{L^2}^2ds\right]\\
&\qquad+2\sum_{k\in K}\sum_{i=1}^N(\lambda_i+\lambda_i^2)\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\langle G^{k,N}(s),e_i\rangle^2ds\right].
\end{aligned}
\tag{56}
$$

Choosing $\epsilon_1=\frac{1}{4}$ and $\epsilon_2=\frac{\nu}{4\alpha^2}$ we arrive at

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]+\frac{\nu}{\alpha^2}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2ds\right]\\
&\quad\le 2\mathbb{E}\left[\|u_0^N\|_*^2\right]\\
&\qquad+4\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\left|\sum_{k\in K}\int_0^t\langle\mathrm{curl}(G^{k,N}(s)),\mathrm{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}dW_s^k\right|\right]\\
&\qquad+\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\frac{8\nu}{\alpha^2}\|\mathrm{curl}\,u^N(s)\|_{L^2}^2ds\right]+\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\frac{8\alpha^2}{\nu}\|\mathrm{curl}\,F^N(s)\|_{L^2}^2ds\right]\\
&\qquad+2\sum_{k\in K}\sum_{i=1}^N(\lambda_i+\lambda_i^2)\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\langle G^{k,N}(s),e_i\rangle^2ds\right]
\end{aligned}
\tag{57}
$$

From Eqs. (48) and (33) we know that

$$
\begin{aligned}
&\sum_{k\in K}\sum_{i=1}^{N}\lambda_i\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\langle G^{k,N}(s),e_i\rangle^2 ds\right]\\
&\quad=\sum_{k\in K}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N} b(\sigma_k,u^N(s),\,P^N(I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u^N(s)))ds\right]\\
&\quad\le\sum_{k\in K}\|\sigma_k\|_{L^\infty}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}\|(I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u^N(s))\|_V ds\right]\\
&\quad\le\sum_{k\in K}\|\sigma_k\|_{L^\infty}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}\|(I-\alpha^2A)^{-1/2}P(\sigma_k\cdot\nabla u^N(s))\|ds\right]\\
&\quad\le\sum_{k\in K}\|\sigma_k\|_{L^\infty}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2 ds\right].
\end{aligned}\tag{58}
$$

Thanks to Eqs. (51), (52), the interpolation estimate (42) and relation (25) we have

$$
\begin{aligned}
&\sum_{k\in K}\sum_{i=1}^{N}\lambda_i^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\langle G^{k,N}(s),e_i\rangle^2 ds\right]\\
&\quad=\sum_{k\in K}\sum_{i=1}^{N}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\langle\tilde{G}^{k,N}(s),e_i\rangle_W^2 ds\right]\\
&\quad\le\sum_{k\in K}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\tilde{G}^{k,N}(s)\|_W^2 ds\right]\\
&\quad\le C\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_{H^2}^2 ds\right]\\
&\quad\le C\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}\|u^N(s)\|_{H^3} ds\right]\\
&\quad\le C\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\\
&\qquad\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}\left(\frac{\alpha^2+1}{\alpha^2}\|\nabla u^N(s)\|_{L^2}+\frac{1}{\alpha^2}\|u^N(s)\|_*\right)ds\right]\\
&\quad\le\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}C\frac{\alpha^2+1+\frac{1}{\epsilon}}{\alpha^2}\|\nabla u^N(s)\|_{L^2}^2+\frac{\epsilon}{\alpha^2}\|u^N(s)\|_*^2 ds\right].
\end{aligned}\tag{59}
$$

Thanks to Burkholder–Davis–Gundy inequality, Eq. (28), the interpolation inequality (42) and relation (25) we get

$$
\begin{aligned}
&4\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\left|\sum_{k\in K}\int_0^t \langle \operatorname{curl}(G^{k,N}(s)), \operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2} dW_s^k\right|\right]\\
&\leq C\mathbb{E}\left[\left(\sum_{k\in K}\int_0^{r\wedge\tau_M^N}\|\operatorname{curl} G^{k,N}(s)\|_{L^2}^2\|u^N(s)\|_*^2 ds\right)^{1/2}\right]\\
&\leq C\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)(s)\|_*\left(\sum_{k\in K}\int_0^{r\wedge\tau_M^N}\|\operatorname{curl} G^{k,N}(s)\|_{L^2}^2 ds\right)^{1/2}\right]\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]+C\mathbb{E}\left[\sum_{k\in K}\int_0^{r\wedge\tau_M^N}\|\operatorname{curl} G^{k,N}(s)\|_{L^2}^2 ds\right]\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]+C\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_{H^2}^2 ds\right]\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]\\
&\quad+\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N} C\frac{\alpha^2+1+\frac{1}{\epsilon}}{\alpha^2}\|\nabla u^N(s)\|_{L^2}^2+\frac{\epsilon}{\alpha^2}\|u^N(s)\|_*^2 ds\right].
\end{aligned}
\tag{60}
$$

Lastly, thanks to Eq. (41) we have

$$
\begin{aligned}
\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\frac{8\alpha^2}{\nu}\|\operatorname{curl} F^N(s)\|_{L^2}^2 ds\right] &\leq \frac{C}{\nu\alpha^2}\left(\sum_{k\in K}\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\right)^2\\
&\quad \mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2 ds\right].
\end{aligned}
\tag{61}
$$

Combining estimates (58),(59),(60),(61) above we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]+\frac{\nu-\epsilon\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2}{\alpha^2}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2 ds\right]\\
&\leq C\left(\mathbb{E}\left[\|u_0^N\|_*^2\right]+\frac{\alpha^2+1+\frac{1}{\epsilon}}{\alpha^2}\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2\right]\right)\\
&\quad+\frac{C}{\nu\alpha^2}\left(\sum_{k\in K}\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\right)^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2 ds\right].
\end{aligned}
\tag{62}
$$

Therefore, choosing ϵ small enough, by Eq. (49) we have

$$\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]+\frac{\nu}{2\alpha^2}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2ds\right]$$
$$\le C_{\nu,\alpha,\{\sigma_k\}_{k\in K}} \text{ independent from } M,\ N.$$

Last inequality proves the Lemma for $p=2$, letting M to $+\infty$ thanks to monotone convergence Theorem. Now we consider $p\ge 4$ and we restart from Eq. (4.79) in [33].

$$\begin{aligned}\|u^N(t)\|_*^p &\le C\|u_0^N\|_*^p + C\Bigg(\int_0^t \|u^N(s)\|_*^{p/2-2}\\ &\times\Bigg(2\langle \operatorname{curl} F^N(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}\\ &+\frac{1}{2}\sum_{k\in K}\sum_{i=1}^N(\lambda_i+\lambda_i^2)\langle G^{k,N}(s),e_i\rangle^2\\ &+\frac{2\nu}{\alpha^2}\langle \operatorname{curl} u^N(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}-\frac{2\nu}{\alpha^2}\|u^N(s)\|_*^2\\ &+\frac{p-4}{p}\sum_{k\in K}\frac{\langle \operatorname{curl} G^{k,N}(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}^2}{\|u^N(s)\|_*^2}\Bigg)ds\Bigg)^2\\ &+\left(\int_0^t\|u^N(s)\|_*^{p/2-2}\sum_{k\in K}\langle \operatorname{curl} G^{k,N}(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}dW_s^k\right)^2.\end{aligned}$$

Let us consider all the terms, one by one. Arguing as before we have

$$\begin{aligned}&\sum_{k\in K}\sum_{i=1}^N(\lambda_i+\lambda_i^2)\langle G^{k,N}(s),e_i\rangle^2\le C_{\epsilon,\alpha,\{\sigma_k\}_{k\in K}}\|u^N(s)\|_V^2+\epsilon\|u^N(s)\|_*^2,\\ &|\langle \operatorname{curl} u^N(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}|\le C_\alpha(1+\|u^N(s)\|_V)(1+\|u^N(s)\|_W),\\ &|\langle \operatorname{curl} F^N(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}\le C_{\alpha,\{\sigma_k\}_{k\in K}}(1+\|u^N(s)\|_V)(1+\|u^N(s)\|_W),\\ &\left|\frac{\langle \operatorname{curl} G^{k,N}(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}^2}{\|u^N(s)\|_*^2}\right|\le\|\operatorname{curl} G^{k,N}(s)\|_{L^2}^2\\ &\le C_{\epsilon,\alpha,\{\sigma_k\}_{k\in K}}(1+\|u^N(s)\|_V)^2+\epsilon\|u^N(s)\|_W^2.\end{aligned}$$

Exploiting the relations above and the continuous embedding $W\hookrightarrow V$ we get

$$\begin{aligned}\|u^N(t)\|_*^p&\le C\|u_0^N\|_*^p+C_{\epsilon,\nu,\alpha,\{\sigma_k\}_{k\in K}}\left(\int_0^t\|u^N(s)\|_*^{p/2-2}(1+\|u^N(s)\|_W)^2ds\right)^2\\ &+\left(\int_0^t\|u^N(s)\|_*^{p/2-2}\sum_{k\in K}\langle \operatorname{curl} G^{k,N}(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}dW_s^k\right)^2.\end{aligned}$$

Thus taking the supremum in time for $t\le r$ and the expected value of this we get the thesis via Grönwall's Lemma arguing exactly as in the proof of Lemma 4.3 in [33] and exploiting previous estimate (60) on $\langle \operatorname{curl} G^{k,N}(s),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}$.

Remark 27 In case of $\nu = 0$, arguing as above we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right] &\le 2\mathbb{E}\left[\|u_0^N\|_*^2\right] + \mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2 ds\right]\\
&+ 4\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\left|\sum_{k\in K}\int_0^t \langle \mathrm{curl}(G^{k,N}(s)), \mathrm{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2} dW_s^k\right|\right]\\
&+ 4\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\mathrm{curl}\, F^N(s)\|_{L^2}^2 ds\right]\\
&+ 2\sum_{k\in K}\sum_{i=1}^N(\lambda_i+\lambda_i^2)\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\langle G^{k,N}(s), e_i\rangle^2 ds\right]. \qquad (63)
\end{aligned}$$

Therefore, thanks to Lemma 25 and estimates (58),(59),(60),(61) we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right] &\le \frac{\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2}{\alpha^2}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2 ds\right]\\
&+ C\left(\mathbb{E}\left[\|u_0^N\|_*^2\right] + \frac{\alpha^2+1}{\alpha^2}\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2\right]\right)\\
&+ \frac{C}{\alpha^4}\left(\sum_{k\in K}\|\sigma_k\|_{L^\infty}\|\sigma_k\|_{W^{1,\infty}}\right)^2\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2 ds\right]\\
&\le C_{\alpha,\{\sigma_k\}_{k\in K}} + C_{\alpha,\{\sigma_k\}_{k\in K}}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2 ds\right]\\
&\le C_{\alpha,\{\sigma_k\}_{k\in K}} + C_{\alpha,\{\sigma_k\}_{k\in K}}\int_0^r \mathbb{E}\left[\|u^N(s)\|_*^2 1_{[0,\tau_M^N]}(s) ds\right] ds. \qquad (64)
\end{aligned}$$

Since $\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right] = \mathbb{E}\left[\sup_{t\le r}\|u^N(t)\|_*^2 1_{[0,\tau_M^N]}(t)\right]$, by Grönwall's Lemma

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_M^N]}\|u^N(t)\|_W^2\right] \le C_{\alpha,\{\sigma_k\}_{k\in K}} \textit{ independent from } M,\ N.$$

Last inequality proves the Lemma for $p = 2$, letting M to ∞ thanks to monotone convergence Theorem. The case $p \ge 4$ can be treated as in the case $\nu > 0$, therefore we do not add other details.

