



Stochastic evolution equations with Lévy noise in the dual of a nuclear space

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Abstract

In this article we give sufficient and necessary conditions for the existence of a weak and mild solution to stochastic evolution equations with (general) Lévy noise taking values in the dual of a nuclear space. As part of our approach we develop a theory of stochastic integration with respect to a Lévy process taking values in the dual of a nuclear space. We also derive further properties of the solution such as the existence of a solution with square moments, the Markov property and path regularity of the solution. In the final part of the paper we give sufficient conditions for the weak convergence of the solutions to a sequence of stochastic evolution equations with Lévy noises.

Keywords Lévy processes · Dual of a nuclear space · Stochastic integrals · Stochastic evolution equations · Weak convergence

Mathematics Subject Classification 60H15 · 60H05 · 60G17 · 60B12.

1 Introduction

The main objective of this paper is to study properties of the solutions to the following class of time-inhomogeneous (linear) stochastic evolution equations

$$\begin{cases} dY_t = A(t)'Y_t dt + \int_{\Phi'} R(t, f)L(dt, df), & t \geq 0, \\ Y_0 = \eta, \end{cases} \quad (1.1)$$

here Φ and Ψ denote two quasi-complete bornological nuclear spaces (e.g. the space of tests functions $\mathcal{D}(\mathbb{R}^d)$, the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, and the spaces of distributions $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, see Sect. 2), L is a Lévy process taking values in the dual Φ' to Φ ,

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$(A(t)' : t \geq 0)$ is the family of dual operators to a family $(A(t) : t \geq 0)$ of closed linear operators which generates a backward evolution system $(U(s, t) : 0 \leq s \leq t < \infty)$ of continuous linear operators on Ψ , and the coefficient R maps $[0, \infty) \times \Omega \times \Phi'$ into the space $\mathcal{L}(\Phi', \Psi')$ of continuous linear operators from Φ' into Ψ' .

A particular case of (1.1) corresponds to the following class of (generalized) Langevin equations

$$dY_t = A(t)'Y_t dt + B'dL_t, \quad t \geq 0. \quad (1.2)$$

here $B' \in \mathcal{L}(\Phi', \Psi')$. Solutions to (1.2) are known as (generalized) Ornstein–Uhlenbeck processes. They arise in many investigations on fluctuations limits of infinite particle systems of independent Brownian motions and different types of diffusion systems (see e.g. [8, 14, 26, 31, 34, 45, 46]). Other applications include the modelling of the dynamics of nerve signals [34], environmental pollution [35], and statistical filtering [37].

Properties of solutions to stochastic evolution equations of the form (1.2) have been studied by many authors (e.g. [9–12, 15, 19, 27, 32, 33, 49, 50, 57]). In most of these works the authors assumed that Φ and Ψ are nuclear Fréchet spaces, and the driving noise L is a Wiener process or a square integrable martingale, although the case where L is assumed to be a semimartingale with independent increments in a nuclear Fréchet space has also been considered (e.g. see [9]).

In the case of quasi-complete bornological nuclear spaces and under the time homogeneous setting $A(t) = A$, the author introduced in [21] sufficient conditions for the existence and uniqueness of weak and mild solutions to non-linear stochastic evolution equations with Lévy driven multiplicative noise. These sufficient conditions requires that A be the generator of a $(C_0, 1)$ -semigroup $(S(t) : t \geq 0)$ on Ψ whose dual semigroup $(S(t)' : t \geq 0)$ is also a $(C_0, 1)$ -semigroup on Ψ' (see definitions in Sect. 4.1) and the coefficients satisfy some growth and Lipschitz type conditions. However, in the case of linear equations the conditions introduced in [21] are rather restrictive and does not cover the time inhomogeneous setting described in (1.1). Further properties of solutions were not studied in [21].

Motivated by the above, in the work we present a complete theory of existence and uniqueness of solutions to the general stochastic evolution equation (1.1), we derive some of the fundamental properties of the solution and its trajectories, and we further explore the problem of finding sufficient conditions for the weak convergence of a sequence of stochastic evolution equations of the form (1.2).

We start in Sect. 2 with some preliminaries on nuclear spaces and their duals, and basic properties of cylindrical and stochastic processes. In Sect. 3 we introduce a theory of stochastic integration for operator-valued processes with respect to Lévy processes in duals of nuclear spaces. The stochastic integral is constructed as a sum of the stochastic integrals with respect to each of the components of the Lévy process according to its Lévy-Itô decomposition developed in [22].

The stochastic integral with respect to the drift term is defined using the regularization theorem (see [20]). In the case of the martingale part of the decomposition, the stochastic integral is defined using the theory of stochastic integration with respect to cylindrical martingale-valued measures developed in [21]. In the case of the large

jumps term in the decomposition the stochastic integral is defined as a finite random sum as in [4]. The class of stochastic integrands (see Definition 3.2) corresponds to families $R : [0, \infty) \times \Omega \times \Phi' \rightarrow \mathcal{L}(\Phi', \Psi')$ satisfying some weak predictability and weak square integrability conditions in terms of the characteristics of the Lévy process in its Lévy-Khintchine formula (see [22]). We are not aware of any other work that considers stochastic integrals with respect to Lévy processes in our general context of quasi-complete bornological nuclear space.

Our study of existence and uniqueness of “weak” solutions to (1.1) is carried out in Sect. 4. We start by applying our recently introduced theory of stochastic integration to define the stochastic convolution of the forward evolution system $(U(t, s)') : 0 \leq s \leq t < \infty)$ (of the dual operators) with respect to the Lévy process L , and by studying some of its properties. The corresponding “mild” solution defined by the stochastic convolution [see (4.17)] is latter shown to be a weak solution to (1.1) (see Theorem 4.7). On the other hand, under a mild assumption on the regularity of the solutions (that holds for example if the solution has finite moments or càdlàg paths) it is shown in Theorem 4.13 that any weak solution to (1.1) is a version of the mild solution. Here it is worth to mention that unlike other works that establish existence and uniqueness of solutions to equations of the form (1.2) (e.g. [9, 32]), we do not require for the family of generators $(A(t) : t \geq 0)$ that each $A(t)$ be a continuous and linear operator on Ψ .

In Sect. 5 we study further properties of the solutions to (1.1). We start by considering the existence of solutions with square moments under the assumption that L has square moments. In particular, in Theorem 5.4 we show that there exists a unique solution $(X_t : t \geq 0)$ to (1.1) which has the property that for each $t > 0$ there exists a Hilbert space Ψ'_ρ embedded in Ψ , Ψ'_ρ equipped with a Hilbertian norm ρ' , such that $\mathbb{E} \int_0^t \rho'(X_s)^2 ds < \infty$. In a second part of this section we will concentrate our efforts in the study of equations of the form (1.2), where we prove that the solution is a Markov process (Proposition 5.5). Moreover, we will show in Theorem 5.7 that under the hypothesis that each $A(t)$ is a continuous and linear operator on Ψ there exists a unique càdlàg solution to (1.2).

Examples and applications of our results will be given in Sect. 6. In particular, we will see that our results generalize to the Lévy noise setting those results obtained in [32] for stochastic equations driven by square-integrable martingale noise in the dual of a Fréchet nuclear space (see Example 6.4 for details).

Later, in Sect. 7 we provide sufficient conditions for the weak convergence of the solutions to the sequence of (generalized) Langevin equations

$$dY_t^n = A^n(t)' Y_t^n dt + (B^n)' dL_t^n, \quad Y_0^n = \eta_0^n.$$

Our main result (Theorem 7.1) generalize those obtained by other authors (see [19, 33, 49]) from the context of Fréchet spaces to quasi-complete bornological nuclear spaces. Furthermore, we also formulate (see Theorem 7.4) sufficient conditions for the weak convergence of solutions in terms of convergence on finite-dimensional distributions of the sequence of Lévy processes L^n and some properties of the sequence of the characteristics of each L^n in its Lévy–Khintchine formula.

Finally, in Sect. 8 we include a literature review of other theories of stochastic integration and stochastic evolution equations with Lévy noise in Banach spaces and comparisons with the theory developed in this article.

2 Preliminaries

Let Φ be a locally convex space (we will only consider vector spaces over \mathbb{R}). Φ is called *quasi-complete* if each bounded and closed subset of it is complete. Φ is called *bornological* (respectively *ultrabornological*) if it is the inductive limit of a family of normed (respectively Banach) spaces. A *barreled space* is a locally convex space such that every convex, balanced, absorbing and closed subset is a neighborhood of zero. Every quasi-complete bornological space is ultrabornological and hence barreled. For further details see [30, 47].

If p is a continuous semi-norm on Φ and $r > 0$, the closed ball of radius r of p given by $B_p(r) = \{\phi \in \Phi : p(\phi) \leq r\}$ is a closed, convex, balanced neighborhood of zero in Φ . A continuous semi-norm (respectively a norm) p on Φ is called *Hilbertian* if $p(\phi)^2 = Q(\phi, \phi)$, for all $\phi \in \Phi$, where Q is a symmetric, non-negative bilinear form (respectively inner product) on $\Phi \times \Phi$. Let Φ_p be the Hilbert space that corresponds to the completion of the pre-Hilbert space $(\Phi/\ker(p), \tilde{p})$, where $\tilde{p}(\phi + \ker(p)) = p(\phi)$ for each $\phi \in \Phi$. The quotient map $\Phi \rightarrow \Phi/\ker(p)$ has an unique continuous linear extension $i_p : \Phi \rightarrow \Phi_p$. Let q be another continuous Hilbertian semi-norm on Φ for which $p \leq q$. In this case, $\ker(q) \subseteq \ker(p)$. Moreover, the inclusion map from $\Phi/\ker(q)$ into $\Phi/\ker(p)$ is linear and continuous, and therefore it has a unique continuous extension $i_{p,q} : \Phi_q \rightarrow \Phi_p$. Furthermore, we have the following relation: $i_p = i_{p,q} \circ i_q$.

Let p and q be continuous Hilbertian semi-norms on Φ such that $p \leq q$. The space of continuous linear operators (respectively Hilbert-Schmidt operators) from Φ_q into Φ_p is denoted by $\mathcal{L}(\Phi_q, \Phi_p)$ (respectively $\mathcal{L}_2(\Phi_q, \Phi_p)$) and the operator norm (respectively Hilbert-Schmidt norm) is denoted by $\|\cdot\|_{\mathcal{L}(\Phi_q, \Phi_p)}$ (respectively $\|\cdot\|_{\mathcal{L}_2(\Phi_q, \Phi_p)}$).

We denote by Φ' the topological dual of Φ and by $\langle f, \phi \rangle$ the canonical pairing of elements $f \in \Phi', \phi \in \Phi$. Unless otherwise specified, Φ' will always be considered equipped with its *strong topology*, i.e. the topology on Φ' generated by the family of semi-norms (η_B) , where for each $B \subseteq \Phi$ bounded we have $\eta_B(f) = \sup\{|\langle f, \phi \rangle| : \phi \in B\}$ for all $f \in \Phi'$. If p is a continuous Hilbertian semi-norm on Φ , then we denote by Φ'_p the Hilbert space dual to Φ_p . The dual norm p' on Φ'_p is given by $p'(f) = \sup\{|\langle f, \phi \rangle| : \phi \in B_p(1)\}$ for all $f \in \Phi'_p$. Moreover, the dual operator i'_p corresponds to the canonical inclusion from Φ'_p into Φ' and it is linear and continuous.

Let $(q_\gamma(\cdot) : \gamma \in \Gamma)$ be a family of seminorms generating the strong topology on Φ' . Fix $T > 0$ and denote by $D_T(\Phi')$ the collection of all càdlàg (i.e. right-continuous with left limits) maps from $[0, T]$ into Φ' . For a given $\gamma \in \Gamma$ we consider the pseudometric d_γ on $D_T(\Phi')$ given by

$$d_\gamma(x, y) = \inf_{\lambda \in \Lambda} \left\{ \sup_{t \in [0, T]} q_\gamma(x(t) - y(\lambda(t))) + \sup_{0 \leq s < t \leq T} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right\}, \tag{2.1}$$

for all $x, y \in D_T(\Phi')$, where Λ denotes the set of all the strictly increasing continuous maps λ from $[0, T]$ onto itself. The family of seminorms $(d_\gamma : \gamma \in \Gamma)$ generates a completely regular topology on $D_T(\Phi')$ that is known as the *Skorokhod topology* (also known as the *J1 topology*).

Let $D_\infty(\Phi')$ denote the space of mappings $x : [0, \infty) \rightarrow \Phi'$ which are càdlàg. For every $s \geq 0$, let $r_s : D(\Phi') \rightarrow D_{s+1}(\Phi')$ be given by

$$r_s(x)(t) = \begin{cases} x(t) & \text{if } t \in [0, s], \\ (s + 1 - t)x(t) & \text{if } t \in [s, s + 1]. \end{cases}$$

For every $\gamma \in \Gamma$ let

$$d_\gamma^\infty(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} \left(1 \wedge d_\gamma^n(r_n(x), r_n(y)) \right),$$

where for each $n \in \mathbb{N}$, d_γ^n is the pseudometric defined in (2.1) for $T = n$. The *Skorokhod topology* in $D_\infty(\Phi')$ is the completely regular topology generated by the family of pseudometrics $(d_\gamma^\infty : \gamma \in \Gamma)$. For further details see [23, 28].

Let us recall that a (Hausdorff) locally convex space (Φ, \mathcal{T}) is called *nuclear* if its topology \mathcal{T} is generated by a family Π of Hilbertian semi-norms such that for each $p \in \Pi$ there exists $q \in \Pi$, satisfying $p \leq q$ and the canonical inclusion $i_{p,q} : \Phi_q \rightarrow \Phi_p$ is Hilbert-Schmidt. Other equivalent definitions of nuclear spaces can be found in [52, 56].

Let Φ be a nuclear space. If p is a continuous Hilbertian semi-norm on Φ , then the Hilbert space Φ_p is separable (see [52], Proposition 4.4.9 and Theorem 4.4.10, p.82). Now, let $(p_n : n \in \mathbb{N})$ be an increasing sequence of continuous Hilbertian semi-norms on (Φ, \mathcal{T}) . We denote by θ the locally convex topology on Φ generated by the family $(p_n : n \in \mathbb{N})$. The topology θ is weaker than \mathcal{T} . We will call θ a (weaker) *countably Hilbertian topology* on Φ and we denote by Φ_θ the space (Φ, θ) and by $\widehat{\Phi}_\theta$ its completion. The space $\widehat{\Phi}_\theta$ is a (not necessarily Hausdorff) separable, complete, pseudo-metrizable (hence Baire and ultrabornological; see Example 13.2.8(b) and Theorem 13.2.12 in [47]) locally convex space and its dual space satisfies $(\widehat{\Phi}_\theta)' = (\Phi_\theta)' = \bigcup_{n \in \mathbb{N}} \Phi'_{p_n}$ (see [20], Proposition 2.4).

The following are all examples of complete, ultrabornological (hence barrelled) nuclear spaces: the spaces of functions $\mathcal{E}_K := \mathcal{C}^\infty(K)$ (K : compact subset of \mathbb{R}^d) and $\mathcal{E} := \mathcal{C}^\infty(\mathbb{R}^d)$, the rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$, and the space of test functions $\mathcal{D}(U) := \mathcal{C}_c^\infty(U)$ (U : open subset of \mathbb{R}^d), as well are the spaces of distributions \mathcal{E}'_K , \mathcal{E}' , $\mathcal{S}'(\mathbb{R}^d)$, and $\mathcal{D}'(U)$. Other examples are the space of harmonic functions $\mathcal{H}(U)$ (U : open subset of \mathbb{R}^d), the space of polynomials \mathcal{P}_n in n -variables and the space of real-valued sequences $\mathbb{R}^\mathbb{N}$ (with direct product topology). For references see [52, 55, 56].

Throughout this work we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and consider a filtration $(\mathcal{F}_t : t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies the *usual conditions*, i.e. it is right continuous and \mathcal{F}_0 contains all subsets of sets of \mathcal{F} of \mathbb{P} -measure zero.

We denote by \mathcal{P}_∞ the predictable σ -algebra on $[0, \infty) \times \Omega$ and for any $T > 0$, let \mathcal{P}_T the restriction of \mathcal{P}_∞ to $[0, T] \times \Omega$.

We denote by $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the space of equivalence classes of real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We always consider the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the topology of convergence in probability and in this case it is a complete, metrizable, topological vector space.

A *cylindrical random variable* in Φ' is a linear map $X : \Phi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ (see [20]). If X is a cylindrical random variable in Φ' , we say that X is *n-integrable* ($n \in \mathbb{N}$) if $\mathbb{E}(|X(\phi)|^n) < \infty, \forall \phi \in \Phi$, and has *zero-mean* if $\mathbb{E}(X(\phi)) = 0, \forall \phi \in \Phi$. The *Fourier transform* of X is the map from Φ into \mathbb{C} given by $\phi \mapsto \mathbb{E}(e^{iX(\phi)})$.

Let X be a Φ' -valued random variable, i.e. $X : \Omega \rightarrow \Phi'$ is a $\mathcal{F}/\mathcal{B}(\Phi')$ -measurable map. For each $\phi \in \Phi$ we denote by $\langle X, \phi \rangle$ the real-valued random variable defined by $\langle X, \phi \rangle(\omega) := \langle X(\omega), \phi \rangle$, for all $\omega \in \Omega$. The linear mapping $\phi \mapsto \langle X, \phi \rangle$ is called the *cylindrical random variable induced/defined by X*.

Let $J = \mathbb{R}_+ := [0, \infty)$ or $J = [0, T]$ for $T > 0$. We say that $X = (X_t : t \in J)$ is a *cylindrical process* in Φ' if X_t is a cylindrical random variable for each $t \in J$. Clearly, any Φ' -valued stochastic processes $X = (X_t : t \in J)$ induces/defines a cylindrical process under the prescription: $\langle X, \phi \rangle = (\langle X_t, \phi \rangle : t \in J)$, for each $\phi \in \Phi$.

If X is a cylindrical random variable in Φ' , a Φ' -valued random variable Y is called a *version* of X if for every $\phi \in \Phi, X(\phi) = \langle Y, \phi \rangle$ \mathbb{P} -a.e. A Φ' -valued process $Y = (Y_t : t \in J)$ is said to be a Φ' -valued *version* of the cylindrical process $X = (X_t : t \in J)$ on Φ' if for each $t \in J, Y_t$ is a Φ' -valued version of X_t .

We will say that a Φ' -valued random variable X is *n-integrable* ($n \in \mathbb{N}$) if the cylindrical random variable induced by X is *n-integrable*, i.e. if $\mathbb{E}(|\langle X, \phi \rangle|^n) < \infty, \forall \phi \in \Phi$. In other words X has weak (in duality sense) *n-moments*. A Φ' -valued process $X = (X_t : t \in J)$ is *n-integrable* if X_t is *n-integrable* for every $t \in J$.

A Φ' -valued process $X = (X_t : t \in J)$ is *continuous* (respectively *càdlàg*) if for \mathbb{P} -a.e. $\omega \in \Omega$, the *sample paths* $t \mapsto X_t(\omega) \in \Phi'$ of X are continuous (respectively right-continuous with left limits).

The Φ' -valued process $X = (X_t : t \geq 0)$ is *adapted* if $\omega \mapsto X_t(\omega)$ is $\mathcal{F}_t/\mathcal{B}(\Phi')$ measurable for each $t \geq 0$. We say X is *predictable* if for each $T \geq 0$ the mapping $(s, \omega) \mapsto X_s(\omega)$ is $\mathcal{P}_T/\mathcal{B}(\Phi')$ -measurable.

A Φ' -valued random variable X is called *regular* if there exists a weaker countably Hilbertian topology θ on Φ such that $\mathbb{P}(\omega : X(\omega) \in (\widehat{\Phi}_\theta)') = 1$. If Φ is barrelled, the property of being regular is equivalent to the property that the law of X is a Radon measure on Φ' (see Theorem 2.10 in [20]).

A Φ' -valued process $X = (X_t : t \geq 0)$ is said to be *regular* if for each $t \geq 0, X_t$ is a regular random variable. In general the topology θ_t with respect to which X_t is regular depend on t . Although there are sufficient conditions on X such that one can find a weaker countably Hilbertian topology θ (independent of t) with respect to which all the random variables X_t are regular (see Theorem 3.2 in [20]).

Assumption 2.1 All through this article Φ and Ψ will denote two quasi-complete, bornological, nuclear spaces.

3 Lévy processes and stochastic integrals

Our main objective in this section is to review some properties on Lévy processes in duals of nuclear spaces and to define stochastic integrals with respect to these processes.

3.1 Lévy processes in duals of nuclear spaces

Recall from [22] that a Φ' -valued process $L = (L_t : t \geq 0)$ is called a *Lévy process* if

- (i) $L_0 = 0$ a.s.,
- (ii) L has *independent increments*, i.e. for any $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \dots < t_n < \infty$ the Φ' -valued random variables $L_{t_1}, L_{t_2} - L_{t_1}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent,
- (iii) L has *stationary increments*, i.e. for any $0 \leq s \leq t$, $L_t - L_s$ and L_{t-s} are identically distributed, and
- (iv) For every $t \geq 0$ the distribution μ_t of L_t is a Radon measure and the mapping $t \mapsto \mu_t$ from \mathbb{R}_+ into the space $\mathfrak{M}_R^1(\Phi')$ of Radon probability measures on Φ' is continuous at 0 when $\mathfrak{M}_R^1(\Phi')$ is equipped with the weak topology.

Observe that by (iv) the Lévy process L is regular. Moreover, L has a càdlàg version. Indeed, by Corollary 3.11 in [22] there exists a weaker countably Hilbertian topology ϑ on Φ (independent of t) such that L has a $(\widehat{\Phi}_\vartheta)'$ -valued càdlàg version $\tilde{L} = (\tilde{L}_t : t \geq 0)$. In particular, \tilde{L} is a Φ' -valued càdlàg Lévy process which is a version of L . From now on we will identify L with \tilde{L} .

Let $N = \{N(t, A) : t \geq 0, A \in \mathcal{B}(\Phi' \setminus \{0\})\}$ be the *Poisson random measure* associated to L , i.e. $N(t, A) = \sum_{0 \leq s \leq t} \mathbb{1}_A(\Delta L_s)$, $\forall t \geq 0, A \in \mathcal{B}(\Phi' \setminus \{0\})$, with respect to the ring \mathcal{A} of all the subsets of $\Phi' \setminus \{0\}$ that are *bounded below* (i.e. $A \in \mathcal{A}$ if $0 \notin \bar{A}$, where \bar{A} is the closure of A). The corresponding compensator measure of N is of the form $\nu(\omega; dt; df) = dt\nu(df)$, where ν is a *Lévy measure* on Φ' in the following sense (see [22], Theorem 4.11):

- (1) $\nu(\{0\}) = 0$,
- (2) for each neighborhood of zero $U \subseteq \Phi'$, the restriction $\nu|_{U^c}$ of ν on the set U^c belongs to the space $\mathfrak{M}_R^b(\Phi')$ of bounded Radon measures on Φ' ,
- (3) there exists a continuous Hilbertian semi-norm ρ on Φ such that

$$\int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) < \infty, \quad \text{and} \quad \nu|_{B_{\rho'}(1)^c} \in \mathfrak{M}_R^b(\Phi'), \tag{3.1}$$

where ρ' is the dual norm of ρ and $B_{\rho'}(1) := B_\rho(1)^0 = \{f \in \Phi' : \rho'(f) \leq 1\}$.

Here is important to stress the fact that the seminorm ρ satisfying (3.1) is not unique. Indeed, any continuous Hilbert semi-norm q on Φ such that $\rho \leq q$ satisfies (3.1).

It is shown in Theorem 4.17 in [22] that relative to a continuous Hilbertian seminorm ρ on Φ satisfying (3.1), for each $t \geq 0$, L_t admits the unique representation

$$L_t = tm + W_t + \int_{B_{\rho'}(1)} f \tilde{N}(t, df) + \int_{B_{\rho'}(1)^c} f N(t, df) \tag{3.2}$$

that is usually called the *Lévy-Itô decomposition* of L . In (3.2), we have that $m \in \Phi'$, $\tilde{N}(dt, df) = N(dt, df) - dt \nu(df)$ is the compensated Poisson random measure, and $(W_t : t \geq 0)$ is a Φ' -valued Lévy process with continuous paths (also called a Φ' -valued Wiener process) with zero-mean (i.e. $\mathbb{E}(\langle W_t, \phi \rangle) = 0$ for each $t \geq 0$ and $\phi \in \Phi$) and *covariance functional* \tilde{Q} satisfying

$$\mathbb{E}(\langle W_t, \phi \rangle \langle W_s, \varphi \rangle) = (t \wedge s) \tilde{Q}(\phi, \varphi), \quad \forall \phi, \varphi \in \Phi, s, t \geq 0. \tag{3.3}$$

Observe that \tilde{Q} is a continuous, symmetric, non-negative bilinear form on $\Phi \times \Phi$. The associated Hilbertian seminorm \mathcal{Q} is defined by $\mathcal{Q}(\phi) = \tilde{Q}(\phi, \phi)^{1/2} \forall \phi \in \Phi$. The process $\int_{B_{\rho'}(1)} f \tilde{N}(t, df), t \geq 0$, is a Φ' -valued zero-mean, square integrable, càdlàg Lévy process with second moments given by $\mathbb{E} \left(\left\| \int_{B_{\rho'}(1)} f \tilde{N}(t, df), \phi \right\|^2 \right) = t \int_{B_{\rho'}(1)} |\langle f, \phi \rangle|^2 \nu(df) \forall t \geq 0$ and $\phi \in \Phi$, and the process $\int_{B_{\rho'}(1)^c} f N(t, df) \forall t \geq 0$ is a Φ' -valued càdlàg Lévy process defined by means of a Poisson integral with respect to the Poisson random measure N of L on the set $B_{\rho'}(1)^c$ (see Section 4.1 in [22] for more information on Poisson integrals in duals of nuclear spaces). It is important to remark that all the random components of the representation (3.2) are independent.

The Fourier transform of the Lévy process $L = (L_t : t \geq 0)$ is given by the *Lévy-Khintchine formula* which characterizes it uniquely (see [22], Theorem 4.18): for each $t \geq 0, \phi \in \Phi$,

$$\begin{aligned} \mathbb{E} \left(e^{i \langle L_t, \phi \rangle} \right) &= e^{t \xi(\phi)}, \quad \text{with} \\ \xi(\phi) &= i \langle m, \phi \rangle - \frac{1}{2} \mathcal{Q}(\phi)^2 + \int_{\Phi'} \left(e^{i \langle f, \phi \rangle} - 1 - i \langle f, \phi \rangle \mathbb{1}_{B_{\rho'}(1)}(f) \right) \nu(df). \end{aligned} \tag{3.4}$$

where the *characteristics* $(m, \mathcal{Q}, \nu, \rho)$ of L are as described above.

3.2 Stochastic integration

Our next objective is to define stochastic integrals with respect to the Φ' -valued Lévy process $L = (L_t : t \geq 0)$. Our plan is to define the stochastic integral of L as a sum of stochastic integrals with respect to each term in the Lévy-Itô decomposition (3.2). We show the existence and properties of these stochastic integrals in the following paragraphs.

Let $R : [0, \infty) \times \Omega \times \Phi' \rightarrow \mathcal{L}(\Phi', \Psi')$ satisfy that for every $\psi \in \Psi, T > 0$, the mapping $(t, \omega, f) \mapsto R(t, \omega, f)' \psi$ is $\mathcal{P}_T \otimes \mathcal{B}(\Phi')/\mathcal{B}(\Phi)$ -measurable.

(1) *Drift stochastic integral* Let $h : [0, \infty) \times \Omega \rightarrow \Phi'$ be such that for every $T > 0$ the mapping $(t, \omega) \mapsto h(t, \omega)$ is $\mathcal{P}_T/\mathcal{B}(\Phi')$ -measurable. Assume moreover that

$$\mathbb{P} \left(\omega \in \Omega : \int_0^T |\langle h(r, \omega), R(r, \omega, 0)' \psi \rangle| dr < \infty \right) = 1, \quad \forall T > 0, \psi \in \Psi.$$

Then there exists a Ψ' -valued, regular, continuous process $\int_0^t R(r, 0)h(r)dr, t \geq 0$, satisfying \mathbb{P} -a.e.

$$\begin{aligned} & \left\langle \int_0^t R(r, \omega, 0)h(r, \omega)dr, \psi \right\rangle \\ &= \int_0^t \langle h(r, \omega), R(r, \omega, 0)' \psi \rangle ds, \quad \forall t \geq 0, \omega \in \Omega, \psi \in \Psi. \end{aligned} \tag{3.5}$$

We will show that such a process exists. To do this we will need the following information on absolutely continuous functions. For $t > 0$, let AC_t denotes the linear space of all absolutely continuous functions on $[0, t]$ which are zero at 0. It is well-known (see Theorem 5.3.6 in [7], p.339) that $G \in AC_t$ if and only if there exists an integrable function g defined on $[0, t]$ such that:

$$G(s) = \int_0^s g(r)dr, \quad \forall s \in [0, t]. \tag{3.6}$$

The space AC_t is a Banach space equipped with the norm $\|\cdot\|_{AC_t}$ given by $\|G\|_{AC_t} = \int_0^t |g(r)|dr$, for $G \in AC_t$ with g satisfying (3.6). Let $X_0(\psi) = 0$ for every $\psi \in \Psi$, and for each $t > 0$, define $X_t : \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t)$ by

$$X_t(\psi)(\omega)(s) = \begin{cases} \int_0^s \langle h(r, \omega), R(r, \omega, 0)' \psi \rangle dr, & \text{for } s \in [0, t], \omega \in \Omega_{t, \psi}, \\ 0, & \text{elsewhere,} \end{cases}$$

for every $\psi \in \Psi$, where $\Omega_{t, \psi} = \{\omega \in \Omega : \int_0^t |\langle h(r, \omega), R(r, \omega, 0)' \psi \rangle| dr < \infty\}$. The mapping X_t is linear. We will show it is sequentially closed, where $L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t)$ is equipped with the topology of convergence in probability in the norm $\|\cdot\|_{AC_t}$. Suppose $\psi_n \rightarrow \psi$ and $X_t(\psi_n) \rightarrow G$ in $L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t)$. There exists a subsequence (ψ_{n_k}) such that $\psi_{n_k} \rightarrow \psi$ and $X_t(\psi_{n_k}) \rightarrow G$ in AC_t \mathbb{P} -a.e. Since $G \in L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t)$, there exists a stochastic process $\{g(r, \omega) : r \in [0, T], \omega \in \Omega\}$ satisfying (3.6) for every $\omega \in \Omega$. Since $\langle h(r, \omega), R(r, \omega, 0)' \psi_{n_k} \rangle \rightarrow \langle h(r, \omega, 0), R(r, \omega)' \psi \rangle$ for all $r \in [0, s], \omega \in \Omega$, then by Fatou’s lemma we have \mathbb{P} -a.e.