Let us now introduce the operator $\hat{A} = (I - \alpha^2 A)^{-1}A$. By Lemmas 16 and 19 the weak formulation satisfied by the Galerkin approximations can be rewritten as

$$\begin{aligned}
\langle u^N(t), e_i\rangle_V - \langle u_0^N, e_i\rangle_V &= \nu\int_0^t \langle \hat{A}u^N(s), e_i\rangle_V ds\\
&- \int_0^t \langle \hat{B}(u^N(s), u^N(s)), e_i\rangle_{W^*,W} ds
\end{aligned}$$

$$+\int_0^t \langle F^N(s), e_i\rangle ds$$
$$+\sum_{k\in K}\int_0^t \langle G^{k,N}(s), e_i\rangle dW_s^k \ \ \mathbb{P}-a.s.$$

Thanks to relations (45),(50) and the continuity of $B,\ F$ and G^k, we know that exists a subsequence of the Galerkin approximations, which we will denote again by u^N just for simplicity, and processes u and $\hat{B}^*$ such that

$$\begin{cases} u^N \overset{*}{\rightharpoonup} u \text{ in } L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0,T;W)), \quad p\geq 2\\ u^N \rightharpoonup u \text{ in } L^p(\Omega, \mathcal{F}, \mathbb{P}; L^q(0,T;V)), \quad p,\ q\geq 2\\ \hat{A}u^N \rightharpoonup \hat{A}u \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;V))\\ \hat{B}(u^N,u^N) \rightharpoonup \hat{B}^* \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;W^*))\\ F(u^N) \rightharpoonup F(u) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;H\cap H^1))\\ G^k(u^N) \rightharpoonup G^k(u) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;H)) \end{cases} \tag{65}$$

The next step will be showing that $\hat{B}^* = \hat{B}(u,u)$. In this way the existence of a solution of Eq. (9) will follow. In fact, we know that $\mathbb{P}-a.s.$ for each $i\in\mathbb{N}$, for each $t\in[0,T]$

$$\langle u(t), e_i\rangle_V - \langle u_0, e_i\rangle_V = \nu\int_0^t \langle \hat{A}u(s), e_i\rangle_V ds - \int_0^t \langle \hat{B}^*(s), e_i\rangle_{W^*,W} ds$$
$$+\int_0^t \langle F(u(s)), e_i\rangle ds$$
$$+\sum_{k\in K}\int_0^t \langle G^k(u(s)), e_i\rangle dW_s^k.$$

For what concerns the continuity in V we can argue in the following way via Itô's formula and Kolmogorov continuity Theorem. From the weak formulation above we get the weak continuity in V of u applying the Kolmogorov continuity Theorem for the SDE satisfied by $\langle u(t), e_i\rangle_V$, applying the Itô's formula to $\|u\|_V^2$ we get

$$d\|u\|_V^2 = -2\nu\|\nabla u\|_{L^2}^2 dt - \langle \hat{B}^*, u\rangle_{W^*,W} dt.$$

From this, we get the continuity of $\|u\|_V^2$ thanks to the integrability properties of u. Weak continuity and continuity of the norm implies strong continuity, thus we have the strong continuity of u as a process taking values in V. Weak continuity of u as a process taking values in W follows from Lemma 1.4, p. 263 in [42]. Alternatively the strong continuity in V of u follows arguing as in [2] or [31].

3.3 Existence, uniqueness and further results

To prove the existence of the solutions of Eq. (9) we need the following Lemma. As stated in Sect. 2, this way of proceed has been introduced in [2] for Navier–Stokes equations.

Lemma 28 *Let*

$$\tau_M = \inf\{t \in [0,T] : \|u(t)\|_V + \|u(t)\|_* \geq M\} \wedge T$$

then

$$1_{t \leq \tau_M}(u^N - u) \to 0 \quad in \quad L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;V)).$$

Proof Let P^N be the projection of W on $W^N = \text{span}\{e_1, \ldots, e_N\}$. Thanks to dominated convergence Theorem,

$$\begin{aligned} P^N w \to w \quad &\text{in } L^r(\Omega, L^q(0,T;W)) \quad \text{if} \quad r,\ q \in [1, +\infty) \quad \text{and} \\ &w \in L^r(\Omega, L^q(0,T;W)). \end{aligned} \tag{66}$$

Consequently we have also convergence in $L^r(\Omega, L^q(0,T;V))$. Moreover, if $w \in W$, $i \leq N$, $\langle P^N w, e_i\rangle_V = \langle w, e_i\rangle_V$. Let $\hat{F}(u) = (I - \alpha^2 A)^{-1} F(u)$, $\hat{G}^k(u) = (I - \alpha^2 A)^{-1} G^k(u)$. From the weak formulation satisfied by u, for each $i \leq N$, we get

$$\begin{aligned} \langle P^N u(t), e_i\rangle_V - \langle u_0^N, e_i\rangle_V = \nu \int_0^t \langle P^N \hat{A} u(s), e_i\rangle_V ds - \int_0^t \langle \hat{B}^*(s), e_i\rangle_{W^*,W} ds \\ + \int_0^t \langle \hat{F}(u(s)), e_i\rangle_V ds \\ + \sum_{k \in K} \int_0^t \langle \hat{G}^k(u(s)), e_i\rangle dW_s^k \ \ \mathbb{P} - a.s. \end{aligned}$$

Exploiting the relation satisfied by u^N, we get

$$\begin{aligned} \langle P^N u(t) - u^N(t), e_i\rangle_V = \nu \int_0^t \langle P^N \hat{A}(u(s) - u^N(s)), e_i\rangle_V ds \\ + \int_0^t \langle \hat{B}(u^N(s), u^N(s)) - \hat{B}^*(s), e_i\rangle_{W^*,W} ds \\ + \int_0^t \langle \hat{F}(u(s)) - \hat{F}^N(s), e_i\rangle_V ds \\ + \sum_{k \in K} \int_0^t \langle \hat{G}^k(u(s)) - \hat{G}^{k,N}(s), e_i\rangle dW_s^k \quad \mathbb{P} - a.s. \end{aligned} \tag{67}$$

Thanks to (67), applying the Itô's formula to $\sigma(t)\|P^N u(t)-u^N(t)\|_V^2$, where $\sigma(t) = exp(-\eta_1 t - \eta_2 \int_0^t \|u(s)\|_W^2 ds)$, we obtain

$$\begin{aligned}
&\sigma(t)\|P^N u(t)-u^N(t)\|_V^2 + 2\nu\int_0^t \sigma(s)\langle \hat{A}(u(s)-u^N(s)), P^N u(s)-u^N(s)\rangle_V ds\\
&=\sum_{i=1}^N\sum_{k\in K}\lambda_i\int_0^t \sigma(s)\langle \hat{G}^k(u(s))-\hat{G}^{k,N}(s), e_i\rangle_V^2 ds\\
&+2\sum_{k\in K}\int_0^t \sigma(s)\langle \hat{G}^k(u(s))-\hat{G}^{k,N}(s), P^N u(s)-u^N(s)\rangle_V dW_s^k\\
&-\eta_1\int_0^t \sigma(s)\|P^N u(s)-u^N(s)\|_V^2 ds - \eta_2\int_0^t \sigma(s)\|P^N u(s)-u^N(s)\|_V^2\|u(s)\|_W^2 ds\\
&+2\int_0^t \sigma(s)\langle \hat{B}(u^N(s),u^N(s))-\hat{B}^*(s), P^N u(s)-u^N(s)\rangle_{W^*,W} ds\\
&+2\int_0^t \sigma(s)\langle \hat{F}(u(s))-\hat{F}^N(s), P^N u(s)-u^N(s)\rangle_V ds. \qquad (68)
\end{aligned}$$

Let us analyze the terms in (68) one by one. We will not add details where the computations are analogous to Lemma 3.9 in [34].

$$\begin{aligned}
&\langle \hat{A}(u(s)-u^N(s)), P^N u(s)-u^N(s)\rangle_V\\
&\quad=\langle \hat{A}(u(s)-u^N(s)), u(s)-u^N(s)\rangle_V - \langle \hat{A}(u(s)-u^N(s)), u(s)-P^N u(s)\rangle_V,\\
&\quad\frac{1}{C_p^2+\alpha^2}\|P^N u(s)-u^N(s)\|_V^2 \le \langle \hat{A}(u(s)-u^N(s)), u(s)-u^N(s)\rangle_V,\\
&\quad\langle \hat{F}(u(s))-\hat{F}^N(s), P^N u(s)-u^N(s)\rangle_V\\
&\quad\overset{\text{equation (40)}}{\le} C^F_{\alpha,\{\sigma_k\}_{k\in K}}\|u(s)-u^N(s)\|_V^2 \le 2C^F_{\alpha,\{\sigma_k\}_{k\in K}}\|u^N(s)-P^N u(s)\|_V^2\\
&\quad+2C^F_{\alpha,\{\sigma_k\}_{k\in K}}\|u(s)-P^N u(s)\|_V^2,\\
&\sum_{i=1}^N\sum_{k\in K}\lambda_i\langle \hat{G}^k(u(s))-\hat{G}^{k,N}(s), e_i\rangle_V^2 = \sum_{k\in K}\|P^N(\hat{G}^k(u(s))-\hat{G}^{k,N}(s))\|_V^2\\
&\quad\overset{\text{equation (27)}}{\le} 2C^G_{\alpha,\{\sigma_k\}_{k\in K}}\|u^N(s)-P^N u(s)\|_V^2 + 2C^G_{\alpha,\{\sigma_k\}_{k\in K}}\|u(s)-P^N u(s)\|_V^2,\\
&2\langle \hat{B}(u^N(s),u^N(s))-\hat{B}^*(s), P^N u(s)-u^N(s)\rangle_{W^*,W}\\
&\le C_B^2\|P^N u(s)-u^N(s)\|_V^2\|P^N(s)u(s)\|_W^2 + \|P^N u(s)-u^N(s)\|_V^2\\
&+2\langle \hat{B}(P^N u(s), P^N u(s))-\hat{B}^*(s), P^N u(s)-u^N(s)\rangle_{W^*,W}.
\end{aligned}$$

Inserting these relations in equality (68) we obtain

$$\begin{aligned}
&\sigma(t)\|P^N u(t)-u^N(t)\|_V^2 + \frac{2\nu}{C_p^2+\alpha^2}\int_0^t \sigma(s)\|P^N u(s)-u^N(s)\|_V^2 ds\\
&\le 2\nu\int_0^t \sigma(s)\langle \hat{A}(u(s)-u^N(s)), u(s)-P^N u(s)\rangle_V ds
\end{aligned}$$

$$+\int_0^t ds\sigma(s)(2C^G_{\alpha,\{\sigma_k\}_{k\in K}}\|u^N(s)-P^Nu(s)\|_V^2+2C^G_{\alpha,\{\sigma_k\}_{k\in K}}\|u(s)-P^Nu(s)\|_V^2)$$
$$+2\sum_{k\in K}\int_0^t \sigma(s)\langle \hat{G}^k(u(s))-\hat{G}^{k,N}(s),\, P^Nu(s)-u^N(s)\rangle_V dW^k_s$$
$$-\eta_1\int_0^t \sigma(s)\|P^Nu(s)-u^N(s)\|_V^2ds-\eta_2\int_0^t \sigma(s)\|P^Nu(s)-u^N(s)\|_V^2\|u(s)\|_W^2ds$$
$$+\int_0^t \sigma(s)\Big(C_B^2\|P^Nu(s)-u^N(s)\|_V^2\|P^Nu(s)\|_W^2+\|P^Nu(s)-u^N(s)\|_V^2$$
$$+2\langle \hat{B}(P^Nu(s),\, P^Nu(s))-\hat{B}^*(s),\, P^Nu(s)-u^N(s)\rangle_{W^*,W}\Big)ds$$
$$+\int_0^t \sigma(s)(4C^F_{\alpha,\{\sigma_k\}_{k\in K}}\|u^N(s)-P^Nu(s)\|_V^2+4C^F_{\alpha,\{\sigma_k\}_{k\in K}}\|u(s)-P^Nu(s)\|_V^2)ds.$$

Taking $\eta_1=2C^G_{\alpha,\{\sigma_k\}_{k\in K}}+4C^F_{\alpha,\{\sigma_k\}_{k\in K}}+1$, $\eta_2=C_B^2$ we get

$$\begin{aligned}&\sigma(t)\|P^Nu(t)-u^N(t)\|_V^2+\frac{2\nu}{C_p^2+\alpha^2}\int_0^t \sigma(s)\|P^Nu(s)-u^N(s)\|_V^2ds\\&\leq 2\nu\int_0^t \sigma(s)\langle \hat{A}(u(s)-u^N(s)),u(s)-P^Nu(s)\rangle_V ds\\&\quad+C\int_0^t \sigma(s)\|u(s)-P^Nu(s)\|_V^2ds\\&\quad+2\sum_{k\in K}\int_0^t \sigma(s)\langle \hat{G}^k(u(s))-\hat{G}^{k,N}(s),\, P^Nu(s)-u^N(s)\rangle_V dW^k_s\\&\quad+2\int_0^t \sigma(s)\langle \hat{B}(P^Nu(s),\, P^Nu(s))-\hat{B}^*(s),\, P^Nu(s)-u^N(s)\rangle_{W^*,W}ds.\end{aligned}\tag{69}$$

Considering the expected value of (69) for $t=\tau_M\wedge r$, $r\in[0,T]$, the stochastic integral cancel out, thus we arrive at

$$\begin{aligned}&\mathbb{E}\left[\sigma(\tau_M\wedge r)\|P^Nu(\tau_M\wedge r)-u^N(\tau_M\wedge r)\|_V^2\right]\\&\quad+\frac{2\nu}{C_p^2+\alpha^2}\mathbb{E}\left[\int_0^{\tau_M\wedge r}\sigma(s)\|P^Nu(s)-u^N(s)\|_V^2ds\right]\\&\leq 2\nu\mathbb{E}\left[\int_0^{\tau_M\wedge r}\sigma(s)\langle \hat{A}(u(s)-u^N(s)),u(s)-P^Nu(s)\rangle_V ds\right]\\&\quad+C\mathbb{E}\left[\int_0^{\tau_M\wedge r}\sigma(s)\|u(s)-P^Nu(s)\|_V^2ds\right]\\&\quad+2\mathbb{E}\left[\int_0^{\tau_M\wedge r}\sigma(s)\langle \hat{B}(P^Nu(s),\, P^Nu(s))-\hat{B}^*(s),\, P^Nu(s)-u^N(s)\rangle_{W^*,W}ds\right].\end{aligned}\tag{70}$$