$$\begin{aligned} \|X_t(\psi)(\omega) - G(\omega)\|_{AC_t} &= \int_0^t \lim_{k \rightarrow \infty} |\langle h(r, \omega), R(r, \omega, 0)' \psi_{n_k} \rangle - g(r, \omega)| dr \\ &\leq \liminf_{k \rightarrow \infty} \int_0^t |\langle h(r, \omega), R(r, \omega, 0)' \psi_{n_k} \rangle - g(r, \omega)| dr \\ &= \lim_{k \rightarrow \infty} \|X_t(\psi_{n_k})(\omega) - G(\omega)\|_{AC_t} = 0. \end{aligned}$$

Since Ψ is ultrabornological and $L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t)$ is a complete, metrizable, topological vector space, the closed graph theorem (Theorem 14.7.3 in [47], p.475) shows that X_t is continuous. Since the projection mapping $\pi_t : L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t) \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P}), \pi_t(Y)(\omega) = Y(\omega)(t) \forall \omega \in \Omega$, is linear

and continuous, then $(\pi_t \circ X_t : t \geq 0)$ defines a cylindrical process in Ψ' such that for each $t \geq 0$, $\pi_t \circ X_t : \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous. Hence, as for each $\psi \in \Psi$, $t \mapsto X_t(\psi)$ is continuous, the regularization theorem for ultrabornological spaces (Corollary 3.11 in [20]) shows that there exists a Ψ' -valued regular continuous process $\int_0^t R(r, 0)h(r)dr$, $t \geq 0$, that is a version of $(\pi_t \circ X_t : t \geq 0)$, i.e. satisfying (3.5). Finally, if we assume that for $n \geq 1$ we have

$$\mathbb{E} \int_0^T |\langle h(r), R(r, 0)' \psi \rangle|^n dr < \infty, \quad \forall T > 0, \psi \in \Psi, \quad (3.7)$$

then $\int_0^t R(r, 0)h(r)dr$, $t \geq 0$, is n -integrable. In effect, given $t \geq 0$ and $\psi \in \Psi$, by (3.5), Jensen's inequality and (3.7) we have

$$\begin{aligned} \mathbb{E} \left[\left| \left\langle \int_0^t R(r, 0)h(r)dr, \psi \right\rangle \right|^n \right] &= \mathbb{E} \left[\left| \int_0^t \langle h(r), R(r, 0)' \psi \rangle ds \right|^n \right] \\ &\leq t \mathbb{E} \int_0^t |\langle h(r), R(r, 0)' \psi \rangle|^n dr < \infty. \end{aligned}$$

- (2) *Wiener and compensated Poisson stochastic integrals* If we assume that for every $T > 0$ and $\psi \in \Psi$ the mapping R satisfies

$$\mathbb{E} \int_0^T \mathcal{Q}(R(r, 0)' \psi)^2 dr < \infty,$$

then we can define the Wiener stochastic integral $\int_0^t R(r, 0)dW_r$, $t \geq 0$, using the theory of (strong) stochastic integration developed in Sect. 5 in [21] with respect to the martingale-valued measure

$$M_1(t, A) = W_t \delta_0(A), \quad \text{for } t \geq 0, A \in \mathcal{B}(\{0\}),$$

(see Example 3.4 in [21]). The Wiener stochastic integral is a Ψ' -valued zero-mean, square integrable martingale with continuous paths.

Let $U \in \mathcal{B}(\Phi')$ be such that $\int_U |\langle f, \phi \rangle|^2 \nu(df) < \infty$ for all $\phi \in \Phi$. If we assume that for every $T > 0$ and $\psi \in \Psi$ the mapping R satisfies

$$\mathbb{E} \int_0^T \int_U |\langle f, R(r, f)' \psi \rangle|^2 \nu(df) dr < \infty,$$

then we can define the compensated Poisson stochastic integral on U , $\int_0^t \int_U R(r, f) \tilde{N}(dr, df)$, $t \geq 0$, using the theory of (strong) stochastic integration developed in Section 5 in [21] with respect to the martingale valued measure

$$M_2(t, A) = \int_A f \tilde{N}(t, df), \quad \text{for } t \geq 0, A \in \mathcal{A} \cap U, \quad (3.8)$$

(see Example 3.5 in [21]). The compensated Poisson stochastic integral on U is a Ψ' -valued zero-mean, square integrable càdlàg martingale.

For all $t \geq 0$, let

$$\int_0^t \int_U R(r, f)M(dr, df) := \int_0^t R(r, 0)dW_r + \int_0^t \int_U R(r, f)\tilde{N}(dr, df).$$

A simple application of Proposition 4.12 in [21] shows that $\int_0^t \int_U R(r, f)M(dr, df)$ can be defined equivalently as the stochastic integral of R with respect to the Lévy martingale-valued measure

$$M(t, A) = W_t\delta_0(A) + \int_{A \setminus \{0\}} f\tilde{N}(t, df), \quad \text{for } t \geq 0, A \in \mathcal{R}, \quad (3.9)$$

where $\mathcal{R} = \{U \cap \Gamma : \Gamma \in \mathcal{A}\} \cup \{\{0\}\}$.

For every $\psi \in \Psi$, \mathbb{P} -a.e. $\forall t \in [0, T]$, we have the following *weak-strong compatibility*:

$$\left\langle \int_0^t \int_U R(r, f)M(dr, df), \psi \right\rangle = \int_0^t \int_U R(r, f)' \psi M(dr, df), \quad (3.10)$$

where the real-valued (weak) stochastic integral on the right-hand side of the above equality is defined by Theorem 4.7 in [21].

- (3) *Poisson stochastic integral* Assume that $V \in \mathcal{A}$, i.e. $0 \notin \bar{V}$. For each $t \geq 0$ we define the Poisson stochastic integral:

$$\int_0^t \int_V R(s, f)N(ds, df)(\omega) = \sum_{0 \leq s \leq t} R(s, \omega, \Delta L_s(\omega))\Delta L_s(\omega)\mathbb{1}_V(\Delta L_s(\omega)), \quad (3.11)$$

which is a finite (random) sum since for \mathbb{P} -a.e. $\omega \in \Omega$ the trajectories $t \mapsto L_t(\omega)$ only have a finite number of jumps on the time interval $[0, t]$ and $0 \notin \bar{V}$ (see the discussion in Section 4.2 in [22]). By definition, the Poisson stochastic integral (3.11) is a Ψ' -valued regular adapted process. Moreover, if $(\tau_n : n \in \mathbb{N} \cup \{\infty\})$ are the arrival times of the Poisson process $(N(t, V) : t \geq 0)$, the sum in (3.11) is a fixed random variable on each interval $[\tau_n, \tau_{n+1})$ and hence the Poisson stochastic integral has càdlàg paths.

For any $\psi \in \Psi$ we define the (weak) Poisson stochastic integral:

$$\begin{aligned} & \int_0^t \int_V R(s, f)' \psi N(ds, df)(\omega) \\ &= \sum_{0 \leq s \leq t} \langle \Delta L_s(\omega), R(s, \omega, \Delta L_s(\omega))' \psi \rangle \mathbb{1}_V(\Delta L_s(\omega)), \end{aligned} \quad (3.12)$$

which is a real-valued adapted càdlàg process by the reasons explained above. We immediately obtain the *weak-strong compatibility* for Poisson integrals:

$$\left\langle \int_0^t \int_V R(s, f)N(ds, df), \psi \right\rangle = \int_0^t \int_V R(s, f)' \psi N(ds, df). \tag{3.13}$$

Remark 3.1 Suppose for $V \in \mathcal{A}$ that $\int_0^T \int_V \left| \langle f, R(r, f)' \psi \rangle \right|^2 \nu(df) dr < \infty$ for every $T > 0$ and $\psi \in \Psi$. We may then define

$$\begin{aligned} & \int_0^t \int_V R(s, f)' \psi \tilde{N}(ds, df) \\ &= \int_0^t \int_V R(s, f)' \psi N(ds, df) - \int_0^t \int_V \langle f, R(s, f)' \psi \rangle \nu(df) ds. \end{aligned}$$

One can check by going to simple processes (see Definition 4.4 and equation (4.7) in [21]) that the above definition is consistent with our earlier definition of the compensated Poisson integral defined with respect to the compensated Poisson random measure (3.8).

We have now all the necessary ingredients to introduce the stochastic integral with respect to a Lévy process. We start by defining the class of integrands for our theory.

Definition 3.2 Let $L = (L_t : t \geq 0)$ be a Φ' -valued Lévy process with Lévy-Itô decomposition (3.2) and characteristics $(m, \mathcal{Q}, \nu, \rho)$. Let $\Lambda(L, \Phi, \Psi)$ denotes the collection of all the mappings $R : [0, \infty) \times \Omega \times \Phi' \rightarrow \mathcal{L}(\Phi', \Psi')$ satisfying the following properties:

- (1) For every $\psi \in \Psi, T > 0$, the mapping $(t, \omega, f) \mapsto R(t, \omega, f)' \psi$ is $\mathcal{P}_T \otimes \mathcal{B}(\Phi')/\mathcal{B}(\Phi)$ -measurable.
- (2) For every $T > 0$ and $\psi \in \Psi$ the mapping R satisfies

$$\mathbb{E} \int_0^T \left[\left| \langle m, R(s, 0)' \psi \rangle \right|^2 + \mathcal{Q}(R(s, 0)' \psi)^2 + \int_{B_{\rho'}(1)} \left| \langle f, R(s, f)' \psi \rangle \right|^2 \nu(df) \right] ds < \infty.$$

We define the *(strong) stochastic integral of R with respect to L* as the Ψ' -valued regular adapted càdlàg process defined for each $t \geq 0$ by

$$\begin{aligned} \int_0^t \int_{\Phi'} R(s, f)L(ds, df) &= \int_0^t R(s, 0)mds + \int_0^t \int_{B_{\rho'}(1)} R(s, f)M(ds, df) \\ &+ \int_0^t \int_{B_{\rho'}(1)^c} R(s, f)N(ds, df). \end{aligned} \tag{3.14}$$

Likewise, for every $t \geq 0$ and $\psi \in \Psi$ we define the (weak) stochastic integral of R with respect to L as the real-valued adapted process:

$$\begin{aligned} & \int_0^t \int_{\Phi'} R(s, f)' \psi L(ds, df) \\ &= \int_0^t \langle m, R(s, 0)' \psi \rangle ds + \int_0^t \int_{B_{\rho'}(1)} R(s, f)' \psi M(ds, df) \\ &+ \int_0^t \int_{B_{\rho'}(1)^c} R(s, f)' \psi N(ds, df). \end{aligned} \tag{3.15}$$

From (3.5), (3.10) and (3.13) we have the following weak-strong compatibility: for every $\psi \in \Psi, \mathbb{P}$ -a.e. $\forall t \in [0, T]$,

$$\left\langle \int_0^t \int_{\Phi'} R(s, f) L(ds, df), \psi \right\rangle = \int_0^t \int_{\Phi'} R(s, f)' \psi L(ds, df). \tag{3.16}$$

Proposition 3.3 *Let Υ be a quasi-complete, bornological, nuclear space and let $S \in \mathcal{L}(\Psi', \Upsilon')$. Then, $SR = (SR(t, \omega, f) : t \geq 0, \omega \in \Omega, f \in \Phi')$ is stochastically integrable with respect to L and we have \mathbb{P} -a.e.*

$$S \left(\int_0^t \int_{\Phi'} R(s, f) L(ds, df) \right) = \int_0^t \int_{\Phi'} SR(s, f) L(ds, df), \quad \forall t \geq 0. \tag{3.17}$$

Proof First, since $S \in \mathcal{L}(\Psi', \Upsilon')$ it is easy to verify that SR satisfies Definition 3.2, hence $SR \in \Lambda(L, \Phi, \Upsilon)$.

To verify (3.17) it is sufficient to check that the equality holds for each term of (3.14). First, from (3.5) we have \mathbb{P} -a.e. $\forall t \geq 0, v \in \Upsilon$:

$$\left\langle S \left(\int_0^t R(s, 0) m ds \right), v \right\rangle = \int_0^t \langle m, R(s, 0)' S' v \rangle ds = \left\langle \int_0^t SR(s, 0) m ds, v \right\rangle.$$

Then, as both $S \left(\int_0^t R(s, 0) m ds \right)$ and $\int_0^t SR(s, 0) m ds$ are regular processes with continuous paths, the above equality shows that $S \left(\int_0^t R(s, 0) m ds \right) = \int_0^t SR(s, 0) m ds$ (see Proposition 2.12 in [20]). For the stochastic integral with respect to the Lévy martingale-valued measure M we have by Proposition 5.18 in [21] that

$$S \left(\int_0^t \int_{B_{\rho'}(1)} R(s, f) M(ds, df) \right) = \int_0^t \int_{B_{\rho'}(1)} SR(s, f) M(ds, df).$$

Finally, by (3.11) we easily conclude that

$$S \left(\int_0^t \int_{B_{\rho'}(1)^c} R(s, f) N(ds, df) \right) = \int_0^t \int_{B_{\rho'}(1)^c} SR(s, f) N(ds, df). \quad \square$$

4 Stochastic evolution equations With Lévy noise

4.1 Evolution systems

For the reader convenience and in order to set our notation, in this section we quickly review some properties of evolution systems and C_0 -semigroups of operators that we will need for our forthcoming arguments.

A family $(U(s, t) : 0 \leq s \leq t < \infty) \subseteq \mathcal{L}(\Psi, \Psi)$ is a *backward evolution system* if $U(t, t) = I$, $U(s, t) = U(s, r)U(r, t)$, $0 \leq s \leq r \leq t$. It is furthermore called *strongly continuous* if for every $\psi \in \Psi$, $s, t \geq 0$, the mappings $[s, \infty) \ni r \mapsto U(s, r)\psi$ and $[0, t] \ni r \mapsto U(r, t)\psi$ are continuous.