We want understand the behavior of the last term in the inequality above. From Lemma 26 and relation (66) we have

$$P^N u - u^N = (P^N u - u) + (u - u^N) \rightharpoonup 0 \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; W)). \tag{71}$$

Instead we have

$$\left\| 1_{[0,\tau_M \wedge r]} \sigma \left(\hat{B}(P^N u, P^N u) - \hat{B}(u, u) \right) \right\|_{L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T;W^*))} \to 0. \tag{72}$$

In fact thanks to relation (66) and the boundedness properties of $\hat{B}$ (20),(21), $\mathbb{P} - a.s.$ for each $t \in [0, T]$ it holds

$$\begin{aligned} &\left\| (1_{[0,\tau_M \wedge r]} \sigma) \left(\hat{B}(P^N u, P^N u) - \hat{B}(u, u) \right) \right\|_{L^2(0,T;W^*)} \\ &\quad \leq C \|u\|_{L^4(0,T;W)} \|(P^N - I)u\|_{L^4(0,T;W)} \to 0. \end{aligned}$$

Moreover

$$\begin{aligned} &\left\| (1_{[0,\tau_M \wedge r]} \sigma) \left(\hat{B}(P^N u, P^N u) - \hat{B}(u, u) \right) \right\|_{L^2(0,T;W^*)} \\ &\quad \leq C \|u(t)\|_W^2 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T)). \end{aligned}$$

By dominated convergence Theorem we have the validity of relation (72). Combing the weak convergence guaranteed by relation (71) and the strong convergence guaranteed by (72) we obtain

$$\begin{aligned} &2\mathbb{E}\left[\int_0^{\tau_M \wedge r} \sigma(s) \langle \hat{B}(P^N u(s), P^N u(s)) - \hat{B}(u(s), u(s)), P^N u(s) - u^N(s) \rangle_{W^*,W} ds \right] \\ &\quad \to 0. \end{aligned}$$

From this relation, by triangle inequality, we can analyze easily the last term in (70)

$$\begin{aligned} &\mathbb{E}\left[\int_0^{\tau_M \wedge r} \sigma(s) \langle \hat{B}(P^N u(s), P^N u(s)) - \hat{B}^*(s), P^N u(s) - u^N(s) \rangle_{W^*,W} ds \right] \\ &\quad = \mathbb{E}\left[\int_0^{\tau_M \wedge r} \sigma(s) \langle \hat{B}(P^N u(s), P^N u(s)) - \hat{B}(u(s), u(s)), P^N u(s) - u^N(s) \rangle_{W^*,W} ds \right] \\ &\qquad + \mathbb{E}\left[\int_0^{\tau_M \wedge r} \sigma(s) \langle \hat{B}(u(s), u(s)) - \hat{B}^*(s), P^N u(s) - u^N(s) \rangle_{W^*,W} ds \right] \to 0. \end{aligned} \tag{73}$$

Thanks to the boundedness of u^N and relation (66)

$$\begin{aligned} &\mathbb{E}\left[\int_0^{\tau_M \wedge r} \sigma(s) \langle \hat{A}(u(s) - u^N(s)), u(s) - P^N u(s) \rangle_V ds \right] \\ &\quad \leq \|u(s) - P^N u(s)\|_{L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V))} \|\hat{A}(u - u^N)\|_{L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V))} \\ &\quad \leq C \|u - P^N u\|_{L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V))} \to 0. \end{aligned} \tag{74}$$

Combining (73) and (74) in relation (70) we obtain

$$\mathbb{E}\left[\sigma(\tau_M \wedge r)\|P^N u(\tau_M \wedge r) - u^N(\tau_M \wedge r)\|_V^2\right] + \frac{2\nu}{C_p^2+\alpha^2}\mathbb{E}\left[\int_0^{\tau_M\wedge r} \sigma(s)\|P^N u(s) - u^N(s)\|_V^2 ds\right] \to 0. \tag{75}$$

From relation (75), $\sigma(t) \geq C_M > 0 \ \forall t \leq \tau_M$ and the properties of P^N via triangle inequality the thesis follows considering $r = T$.

Remark 29 The proof presented above works only in the case $\nu > 0$. In order to treat the case $\nu = 0$ we start from relation (75). Then, triangle inequality allows to prove

$$\mathbb{E}\left[\sigma(\tau_M \wedge r)\|u(\tau_M \wedge r) - u^N(\tau_M \wedge r)\|_V^2\right] \to 0 \quad \forall r \in [0, T]. \tag{76}$$

By dominated convergence theorem we can improve the pointwise convergence of relation (76) in order to obtain Lemma 28. We omit the easy details at this stage, since this argument will be described in full details in the proof of Corollary 30 below.

Combing Lemma 28 and the moment estimates for u and u^N we get the following Corollary.

Corollary 30 *The subsequence u^N satisfies*

$$\lim_{N\to+\infty} \mathbb{E}\left[\|u^N(t) - u(t)\|_V^2\right] = 0, \tag{77}$$

$$\lim_{N\to+\infty} \int_0^T \mathbb{E}\left[\|u^N(t) - u(t)\|_V^2\right] dt = 0. \tag{78}$$

Proof By relation (75) and triangle inequality we already know that

$$\lim_{N\to+\infty} \mathbb{E}\left[\|u^N(t\wedge\tau_M) - u(t\wedge\tau_M)\|_V^2\right] = 0, \tag{79}$$

$$\lim_{N\to+\infty} \mathbb{E}\left[\int_0^{T\wedge\tau_M} \|u^N(t) - u(t)\|_V^2 dt\right] = 0. \tag{80}$$

We start proving convergence (77). By definition of τ_M, Lemma 26 and the weak-$*$ convergence of u^N to u described by relation (65) and Markov's inequality it follows that

$$\begin{aligned}
&\mathbb{E}\left[\|u^N(t) - u(t)\|_V^2\right] \\
&= \mathbb{E}\left[\|u^N(t) - u(t)\|_V^2 1_{\tau_M \geq t}\right] + \mathbb{E}\left[\|u^N(t) - u(t)\|_V^2 1_{\tau_M < t}\right] \\
&= \mathbb{E}\left[\|u^N(t\wedge\tau_M) - u(t\wedge\tau_M)\|_V^2 1_{\tau_M \geq t}\right] + \mathbb{E}\left[\|u^N(t) - u(t)\|_V^2 1_{\tau_M < t}\right] \\
&\leq \mathbb{E}\left[\|u^N(t\wedge\tau_M) - u(t\wedge\tau_M)\|_V^2\right] + \mathbb{E}\left[\|u^N(t) - u(t)\|_V^4\right]^{1/2} \mathbb{P}(\tau_M < t)^{1/2}
\end{aligned}$$

$$\leq \mathbb{E}\left[\|u^N(t\wedge\tau_M)-u(t\wedge\tau_M)\|_V^2\right]$$
$$+ C\,\sup_{N\in\mathbb{N}}\mathbb{E}\left[\|u^N(t)\|_W^4\right]^{1/2}\mathbb{P}(\sup_{t\in[0,T]}\|u(t)\|_W > M)^{1/2}$$
$$\leq \mathbb{E}\left[\|u^N(t\wedge\tau_M)-u(t\wedge\tau_M)\|_V^2\right]+\frac{C}{M^2}\sup_{N\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^N(t)\|_W^4\right]$$
$$\leq \mathbb{E}\left[\|u^N(t\wedge\tau_M)-u(t\wedge\tau_M)\|_V^2\right]+\frac{C_{\nu,\alpha,u_0,\{\sigma_k\}_{k\in K}}}{M^2},$$

where $C_{\nu,\alpha,u_0,\{\sigma_k\}_{k\in K}}$ is a constant independent from M and N. If we fix $\epsilon>0$ and choose M large enough such that $\frac{C_{\nu,\alpha,u_0,\{\sigma_k\}_{k\in K}}}{M^2}\leq\epsilon$ then by relation (79) we have

$$\limsup_{N\to+\infty}\mathbb{E}\left[\|u^N(t)-u(t)\|_V^2\right]\leq\epsilon.$$

From the arbitrariness of ϵ, the first thesis follows. In order to obtain the other convergence we apply dominated convergence Theorem. Indeed, by relation (77) we already know that for each $t\in[0,T]$

$$\mathbb{E}\left[\|u^N(t)-u(t)\|_V^2\right]\to 0.$$

Moreover, by Lemma 25, for each N

$$\mathbb{E}\left[\|u^N(t)-u(t)\|_V^2\right]\leq 2\mathbb{E}\left[\|u^N(t)\|_V^2\right]+2\mathbb{E}\left[\|u(t)\|_V^2\right]$$
$$\leq C_{p,\alpha,u_0\{\sigma_k\}_{k\in K}}+2\mathbb{E}\left[\|u(t)\|_V^2\right]\in L^1(0,T).$$

Therefore convergence (78) follows.

From Lemma 28, without any change with respect to the proof of Lemma 3.8 in [34], we have that the Lemma below holds, thus u is a solution of problem (9) in the sense of Definition 3.

Lemma 31 $\hat{B}^*=\hat{B}(u,u)$ *in* $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;W^*))$

Now we can prove the uniqueness.

Theorem 32 *The solution of problem* (9) *in the sense of Definition* 3 *is unique.*

Proof Let u_1 and u_2 be two solutions. Let w be their difference, then for each $\phi\in W$ and $t>0$

$$\langle w(t),\phi\rangle_V=\nu\int_0^t\langle\nabla w(s),\nabla\phi\rangle_{L^2}ds-\int_0^t b(u_1(s),u_1(s)-\alpha^2\Delta u_1(s),\phi)ds$$
$$-\alpha^2\int_0^t b(\phi,\Delta u_1(s),u_1(s))ds+\int_0^t b(u_2(s),u_2(s)-\alpha^2\Delta u_2(s),\phi)ds$$

$$+\alpha^2\int_0^t b(\phi,\Delta u_2(s),u_2(s))ds+\int_0^t\langle F(w(s)),\phi\rangle ds$$
$$+\sum_{k\in K}\int_0^t\langle G^k(w(s)),\phi\rangle dW_s^k\ \ \mathbb{P}-a.s.$$

Now we apply the Itô's formula to compute $\|w\|_V^2$. Arguing as in the first part of the proof of Lemma 35 we obtain

$$d\|w\|_V^2=-2\nu\|\nabla w\|_{L^2}^2dt+(b(w,w-\alpha\Delta w,u_2)-b(u_2,w-\alpha\Delta w,w))dt.$$

Let us consider $exp(-\int_0^t\|u_2(s)\|_W^2ds)\|w(t)\|_V^2:=\sigma(t)\|w(t)\|_V^2$, via Itô's formula we get

$$d(\sigma\|w\|_V^2)+2\nu\sigma\|\nabla w\|_{L^2}^2dt=-\sigma\|u_2\|_W^2\|w\|_V^2dt$$
$$+\sigma(b(w,w-\alpha\Delta w,u_2)-b(u_2,w-\alpha\Delta w,w))dt$$

Combining relations (16) and (18) it follows that

$$|b(w,w-\alpha^2\Delta w,u_2)-b(u_2,w-\alpha^2\Delta w,w)|\leq C\|w\|_V^2\|u_2\|_W.$$

Therefore

$$d(\sigma\|w\|_V^2)\leq-\sigma\|u_2\|_W^2\|w\|_V^2dt+C\sigma\|w\|_V^2\|u_2\|_Wdt\leq C_\epsilon\sigma\|w\|_V^2,$$

where in the last step we applied Young's inequality. From the last chain of inequalities, via Grönwall's Lemma we get the thesis.

Theorem 33 *The entire Galerkin's sequence u^N satisfies*

$$\lim_{N\to+\infty}\mathbb{E}\left[\|u^N(t)-u(t)\|_V^2\right]=0,$$
$$\lim_{N\to+\infty}\int_0^T\mathbb{E}\left[\|u^N(t)-u(t)\|_V^2\right]dt=0.$$

Proof Each subsequence u^{N_k} has a converging sub-subsequence $u^{N_{k,k}}$ which satisfies all previous Lemmas. By uniqueness of the solution of Eq. (9) and Corollary 30 then the thesis follows.

Remark 34 Theorem 33 plays no role concerning the well-posedness of Eq. (9), but it will be crucial for obtaining the energy estimates of Sect. 4, and thus for proving Theorem 9.

4 Energy estimates

Now we start considering Eq. (14) and assuming also Hypothesis 6. The goal of this section is to prove the following lemma:

Lemma 35 *Under Hypothesis 2–6, if u^α is the solution of problem (14) in the sense of Definition 3, then*

$$\|u^\alpha(t)\|^2 + \alpha^2\|\nabla u^\alpha(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds = \|u_0^\alpha\|^2 + \alpha^2\|\nabla u_0^\alpha\|_{L^2}^2; \quad (81)$$

$$\mathbb{E}\left[\alpha^6 \sup_{t\in[0,T]}\|u^\alpha(t)\|_{H^3}^2\right] = O(1). \quad (82)$$

Proof For the sake of simplicity we write u and u_0 instead of u^α, u_0^α since α is fixed in this proof. Therefore all the asymptotic expansions and limits will be considering $N \to +\infty$.

- Let $\tilde{e}_i$ be the eigenfunctions of the Stokes operator $-A$, and $\tilde{\lambda}_i$ the corresponding eigenvalues introduced in Lemma 20. Let, moreover, $\tilde{u}^N = \sum_{i=1}^N \langle u, \tilde{e}_i\rangle \tilde{e}_i = \tilde{P}^N u$. Exploiting the weak formulation with test functions $\tilde{e}_i$ we get

$$\begin{aligned}
&\langle u(t), \tilde{e}_i\rangle - \alpha^2\langle u(t), A\tilde{e}_i\rangle - \langle u_0, \tilde{e}_i\rangle + \alpha^2\langle u_0, A\tilde{e}_i\rangle \\
&= \nu\int_0^t \langle u(s), A\tilde{e}_i\rangle ds - \int_0^t b(u(s), u(s) - \alpha^2\Delta u(s), \tilde{e}_i) ds \\
&\quad - \alpha^2\int_0^t b(\tilde{e}_i, \Delta u(s), u(s)) ds + \tilde{\nu}\int_0^t \langle F(u), \tilde{e}_i\rangle ds \\
&\quad + \sqrt{\tilde{\nu}}\sum_{k\in K}\int_0^t \langle G^k(u(s)), \tilde{e}_i\rangle dW_s^k \;\; \mathbb{P}-a.s.
\end{aligned}$$

Multiplying each equation by $\tilde{e}_i$ and summing up, we get

$$\begin{aligned}
d(\tilde{u}^N - \alpha^2 A\tilde{u}^N) &= \nu A\tilde{u}^N dt - \sum_{i=1}^N b(u, u - \alpha^2\Delta u, \tilde{e}_i)\, dt \\
&\quad - \alpha^2\sum_{i=1}^N \int_0^t b(\tilde{e}_i, \Delta u, u)\tilde{e}_i\, dt + \tilde{\nu}\sum_{i=1}^N \langle F(u), \tilde{e}_i\rangle \tilde{e}_i\, dt \\
&\quad + \sqrt{\tilde{\nu}}\sum_{k\in K}\sum_{i=1}^N \langle G^k(u), \tilde{e}_i\rangle \tilde{e}_i\, dW_t^k.
\end{aligned}$$

Now we can apply the Itô's formula to the process

$$\frac{1}{2}(\|\tilde{u}^N(t)\|^2 + \alpha^2\|\nabla\tilde{u}^N(t)\|_{L^2}^2) = \frac{1}{2}\langle (I - \alpha^2 A)\tilde{u}^N(t), \tilde{u}^N(t)\rangle$$

obtaining

$$\frac{\|\tilde{u}^N(t)\|^2+\alpha^2\|\nabla\tilde{u}^N(t)\|}{2}=\frac{\|\tilde{u}_0^N\|^2+\alpha^2\|\nabla\tilde{u}_0^N\|}{2}-\nu\int_0^t\langle\nabla\tilde{u}^N(s),\nabla u^N(s)\rangle_{L^2}ds$$
$$-\int_0^t b(u(s),u(s)-\alpha^2\Delta u(s),\tilde{u}^N(s))-\alpha^2\int_0^t b(\tilde{u}^N(s),\Delta u(s),u(s))ds$$
$$+\frac{\tilde{\nu}}{2}\int_0^t\sum_{k\in K}\langle P(\sigma_k\cdot\nabla((I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u(s)))),\tilde{u}^N(s)\rangle ds$$
$$+\sqrt{\tilde{\nu}}\sum_{k\in K}\int_0^t\langle P(\sigma_k\cdot\nabla u(s)),\tilde{u}^N\rangle dW_s^K$$
$$+\frac{\tilde{\nu}}{2}\sum_{k\in K}\int_0^t\sum_{i=1}^N\langle P(\sigma_k\cdot\nabla u(s)),\tilde{e}_i\rangle^2\langle\tilde{e}_i,(I-\alpha^2A)^{-1}\tilde{e}_i\rangle ds.$$

Thanks to the properties of the projector $\tilde{P}^N$ we get easily the first relation. The only thing we need to prove is that

$$\sum_{i=1}^N\langle P(\sigma_k\cdot\nabla u),\tilde{e}_i\rangle^2\langle\tilde{e}_i,(I-\alpha^2A)^{-1}\tilde{e}_i\rangle$$
$$+\langle P(\sigma_k\cdot\nabla((I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u))),\tilde{u}^N\rangle\to 0.$$

The last relation is true, in fact

$$\begin{aligned}&\sum_{i=1}^N\langle P(\sigma_k\cdot\nabla u),\tilde{e}_i\rangle^2\langle\tilde{e}_i,(I-\alpha^2A)^{-1}\tilde{e}_i\rangle\\&\quad+\langle P(\sigma_k\cdot\nabla((I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u))),\tilde{u}^N\rangle\\&=\sum_{i=1}^N\langle P(\sigma_k\cdot\nabla u),(I-\alpha^2A)^{-1/2}\tilde{e}_i\rangle^2\\&\quad+\langle P(\sigma_k\cdot\nabla((I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u))),\tilde{u}^N\rangle\\&\to\langle(I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u),P(\sigma_k\cdot\nabla u)\rangle\\&\quad+\langle P(\sigma_k\cdot\nabla((I-\alpha^2A)^{-1}P(\sigma_k\cdot\nabla u))),u\rangle\\&=0.\end{aligned}$$

- From Theorem 33 and Eq. (5), we know that

$$\begin{aligned}\int_0^T\mathbb{E}\left[\|\nabla u^N(s)\|_{L^2}^2\right]ds&\le\frac{1}{\alpha^2}\int_0^T\mathbb{E}\left[\|u^N(s)\|_V^2\right]ds\\&=\frac{1}{\alpha^2}\int_0^T\mathbb{E}\left[\|u(s)\|_V^2\right]ds+o(1).\end{aligned}$$

Thus, from the Itô formula (81) the following relations hold true:

$$\int_0^T \mathbb{E}\left[\|\nabla u^N(s)\|_{L^2}^2\right] ds \le \frac{C}{\alpha^2}\mathbb{E}\left[\|u_0\|^2\right] + C\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] + o(1) \tag{83}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2\right] \le \frac{1}{\alpha^2}\mathbb{E}\left[\|u_0\|^2\right] + \mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] \tag{84}$$

According to inequality (25), in order to prove relation (82), it remains to study

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u^N(t)\|_*^2\right].$$

Before going on we recall some notation. For each $N \in \mathbb{N}$

$$\tau_M^N = \inf\{t : \|u^N(t)\|_V + \|u^N(t)\|_* \ge M\} \wedge T,$$

Thanks to the scaling factor $\sqrt{\tilde{\nu}}$ appearing in front of the noise and exploiting the asymptotic relation between ν, $\tilde{\nu}$ and α^2 described by Hypothesis 6, if we choose

$$\epsilon = \frac{1}{2\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2},$$

Equation (62) in Lemma 26 becomes

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right] + \mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2 ds\right] \\
&\quad\le C\left(\mathbb{E}\left[\|u_0^N\|_*^2\right] + \left(\alpha^2+1\right)\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2\right]\right).
\end{aligned} \tag{85}$$

Therefore, thanks to Eq. (83), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\wedge\tau_M^N}\|u^N(t)\|_*^2\right] + \mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2 ds\right] \\
&\quad\le C\left(\mathbb{E}\left[\|u_0^N\|_*^2\right] + \left(\alpha^2+1\right)\left(\mathbb{E}\left[\frac{\|u_0\|^2}{\alpha^2}\right] + \mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]\right)\right) + o(1).
\end{aligned} \tag{86}$$

So far we showed that $u^N \in L^2(\Omega; L^2([0,T]; H^1))$, $\operatorname{curl}(u^N - \alpha^2\Delta u^N) \in L^2(\Omega, L^\infty([0,T]; L^2))$. By monotone convergence Theorem, we can remove the dependence from M in relation (86). Therefore

$$\mathbb{E}\left[\sup_{t\le T}\|u^N(t)\|_*^2\right] \le C\left(\mathbb{E}\left[\|u_0^N\|_*^2\right]\right.$$

$$+\left(\alpha^2+1\right)\left(\mathbb{E}\left[\frac{\|u_0\|^2}{\alpha^2}\right]+\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]\right)\bigg)$$
$$+o(1). \tag{87}$$

Thus, by Theorem 33 and the uniform bound (87) there exists a subsequence N_k such that

$$u^{N_k}\to u \text{ in } L^2(\Omega;L^2([0,T];H^1))$$
$$\operatorname{curl}(u^{N_k}-\alpha^2\Delta u^{N_k})\overset{*}{\rightharpoonup} g \text{ in } L^2(\Omega,L^\infty([0,T];L^2)).$$

If we take a test function $\phi\in L^2(\Omega;L^2(0,T;C_c^\infty(D)))$, we get easily

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\langle\phi(s),g(s)\rangle_{L^2}ds\right]\\
&=\lim_{k\to+\infty}\mathbb{E}\left[\int_0^T\langle\phi(s),\operatorname{curl}(u^{N_k}(s)-\alpha^2\Delta u^{N_k}(s))\rangle_{L^2}ds\right]\\
&=\lim_{k\to+\infty}\mathbb{E}\left[\int_0^T\langle(I-\alpha^2\Delta)\nabla^\perp\phi(s),u^{N_k})(s)\rangle_{L^2}ds\right]\\
&=\mathbb{E}\left[\int_0^T\langle(I-\alpha^2\Delta)\nabla^\perp\phi(s),u(s))\rangle_{L^2}ds\right].
\end{aligned}$$

Therefore $g=\operatorname{curl}(u-\alpha^2\Delta u)\in L^2(\Omega,L^\infty([0,T];L^2))$ and the following inequality holds true

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}\|u(t)\|_*^2\right]\le C\bigg(&\liminf_{k\to+\infty}\mathbb{E}\left[\|u_0^{N_k}\|_*^2\right]\\
&+\left(\alpha^2+1\right)\left(\mathbb{E}\left[\frac{\|u_0\|^2}{\alpha^2}\right]+\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]\right)\bigg).
\end{aligned}\tag{88}$$

Let us analyze better the first term. We denote by $u_0^{N,\infty}=u_0-u_0^N$.

$$\begin{aligned}
\|u_0^{N_k}\|_*^2&=\|u_0^{N_k}\|_W^2-\|u_0^{N_k}\|_V^2\\
&\le\|u_0\|_W^2-\|u_0^{N_k}\|_V^2\\
&\le\|u_0\|_*^2+\|u_0^{N_k,\infty}\|_V^2\\
&\le\|u_0\|_*^2+\|u_0\|_V^2\\
&\le C(\|\nabla u_0\|_{L^2}^2+\alpha^4\|\operatorname{curl}\Delta u_0\|_{L^2}^2+\|u_0\|^2+\alpha^2\|\nabla u_0\|_{L^2}^2)\\
&\le C(\|u_0\|^2+(1+\alpha^2)\|\nabla u_0\|_{L^2}^2+\alpha^4\|u_0\|_{H^3}^2).
\end{aligned}$$

In conclusion, combining the observation above, relations (25), (84) and (88) we get

$$\mathbb{E}\left[\sup_{t\leq T}\|u(t)\|_{H^3}^2\right]\leq C\left(\frac{\alpha^4+\alpha^2+1}{\alpha^6}\mathbb{E}\left[\|u_0\|^2\right]+\frac{1+\alpha^2+\alpha^4}{\alpha^4}\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]\right.$$
$$\left.+\mathbb{E}\left[\|u_0\|_{H^3}^2\right]\right). \tag{89}$$

Thanks to the assumptions on u_0^α, see Hypothesis 6, the thesis follows.