Let $A = (A_t : t \geq 0)$ be a family of linear, closed operators on Ψ with domains $\text{Dom}(A_t)$, $t \geq 0$. We say that A *generates* the backward evolution system $(U(s, t) : 0 \leq s \leq t < \infty)$ if there are dense subspaces $(D_t : t \geq 0)$ of Ψ (known as *regularity spaces*), such that $U(s, t)D_t \subseteq D_s \subseteq \text{Dom}(A_s)$ for $0 \leq s \leq t$ and if the following relations are satisfied:

$$\frac{d}{dt}U(s, t)\psi = U(s, t)A(t)\psi, \quad \forall \psi \in D_t, s \leq t. \tag{4.1}$$

$$\frac{d}{ds}U(s, t)\psi = -A(s)U(s, t)\psi, \quad \forall \psi \in D_t, s \leq t. \tag{4.2}$$

Equations (4.1) and (4.2) are called the *forward and backward equations*. From these equations we can conclude the following useful result (Lemma 1.1 in [32]):

$$U(u, t)\psi = \psi + \int_u^t U(u, s)A(s)\psi ds, \quad \forall \psi \in D_t, 0 \leq u \leq t. \tag{4.3}$$

$$U(u, t)\psi = \psi + \int_u^t A(s)U(s, t)\psi ds, \quad \forall \psi \in D_t, 0 \leq u \leq t. \tag{4.4}$$

The integrals in (4.3) and (4.4) are Riemann integrals in Ψ . Further properties of Riemann integrals for functions with real domain and with values in locally convex spaces can be consulted in [1, 18].

The backward evolution system $(U(s, t) : 0 \leq s \leq t < \infty)$ is called $(C_0, 1)$ if for each $T > 0$ and each continuous seminorm p on Ψ there exists some $\vartheta_p \geq 0$ and a continuous seminorm q on Ψ such that $p(U(s, t)\psi) \leq e^{\vartheta_p(t-s)}q(\psi)$, for all $0 \leq s \leq t \leq T$ and $\psi \in \Psi$.

Observe that since Ψ is reflexive, the family of dual operators $(U(t, s)' : 0 \leq s \leq t < \infty) \subseteq \mathcal{L}(\Psi', \Psi')$ defines a *forward evolution system*, i.e. $U(t, t)' = I$, $U(t, s)' = U(t, r)'U(r, s)'$, $0 \leq s \leq r \leq t$. Moreover, the dual family is *strongly continuous*, i.e. the mapping $\{(s, t) : s \leq t\} \ni (s, t) \mapsto U(s, t)'$ is continuous for each $f \in \Psi'$. Examples of backwards and forward evolution systems that appear on the study of stochastic evolution equations in duals of nuclear spaces can be found in [31, 32, 46] (see also Sect. 6).

A family $(S(t) : t \geq 0) \subseteq \mathcal{L}(\Psi, \Psi)$ is a C_0 -semigroup on Ψ if (i) $S(0) = I$, $S(t)S(s) = S(t + s)$ for all $t, s \geq 0$, and (ii) $\lim_{t \rightarrow s} S(t)\psi = S(s)\psi$, for all $s \geq 0$

and any $\psi \in \Psi$. The *infinitesimal generator* A of $(S(t) : t \geq 0)$ is the linear operator on Ψ defined by $A\psi = \lim_{h \downarrow 0} \frac{S(h)\psi - \psi}{h}$ (limit in Ψ), whenever the limit exists; the domain of A being the set $\text{Dom}(A) \subseteq \Psi$ for which the above limit exists. Since our assumptions imply that Ψ is reflexive, then the family $(S(t)' : t \geq 0)$ of dual operators is a C_0 -semigroup on Ψ' with generator A' , that we call the *dual semigroup* and the *dual generator* respectively.

A C_0 -semigroup $(S(t) : t \geq 0)$ is called a $(C_0, 1)$ -semigroup if for each continuous seminorm p on Ψ there exists some $\vartheta_p \geq 0$ and a continuous seminorm q on Ψ such that $p(S(t)\psi) \leq e^{\vartheta_p t} q(\psi)$, for all $t \geq 0$ and $\psi \in \Psi$. If in the above inequality $\vartheta_p = \omega$ with ω a positive constant (independent of p) $(S(t) : t \geq 0)$ is called *quasiequicontinuous*, and if $\omega = 0$ $(S(t) : t \geq 0)$ is called *equicontinuous*. It is worth to mention that even under our assumption that Ψ is reflexive, the dual semigroup $(S(t)' : t \geq 0)$ to a $(C_0, 1)$ -semigroup $(S(t) : t \geq 0)$ is not in general a $(C_0, 1)$ -semigroup on Ψ' (see [6], Section 6). However, if $(S(t) : t \geq 0)$ is equicontinuous then $(S(t)' : t \geq 0)$ is also equicontinuous (see [36], Theorem 1 and its Corollary). Further properties of $(C_0, 1)$ -semigroup and its generator can be consulted in [6].

Note that if $(S(t) : t \geq 0)$ is a C_0 -semigroup with generator A , then $U(s, t) = S(t - s)$ for $0 \leq s \leq t$ defines a backward evolution system with family of generators $A(t) = A$ and $\text{Dom}(A_t) = D_t = \text{Dom}(A)$ for every $t \geq 0$. If $(S(t) : t \geq 0)$ is a $(C_0, 1)$ -semigroup, then $U(s, t) = S(t - s)$ for $0 \leq s \leq t$ is $(C_0, 1)$.

4.2 Stochastic convolution For Lévy processes

The following is the main result of this section which guarantees the existence of the stochastic convolution and some of its properties.

Theorem 4.1 *Let $L = (L(t) : t \geq 0)$ be a Φ' -valued Lévy process with Lévy-Itô decomposition (3.2), $R \in \Lambda(L, \Phi, \Psi)$, and $(U(s, t) : 0 \leq s \leq t < \infty) \subseteq \mathcal{L}(\Psi, \Psi)$ be a $(C_0, 1)$ -backward evolution system. The stochastic convolution process*

$$X_{U,R}(t) := \int_0^t \int_{\Phi'} U(t, s)' R(s, f) L(ds, df), \quad \forall t \geq 0.$$

is the Ψ' -valued regular adapted process defined by

$$\begin{aligned} \int_0^t \int_{\Phi'} U(t, s)' R(s, f) L(ds, df) &= \int_0^t U(t, s)' R(s, 0) m ds \\ &+ \int_0^t \int_{B_{\rho'}(1)} U(t, s)' R(s, f) M(ds, df) \\ &+ \int_0^t \int_{B_{\rho'}(1)^c} U(t, s)' R(s, f) N(ds, df). \end{aligned} \tag{4.5}$$

Proof We define $X_{U,R}(0) = 0$. Now fix $t > 0$ and define $G : [0, \infty) \times \Omega \times \Phi' \rightarrow \mathcal{L}(\Phi', \Psi')$ by $G(s, \omega, f) = \mathbb{1}_{[0,t]}(s) U(t, s)'R(s, \omega, f)$. We must show that $G \in \Lambda(L, \Phi, \Psi)$.

In effect, Definition 3.2(1) and the strong continuity of the evolution system $(U(s, t) : 0 \leq s \leq t < \infty)$ imply that for every $\psi \in \Psi$, the mapping $(s, \omega, f) \mapsto R(s, \omega, f)'U(s, t)\psi$ defined on $[0, t] \times \Omega \times \Phi'$ is $\mathcal{P}_t \otimes \mathcal{B}(\Phi')/\mathcal{B}(\Phi)$ -measurable.

Now we must show the integrability condition of Definition 3.2(2). Let $X_s(\omega) = R(s, \omega, 0)m$ for $(s, \omega) \in [0, t] \times \Omega$. The Ψ' -valued process $(X_s : s \in [0, t])$ is (weakly) predictable by Definition 3.2(1). Since $\mathbb{E} \int_0^t |\langle X_s, \psi \rangle|^2 ds = \mathbb{E} \int_0^t |\langle m, R(s, 0)' \psi \rangle|^2 ds < \infty$ for each $\psi \in \Psi$, from a mild modification of the arguments used in the proof of Lemma 6.11 in [21] we can show that there exists a continuous Hilbertian seminorm ϱ on Ψ such that

$$\mathbb{E} \int_0^t \varrho'(R(s, 0)m)^2 ds = \mathbb{E} \int_0^t \varrho'(X_s)^2 ds < \infty. \tag{4.6}$$

Now, since $(U(s, t) : 0 \leq s \leq t < \infty)$ is a $(C_0, 1)$ -evolution system, there exist $\vartheta_\varrho \geq 0$ and a continuous seminorm q on Ψ such that $\varrho(U(s, t)\psi) \leq e^{\vartheta_\varrho(t-s)}q(\psi)$, for all $0 \leq s \leq t, \psi \in \Psi$. Then it follows from (4.6) that

$$\begin{aligned} \mathbb{E} \int_0^t |\langle m, R(s, 0)'U(s, t)\psi \rangle|^2 ds &\leq \mathbb{E} \int_0^t \varrho'(R(s, 0)m)^2 \varrho(U(s, t)\psi)^2 ds \\ &\leq e^{2\vartheta_\varrho t} q(\psi)^2 \mathbb{E} \int_0^t \varrho'(R(s, 0)m)^2 ds < \infty. \end{aligned} \tag{4.7}$$

Now, since $R \in \Lambda(L, \Phi, \Psi)$, on the time interval $[0, t]$ we have that R is integrable with respect to the Lévy martingale-valued measure (3.9). Theorem 5.11 in [21] shows that there exists a continuous Hilbertian seminorm p on Ψ and $\tilde{R} : [0, t] \times \Omega \times B_{\rho'}(1) \rightarrow \mathcal{L}(\Phi', \Psi'_p)$ such that $R(s, \omega, f) = i'_p \tilde{R}(s, \omega, f)$ for $\text{Leb} \otimes \mathbb{P} \otimes \nu$ -a.e. $(s, \omega, f) \in [0, t] \times \Omega \times B_{\rho'}(1)$ and

$$\mathbb{E} \int_0^t \left\| \tilde{R}(s, 0) \right\|_{\mathcal{L}_2(\Phi'_\varrho, \Psi'_p)}^2 ds + \mathbb{E} \int_0^t \int_{B_{\rho'}(1)} p'(\tilde{R}(s, f)f)^2 \nu(df) ds < \infty. \tag{4.8}$$

Let $\vartheta_p \geq 0$ and q a continuous seminorm on Ψ such that $p(U(s, t)\psi) \leq e^{\vartheta_p(t-s)}q(\psi)$ for all $0 \leq s \leq t$ and $\forall \psi \in \Psi$. For every $\psi \in \Psi$ we have by (4.8) that

$$\begin{aligned} \mathbb{E} \int_0^t \varrho(R(s, 0)'U(s, t)\psi)^2 ds &\leq \mathbb{E} \int_0^t \left\| \tilde{R}(s, 0) \right\|_{\mathcal{L}_2(\Psi_p, \Phi_\varrho)}^2 p(i_p U(s, t)\psi)^2 ds \\ &\leq q(\psi)^2 e^{2\vartheta_p t} \mathbb{E} \int_0^t \left\| \tilde{R}(s, 0) \right\|_{\mathcal{L}_2(\Phi'_\varrho, \Psi'_p)}^2 ds < \infty. \end{aligned} \tag{4.9}$$

Similarly, for every $\psi \in \Psi$ we have by (4.8) that

$$\begin{aligned} & \mathbb{E} \int_0^t \int_{B_{\rho'}(1)} \left| \langle f, R(s, f)'U(s, t)\psi \rangle \right|^2 \nu(df) ds \\ & \leq \mathbb{E} \int_0^t \int_{B_{\rho'}(1)} p'(\tilde{R}(s, f)f)^2 p(i_p U(s, t)\psi)^2 \nu(df) ds \\ & \leq q(\psi)^2 e^{2\theta_p t} \mathbb{E} \int_0^t \int_{B_{\rho'}(1)} p'(\tilde{R}(s, f)f)^2 \nu(df) ds < \infty. \end{aligned} \tag{4.10}$$

If we collect the estimates (4.7), (4.9) and (4.10) we conclude that G satisfies Definition 3.2(2) and hence $G \in \Lambda(L, \Phi, \Psi)$. Then we define $X_{U,R}(t) = \int_0^t \int_{\Phi'} G(s, f) L(ds, df)$, showing the existence of the stochastic convolution process defined by (4.5). \square

As a direct consequence of the results in Theorem 4.1 and from (3.16), for any given $\psi \in \Psi$ and $t \geq 0$ we have \mathbb{P} -a.e.

$$\left\langle \int_0^t \int_{\Phi'} U(t, s)'R(s, f) L(ds, df), \psi \right\rangle = \int_0^t \int_{\Phi'} R(s, f)'U(s, t)\psi L(ds, df), \tag{4.11}$$

where from (3.15) it follows that

$$\begin{aligned} & \int_0^t \int_{\Phi'} R(s, f)'U(s, t)\psi L(ds, df) \\ & = \int_0^t \langle m, R(s, f)'U(s, t)\psi \rangle ds \\ & \quad + \int_0^t \int_{B_{\rho'}(1)} R(s, f)'U(s, t)\psi M(ds, df) \\ & \quad + \int_0^t \int_{B_{\rho'}(1)^c} \langle f, R(s, f)'U(s, t)\psi \rangle N(ds, df). \end{aligned} \tag{4.12}$$

Remark 4.2 If we have $U(s, t) = S(t - s)$ where $(S(t) : t \geq 0)$ is a $(C_0, 1)$ -semigroup, following the standard practice we will use the notation $\int_0^t \int_{\Phi'} S(t - s)'R(s, f) L(ds, df)$ and $\int_0^t \int_{\Phi'} R(s, f)'S(t - s)\psi L(ds, df)$ for the stochastic convolution (in the strong and weak sense respectively). Likewise in (4.5) we replace $U(t, s)$ with $S(t - s)$ and in (4.12) we replace $U(s, t)$ with $S(t - s)$.

In the next result we show that the stochastic convolution possesses almost surely finite square moments. Recall the definition of the regularity spaces D_t in Sect. 4.1.

Proposition 4.3 *For every $t > 0$ and $\psi \in \Psi$ we have \mathbb{P} -a.e.*

$$\int_0^t \left(\left\langle \int_0^s \int_{\Phi'} U(s, r)'R(r, f) L(dr, df), \psi \right\rangle \right)^2 ds < \infty. \tag{4.13}$$

Assume moreover that for every $t > 0$ and $\psi \in D_t$, the mapping $s \mapsto A(s)\psi$ is continuous on $[0, t]$. Then for every $t > 0$ and $\psi \in D_t$ we have \mathbb{P} -a.e.

$$\int_0^t \left(\left\langle \int_0^s \int_{\Phi'} U(s, r)' R(r, f) L(dr, df), A(s)\psi \right\rangle \right)^2 ds < \infty. \tag{4.14}$$

Proof Let $t > 0$ and $\psi \in \Psi$. From (4.11) it suffices to verify that each term in the right-hand side of (4.12) has \mathbb{P} -a.e. a finite second moment with respect to the Lebesgue measure on $[0, t]$.