Remark 36 In the case $\nu=0$, relation (81) follows without any change with respect to the main proof. For what concerns relation (82), Eq. (85) above is false in this framework. However, introducing the proper scaling in front of the noise we can restart from relation (63) obtaining

$$\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]\leq 2\mathbb{E}\left[\|u_0^N\|_*^2\right]+\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2ds\right]$$
$$+4\sqrt{\tilde{\nu}}\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\left|\sum_{k\in K}\int_0^t\langle \operatorname{curl}(G^{k,N}(s)),\operatorname{curl}(u^N(s)-\alpha^2\Delta u^N(s))\rangle_{L^2}dW_s^k\right|\right]$$
$$+4\tilde{\nu}\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\operatorname{curl}F^N(s)\|_{L^2}^2ds\right]+2\tilde{\nu}\sum_{k\in K}\sum_{i=1}^N(\lambda_i+\lambda_i^2)\mathbb{E}$$
$$\left[\int_0^{r\wedge\tau_M^N}\langle G^{k,N}(s),e_i\rangle^2ds\right]. \tag{90}$$

Therefore, combining estimates (58),(59),(60),(61), exploiting the asymptotic relation between $\tilde{\nu}$ and α^2 described by Hypothesis 6 and choosing $\epsilon=\frac{1}{\sum_{k\in K}\|\sigma_k\|_{W^{1,\infty}}^2}$, we obtain

$$\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]\leq 4\mathbb{E}\left[\|u_0^N\|_*^2\right]+6\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2ds\right]$$
$$+C_{\{\sigma_k\}_{k\in K}}\left(\alpha^2+1\right)\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|\nabla u^N(s)\|_{L^2}^2ds\right]. \tag{91}$$

Therefore, thanks to Eq. (83), we have

$$\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]\leq 4\mathbb{E}\left[\|u_0^N\|_*^2\right]+6\mathbb{E}\left[\int_0^{r\wedge\tau_M^N}\|u^N(s)\|_*^2ds\right]$$
$$+C_{\{\sigma_k\}_{k\in K}}\left(\alpha^2+1\right)\left(\mathbb{E}\left[\frac{\|u_0\|^2}{\alpha^2}\right]+\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]\right)$$
$$+o(1). \tag{92}$$

Arguing as in Remark 27, we can apply Grönwall's Lemma in inequality (92) obtaining

$$\mathbb{E}\left[\sup_{t\leq r\wedge\tau_M^N}\|u^N(t)\|_*^2\right]\leq C_{\{\sigma_k\}_{k\in K}}\left(\mathbb{E}\left[\|u_0^N\|_*^2\right]+\left(\alpha^2+1\right)\right.\\ \left.\left(\mathbb{E}\left[\frac{\|u_0\|^2}{\alpha^2}\right]+\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]\right)\right)+o(1). \tag{93}$$

Relation (93) is completely analogous to relation (86) above. Therefore we can follow the same argument of the main proof in order to obtain estimate (82) and we omit the details.

5 Proof of Theorem 9

In order to prove Theorem 9, we will follow the ideas of [27, 28]. We will start with a weaker result with the supremum in time outside the expected value and then we will move to the stronger one with the supremum in time inside the expected value.

Proof of Theorem 9 Let $W^\alpha=u^\alpha-\bar{u}$, it satisfies $\mathbb{P}-a.s.$ for each $\phi\in H$ and $t\in[0,T]$

$$\begin{aligned}&\langle W^\alpha(t),\phi\rangle-\langle W_0^\alpha,\phi\rangle\\&=\alpha^2\langle Au^\alpha(t),\phi\rangle-\alpha^2\langle Au_0^\alpha,\phi\rangle+\nu\int_0^t\langle Au^\alpha(s),\phi\rangle ds\\&\quad-\int_0^t b(u^\alpha(s),W^\alpha(s),\phi)ds-\int_0^t b(W^\alpha(s),\bar{u}(s),\phi)ds\\&\quad-\alpha^2\int_0^t b(\phi,\Delta u^\alpha(s),u^\alpha(s))ds+\alpha^2\int_0^t b(u^\alpha(s),\Delta u^\alpha(s),\phi)ds\\&\quad\tilde{\nu}\int_0^t\langle F(u^\alpha)(s),\phi\rangle ds+\sqrt{\tilde{\nu}}\sum_{k\in K}\int_0^t\langle G^k(u^\alpha(s)),\phi\rangle dW_s^k.\end{aligned}$$

Following the idea of [21], let v the corrector of the boundary layer of width δ, i.e. a divergence free vector field with support in a strip of the boundary of width δ such that $\bar{u}-v\in V$ and

$$\sup_{t\in[0,T]}\|\partial_t^l v\|\lesssim\delta^{1/2},\quad\sup_{t\in[0,T]}\|\partial_t^l\nabla v\|\lesssim\delta^{-1/2},\ l\in\{0,1\}. \tag{94}$$

Let $\delta=\delta(\alpha)$ such that

$$\lim_{\alpha\to0}\delta=0,\quad\lim_{\alpha\to0}\frac{\alpha^2}{\delta}=0. \tag{95}$$

We want to write the Itô's formula for $\|W^\alpha(t)\|^2$. Let us take an orthonormal basis of H, $\{\tilde{e}_i\}$ made by eigenvectors of A, let $\{-\tilde{\lambda}_i\}$ the corresponding eigenvalues. Let us consider the weak formulation with test functions $\phi=\tilde{e}_i$, let us call $W^{\alpha,n}=$

$\sum_{i=1}^{n}\langle W^\alpha, \tilde{e}_i\rangle\tilde{e}_i, u^{\alpha,n} = \sum_{i=1}^{n}\langle u^\alpha, \tilde{e}_i\rangle\tilde{e}_i, \bar{u}^n = \sum_{i=1}^{n}\langle \bar{u}, \tilde{e}_i\rangle\tilde{e}_i$ e $v^n = \sum_{i=1}^{n}\langle v, \tilde{e}_i\rangle\tilde{e}_i$, then, arguing as in the proof of Lemma 35, we get

$$\begin{aligned}
W^{\alpha,n}(t) - W_0^{\alpha,n} &= \alpha^2 Au^{\alpha,n}(t) - \alpha^2 Au_0^{\alpha,n} + \nu\int_0^t Au^{\alpha,n}(s)ds \\
&\quad - \int_0^t \sum_{i=1}^{n} b(u^\alpha, W^\alpha(s), \tilde{e}_i)\tilde{e}_i ds - \int_0^t \sum_{i=1}^{n} b(W^\alpha(s), \bar{u}(s), \tilde{e}_i)\tilde{e}_i ds \\
&\quad - \alpha^2 \int_0^t \sum_{i=1}^{n} b(\tilde{e}_i, \Delta u^\alpha(s), u^\alpha(s))ds \\
&\quad + \alpha^2 \int_0^t \sum_{i=1}^{n} b(u^\alpha(s), \Delta u^\alpha(s), \tilde{e}_i)ds \\
&\quad + \tilde{\nu}\int_0^t \sum_{i=1}^{n}\langle F(u^\alpha(s)), \tilde{e}_i\rangle\tilde{e}_i ds \\
&\quad + \sqrt{\tilde{\nu}}\sum_{k\in K}\int_0^t \sum_{i=1}^{n}\langle G^k(u^\alpha(s)), \tilde{e}_i\rangle\tilde{e}_i dW_s^k. \qquad (96)
\end{aligned}$$

Therefore

$$\begin{aligned}
d\|W^{\alpha,n}\|^2 &= 2\langle W^{\alpha,n}, dW^{\alpha,n}\rangle + \alpha^4 d\langle\langle Au^{\alpha,n}, Au^{\alpha,n}\rangle\rangle_t \\
&\quad + \tilde{\nu}\sum_{k\in K}\sum_{i=1}^{n}\langle G^k(u^\alpha), \tilde{e}_i\rangle^2 dt \\
&\quad + \alpha^2\sqrt{\tilde{\nu}}\sum_{k\in K}\sum_{i=1}^{n}\langle G^k(u^\alpha), \tilde{e}_i\rangle d\langle\langle \tilde{e}_i W^k, Au^{\alpha,n}\rangle\rangle_t. \qquad (97)
\end{aligned}$$

In the same way, considering the weak formulation satisfied by u^α, we get

$$\begin{aligned}
dAu^{\alpha,n} &= \left(\nu A(I-\alpha^2 A)^{-1}Au^{\alpha,n} - \sum_{i=1}^{n} b(u^{\alpha,n}, u^{\alpha,n}, \tilde{e}_i)A(I-\alpha^2 A)^{-1}\tilde{e}_i\right)dt \\
&\quad - \alpha^2\sum_{i=1}^{n} b(\tilde{e}_i, \Delta u, u)A(I-\alpha^2 A)^{-1}\tilde{e}_i dt \\
&\quad + \tilde{\nu}\sum_{i=1}^{n}\langle F(u^\alpha), \tilde{e}_i\rangle A(I-\alpha^2 A)^{-1}\tilde{e}_i dt \\
&\quad + \sqrt{\tilde{\nu}}\sum_{k\in K}\sum_{i=1}^{n}\langle G^k(u^\alpha), \tilde{e}_i\rangle A(I-\alpha^2 A)^{-1}\tilde{e}_i dW_t^k. \qquad (98)
\end{aligned}$$

Combining relation (96), (97), (98) we obtain

$$\begin{aligned}
d\|W^{\alpha,n}\|^2 &= 2\alpha^2\langle W^{\alpha,n}, dAu^{\alpha,n}\rangle + 2\nu\langle W^{\alpha,n}, Au^{\alpha,n}\rangle dt \\
&\quad - 2\left\langle W^{\alpha,n}, \sum_{i=1}^{n} b(W^{\alpha}, \bar{u}, \tilde{e}_i)\tilde{e}_i\right\rangle dt - 2\left\langle W^{\alpha,n}, \sum_{i=1}^{n} b(u^{\alpha}, W^{\alpha}, \tilde{e}_i)\tilde{e}_i\right\rangle dt \\
&\quad - 2\alpha^2\left\langle W^{\alpha,n}, \sum_{i=1}^{n} b(\tilde{e}_i, \Delta u^{\alpha}, u^{\alpha})\tilde{e}_i\right\rangle dt \\
&\quad + 2\alpha^2\left\langle W^{\alpha,n}, \sum_{i=1}^{n} b(u^{\alpha}, \Delta u^{\alpha}, \tilde{e}_i)\tilde{e}_i\right\rangle dt \\
&\quad + 2\tilde{\nu}\sum_{i=1}^{n}\langle F(u^{\alpha}), \tilde{e}_i\rangle\langle W^{\alpha,n}, \tilde{e}_i\rangle dt \\
&\quad + 2\sqrt{\tilde{\nu}}\sum_{k\in K}\sum_{i=1}^{n}\langle G^k(u), \tilde{e}_i\rangle\langle W^{\alpha,n}, \tilde{e}_i\rangle dW^k_t \\
&\quad + \alpha^4\tilde{\nu}\sum_{k\in K}\sum_{i=1}^{n}\langle G^k(u^{\alpha}), \tilde{e}_i\rangle^2\|A(I-\alpha^2 A)^{-1}\tilde{e}_i\|^2\, dt \\
&\quad + \tilde{\nu}\sum_{k\in K}\sum_{i=1}^{n}\langle G^k(u^{\alpha}), \tilde{e}_i\rangle^2 dt \\
&\quad + \alpha^2\tilde{\nu}\sum_{k\in K}\sum_{i=1}^{n}\langle G^k(u^{\alpha}), \tilde{e}_i\rangle^2\langle A(I-\alpha^2 A)^{-1}\tilde{e}_i, \tilde{e}_i\rangle dt.
\end{aligned}$$

Let us rewrite $\langle W^{\alpha,n}, dAu^{\alpha,n}\rangle$ in a different way

$$\begin{aligned}
&\langle W^{\alpha,n}, dAu^{\alpha,n}\rangle \\
&\quad = \langle u^{\alpha,n}-\bar{u}^n, dAu^{\alpha,n}\rangle = \langle u^{\alpha,n}, dAu^{\alpha,n}\rangle - \langle \bar{u}^n - v^n, dAu^{\alpha,n}\rangle - \langle v^n, dAu^{\alpha,n}\rangle \\
&\quad = -\langle(-A)^{1/2}u^{\alpha,n}, d(-A)^{1/2}u^{\alpha,n}\rangle \\
&\qquad + \langle(-A)^{1/2}(\bar{u}^n - v^n), d(-A)^{1/2}u^{\alpha,n}\rangle - \langle v^n, dAu^{\alpha,n}\rangle \\
&\quad = -\frac{d\|(-A)^{1/2}u^{\alpha,n}\|^2}{2} + \frac{d\langle\langle(-A)^{1/2}u^{\alpha,n}, (-A)^{1/2}u^{\alpha,n}\rangle\rangle_t}{2} \\
&\qquad + d\langle(-A)^{1/2}(\bar{u}^n - v^n), (-A)^{1/2}u^{\alpha,n}\rangle \\
&\qquad - \langle(-A)^{1/2}\partial_t(\bar{u}^n - v^n), (-A)^{1/2}u^{\alpha,n}\rangle - d\langle v^n, Au^{\alpha,n}\rangle + \langle\partial_t v^n, Au^{\alpha,n}\rangle.
\end{aligned}$$

Therefore, we arrive to this final expression

$$\begin{aligned}
\|W^{\alpha,n}(t)\|^2 &= \|W^{\alpha,n}_0\|^2 - \alpha^2\|\nabla u^{\alpha,n}(t)\|^2_{L^2} + \alpha^2\|\nabla u^{\alpha,n}_0\|^2_{L^2} \\
&\qquad + \tilde{\nu}\alpha^2\sum_{k\in K}\int_0^t\sum_{i=1}^{n}\langle G^k(u^{\alpha}(s)), \tilde{e}_i\rangle^2\|(-A)^{1/2}(I-\alpha^2 A)^{-1}\tilde{e}_i\|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha^2 \langle \nabla (\bar{u}^n - v^n)(t), \nabla u^{\alpha,n}(t)\rangle_{L^2} - 2\alpha^2 \langle \nabla (\bar{u}^n - v^n)_0, \nabla u_0^{\alpha,n}\rangle_{L^2} \\
&- 2\alpha^2 \int_0^t \langle \nabla \partial_s (\bar{u}^n(s) - v^n(s)), \nabla u^{\alpha,n}(s)\rangle_{L^2} ds \\
&- 2\alpha^2 \langle v^n(t), \Delta u^{\alpha,n}(t)\rangle_{L^2} + 2\alpha^2 \langle v_0^n, \Delta u_0^{\alpha,n}\rangle_{L^2} \\
&+ 2\alpha^2 \int_0^t \langle \partial_s v^n(s), \Delta u^{\alpha,n}(s)\rangle_{L^2} ds + 2\nu \int_0^t \langle W^{\alpha,n}(s), A u^{\alpha,n}(s)\rangle ds \\
&- 2 \int_0^t \langle W^{\alpha,n}(s), \sum_{i=1}^n b(W^\alpha(s), \bar{u}(s), \tilde{e}_i)\tilde{e}_i\rangle ds \\
&- 2 \int_0^t \langle W^{\alpha,n}(s), \sum_{i=1}^n b(u^\alpha(s), W^\alpha(s), \tilde{e}_i)\tilde{e}_i\rangle ds \\
&- 2\alpha^2 \int_0^t \langle W^{\alpha,n}(s), \sum_{i=1}^n b(\tilde{e}_i, \Delta u^\alpha(s), u^\alpha(s))\tilde{e}_i\rangle ds \\
&+ 2\alpha^2 \int_0^t \langle W^{\alpha,n}(s), \sum_{i=1}^n b(u^\alpha(s), \Delta u^\alpha(s), \tilde{e}_i)\tilde{e}_i\rangle ds \\
&+ 2\tilde{\nu} \sum_{i=1}^n \int_0^t \langle F(u^\alpha(s)), \tilde{e}_i\rangle \langle W^{\alpha,n}(s), \tilde{e}_i\rangle ds \\
&+ 2\sqrt{\tilde{\nu}} \sum_{k\in K} \int_0^t \sum_{i=1}^n \langle G^k(u^\alpha(s)), \tilde{e}_i\rangle \langle W^{\alpha,n}(s), \tilde{e}_i\rangle dW_s^k \\
&+ \alpha^4 \tilde{\nu} \sum_{k\in K} \int_0^t \sum_{i=1}^n \langle G^k(u^\alpha(s)), \tilde{e}_i\rangle^2 \| A(I - \alpha^2 A)^{-1}\tilde{e}_i\|^2 ds \\
&+ \tilde{\nu} \sum_{k\in K} \int_0^t \sum_{i=1}^n \langle G^k(u^\alpha(s)), \tilde{e}_i\rangle^2 ds \\
&+ \alpha^2 \tilde{\nu} \sum_{k\in K} \int_0^t \sum_{i=1}^n \langle G^k(u^\alpha(s)), \tilde{e}_i\rangle^2 \langle A(I - \alpha^2 A)^{-1}\tilde{e}_i, \tilde{e}_i\rangle ds.
\end{aligned}
$$

Now, letting $n \to +\infty$, exploiting the regularity of u^α, $\bar{u}$, v and the continuity of the trilinear form b we arrive to the formula below

$$
\begin{aligned}
\| W^\alpha(t)\|^2 + \alpha^2 \|\nabla u^\alpha(t)\|_{L^2}^2 = \| W_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2 \\
&+ \alpha^2 \tilde{\nu} \sum_{k\in K} \int_0^t \|(-A)^{1/2}(I - \alpha^2 A)^{-1} G^k(u^\alpha(s))\|^2 ds \\
&+ 2\alpha^2 \langle \nabla(\bar{u} - v)(t), \nabla u^\alpha(t)\rangle_{L^2} \\
&- 2\alpha^2 \langle \nabla(\bar{u} - v)_0, \nabla u_0^\alpha\rangle_{L^2} \\
&- 2\alpha^2 \int_0^t \langle \nabla \partial_s (\bar{u}(s) - v(s)), \nabla u^\alpha(s)\rangle_{L^2} ds
\end{aligned}
$$

$$
\begin{aligned}
&- 2\alpha^2 \langle v(t), \Delta u^\alpha(t)\rangle_{L^2} + 2\alpha^2 \langle v_0, \Delta u_0^\alpha\rangle_{L^2} \\
&+ 2\alpha^2 \int_0^t \langle \partial_s v(s), \Delta u^\alpha(s)\rangle_{L^2} ds \\
&+ 2\nu \int_0^t \langle W^\alpha(s), Au^\alpha(s)\rangle ds \\
&- 2 \int_0^t b(W^\alpha(s), \bar{u}(s), W^\alpha(s))\rangle ds \\
&- 2\alpha^2 \int_0^t b(W^\alpha(s), \Delta u^\alpha(s), u^\alpha(s)) ds \\
&+ 2\alpha^2 \int_0^t b(u^\alpha(s), \Delta u^\alpha(s), W^\alpha(s)) ds \\
&+ 2\tilde{\nu} \int_0^t \langle F(u^\alpha(s)), W^\alpha(s)\rangle ds \\
&+ 2\sqrt{\tilde{\nu}} \sum_{k\in K} \int_0^t \langle G^k(u^\alpha(s)), W^\alpha(s)\rangle dW_s^k \\
&+ \alpha^4 \tilde{\nu} \sum_{k\in K} \int_0^t \|A(I-\alpha^2 A)^{-1} G^k(u^\alpha(s))\|^2 ds \\
&+ \tilde{\nu} \sum_{k\in K} \int_0^t \|G^k(u^\alpha(s))\|^2 ds \\
&+ \alpha^2 \tilde{\nu} \sum_{k\in K} \int_0^t \langle A(I-\alpha^2 A)^{-1} G^k \\
&(u^\alpha(s)), G^k(u^\alpha(s))\rangle ds \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + M(t),
\end{aligned}
$$

where:

$$
\begin{aligned}
I_1(t) &= \|W_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2 + 2\alpha^2 \langle \nabla(\bar{u}-v)(t), \nabla u^\alpha(t)\rangle_{L^2} \\
&\quad - 2\alpha^2 \langle \nabla(\bar{u}-v)_0, \nabla u_0^\alpha\rangle_{L^2} - 2\alpha^2 \langle v(t), \Delta u^\alpha(t)\rangle_{L^2} + 2\alpha^2 \langle v_0, \Delta u_0^\alpha\rangle_{L^2}, \\
I_2(t) &= \alpha^2 \tilde{\nu} \sum_{k\in K} \int_0^t \|(-A)^{1/2}(I-\alpha^2 A)^{-1} G^k(u^\alpha)\|^2 ds \\
&\quad + \alpha^4 \tilde{\nu} \sum_{k\in K} \int_0^t \|A(I-\alpha^2 A)^{-1} G^k(u^\alpha(s))\|^2 ds \\
&\quad + \tilde{\nu} \sum_{k\in K} \int_0^t \|G^k(u^\alpha(s))\|^2 ds \\
&\quad + \alpha^2 \tilde{\nu} \sum_{k\in K} \int_0^t \langle A(I-\alpha^2 A)^{-1} G^k(u^\alpha(s)), G^k(u^\alpha(s))\rangle ds \\
&\quad + 2\tilde{\nu} \int_0^t \langle F(u^\alpha(s)), W^\alpha(s)\rangle ds,
\end{aligned}
$$

$$I_3(t) = -2\alpha^2 \int_0^t \langle \nabla \partial_s (\bar{u}(s) - v(s)), \nabla u^\alpha(s) \rangle_{L^2} ds$$
$$+ 2\alpha^2 \int_0^t \langle \partial_s v(s), \Delta u^\alpha(s) \rangle_{L^2} ds,$$
$$I_4(t) = 2\nu \int_0^t \langle W^\alpha(s), A u^\alpha(s) \rangle ds,$$
$$I_5(t) = -2 \int_0^t b(W^\alpha(s), \bar{u}(s), W^\alpha(s)) \rangle ds,$$
$$I_6(t) = -2\alpha^2 \int_0^t b(W^\alpha(s), \Delta u^\alpha(s), u^\alpha(s)) ds$$
$$+ 2\alpha^2 \int_0^t b(u^\alpha(s), \Delta u^\alpha(s), W^\alpha(s)) ds,$$
$$M(t) = 2\sqrt{\tilde{\nu}} \sum_{k \in K} \int_0^t \langle G^k(u^\alpha(s)), W^\alpha(s) \rangle d W_s^k.$$

Our approach is almost completely pathwise. Therefore we need to estimate the terms $I_i(t),\ t \in \{1, \ldots, 6\}$. The analysis of $I_1(t)$ follows by Young's inequality, the estimates on the boundary layer corrector (94) and the interpolation estimate (42)

$$\begin{aligned} I_1(t) &\le \|W_0^\alpha\|^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2 + C\alpha^2 \delta^{1/2} \|\nabla u_0^\alpha\|_{L^2}^{1/2} \|u_0^\alpha\|_{H^3}^{1/2} \\ &\quad + C\alpha^2 (1 + \delta^{-1/2}) \|\nabla u^\alpha(t)\|_{L^2} + C\alpha^2 \delta^{1/2} \|\nabla u(t)^\alpha\|_{L^2}^{1/2} \|u(t)^\alpha\|_{H^3}^{1/2} \\ &\quad + C\alpha^2 (1 + \delta^{-1/2}) \|\nabla u_0^\alpha\|_{L^2} \\ &\le \|W_0^\alpha\|^2 + C\alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2 + C\alpha^2 (1 + \delta^{-1}) + C\alpha^6 \delta (\|u_0^\alpha\|_{H^3}^2 + \|u(t)^\alpha\|_{H^3}^2) \\ &\quad + C\delta^{1/2} + \frac{\alpha^2}{2} \|\nabla u^\alpha(t)\|_{L^2}^2. \end{aligned} \tag{99}$$

The analysis of $I_2(t)$ follows by Young's inequality and the results of Lemma 21, Corollary 22. Indeed it holds

$$\begin{aligned} I_2(t) &\le C\tilde{\nu} \sum_{k \in K} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds \\ &\quad + \frac{\tilde{\nu}}{\alpha} \sum_{k \in K} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|W^\alpha(s)\| \|\nabla u^\alpha(s)\|_{L^2} ds \\ &\le C\tilde{\nu} \sum_{k \in K} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds + \sum_{k \in K} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|W^\alpha(s)\|^2 ds \\ &\quad + \left(\frac{\tilde{\nu}}{\alpha}\right)^2 \sum_{k \in K} \|\sigma_k\|_{L^\infty}^2 \int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds. \end{aligned} \tag{100}$$

The analysis of $I_3(t)$ follows by Young's inequality, the estimates on the boundary layer corrector (94) and the interpolation estimate (42)

$$\begin{aligned} I_3(t) &\leq C\alpha^2(1+\delta^{-1/2})\int_0^t \|\nabla u^\alpha(s)\|_{L^2}ds + C\alpha^2\delta^{1/2}\int_0^t \|\nabla u^\alpha(s)\|_{L^2}^{1/2}\|u^\alpha(s)\|_{H^3}^{1/2} \\ &\leq C\delta^{1/2} + C\alpha^2(1+\delta^{-1}) + C\alpha^2\int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds \\ &\quad + C\alpha^6\delta\int_0^t \|u^\alpha(s)\|_{H^3}^2 ds. \end{aligned} \tag{101}$$

The analysis of $I_4(t)$ is analogous to Eqs. (3.20)–(3.22) in [27], it implies:

$$\begin{aligned} 2\nu\int_0^t \langle W^\alpha(s), Au^\alpha(s)\rangle ds \leq -2\nu\int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds \\ + C\frac{\nu}{\alpha}(1+\delta^{-1/2})\int_0^t \alpha\|\nabla u^\alpha(s)\|_{L^2}ds \\ + \frac{C\nu\delta^{1/2}}{\alpha^2}\int_0^t \alpha^2\|\Delta u^\alpha(s)\|_{L^2}ds. \end{aligned}$$