From Jensen’s inequality, (4.7), and Fubini’s theorem, it follows that

$$\begin{aligned} & \mathbb{E} \int_0^t \left(\int_0^s \langle \mathfrak{m}, R(r, 0)' U(r, s)\psi \rangle dr \right)^2 ds \\ & \leq \mathbb{E} \int_0^t \left(s \int_0^s |\langle \mathfrak{m}, R(r, 0)' U(r, s)\psi \rangle|^2 dr \right) ds \\ & \leq q(\psi)^2 \mathbb{E} \int_0^t s e^{2\vartheta e^s} \left(\int_0^s \varrho'(R(r, 0)\mathfrak{m})^2 dr \right) ds \\ & \leq q(\psi)^2 \left(\mathbb{E} \int_0^t \varrho'(R(r, 0)\mathfrak{m})^2 dr \right) \left(\int_0^t s e^{2\vartheta e^s} ds \right). \end{aligned}$$

Hence we conclude that the first term in the right-hand side of (4.12) has \mathbb{P} -a.e. a finite square moment.

Now from the Itô isometry (Theorem 4.7 in [21]), (4.8), (4.9) and (4.10), we have for every $s \in [0, t]$,

$$\begin{aligned} & \mathbb{E} \left(\int_0^s \int_{B_{\rho'}(1)} R(r, f)' U(r, s)\psi M(dr, df) \right)^2 \\ & = \mathbb{E} \int_0^s \mathcal{Q}(R(r, 0)' U(r, s)\psi)^2 dr + \mathbb{E} \int_0^s \int_{B_{\rho'}(1)} |\langle f, R(r, f)' U(r, s)\psi \rangle|^2 \nu(df) dr \\ & \leq q(\psi)^2 e^{2\vartheta \rho^s} \left(\mathbb{E} \int_0^t \left\| \tilde{R}(r, 0) \right\|_{\mathcal{L}_2(\Phi'_\mathcal{Q}, \Psi'_\rho)}^2 dr + \mathbb{E} \int_0^t \int_{B_{\rho'}(1)} p(\tilde{R}(r, f)f)^2 \nu(df) dr \right). \end{aligned}$$

The above inequality immediately implies that

$$\mathbb{E} \int_0^t \left(\int_0^s \int_{B_{\rho'}(1)} R(r, f)' U(r, s)\psi M(dr, df) \right)^2 ds < \infty.$$

Thus the second term in the right-hand side of (4.12) has \mathbb{P} -a.e. a finite second moment.

Finally, since for every $\omega \in \Omega$ the integral $\int_0^s \int_{B_{\rho'}(1)^c} \langle f, R(r, f)' U(r, s)\psi \rangle N(dr, df)$ is defined as a finite sum, it is immediate that we have

$$\int_0^t \left| \int_0^s \int_{B_{\rho'}(1)^c} \langle f, R(r, f)' U(r, s)\psi \rangle^2 N(dr, df) \right| ds < \infty.$$

Therefore the third term in the right-hand side of (4.12) has \mathbb{P} -a.e. a finite second moment. We have shown that (4.13) holds.

The proof of (4.14) can be carried out from similar arguments to those used above for (4.13) replacing ψ by $A(s)\psi$ and using the fact that the hypothesis of continuity on $[0, t]$ of the mapping $s \mapsto A(s)\psi$ implies that $\sup_{0 \leq s \leq t} q(A(s)\psi) < \infty$ for every continuous seminorm q on Ψ . \square

4.3 Existence and uniqueness of solutions

In this section we show the existence and uniqueness of solutions to the following class of (time-inhomogeneous) stochastic evolution equations,

$$dY_t = A(t)'Y_t dt + \int_{\Phi'} R(t, f)L(dt, df), \quad t \geq 0, \tag{4.15}$$

with initial condition $Y_0 = \eta$ \mathbb{P} -a.e., where we will assume the following:

Assumption 4.4

- (A1) η is a \mathcal{F}_0 -measurable Φ' -valued regular random variable.
- (A2) $(U(s, t) : 0 \leq s \leq t < \infty) \subseteq \mathcal{L}(\Psi, \Psi)$ is a $(C_0, 1)$ backward evolution system with family of generators $(A(t) : t \geq 0)$ with corresponding regularity spaces $(D_t : t \geq 0)$. We will make the following additional assumptions:
 - (a) For every $t > 0$ and $\psi \in D_t$, the mapping $s \mapsto A(s)\psi$ is continuous on $[0, t]$.
 - (b) For every $0 \leq s < t$ and $\psi \in D_t$ we have $A(s)U(s, t)\psi \in D_s$.
- (A3) $L = (L(t) : t \geq 0)$ is a Φ' -valued Lévy process with Lévy-Itô decomposition (3.2).
- (A4) $R \in \Lambda(L, \Phi, \Psi)$.

Remark 4.5 The conditions (a),(b) in Assumption 4.4(A2) are fulfilled in the following two important cases:

- (1) Suppose that $U(s, t) = S(t - s)$, $0 \leq s \leq t$, for a $(C_0, 1)$ -semigroup $(S(t) : t \geq 0)$. Since $A(t) = A$ and $\text{Dom}(A_t) = D_t = \text{Dom}(A)$ for every $t \geq 0$ we immediately get Assumption 4.4(A2)(a). Moreover, basic properties of C_0 -semigroups (see e.g. [36]) shows that $A(s)U(s, t)\psi = AS(t - s)\psi = S(t - s)A\psi \in \text{Dom}(A)$ for every $\psi \in \Psi$, which shows Assumption 4.4(A2)(b).
- (2) Suppose that $A(t) \in \mathcal{L}(\Psi, \Psi)$ for each $t \geq 0$ and the mapping $t \mapsto A(t)\psi$ is continuous from $[0, \infty)$ into Ψ for every $\psi \in \Psi$. The latter assumption is stronger than Assumption 4.4(A2)(a). From the former assumption we have $A(s)U(s, t)\psi \in D_s = \text{Dom}(A(s)) = \Psi$ for every $0 \leq s < t$ and $\psi \in \Psi$. So we obtain Assumption 4.4(A2)(b).

In this work we are interested in the existence and uniqueness of weak solutions.

Definition 4.6 A Ψ' -valued regular adapted process $Y = (Y_t : t \geq 0)$ is called a *weak solution* to (4.15) if for any given $t \geq 0$, for each $\psi \in D_t$ we have

$\int_0^t |\langle Y_s, A(s)\psi \rangle| ds < \infty$ \mathbb{P} -a.e. and

$$\langle Y_t, \psi \rangle = \langle \eta, \psi \rangle + \int_0^t \langle Y_s, A(s)\psi \rangle ds + \int_0^t \int_{\Phi'} R(s, f)' \psi L(ds, df). \quad (4.16)$$

We start by considering the existence of a weak solution to (4.15).

Theorem 4.7 *The stochastic evolution equation (4.15) has a weak solution given by the mild (or evolution) solution*

$$X_t = U(t, 0)' \eta + \int_0^t \int_{\Phi'} U(t, s)' R(s, f) L(ds, df), \quad \forall t \geq 0. \quad (4.17)$$

The key step in the proof of Theorem 4.7 is the following result on iterated integration for the stochastic convolution.

Lemma 4.8 *Given $t > 0$, for every $\psi \in D_t$, we have \mathbb{P} -a.e.*

$$\begin{aligned} & \int_0^t \left(\int_0^s \int_{\Phi'} R(r, f)' U(r, s) A(s) \psi L(dr, df) \right) ds \\ &= \int_0^t \int_{\Phi'} R(r, f)' U(r, t) \psi L(dr, df) - \int_0^t \int_{\Phi'} R(r, f)' \psi L(dr, df) \\ &= \int_0^t \left(\int_0^s \int_{\Phi'} R(r, f)' A(s) U(s, t) \psi L(dr, df) \right) ds. \end{aligned} \quad (4.18)$$

Proof Let $t > 0$ and consider any $\psi \in D_t$. We start by showing the equality of the first two lines in (4.18). It suffices to check that the equality holds for each term in (4.12).

For the first term in (4.12), observe that from Fubini's theorem and the forward equation (4.3), ω -wise we have that

$$\begin{aligned} & \int_0^t \left(\int_0^s \langle \mathfrak{m}, R(r, 0)' U(r, s) A(s) \psi \rangle dr \right) ds \\ &= \int_0^t \left(\int_r^t \langle R(r, 0) \mathfrak{m}, U(r, s) A(s) \psi \rangle ds \right) dr \\ &= \int_0^t \langle R(r, 0) \mathfrak{m}, U(r, t) \psi - \psi \rangle dr \\ &= \int_0^t \langle \mathfrak{m}, R(r, 0)' U(r, t) \psi \rangle dr - \int_0^t \langle \mathfrak{m}, R(r, 0)' \psi \rangle dr. \end{aligned}$$

The second term in (4.12), corresponds to the (weak) stochastic integral with respect to the Lévy martingale-valued measure M of (3.9). From the stochastic Fubini theorem

(Theorem 4.24 in [21]) and the forward equation (4.3) it follows that \mathbb{P} -a.e.

$$\begin{aligned} & \int_0^t \left(\int_0^s \int_{B_{\rho'}(1)} R(r, f)' U(r, s) A(s) \psi M(dr, df) \right) ds \\ &= \int_0^t \int_{B_{\rho'}(1)} \left(\int_r^t R(r, f)' U(r, s) A(s) \psi ds \right) M(dr, df) \\ &= \int_0^t \int_{B_{\rho'}(1)} R(r, f)' U(r, t) \psi - R(r, f)' \psi M(dr, df) \\ &= \int_0^t \int_{B_{\rho'}(1)} R(r, f)' U(r, t) \psi M(dr, df) - \int_0^t \int_{B_{\rho'}(1)} R(r, f)' \psi M(dr, df). \end{aligned}$$

For the third term in (4.12), from (3.12), Fubini’s theorem and the forward equation (4.3), ω -wise we have that

$$\begin{aligned} & \int_0^t \left(\int_0^s \int_{B_{\rho'}(1)^c} \langle f, R(r, f)' U(r, s) A(s) \psi \rangle N(dr, df) \right) ds \\ &= \int_0^t \left(\sum_{0 \leq r \leq s} \langle \Delta L_r, R(r, \Delta L_r)' U(r, s) A(s) \psi \rangle \mathbb{1}_{B_{\rho'}(1)^c}(\Delta L_r) \right) ds \\ &= \sum_{0 \leq r \leq t} \left(\int_r^t \langle R(r, \Delta L_r) \Delta L_r, U(r, s) A(s) \psi \rangle ds \right) \mathbb{1}_{B_{\rho'}(1)^c}(\Delta L_r) \\ &= \sum_{0 \leq r \leq t} (\langle R(r, \Delta L_r) \Delta L_r, U(r, t) \psi - \psi \rangle) \mathbb{1}_{B_{\rho'}(1)^c}(\Delta L_r) \\ &= \int_0^t \int_{B_{\rho'}(1)^c} \langle f, R(r, f)' U(r, t) \psi \rangle N(dr, df) \\ &\quad - \int_0^t \int_{B_{\rho'}(1)^c} \langle f, R(r, f)' \psi \rangle N(dr, df). \end{aligned}$$

The equality of the second and third lines in (4.18) can be proved following completely analogous arguments to those used in the above paragraphs but this time using the backward equation (4.4). Details are left to the reader. \square

Corollary 4.9 *Given $t > 0$, for every $\psi \in D_t$, we have \mathbb{P} -a.e.*

$$\begin{aligned} & \int_0^t \left(\left\langle \int_0^s \int_{\Phi'} U(t, r)' R(r, f) L(dr, df), A(s) \psi \right\rangle \right) ds \\ &= \left\langle \int_0^t \int_{\Phi'} U(t, r)' R(r, f) L(dr, df), \psi \right\rangle - \left\langle \int_0^t \int_{\Phi'} R(r, f) L(dr, df), \psi \right\rangle, \end{aligned} \tag{4.19}$$

Proof The equality in (4.19) follows from (4.18), by considering the compatibility between weak and strong integrals in (3.16) and (4.11). \square

Proof of Theorem 4.7 Let $(X_t : t \geq 0)$ be as defined in (4.17). We already know from Theorem 4.1 that the stochastic convolution is a Ψ' -valued regular adapted process. The strong continuity of the forward evolution system $(U(t, s)' : 0 \leq s \leq t)$ and our assumptions on η imply that $(U(t, 0)' \eta : t \geq 0)$ is a Ψ' -valued regular adapted process, so is $(X_t : t \geq 0)$.

Let $t \geq 0$ and $\psi \in D_t$. We must show that $\int_0^t |\langle X_s, A(s)\psi \rangle| ds < \infty$ \mathbb{P} -a.e. In effect, from our assumptions the mapping $s \mapsto \langle U(s, 0)' \eta, A(s)\psi \rangle$ has continuous paths on $[0, t]$. Hence, we have $\int_0^t |\langle U(s, 0)' \eta, A(s)\psi \rangle| ds < \infty$ \mathbb{P} -a.e. Then from (4.14) and (4.17) we conclude that $\int_0^t |\langle X_s, A(s)\psi \rangle| ds < \infty$ \mathbb{P} -a.e.

Our next objective is to show that $(X_t : t \geq 0)$ satisfies (4.16). Fix $t \geq 0$ and choose any $\psi \in D_t$. From (4.17) for each $s \in [0, t]$ we have

$$\langle X_s, A(s)\psi \rangle = \langle U(s, 0)' \eta, A(s)\psi \rangle + \left\langle \int_0^s \int_{\Phi'} U(s, r)' R(r, f) L(dr, df), A(s)\psi \right\rangle.$$

Integrating both sides on $[0, t]$ with respect to the Lebesgue measure, then from (4.19) and the forward equation (4.3), we have \mathbb{P} -a.e.

$$\begin{aligned} & \int_0^t \langle X_s, A(s)\psi \rangle ds \\ &= \int_0^t \langle \eta, U(0, s)A(s)\psi \rangle ds + \int_0^t \left\langle \int_0^s \int_{\Phi'} U(s, r)' R(r, f) L(dr, df), A(s)\psi \right\rangle ds \\ &= \langle \eta, U(0, t)\psi - \psi \rangle + \left\langle \int_0^t \int_{\Phi'} U(t, r)' R(r, f) L(dr, df), \psi \right\rangle \\ & \quad - \left\langle \int_0^t \int_{\Phi'} R(r, f) L(dr, df), \psi \right\rangle \\ &= \langle X_t, \psi \rangle - \langle \eta, \psi \rangle - \left\langle \int_0^t \int_{\Phi'} R(r, f) L(dr, df), \psi \right\rangle. \end{aligned}$$

Hence, the process $X = (X_t : t \geq 0)$ defined by (4.17) is a weak solution to (4.15). \square

The mild solution possesses almost surely square integrable paths as the next result shows.

Proposition 4.10 *Let $X = (X_t : t \geq 0)$ be the mild solution (4.17) to (4.15). Then, X has almost surely finite square moments, i.e. $\forall t > 0, \psi \in \Psi$, we have \mathbb{P} -a.e. $\int_0^t |\langle X_s, \psi \rangle|^2 ds < \infty$.*

Proof From Proposition 4.3 the stochastic convolution has almost surely finite square moments. The same holds true for the process $(U(t, 0)' \eta : t \geq 0)$ as a direct consequence of the strong continuity of the forward evolution system $(U(t, s)' : 0 \leq s \leq t < \infty)$. From (4.17) we have the corresponding result for X . \square

Our next main issue in this section concerns the uniqueness of solutions to (4.15). We will require some extra regularity on the trajectories of the solution which is explained in the following definition.