Therefore by the interpolation inequality (42) and Young's inequality we have

$$\begin{aligned} 2\nu\int_0^t \langle W^\alpha(s), Au^\alpha(s)\rangle ds &\leq -2\nu\int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds + C\alpha^2\int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds \\ &\quad + C\alpha^6\delta\int_0^t \|u^\alpha(s)\|_{H^3}^2 ds + C\left(\frac{\nu}{\alpha^2}\right)^2\delta^{1/2} \\ &\quad + C\left(\frac{\nu}{\alpha}\right)^2(1+\delta^{-1}). \end{aligned} \tag{102}$$

The analysis of $I_5(t)$ follows immediately by Hölder's inequality:

$$I_5(t) \leq \|\bar{u}\|_{L^\infty(0,T;H^3)}\int_0^t \|W^\alpha(s)\|^2 ds. \tag{103}$$

For what concerns the analysis of $I_6(t)$, preliminary we observe that

$$\begin{aligned} &-2\alpha^2 b(W^\alpha, \Delta u^\alpha, u^\alpha) + 2\alpha^2 b(u^\alpha, \Delta u^\alpha, W^\alpha) \\ &= 2\alpha^2 b(\bar{u}, \Delta u^\alpha, u^\alpha) - 2\alpha^2 b(u^\alpha, \Delta u^\alpha, u^\alpha) \\ &\quad + 2\alpha^2 b(u^\alpha, \Delta u^\alpha, u^\alpha) - 2\alpha^2 b(u^\alpha, \Delta u^\alpha, \bar{u}). \end{aligned}$$

Arguing as in [24], Equations (4.18)–(4.19) we get

$$I_6(t) \leq C\alpha^2\left(1+\|\bar{u}\|_{L^\infty(0,T;H^3)}\right)\int_0^t \|\nabla u^\alpha(s)\|_{L^2}^2 ds + C\alpha^2\|\bar{u}\|_{L^\infty(0,T;H^3)}^4$$

$$\int_0^t \|u^\alpha(s)\|^2 ds. \tag{104}$$

Combining Eqs. (99), (100), (101), (102), (103), (104) and exploiting our assumptions on the behavior of ν, $\tilde{\nu}$, α^2, see Hypothesis 6, we have the integral relation below:

$$\begin{aligned}
&\|W^\alpha(t)\|^2 + \frac{\alpha^2}{2}\|\nabla u^\alpha(t)\|_{L^2}^2 \\
&\quad \leq M(t) + C\alpha^2(1+\delta^{-1}) + C\delta^{1/2} + \|W_0^\alpha\|^2 + C\alpha^2\|\nabla u_0^\alpha\|_{L^2}^2 \\
&\qquad + C\alpha^6\delta(\|u_0^\alpha\|_{H^3}^2 + \|u(t)^\alpha\|_{H^3}^2) \\
&\qquad + C_{\{\sigma_k\}_{k\in K}} \int_0^t \|W^\alpha(s)\|^2 + \alpha^2\|\nabla u^\alpha(s)\|_{L^2}^2 ds \\
&\qquad + C\alpha^6\delta \int_0^t \|u^\alpha(s)\|_{H^3}^2 ds + C\alpha^2 \int_0^t \|u^\alpha(s)\|^2 ds.
\end{aligned} \tag{105}$$

By the stochastic Grönwall's Lemma 24 above we have:

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left[\|W^\alpha(t)\|^2\right] + \alpha^2 \sup_{t\in[0,T]} \mathbb{E}\left[\|\nabla u^\alpha(t)\|_{L^2}^2\right] \\
&\leq C_{\{\sigma_k\}_{k\in K}} \left(\alpha^2\mathbb{E}\left[\int_0^T \|u^\alpha(s)\|^2 ds\right] + \alpha^6\delta \int_0^T \|u^\alpha(s)\|_{H^3}^2 ds\right. \\
&\quad \left. +\alpha^6\delta\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\alpha(t)\|_{H^3}^2\right]\right) + C_{\{\sigma_k\}_{k\in K}} \\
&\quad \left(\alpha^2(1+\delta^{-1}) + \delta^{1/2} + \mathbb{E}\left[\|W_0^\alpha\|^2 + \alpha^2\|\nabla u_0^\alpha\|_{L^2}^2 + \alpha^6\delta\|u_0^\alpha\|_{H^3}^2\right]\right).
\end{aligned} \tag{106}$$

Thanks to Hypothesis 6 and our assumptions on δ, see Eq. (95), we have that

$$\alpha^2(1+\delta^{-1}) + \delta^{1/2} + \mathbb{E}\left[\|W_0^\alpha\|^2 + \alpha^2\|\nabla u_0^\alpha\|_{L^2}^2 + \alpha^6\delta\|u_0^\alpha\|_{H^3}^2\right] \to 0. \tag{107}$$

Thanks to Lemma 35, we have that

$$\begin{aligned}
\alpha^2\mathbb{E}\left[\int_0^T \|u^\alpha(s)\|^2 ds\right] + \alpha^6\delta \int_0^T \|u^\alpha(s)\|_{H^3}^2 ds \\
+ \alpha^6\delta\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\alpha(t)\|_{H^3}^2\right] \to 0.
\end{aligned} \tag{108}$$

Therefore

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|W^\alpha(t)\|^2\right] + \alpha^2 \sup_{t\in[0,T]} \mathbb{E}\left[\|\nabla u^\alpha(t)\|_{L^2}^2\right] \to 0. \tag{109}$$

Restarting from Eq. (105) and considering the expected value of the supremum of both the terms in the left hand side we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|W^\alpha(t)\|^2\right] + \alpha^2\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla u^\alpha(t)\|_{L^2}^2\right]$$

$$
\begin{aligned}
&\leq C\left(\alpha^2\mathbb{E}\left[\int_0^T\|u^\alpha(s)\|^2ds\right]+\alpha^6\delta\int_0^T\|u^\alpha(s)\|_{H^3}^2ds\right.\\
&\quad\left.+\alpha^6\delta\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\alpha(t)\|_{H^3}^2\right]\right)\\
&\quad+C\left(\alpha^2(1+\delta^{-1})+\delta^{1/2}+\mathbb{E}\left[\|W_0^\alpha\|^2+\alpha^2\|\nabla u_0^\alpha\|_{L^2}^2+\alpha^6\delta\|u_0^\alpha\|_{H^3}^2\right]\right)\\
&\quad+C\mathbb{E}\left[\sup_{t\in[0,T]}M(t)\right]+C_{\{\sigma_k\}_{k\in K}}\mathbb{E}\left[\int_0^T\|W^\alpha(s)\|^2+\alpha^2\|\nabla u^\alpha(s)\|_{L^2}^2ds\right].
\end{aligned}
\tag{110}
$$

We already proved that almost all the terms in the right hand side of Eq. (110) go to 0. Therefore in order to complete the proof we left to show that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}M(t)\right]+\mathbb{E}\left[\int_0^t\|W^\alpha(s)\|^2+\alpha^2\|\nabla u^\alpha(s)\|_{L^2}^2ds\right]\to 0.
$$

By the weaker convergence described by Eq. (109) and Fubini Theorem

$$
\mathbb{E}\left[\int_0^t\|W^\alpha(s)\|^2+\alpha^2\|\nabla u^\alpha(s)\|_{L^2}^2ds\right]\to 0.
$$

For what concerns the other, the convergence follows by Burkholder–Davis–Gundy inequality, Hypothesis 6, Eq. (109), Fubini Theorem and relation (27). Indeed

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}M(t)\right]\\
&\leq C\sqrt{\tilde{\nu}}\mathbb{E}\left[\left(\sum_{k\in K}\int_0^T\|G^k(u^\alpha(s))\|^2\|W^\alpha(s)\|^2ds\right)^{1/2}\right]\\
&\leq C\left(\sum_{k\in K}\|\sigma_k\|_{L^\infty}^2\right)^{1/2}\sqrt{\tilde{\nu}}\mathbb{E}\left[\left(\int_0^T\|\nabla u^\alpha(s)\|_{L^2}^2\|W^\alpha(s)\|^2ds\right)^{1/2}\right]\\
&\leq C\left(\sum_{k\in K}\|\sigma_k\|_{L^\infty}^2\right)^{1/2}\sqrt{\tilde{\nu}}\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla u^\alpha(t)\|_{L^2}\left(\int_0^T\|W^\alpha(s)\|^2ds\right)^{1/2}\right]\\
&\leq C\left(\sum_{k\in K}\|\sigma_k\|_{L^\infty}^2\right)^{1/2}\mathbb{E}\left[\int_0^T\|W^\alpha(s)\|^2ds\right]^{1/2}\left(\mathbb{E}\left[\alpha^2\sup_{t\in[0,T]}\|\nabla u^\alpha(t)\|_{L^2}^2\right]\right)^{1/2}\\
&\to 0.
\end{aligned}
$$

Now the proof is complete.

Remark 37 Combining Lemma 35 and Theorem 9 we understand that, if $\nu = O(\alpha^2)$ and $\tilde{\nu} = O(\alpha^2)$, the assumptions on the behavior of the initial conditions u_0^α in norm $H,\ H^1$ and H^3 are satisfied also for $t \in [0, T]$.

6 The case of additive noise

For what concerns the case with additive noise, as stated in Sect. 2, the well-posedness is a well-known fact in case of $\nu > 0$ and we can prove a result completely analogous to Theorem 9, following exactly the same argument. However, the restriction $\nu > 0$ can be omitted modifying slightly the proof of [34] as described in Remarks 27, 29 and 36. However, we do not stress this assumption in this section, therefore $\nu > 0$ in what follows. What was crucial for the proof of Theorem 9 were the energy estimates of Lemma 35. Thus in this section we want to explain a different approach to prove these energy estimates in the case of additive noise. These computations are more similar to what happens in the deterministic framework. We keep previous assumptions on the coefficients σ_k and the Brownian motions W^k. For generality reasons we consider the equations without any scaling factor on the noise. Thus we consider

$$\begin{cases} dv = (\nu\Delta u - \mathrm{curl}(v)\times u + \nabla p)dt + \sum_{k\in K}\sigma_k dW_t^k \\ \mathrm{div}\, u = 0 \\ v = u - \alpha^2\Delta u \\ u|_{\partial D} = 0 \\ u(0) = u_0 \end{cases} \tag{111}$$

Before going on, we need to recall a result of [24].

Lemma 38 *Let $q \in L^2(D)$, there exists a unique $\phi \in H^2_0(D)$ solution of*

$$\begin{cases} \Delta\phi - \alpha^2\Delta^2\phi = q \\ \phi|_{\partial D} = \partial_n\phi|_{\partial D} = 0 \end{cases}$$

which satisfies

$$\langle \nabla\phi, \nabla v\rangle_{L^2} + \alpha^2\langle \Delta\phi, \Delta v\rangle_{L^2} = -\langle q, v\rangle \textit{ for each } v \in H^2_0.$$

Moreover, the solution map is continuous from $L^2(D)$ to $H^2_0(D)\cap H^4(D)$.

Thanks to this Lemma, we can define an operator $\mathbb{K} : L^2(D) \to H^3(D)\cap W^{1,\infty}_0(D)$ which associates to each $q \in L^2(D)$ the vector field $u = \nabla^{\perp}\phi$, where ϕ is the solution of the equation of Lemma 38.

Definition 39 A stochastic process u weakly continuous with values in W and continuous with values in V is a weak solution of Eq. (111) if

$$u \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; W)),\ \forall p \geq 2.$$

and $\mathbb{P}-a.s.$ for every $t \in [0,T]$ and $\phi \in D(A)$ we have

$$\begin{aligned}
&\langle u(t), (I-\alpha^2 A)\phi\rangle - \langle u_0, (I-\alpha^2 A)\phi\rangle \\
&= \nu \int_0^t \langle u(s), A\phi\rangle ds - \int_0^t b(u(s), u(s)-\alpha^2\Delta u(s), \phi) ds \\
&\quad - \alpha^2 \int_0^t b(\phi, \Delta u(s), u(s)) ds + \sum_{k\in K} \langle \sigma_k, \phi\rangle W_t^k.
\end{aligned}$$

Arguing as in the first part of the proof of Lemma 35 we can prove the following result.