Definition 4.11 We say that a Ψ' -valued adapted process $Y = (Y_t : t \geq 0)$ has *almost surely locally Bochner integrable trajectories* if for each $t > 0$ there exists $\Omega_t \subseteq \Omega$ with $\mathbb{P}(\Omega_t) = 1$ and such that for each $\omega \in \Omega_t$ there exists a continuous seminorm $p = p(t, \omega)$ on Ψ such that $Y_s(\omega) \in \Psi'_p$ a.e. on $[0, t]$ and $\int_0^t p'(Y_s(\omega)) ds < \infty$.

The property of almost surely locally Bochner integrable trajectories is implied by other stronger regularity properties of the paths as for example are the existence of finite moments (see Theorem 5.4) and for processes having càdlàg trajectories as the next result shows.

Proposition 4.12 *Let $Y = (Y_t : t \geq 0)$ be a Ψ' -valued regular adapted càdlàg process. Then Y has almost surely locally Bochner integrable trajectories.*

Proof Since Y is càdlàg there exists $\Omega_0 \subseteq \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and for each $\omega \in \Omega$ we have that $s \mapsto X_s(\omega) \in D_\infty(\Psi')$. As Ψ is barrelled, given any $t > 0$ and $\omega \in \Omega_0$ there exists a continuous seminorm p on Ψ such that $s \mapsto Y_s(\omega) \in D_t(\Psi'_p)$ (Remark 3.6 in [23]). Hence, we have

$$\int_0^t p'(Y_s(\omega)) ds \leq t \sup_{0 \leq s \leq t} p'(Y_s(\omega)) < \infty.$$

Therefore Y has almost surely locally Bochner integrable trajectories. □

We are ready for our main result on the uniqueness of solutions to (4.15).

Theorem 4.13 *Let $Y = (Y_t : t \geq 0)$ be a weak solution to (4.15) which has almost surely locally Bochner integrable trajectories. Then for each $t \geq 0$, Y_t is given \mathbb{P} -a.e. by*

$$Y_t = U(t, 0)' \eta + \int_0^t \int_{\Phi'} U(t, r)' R(r, f) L(dr, df).$$

Proof Let $t \geq 0$ and $\psi \in D_t$. We start by showing that \mathbb{P} -a.e.

$$\int_0^t \left(\int_0^t |\mathbb{1}_{[0,s]}(r) \langle Y_r, A(r)A(s)U(s, t)\psi \rangle| dr \right) ds < \infty. \tag{4.20}$$

Let $\omega \in \Omega_t$ and $p = p(t, \omega)$ as in Definition 4.11. From Assumption 4.4(A2) it is clear that the set $\{A(r)A(s)U(s, t)\psi : 0 \leq r \leq s \leq t\}$ is bounded in Ψ , hence bounded under p . Then we have

$$\begin{aligned} & \int_0^t \left(\int_0^t |\mathbb{1}_{[0,s]}(r) \langle Y_r(\omega), A(r)A(s)U(s, t)\psi \rangle| dr \right) ds \\ & \leq t \sup_{0 \leq r \leq s \leq t} p(A(r)A(s)U(s, t)\psi) \int_0^t p'(Y_s(\omega)) ds < \infty. \end{aligned}$$

Now from an application of Fubini’s theorem (which we can apply thanks to (4.20)), then using that $A(r)$ is a closed operator for every $r \in [0, t]$, properties of the Riemann integral, and then using the backward equation (4.4) we have for $\omega \in \Omega_t$,

$$\begin{aligned} & \int_0^t \left(\int_0^s \langle Y_r(\omega), A(r)A(s)U(s, t)\psi \rangle dr \right) ds \\ &= \int_0^t \left(\int_r^t \langle Y_r(\omega), A(r)A(s)U(s, t)\psi \rangle ds \right) dr \\ &= \int_0^t \left\langle Y_r(\omega), A(r) \int_r^t A(s)U(s, t)\psi ds \right\rangle dr \\ &= \int_0^t \langle Y_r(\omega), A(r)(U(r, t)\psi - \psi) \rangle dr \end{aligned} \tag{4.21}$$

Given any $s \in [0, t]$, from the definition of weak solution (with ψ replaced by $A(s)U(s, t)\psi$), we have \mathbb{P} -a.e.

$$\begin{aligned} & \int_0^s \int_{\Phi'} R(r, f)' A(s)U(s, t)\psi L(dr, df) \\ &= \langle Y_s - \eta, A(s)U(s, t)\psi \rangle - \int_0^s \langle Y_r, A(r)A(s)U(s, t)\psi \rangle dr. \end{aligned}$$

Integration both sides on $[0, t]$ with respect to the Lebesgue measure and then using (4.21) and the backward equation (4.4) it follows that

$$\begin{aligned} & \int_0^t \left(\int_0^s \int_{\Phi'} R(r, f)' A(s)U(s, t)\psi L(dr, df) \right) ds \\ &= \int_0^t \langle Y_s, A(s)U(s, t)\psi \rangle ds - \int_0^t \langle \eta, A(s)U(s, t)\psi \rangle ds \\ &\quad - \int_0^t \left(\int_0^s \langle Y_r, A(r)A(s)U(s, t)\psi \rangle dr \right) ds \\ &= \int_0^t \langle Y_s, A(s)\psi \rangle ds - (\langle \eta, U(0, t)\psi \rangle - \langle \eta, \psi \rangle). \end{aligned}$$

Reordering, and then from (4.18) we get that

$$\begin{aligned} & \langle \eta, \psi \rangle + \int_0^t \langle Y_s, A(s)\psi \rangle ds \\ &= \langle U(t, 0)' \eta, \psi \rangle + \int_0^t \left(\int_0^s R(r, f)' A(s)U(s, t)\psi L(dr, df) \right) ds \\ &= \langle U(t, 0)' \eta, \psi \rangle + \int_0^t \int_{\Phi'} R(r, f)' U(r, t)\psi L(dr, df) \\ &\quad - \int_0^t \int_{\Phi'} R(r, f)' \psi L(dr, df). \end{aligned}$$

Summing the term $\int_0^t \int_{\Phi'} R(r, f)' \psi L(dr, df)$ at both sides, then using (4.11) and the fact that Y is a weak solution we conclude that \mathbb{P} -a.e

$$\langle Y_t, \psi \rangle = \langle U(t, 0)' \eta, \psi \rangle + \left\langle \int_0^t \int_{\Phi'} U(t, r)' R(r, f) L(dr, df), \psi \right\rangle. \tag{4.22}$$

Since (4.22) is valid for each ψ in the dense subset D_t of Ψ , we conclude from (4.11) that

$$Y_t = U(t, 0)' \eta + \int_0^t \int_{\Phi'} U(t, r)' R(r, f) L(dr, df).$$

□

We can combine the results of Theorems 4.7 and 4.13 to obtain the following result on existence and uniqueness of solutions to stochastic evolution equations in the time-homogeneous setting, i.e. when $A(t) = A$.

Corollary 4.14 *Suppose that A is the generator of a $(C_0, 1)$ -semigroup $(S(t) : t \geq 0)$ on Ψ . The stochastic evolution equation*

$$dY_t = AY_t dt + \int_{\Phi'} R(t, f) L(dt, df), \quad t \geq 0, \tag{4.23}$$

with initial condition $Y_0 = \eta$ \mathbb{P} -a.e. has a weak solution given by the mild solution

$$X_t = S(t)' \eta + \int_0^t \int_{\Phi'} S(t - s)' R(s, f) L(ds, df), \quad \forall t \geq 0.$$

Moreover, any weak solution $Y = (Y_t : t \geq 0)$ to (4.23) which has almost surely locally Bochner integrable trajectories is a version of $X = (X_t : t \geq 0)$.

5 Properties of the solution

In this section we study different properties of the solution to (4.15). Apart from Assumption 4.4, some other assumptions might be considered in each subsection.

5.1 Square integrable solutions

In this section we consider the existence of solutions with (strong) square moments. For this purpose, we naturally require the Lévy process $L = (L_t : t \geq 0)$ to be square integrable (recall the definition of square moments in Sect. 2). In such a case by the Lévy-Itô decomposition (3.2) we must have that the Poisson integral process $\int_{B_{\rho'}(1)^c} f N(t, df)$, $t \geq 0$, corresponding to the large jumps of L is square integrable. Thus we have $\int_{B_{\rho'}(1)^c} | \langle f, \phi \rangle |^2 \nu(df) < \infty, \forall \phi \in \Phi$. In particular, it follows that $\mathfrak{n} \in \Phi'$ where $\langle \mathfrak{n}, \phi \rangle = \int_{B_{\rho'}(1)^c} \langle f, \phi \rangle \nu(df)$ for every

$\phi \in \Phi$. Moreover, the Φ' -valued compensated Poisson integral $\int_{B_{\rho'}(1)^c} f \tilde{N}(t, df) = \int_{B_{\rho'}(1)^c} f N(t, df) - t\mathfrak{n}, t \geq 0$, is a zero-mean, square integrable càdlàg Lévy process and $\mathbb{E} \left(\left| \int_{B_{\rho'}(1)^c} f \tilde{N}(t, df), \phi \right|^2 \right) = t \int_{B_{\rho'}(1)^c} |\langle f, \phi \rangle|^2 \nu(df), \forall \phi \in \Phi, t \geq 0$ (see Section 4.1 in [22]).

From the above arguments and (3.2) we can decompose L as

$$L_t = t\tilde{\mathfrak{m}} + W_t + \int_{\Phi'} f \tilde{N}(t, df), \tag{5.1}$$

for $\tilde{\mathfrak{m}} = \mathfrak{m} + \mathfrak{n} \in \Phi'$ and where

$$\int_{\Phi'} f \tilde{N}(t, df) := \int_{B_{\rho'}(1)} f \tilde{N}(t, df) + \int_{B_{\rho'}(1)^c} f \tilde{N}(t, df).$$

The decomposition (5.1) of L motivates the introduction of the following class of integrands.

Definition 5.1 Let $\Lambda^2(L, \Phi, \Psi)$ denotes the collection of all $R \in \Lambda(L, \Phi, \Psi)$ satisfying for every $T > 0$ and $\psi \in \Psi$ that

$$\mathbb{E} \int_0^T \left[|\langle \mathfrak{m}, R(s, 0)' \psi \rangle|^2 + \mathcal{Q}(R(s, 0)' \psi)^2 + \int_{\Phi'} |\langle f, R(s, f)' \psi \rangle|^2 \nu(df) \right] ds < \infty.$$

Assume $R \in \Lambda^2(L, \Phi, \Psi)$. By Remark 3.1 (for $V = B_{\rho'}(1)^c$) and (4.12), for each $t \geq 0$ and $\psi \in \Psi$ we have \mathbb{P} -a.e.

$$\begin{aligned} & \int_0^t \int_{\Phi'} R(s, f)' U(s, t) \psi L(ds, df) \\ &= \int_0^t \langle \mathfrak{m}, R(s, f)' U(s, t) \psi \rangle ds \\ & \quad + \int_0^t \int_{B_{\rho'}(1)^c} \langle f, R(s, f)' U(s, t) \psi \rangle \nu(df) ds \\ & \quad + \int_0^t \int_{\Phi'} R(s, f)' U(s, t) \psi M(ds, df), \end{aligned} \tag{5.2}$$

where the stochastic integral with respect to the Lévy martingale-valued measure M on Φ' is well-defined since by our square integrability assumption on L we have

$\int_{\Phi'} |\langle f, \phi \rangle|^2 \nu(df) < \infty, \forall \phi \in \Phi$. In (5.2) we have used that \mathbb{P} -a.e.

$$\begin{aligned} \int_0^t \int_{\Phi'} R(s, f)' U(s, t) \psi M(ds, df) &= \int_0^t \int_{B_{\rho'}(1)} R(s, f)' U(s, t) \psi M(ds, df) \\ &+ \int_0^t \int_{B_{\rho'}(1)^c} R(s, f)' U(s, t) \psi M(ds, df), \end{aligned}$$

which can be justified using Proposition 4.12 in [21]. We will apply (5.2) to obtain the existence of strong second moments for the stochastic convolution:

Proposition 5.2 *Suppose that L is square integrable and $R \in \Lambda^2(L, \Phi, \Psi)$. For every $t > 0$, there exists a continuous Hilbertian seminorm ϱ on Ψ such that*

$$\mathbb{E} \int_0^t \varrho' \left(\int_0^s \int_{\Phi'} U(s, r)' R(r, f) L(dr, df) \right)^2 ds < \infty. \tag{5.3}$$

Proof Let $t > 0$. Our first task will be to demonstrate that

$$\mathbb{E} \int_0^t \left(\int_0^s \int_{\Phi'} R(r, f)' U(r, s) \psi L(dr, df) \right)^2 ds < \infty, \quad \psi \in \Psi. \tag{5.4}$$

To prove (5.4), observe that by the arguments used in the proof of Proposition 4.3 we have

$$\begin{aligned} \mathbb{E} \int_0^t \left(\int_0^s \langle m, R(r, 0)' U(r, s) \psi \rangle dr \right)^2 ds &< \infty, \\ \mathbb{E} \int_0^t \left(\int_0^s \int_{\Phi'} R(r, f)' U(r, s) \psi M(dr, df) \right)^2 ds &< \infty. \end{aligned}$$

Hence in view of (5.2) to prove (5.4) it is sufficient to check that

$$\mathbb{E} \int_0^t \left(\int_0^s \int_{B_{\rho'}(1)^c} \langle f, R(r, f)' U(r, s) \psi \rangle \nu(df) dr \right)^2 ds < \infty.$$

To do this, note that our assumption that $R \in \Lambda^2(L, \Phi, \Psi)$ implies that $\xi : \Psi \rightarrow [0, \infty)$ given by

$$\xi(\psi) = \left(\mathbb{E} \int_0^t \int_{B_{\rho'}(1)^c} |\langle f, R(r, f)' \psi \rangle|^2 \nu(df) dr \right)^{1/2}, \quad \forall \psi \in \Psi,$$

is a well-defined Hilbertian seminorm on Ψ . Moreover, since the mapping $\psi \mapsto |\langle f, R(r, \omega, f)' \psi \rangle|^2$ is continuous on Ψ for each $(r, \omega, f) \in [0, t] \times \Omega \times B_{\rho'}(1)^c$, from an application of Fatou’s Lemma we can check that ξ is lower-semicontinuous.

Then Proposition 5.7 in [23] shows that ξ is continuous on Ψ . As Ψ is a nuclear space, there exists a continuous Hilbertian seminorm q on Ψ such that $\xi \leq q$ and $i_{\xi,q}$ is Hilbert-Schmidt. Then if $(\psi_j^q : j \in \mathbb{N}) \subseteq \Psi$ is a complete orthonormal system in Ψ_q , we have

$$\begin{aligned} & \mathbb{E} \int_0^t \int_{B_{\rho'}(1)^c} q'(R(r, f), f)^2 \nu(df) dr \\ &= \mathbb{E} \int_0^t \int_{B_{\rho'}(1)^c} \sum_{j \in \mathbb{N}} \left| \langle f, R(r, f)' \psi_j^q \rangle \right|^2 \nu(df) dr \\ &= \sum_{j \in \mathbb{N}} \xi(\psi_j^q)^2 = \|i_{\xi,q}\|_{\mathcal{L}_2(\Psi_q, \Psi_\xi)}^2 < \infty. \end{aligned}$$

Since $(U(s, t) : 0 \leq s \leq t < \infty)$ is a $(C_0, 1)$ -evolution system, there exist $\vartheta_q \geq 0$ and a continuous seminorm p on Ψ such that $q(U(r, s)\psi) \leq e^{\vartheta_q(s-r)} p(\psi)$, for all $0 \leq r \leq s, \psi \in \Psi$. Then using the above estimates, Jensen’s inequality and Fubini’s theorem we have for each $\psi \in \Psi$ that

$$\begin{aligned} & \mathbb{E} \int_0^t \left(\int_0^s \int_{B_{\rho'}(1)^c} \langle f, R(r, f)' U(r, s)\psi \rangle \nu(df) dr \right)^2 ds \\ & \leq \mathbb{E} \int_0^t \left(s \nu(B_{\rho'}(1)^c) \int_0^s \int_{B_{\rho'}(1)^c} \left| \langle f, R(r, f)' U(r, s)\psi \rangle \right|^2 \nu(df) dr \right) ds \\ & \leq p(\psi)^2 \nu(B_{\rho'}(1)^c) \mathbb{E} \int_0^t s e^{2\vartheta_q s} \left(\int_0^s \int_{B_{\rho'}(1)^c} q'(R(r, f), f)^2 \nu(df) dr \right) ds \\ & \leq p(\psi)^2 \nu(B_{\rho'}(1)^c) \left(\mathbb{E} \int_0^t \int_{B_{\rho'}(1)^c} q'(R(r, f), f)^2 \nu(df) dr \right) \left(\int_0^t s e^{2\vartheta_q s} ds \right) \\ & < \infty. \end{aligned}$$

This proves (5.4). Now we prove the existence of a continuous Hilbertian seminorm on Ψ for which (5.3) is satisfied.

Define $k : \Psi \rightarrow [0, \infty)$ by

$$k(\psi) = \left(\mathbb{E} \int_0^t \left| \left\langle \int_0^s \int_{\Phi'} U(s, r)' R(r, f) L(dr, df), \psi \right\rangle \right|^2 ds \right)^{1/2}, \quad \forall \psi \in \Psi.$$

By (4.11) and (5.4) k is a well-defined Hilbertian seminorm on Ψ . From an application of Fatou’s Lemma we can check that k is lower-semicontinuous and hence from Proposition 5.7 in [23] we have that k is continuous on Ψ . As Ψ is a nuclear space, there exists a continuous Hilbertian seminorm ϱ on Ψ such that $k \leq \varrho$ and $i_{k,\varrho}$ is Hilbert-Schmidt. Then if $(\psi_j^\varrho : j \in \mathbb{N}) \subseteq \Psi$ is a complete orthonormal system in Ψ_ϱ , we have

$$\begin{aligned} & \mathbb{E} \int_0^t \varrho' \left(\int_0^s \int_{\Psi'} U(s, r)' R(r, f) L(dr, df) \right)^2 ds \\ &= \mathbb{E} \int_0^t \sum_{j \in \mathbb{N}} \left| \left\langle \int_0^s \int_{\Psi'} U(s, r)' R(r, f) L(dr, df), \psi_j^\varrho \right\rangle \right|^2 ds \\ &= \sum_{j \in \mathbb{N}} k(\psi_j^\varrho)^2 = \|i_{k, \varrho}\|_{\mathcal{L}_2(\Psi_\varrho, \Psi_k)}^2 < \infty. \end{aligned}$$

□

The result of Proposition 5.2 together with the following result will allow us to conclude that the mild solution to (4.15) has strong second moments.

Lemma 5.3 *Suppose η is square integrable. Then, for every $t > 0$ there exists a continuous Hilbertian seminorm ϱ on Ψ such that*

$$\mathbb{E} \int_0^t \varrho' (U(s, 0)' \eta)^2 ds < \infty. \tag{5.5}$$

Proof Let $k : \Psi \rightarrow [0, \infty)$ be given by $k(\psi)^2 = \mathbb{E} (|\langle \eta, \psi \rangle|^2)$ for each $\psi \in \Psi$. Our hypothesis of square integrability of η guarantees that k is well-defined and is clearly a Hilbertian seminorm on Ψ . Moreover, k is continuous since η is regular (here apply Theorem 2.10 in [20]). As Ψ is nuclear, similar arguments to those used in the proof of Proposition 5.2 shows the existence of a continuous Hilbertian seminorm ξ on Ψ such that $k \leq \xi$ and $\mathbb{E} (\xi'(\eta)^2) < \infty$.

Let $\vartheta_\xi \geq 0$ and q a continuous seminorm on Ψ such that $\xi(U(r, s)\psi) \leq e^{\vartheta_\xi(s-r)} q(\psi)$, for all $0 \leq r \leq s, \psi \in \Psi$. Then, for each $\psi \in \Psi$ we have

$$\mathbb{E} \int_0^t |\langle U(s, 0)' \eta, \psi \rangle|^2 ds \leq q(\psi)^2 \mathbb{E} (\xi'(\eta)^2) \int_0^t s e^{2\vartheta_\xi s} ds < \infty.$$

Define $p : \Psi \rightarrow [0, \infty)$ by

$$p(\psi) = \left(\mathbb{E} \int_0^t |\langle U(s, 0)' \eta, \psi \rangle|^2 ds \right)^{1/2}, \quad \forall \psi \in \Psi.$$

Similar arguments to those used in the proof of Proposition 5.2 show that p is a continuous Hilbertian seminorm on Ψ and that there exists a continuous Hilbertian seminorm ϱ on $\Psi, p \leq \varrho$, such that $i_{p, \varrho}$ is Hilbert-Schmidt and that

$$\mathbb{E} \int_0^t \varrho' (U(s, 0)' \eta)^2 ds = \|i_{p, \varrho}\|_{\mathcal{L}_2(\Psi_\varrho, \Psi_p)}^2 < \infty.$$

So we have proved (5.5). □

Finally we have the main result of this section concerning the existence and uniqueness of a solution which is (strongly) square integrable.

Theorem 5.4 *Suppose that η and the Lévy process L are square integrable, and $R \in \Lambda^2(L, \Phi, \Psi)$. Then the mild solution $X = (X_t : t \geq 0)$ to (4.15) is the unique (up to modifications) weak solution to (4.15) satisfying that for every $t > 0$ there exists a continuous Hilbertian seminorm ϱ on Ψ such that $\mathbb{E} \int_0^t \varrho'(X_s)^2 ds < \infty$.*

Proof We already know by Theorem 4.7 that the mild solution $X = (X_t : t \geq 0)$ is a weak solution to (4.15). By (4.17), Proposition 5.2 and Lemma 5.3 for every $t > 0$ there exists a continuous Hilbertian seminorm ϱ on Ψ such that $\mathbb{E} \int_0^t \varrho'(X_s)^2 ds < \infty$. This proves the existence of a weak solution satisfying the conditions in the statement of Theorem 5.4.

To prove uniqueness, assume that $Y = (Y_t : t \geq 0)$ is a any weak solution to (4.15) satisfying that for every $t > 0$ there exists a continuous Hilbertian seminorm ϱ on Ψ such that $\mathbb{E} \int_0^t \varrho'(Y_s)^2 ds < \infty$. It is clear that in such a case Y has almost surely Bochner (square) integrable trajectories. Then by Theorem 4.13 we have that Y is a version of the mild solution X to (4.15). \square

5.2 Markov property

In this and the next section we will study properties of stochastic evolution equations of the form

$$dY_t = A(t)'Y_t dt + B'dL_t, \quad t \geq 0, \quad (5.6)$$

with initial condition $Y_0 = \eta$ \mathbb{P} -a.e., for $\eta, A(t), U(s, t), L$ as in Assumption 4.4(A1)-(A3) and for $B \in \mathcal{L}(\Psi, \Phi)$. In accordance with the literature the stochastic evolution equation (5.6) is called a (*generalized*) *Langevin equation* with Lévy noise.

A weak solution to (5.6) is a Ψ' -valued regular adapted process $Y = (Y_t : t \geq 0)$ satisfying that for any given $t \geq 0$, for each $\psi \in D_t$ we have $\int_0^t |\langle Y_s, A(s)\psi \rangle| ds < \infty$ \mathbb{P} -a.e. and

$$\langle Y_t, \psi \rangle = \langle \eta, \psi \rangle + \int_0^t \langle Y_s, A(s)\psi \rangle ds + \langle B'L_t, \psi \rangle.$$

Let $R(t, \omega, f) = B'$ for $t \geq 0, \omega \in \Omega, f \in \Phi'$. We have that $R \in \Lambda(L, \Phi, \Psi)$ since for every $T > 0$ and $\psi \in \Psi$ we have

$$\begin{aligned} & \mathbb{E} \int_0^T \left[|\langle m, R(s, 0)\psi \rangle|^2 + \mathcal{Q}(R(s, 0)\psi)^2 + \int_{B_{\rho'}(1)} |\langle f, R(s, f)\psi \rangle|^2 \nu(df) \right] ds \\ & \leq T \left(|\langle m, B\psi \rangle|^2 + \mathcal{Q}(B\psi)^2 + \rho(B\psi)^2 \int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) \right) < \infty. \end{aligned}$$

Moreover, it is easy to verify that $\int_0^t \int_{\Phi'} B\psi L(dr, df) = \langle B'L_t, \psi \rangle$ for every $\psi \in \Psi$ and hence (5.6) can be interpreted as a particular case of (4.15) for the coefficient

$R(t, \omega, f) = B'$. Then Theorem 4.7 shows that

$$X_t = U(t, 0)' \eta + \int_0^t \int_{\Phi'} U(t, s)' B' L(ds, df), \quad \forall t \geq 0, \tag{5.7}$$

is a weak solution to (5.6), which is known as a (generalized) Ornstein–Uhlenbeck process with Lévy noise.

Our main objective in this section is to study the flow property and the Markov property of the mild solution (5.7) to (5.6). For $0 \leq s \leq t < \infty$, define $\Gamma_{s,t} : \Psi' \times \Omega \rightarrow \Psi'$ by

$$\Gamma_{s,t}(\psi) = U(t, s)' \psi + \int_s^t \int_{\Phi'} U(t, r)' B' L(dr, df), \quad \forall \psi \in \Psi.$$

Proposition 5.5 *Let $(X_t : t \geq 0)$ be the Ornstein–Uhlenbeck process.*

- (1) *The family $(\Gamma_{s,t} : 0 \leq s \leq t < \infty)$ is a stochastic flow: $\Gamma_{s,s} = Id$ and $\Gamma_{s,t} \circ \Gamma_{r,s} = \Gamma_{r,t} \forall 0 \leq r \leq s \leq t < \infty$.*
- (2) *For every $0 \leq s \leq t < \infty$, $\int_s^t \int_{\Phi'} U(t, r)' B' L(dr, df)$ is independent of \mathcal{F}_s .*
- (3) *For each $\psi \in \Psi, s, t \geq 0$,*

$$\mathbb{E} \left(e^{i \langle X_{s+t}, \psi \rangle} \middle| \mathcal{F}_s \right) = \mathbb{E} \left(e^{i \langle X_s, U(s, s+t) \psi \rangle} \right) H(s, t, \psi),$$

for some deterministic complex-valued function $H(s, t, \psi)$.

- (4) *$(X_t : t \geq 0)$ is a Markov process with respect to the filtration (\mathcal{F}_t) .*

Proof We adapt to our context the arguments used in the proof of Proposition 4.1 in [4] and Theorem 2.2 in [9].

To prove (1). Let $0 \leq r \leq s \leq t < \infty$. From the semigroup property of the forward evolution system $(U(t, s)' : 0 \leq s \leq t < \infty)$ and Proposition 3.3 we have

$$\begin{aligned} &\Gamma_{s,t}(\Gamma_{r,s}(\phi)) \\ &= U(t, s)' \Gamma_{r,s}(\phi) + \int_s^t \int_{\Phi'} U(t, u)' B' L(du, df) \\ &= U(t, s)' U(s, r)' \phi + U(t, s)' \int_r^s \int_{\Phi'} U(s, u)' B' L(du, df) \\ &\quad + \int_s^t \int_{\Phi'} U(t, u)' B' L(du, df) \\ &= U(t, r)' \phi + \int_r^s \int_{\Phi'} U(t, u)' B' L(du, df) + \int_s^t \int_{\Phi'} U(t, u)' B' L(du, df) \\ &= U(t, r)' \phi + \int_r^t \int_{\Phi'} U(t, u)' B' L(du, df) = \Gamma_{r,t}(\phi). \end{aligned}$$

To prove (2), let $0 \leq s \leq t < \infty$. Observe that since $\int_s^t \int_{\Phi'} U(t, r)' B' L(dr, df)$ is regular and in view of (4.11), to show that it is independent of \mathcal{F}_s it is sufficient to check

that for each $\psi \in \Psi$, $\int_s^t \int_{\Phi'} BU(t, r)\psi L(dr, df)$ is independent of \mathcal{F}_s . In effect, the first and third stochastic integrals in the right-hand side of (4.12) are easily seen to be independent of \mathcal{F}_s by the independent increments of L . The second stochastic integral in the right-hand side of (4.12) was defined via an Itô type integration theory (see Sect. [21]) and then standard arguments and the independent increments of L show that these stochastic integrals are independent of \mathcal{F}_s . This shows (2).

To prove (3), let $\psi \in \Psi$ and $s, t \geq 0$. From (5.7) we have $X_t = \Gamma_{0,t}(\eta)$ and then from part (1) we have $X_{t+s} = \Gamma_{s,s+t} \circ \Gamma_{0,s}(\eta) = \Gamma_{s,s+t}(X_s)$. Using the above identity and from part (2) we have that

$$\begin{aligned} & \mathbb{E} \left(e^{i \langle X_{s+t}, \psi \rangle} \middle| \mathcal{F}_s \right) \\ &= \mathbb{E} \left(e^{i \langle \Gamma_{s,s+t}(X_s), \psi \rangle} \middle| \mathcal{F}_s \right) \\ &= \mathbb{E} \left(\exp \left\{ i \left\langle U(s+t, s)' X_s + \int_s^{s+t} \int_{\Phi'} U(s+t, r)' B' L(dr, df), \psi \right\rangle \right\} \middle| \mathcal{F}_s \right) \\ &= \mathbb{E} \left(e^{i \langle U(s+t, s)' X_s, \psi \rangle} \right) \cdot \mathbb{E} \left(\exp \left\{ i \int_s^{s+t} \int_{\Phi'} BU(r, s+t)\psi L(dr, df) \right\} \right) \\ &= \mathbb{E} \left(e^{i \langle X_s, U(s, s+t)\psi \rangle} \right) H(s, t, \psi). \end{aligned}$$

Finally, (4) is a direct consequence of (3). □

5.3 Time regularity of solutions

In this section we give sufficient conditions for the existence and uniqueness of a càdlàg solution to stochastic evolution equations of the form (5.6). We will need the following result on existence of some types of stochastic convolution integrals whose paths are continuous.