Lemma 40 *Let u be a weak solution of problem (111) in the sense of Definition 39, then the following relations hold true*

1. $$d\|u\|^2 + \alpha^2 d\|\nabla u\|_{L^2}^2 = \left(-2\nu\|\nabla u\|_{L^2}^2 + \sum_{k\in K}\langle \sigma_k, (I-\alpha^2 A)^{-1}\sigma_k\rangle\right) dt + 2\sum_{k\in K}\langle \sigma_k, u\rangle dW_t^k$$

2. $$\begin{aligned}
&\mathbb{E}\left[\|u(t)\|^2\right] + \alpha^2\mathbb{E}\left[\|\nabla u(t)\|_{L^2}^2\right] + 2\nu\int_0^t \mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right] ds \\
&= \mathbb{E}\left[\|u_0\|^2\right] + \alpha^2\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] + t\sum_{k\in K}\langle \sigma_k, (I-\alpha^2 A)^{-1}\sigma_k\rangle
\end{aligned}$$

3. $$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\|u(t)\|^2\right] + \alpha^2\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2\right] + 2\nu\int_0^T \mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right] ds \\
&\leq C\left(\mathbb{E}\left[\|u_0\|^2\right] + \alpha^2\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] + T\sum_{k\in K}\langle \sigma_k, (I-\alpha^2 A)^{-1}\sigma_k\rangle \right.\\
&\quad \left. + \mathbb{E}\left[\left(\int_0^T \sum_{k\in K}\langle \sigma_k, u(s)\rangle^2 ds\right)^{1/2}\right]\right)
\end{aligned}$$

Let us introduce the vorticity formulation of (111), we denote $s_k = \operatorname{curl}\sigma_k$

$$\begin{cases}
dq + \left(\frac{\nu}{\alpha^2}(q-\operatorname{curl} u) + u\cdot\nabla q\right) dt = \sum_{k\in K} s_k dW_t^k \\
\operatorname{div} u = 0 \\
q = \operatorname{curl}(u-\alpha^2\Delta u) \\
q(0) = q_0 := \operatorname{curl}(u_0 - \alpha^2\Delta u_0) \\
u|_{\partial D} = 0
\end{cases} \tag{112}$$

Definition 41 A stochastic process q, which is weakly continuous with values in $L^2(D)$ and continuous with values in $H^{-1}(D)$, is a weak solution of Eq. (112) if

$$q \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0,T; L^2(D))), \quad \forall p \geq 2.$$

and $\mathbb{P}-a.s.$ for every $t \in [0, T]$ and $\phi \in H_0^2(D)$ we have

$$\begin{aligned}\langle q(t), \phi\rangle - \langle q_0, \phi\rangle = &\int_0^t \int_D u(s) \cdot \nabla \phi q(s)\, dxds \\ &- \frac{\nu}{\alpha^2} \int_0^t \int_D (q(s) - \operatorname{curl} u(s))\phi\, dxds \\ &+ \sum_{k \in K} \langle s_k, \phi\rangle W_t^k \;\; \mathbb{P}-a.s.\end{aligned}$$

$u = \nabla^{\perp}\varphi$, φ obtained by Lemma 38, $u \in W$.

Let us obtain a result about the equivalence between the solutions of these two problems. Since we know from the results of [34] that problem (111) is well-posed, then problem (112) is well-posed as well.

Proposition 42 *Let u be a solution of* (111) *in the sense of Definition* 39*, then* $q := \operatorname{curl}(u - \alpha^2\Delta u)$ *is a solution of* (112) *in the sense of Definition* 41*. Conversely, if q is a solution of* (112) *in the sense of Definition* 41 *then $u = \nabla^{\perp}\varphi$, φ obtained by Lemma* 38*, is a solution of* (111) *in the sense of Definition* 39*.*

Proof *Definition* 39 $\Longrightarrow$ *Definition* 41 is immediate taking $\phi = -\nabla^{\perp}\tilde{\phi}$, $\tilde{\phi} \in H_0^2(D)$ as test function for problem (111).

Therefore it remains to show that *Definition* 41 $\Longrightarrow$ *Definition* 39. We take $u = \nabla^{\perp}\varphi$, $v = u - \alpha^2\Delta u$, where φ is obtained by Lemma 38 and $\phi = -\nabla^{\perp}\tilde{\phi}$, where $\tilde{\phi} \in H_0^2(D)$. Then integrating by parts and exploiting that $\operatorname{curl}\nabla^{\perp} = \Delta$, $\Delta\varphi - \alpha^2\Delta^2\varphi = q$ and q is a solution of (112) in the sense of Definition 41 we get

$$\begin{aligned}&-\langle (I - \alpha^2\Delta)u(t), \nabla^{\perp}\tilde{\phi}\rangle_{L^2} + \langle (I - \alpha^2\Delta)u_0, \nabla^{\perp}\tilde{\phi}\rangle_{L^2} \\ &\quad - \frac{\nu}{\alpha^2}\int_0^t \langle v(s) - u(s), \nabla^{\perp}\tilde{\phi}\rangle ds \\ &\quad + \int_0^t \int_D (u(s)\cdot\nabla)\nabla^{\perp}\tilde{\phi} v(s)\, dxds - \alpha^2 \int_0^t \int_D (\nabla^{\perp}\tilde{\phi}\cdot\nabla)\Delta u(s) u(s) dxds \\ &\quad + \sum_{k\in K}\langle \sigma_k, \nabla^{\perp}\tilde{\phi}\rangle W_t^k = 0 \;\; \mathbb{P}-a.s.\end{aligned}$$

From the last relation the thesis follows if we are able to prove the continuity properties of u. The weak continuity of u with values in W follows immediately from the regularity of q and Lemma 38. Again by Lemma 38 we get the strong continuity of u with values in V. In fact, via Lax–Milgram Lemma we get the regularity of the solution mapping of the problem described in Lemma 38 between $H^{-2}(D)$ and $H_0^2(D)$. Via interpolation techniques we recover the regularity of the solution mapping between $H^{-1}(D)$ and $H^3(D) \cap H_0^2(D)$, therefore the required regularity for u.

Approximating the process $q(t)$ solution of (112) by the eigenvectors of the Laplacian with Dirichlet boundary conditions and then arguing as in the first part of the proof of Lemma 35, we can obtain some Itô's formula and energy estimates. Moreover, if $u \in V$ we have $\|\nabla u\|_{L^2}^2 = \|\operatorname{curl} u\|_{L^2}^2$. Thanks to Proposition 42, we know that u appearing in problem (112) is a solution of problem (111). Therefore, thanks to Lemma 40 we know that

$$2\nu \int_0^t \mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right] ds \leq \mathbb{E}\left[\|u_0\|^2\right] + \alpha^2 \mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] + t \sum_{k\in K} \langle \sigma_k, (I-\alpha^2 A)^{-1}\sigma_k\rangle, \tag{113}$$

$$\alpha^2 \mathbb{E}\left[\|\nabla u(t)\|_{L^2}^2\right] \leq \mathbb{E}\left[\|u_0\|^2\right] + \alpha^2 \mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] + t \sum_{k\in K} \langle \sigma_k, (I-\alpha^2 A)^{-1}\sigma_k\rangle \tag{114}$$

and we can obtain the following energy relations.

Lemma 43 *Let q be a weak solution of problem (112) in the sense of Definition 41, then the following relations hold true*

1. $$d\|q\|^2 = -\frac{2\nu}{\alpha^2}\langle q - \operatorname{curl} u, q\rangle dt + \sum_{k\in K}\|s_k\|^2\, dt + 2\sum_{k\in K}\langle s_k, q\rangle dW_t^k$$
2. $$\mathbb{E}\left[\|q(t)\|^2\right] \leq e^{-\frac{\nu}{\alpha^2}t}\mathbb{E}\left[\|q_0\|^2\right] + \frac{\alpha^2}{\nu}(1-e^{-\frac{\nu t}{\alpha^2}})\sum_{k\in K}\|s_k\|^2 + \frac{1}{2\nu}\left(\mathbb{E}\left[\|u_0\|^2\right] + \alpha^2\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] + T\sum_{k\in K}\langle \sigma_k, (I-\alpha^2 A)^{-1}\sigma_k\rangle\right)$$
3. $$\mathbb{E}\left[\sup_{t\in[0,T]}\|q(t)\|^2\right] \leq \mathbb{E}\left[\|q_0\|^2\right] + \sum_{k\in K}\|s_k\|^2 T + C\mathbb{E}\left[\left(\sum_{k\in K}\int_0^T \langle s_k, q(s)\rangle^2 ds\right)^{1/2}\right] + \frac{1}{2\alpha^2}\left(\mathbb{E}\left[\|u_0\|^2\right] + \alpha^2\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right] + T\sum_{k\in K}\langle \sigma_k, (I-\alpha^2 A)^{-1}\sigma_k\rangle\right).$$

Remark 44 We can control the H^3 norm of u via the H^1 norm of u and the L^2 norm of q in the following way

$$\|u(t)\|_{H^3} \leq C\left(\|\nabla u(t)\|_{L^2} + \|\operatorname{curl} \Delta u(t)\|_{L^2}\right) \leq C\left(\frac{\|q(t)\|}{\alpha^2} + \frac{\|\operatorname{curl} u(t)\|_{L^2}}{\alpha^2} + \|\nabla u(t)\|_{L^2}\right)$$

$$= C\left(\frac{\|q(t)\|}{\alpha^2}+\frac{\|\nabla u(t)\|_{L^2}}{\alpha^2}+\|\nabla u(t)\|_{L^2}\right). \quad (115)$$

Therefore, thanks to Lemma 43 it holds

$$\begin{aligned}\mathbb{E}\left[\|u(t)\|_{H^3}^2\right] &\lesssim \frac{e^{-\frac{\nu}{\alpha^2}t}}{\alpha^4}\mathbb{E}\left[\|q_0\|^2\right]+\frac{1}{\nu\alpha^2}(1-e^{-\frac{\nu t}{\alpha^2}})\sum_{k\in K}\|s_k\|^2\\ &+\left(\frac{1}{\alpha^2}+\frac{1}{\alpha^6}+\frac{1}{\nu\alpha^4}\right)\left(\mathbb{E}\left[\|u_0\|^2\right]\right.\\ &\left.+\alpha^2\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]+T\sum_{k\in K}\langle\sigma_k,(I-\alpha^2A)^{-1}\sigma_k\rangle\right) \quad (116)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\|u(t)\|_{H^3}^2\right] &\lesssim \frac{1}{\alpha^4}\mathbb{E}\left[\|q_0\|^2\right]+\frac{1}{\alpha^4}\sum_{k\in K}\|s_k\|^2T\\ &+\frac{1}{\alpha^4}\mathbb{E}\left[\left(\sum_{k\in K}\int_0^T\langle s_k,q(s)\rangle^2ds\right)^{1/2}\right]\\ &+\left(\frac{1}{\alpha^6}+\frac{1}{\alpha^4}\right)\left(\mathbb{E}\left[\|u_0\|^2\right]+\alpha^2\mathbb{E}\left[\|\nabla u_0\|_{L^2}^2\right]\right.\\ &\left.+T\sum_{k\in K}\langle\sigma_k,(I-\alpha^2A)^{-1}\sigma_k\rangle\right)\\ &+\left(\frac{1}{\alpha^6}+\frac{1}{\alpha^4}\right)\left(\sum_{k\in K}\mathbb{E}\left[\int_0^T\langle\sigma_k,u(s)\rangle^2ds\right]^{1/2}\right) \quad (117)\end{aligned}$$

Remark 45 If we consider the scaled equations with $\sqrt{\tilde{\nu}}$ in front of the noise, then each σ_k and s_k is multiplied by $\sqrt{\tilde{\nu}}$ in Lemmas 40, 43 and Remark 44.

Thanks to Remark 44, if we consider the scaled equation with additive noise and initial condition u_0^α satisfying Hypothesis 6, then the following result follows immediately.

Lemma 46 *If we consider the stochastic second-grade fluid equations with additive noise* (111) *scaled by $\sqrt{\tilde{\nu}}$, under Hypothesis* 2–6*, if u^α is the solution in the sense of Definition* 39 *of the problem with initial condition u_0^α, then*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\|u(t)\|^2\right]+\alpha^2\mathbb{E}\left[\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2\right]+2\nu\int_0^T\mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right]ds\\ &=O(1),\\ &\mathbb{E}\left[\alpha^6\sup_{t\in[0,T]}\|u^\alpha(t)\|_{H^3}^2\right]=O(1).\end{aligned}$$

Looking carefully at the proof of Theorem 9, Lemma 46 contains the crucial bounds on the norm of the solutions to obtain the inviscid limit. Therefore, following the same ideas of Sect. 5, one can prove that the inviscid limit holds:

Theorem 47 *Under Hypotheses 1–6, calling u^α the solution of the stochastic second-grade fluid equations with additive noise (111) scaled by $\sqrt{\bar{\nu}}$ and $\bar{u}$ the solution of (15), then*

$$\lim_{\alpha\to 0} \mathbb{E}\left[\sup_{t\in[0,T]} \|u^\alpha(t) - \bar{u}(t)\|^2\right] = 0.$$

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Declarations

Conflict of interest The author declares no conflict of interest.

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