Lemma 5.6 *Let $X = (X_t : t \geq 0)$ be a Ψ' -valued adapted process with càdlàg paths. Consider a family $(G(s, t) : 0 \leq s \leq t < \infty) \subseteq \mathcal{L}(\Psi, \Psi)$ with the property that for every $\psi \in \Psi$, $s, t \geq 0$, the mappings $[s, \infty) \ni r \mapsto G(s, r)\psi$ and $[0, t] \ni r \mapsto G(r, t)\psi$ are continuous.*

Then there exists a unique Ψ' -valued regular adapted process $\int_0^t G(t, s)' X_s ds$, $t \geq 0$, with continuous paths and satisfying \mathbb{P} -a.e.

$$\left\langle \int_0^t G(t, s)' X_s ds, \psi \right\rangle = \int_0^t \langle X_s, G(s, t)\psi \rangle ds, \quad \forall t \geq 0, \psi \in \Psi. \quad (5.8)$$

Proof We follow arguments used in the proof of Lemma 1.1 in [19].

Let $\psi \in \Psi$ and $t \geq 0$. Define $Y_t(\psi)(\omega) := \int_0^t \langle X_s(\omega), G(s, t)\psi \rangle ds$. We will check that $Y_t(\psi)$ is a real-valued random variable. In effect, since X is càdlàg (for \mathbb{P} -a.e. $\omega \in \Omega$) we have that $s \mapsto X_s(\omega) \in D_\infty(\Psi')$ and hence from the continuity of the mapping $[0, t] \ni r \mapsto G(r, t)\psi$ we can conclude that the

mapping $s \mapsto \mathbb{1}_{[0,t]}(s) \langle X_s(\omega), G(s, t)\psi \rangle$ is càdlàg on $[0, t]$. Hence, the integral $\int_0^t \langle X_s(\omega), G(s, t)\psi \rangle ds$ is well-defined for \mathbb{P} -a.e. $\omega \in \Omega$. Moreover, Fubini’s theorem shows that $\omega \mapsto Y_t(\psi)$ is \mathcal{F}_t -measurable. The process $(Y_t : t \geq 0)$ is hence adapted.

Now, given $\omega \in \Omega$ we have $s \mapsto X_s(\omega) \in D_t(\Psi')$ and because Ψ is barrelled, there exists a continuous seminorm $p = p(t, \omega)$ on Ψ such that $s \mapsto X_s(\omega) \in D_t(\Psi'_p)$ (see Remark 3.6 in [23]). Since $[0, t] \ni r \mapsto G(r, t)\psi$ is continuous, the set $\{G(r, t)\psi : 0 \leq r \leq t\}$ is bounded in Ψ , hence under the seminorm p . Then, for each $\psi \in \Psi$, it follows that

$$\left| \int_0^t \langle X_s(\omega), G(s, t)\psi \rangle ds \right| \leq t \sup_{0 \leq s \leq t} p'(X_s(\omega)) \sup_{0 \leq s \leq t} p(G(s, t)).$$

We have therefore shown that for each $t \geq 0$ and $\omega \in \Omega$, $\psi \mapsto Y_t(\psi)(\omega) \in \Psi'$. Hence, for each $t \geq 0$, Y_t is a cylindrical random variable in Ψ' such that the mapping $Y_t : \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous. Observe moreover that for each $\omega \in \Omega$ and $\psi \in \Psi$, the mapping $t \mapsto Y_t(\psi)(\omega)$ is continuous. In effect, given $0 \leq r_1 \leq r_2 \leq t$, we have

$$\begin{aligned} & \left| \int_0^{r_1} \langle X_s(\omega), G(s, r_1)\psi \rangle ds - \int_0^{r_2} \langle X_s(\omega), G(s, r_2)\psi \rangle ds \right| \\ & \leq \int_0^{r_1} |\langle X_s(\omega), G(s, r_1)\psi - G(s, r_2)\psi \rangle| ds + \int_{r_1}^{r_2} \langle X_s(\omega), G(s, r_2)\psi \rangle ds, \end{aligned}$$

which converges to 0 as $r_1 \rightarrow r_2$ or $r_2 \rightarrow r_1$ by using the strong continuity of the family $(G(s, t) : 0 \leq s \leq t < \infty)$ and dominated convergence. The version of the regularization theorem for cylindrical processes in the dual of an ultrabornological nuclear space (Corollary 3.11 in [20]) shows the existence of a unique Ψ' -valued regular continuous process $\int_0^t G(t, s)'X_s ds, t \geq 0$, that is a version of $(Y_t : t \geq 0)$, i.e. satisfying (5.8) \mathbb{P} -a.e. Since $\int_0^t G(t, s)'X_s ds, t \geq 0$ is regular and weakly adapted it is (strongly) adapted. □

The following theorem shows the existence of a unique càdlàg version under the assumption that each $A(t)$ is a continuous linear operator on Ψ .

Theorem 5.7 *Suppose that $A(t) \in \mathcal{L}(\Psi, \Psi)$ for each $t \geq 0$ and the mapping $t \mapsto A(t)\psi$ is continuous from $[0, \infty)$ into Ψ for every $\psi \in \Psi$. Then the Ornstein–Uhlenbeck process (5.7) has a unique (up to indistinguishable versions) Ψ' -valued regular càdlàg version $(Z_t : t \geq 0)$ given by*

$$Z_t = U(t, 0)' \eta + \int_0^t U(t, s)' A(s)' B' L_s ds + B' L_t, \quad \forall t \geq 0, \tag{5.9}$$

where $\int_0^t U(t, s)' A(s)' B' L_s ds, t \geq 0$, is a Ψ' -valued regular adapted process with continuous paths satisfying \mathbb{P} -a.e.

$$\left\langle \int_0^t U(t, s)' A(s)' B' L_s ds, \psi \right\rangle = \int_0^t \langle L_s, BA(s)U(s, t)\psi \rangle ds, \quad \forall t \geq 0, \psi \in \Psi. \tag{5.10}$$

Moreover if $(L_t : t \geq 0)$ has continuous paths, then $(Z_t : t \geq 0)$ has continuous paths too.

Proof We define the stochastic integral $\int_0^t U(t, s)' A(s)' B' L_s ds$, by applying Lemma 5.6 to the Ψ' -valued adapted càdlàg process $X_t = L_t$ and to the family $G(s, t) = BA(s)U(s, t)$ which by our assumptions is strongly continuous in each of its variables. Lemma 5.6 then shows that $\int_0^t U(t, s)' A(s)' B' L_s ds, t \geq 0$, is a Ψ' -valued regular adapted process with continuous paths satisfying (5.10) \mathbb{P} -a.e.

Now observe that since $B' \in \mathcal{L}(\Phi', \Psi')$, then $(B' L_t : t \geq 0)$ is a Ψ' -valued regular adapted process with càdlàg paths. Likewise, the strong continuity of the forward evolution system $(U(t, s)' : 0 \leq s \leq t)$ shows that $(U(t, 0)' \eta : t \geq 0)$ is a Ψ' -valued regular adapted process with continuous paths. Then $(Z_t : t \geq 0)$ defined by (5.9) is a Ψ' -valued regular adapted process with càdlàg paths. It is clear that if $(L_t : t \geq 0)$ has continuous paths the same is satisfied for $(B' L_t : t \geq 0)$ and hence $(Z_t : t \geq 0)$ has continuous paths by the arguments given above.

To show that $(Z_t : t \geq 0)$ is a version of the Ornstein–Uhlenbeck process (5.7) it is sufficient to show that $(Z_t : t \geq 0)$ is a weak solution to (5.6). In such a case, Theorem 4.13 together with Proposition 4.12 shows that $(Z_t : t \geq 0)$ is the unique càdlàg version of (5.7).

Let $t \geq 0$ and $\psi \in \Psi$. Using (5.9), (5.10), then (4.3) and finally using again (5.9) we have \mathbb{P} -a.e.

$$\begin{aligned} \langle Z_t, \psi \rangle &= \langle U(t, 0)' \eta, \psi \rangle + \left\langle \int_0^t U(t, s)' A(s)' B' L_s ds, \psi \right\rangle + \langle B' L_t, \psi \rangle \\ &= \langle \eta, U(0, t)\psi \rangle + \int_0^t \langle L_s, BA(s)U(s, t)\psi \rangle ds + \langle B' L_t, \psi \rangle \\ &= \langle \eta, \psi \rangle + \int_0^t \langle \eta, U(0, r)A(r)\psi \rangle dr + \langle B' L_t, \psi \rangle \\ &\quad + \int_0^t \left(\langle L_s, BA(s)\psi \rangle + \int_s^t \langle L_s, BA(s)U(s, r)A(r)\psi \rangle dr \right) ds \\ &= \langle \eta, \psi \rangle + \langle B' L_t, \psi \rangle \\ &\quad + \int_0^t (\langle U(r, 0)' \eta, A(r)\psi \rangle + \langle L_r, BA(r)\psi \rangle \\ &\quad + \int_0^r \langle L_s, BA(s)U(s, r)A(r)\psi \rangle ds) dr \\ &= \langle \eta, \psi \rangle + \langle B' L_t, \psi \rangle + \int_0^t \langle Z_r, A(r)\psi \rangle dr. \end{aligned}$$

Thus $(Z_t : t \geq 0)$ is a weak solution to (5.6). □

Corollary 5.8 *Suppose that A is the generator of a $(C_0, 1)$ -semigroup $(S(t) : t \geq 0)$ on Ψ . If $A \in \mathcal{L}(\Psi, \Psi)$, the Ornstein–Uhlenbeck process*

$$X_t = S(t)' \eta + \int_0^t \int_{\Phi'} S(t-s)' B L(ds, df), \quad \forall t \geq 0,$$

has a unique (up to indistinguishable versions) Ψ' -valued regular càdlàg version $(Z_t : t \geq 0)$ satisfying $\forall t \geq 0, \psi \in \Psi$,

$$\langle Z_t, \psi \rangle = \langle \eta, S(t)\psi \rangle + \int_0^t \langle L_s, BAS(t-s)\psi \rangle ds + \langle L_t, B\psi \rangle. \quad (5.11)$$

Moreover if $(L_t : t \geq 0)$ has continuous paths, then $(Z_t : t \geq 0)$ has continuous paths too.

6 Examples and applications

In this section we consider examples and applications of our theory of stochastic evolutions equations developed in the last two sections.

Example 6.1 Let $\Phi = \Psi = \mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions. Let $L = (L_t : t \geq 0)$ be a Lévy process in the space of tempered distributions $\mathcal{S}(\mathbb{R})'$ and $B \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$. Consider the stochastic evolution equation on $\mathcal{S}(\mathbb{R})'$:

$$dY_t = A'Y_t + B'dL_t, \quad t \geq 0, \quad (6.1)$$

with initial condition $Y_0 = \eta$, for a $\mathcal{S}(\mathbb{R})'$ -valued \mathcal{F}_0 -measurable random variable, and here A is the continuous linear operator on $\mathcal{S}(\mathbb{R})$ defined by $(A\psi)(x) = x\psi'(x)$, for all $x \in \mathbb{R}, \psi \in \mathcal{S}(\mathbb{R})$. It is shown in Section 6 in [6] that A is the infinitesimal generator of the $(C_0, 1)$ -semigroup on $\mathcal{S}(\mathbb{R})$ defined by $(S(t)\psi)(x) = \psi(e^{tx})$, for all $t \geq 0, x \in \mathbb{R}, \psi \in \mathcal{S}(\mathbb{R})$.

Then Corollary 5.8 shows that (6.1) has a unique càdlàg solution $(Z_t : t \geq 0)$ satisfying for any $t \geq 0$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$:

$$\langle Z_t, \psi \rangle = \left\langle \eta, \hat{\psi}_t \right\rangle + \int_0^t \left\langle L_s, B\tilde{\psi}_{t-s} \right\rangle ds + \langle L_t, B\psi \rangle,$$

where $\hat{\psi}_t(x) := (S(t)\psi)(x) = \psi(e^{tx})$ and $\tilde{\psi}_t(x) := AS(t)\psi(x) = xte^{tx}\psi'(e^{tx})$ for every $t \geq 0$ and $x \in \mathbb{R}$. The process $(Z_t : t \geq 0)$ is Markov by Proposition 5.5.

Example 6.2 Let $\Phi = \Psi = \mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing functions, $d \geq 1$. Let $L = (L_t : t \geq 0)$ be a Lévy process in the space of tempered

distributions $\mathcal{S}(\mathbb{R}^d)'$ and $B \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$. Consider the stochastic evolution equation on $\mathcal{S}(\mathbb{R}^d)'$:

$$dY_t = \Delta Y_t + B'dL_t, \quad t \geq 0, \quad (6.2)$$

with initial condition $Y_0 = \eta$, for a $\mathcal{S}(\mathbb{R}^d)'$ -valued \mathcal{F}_0 -measurable random variable, and here Δ is the Laplace operator on $\mathcal{S}(\mathbb{R}^d)'$.

It is well-known that the Laplace operator Δ is the infinitesimal generator of the *heat semigroup* $(S(t) : t \geq 0)$ which is the C_0 -semigroup on $\mathcal{S}(\mathbb{R}^d)$ defined as: $S(0) = I$ and for each $t > 0$,

$$(S(t)\psi)(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\|x-y\|^2/4t} \psi(y) dy, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d.$$

If we let $\mu_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\|x\|^2/4t}$, then $\mu_t \in \mathcal{S}(\mathbb{R}^d)$ and $S(t)\psi = \mu_t * \psi$ for each $\psi \in \mathcal{S}(\mathbb{R}^d)$.

The heat semigroup is equicontinuous hence a $(C_0, 1)$ -semigroup. Since $\Delta \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$, Corollary 5.8 shows that (6.2) has a unique càdlàg solution $(Z_t : t \geq 0)$ satisfying for any $t \geq 0$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$:

$$\langle Z_t, \psi \rangle = \langle \eta, \mu_t * \psi \rangle + \int_0^t \langle L_s, B\Delta(\mu_{t-s} * \psi) \rangle ds + \langle L_t, B\psi \rangle. \quad (6.3)$$

The existence of solutions to (6.2) in the case where L is a Wiener process was studied by Üstünel in [57] by following a different approach. It is worth to mention however that our solution (6.3) to (6.2) coincides with that obtained in [57].

Example 6.3 In this example we provide a situation in which we have a Langevin problem defined on a Hilbert space with cylindrical Lévy noise and an appropriate nuclear space Φ can be constructed such that the Langevin problem is solved in the dual space Φ' .

Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable Hilbert space and $-J$ be a closed densely defined self-adjoint operator on H such that $\langle -J\phi, \phi \rangle_H \leq 0$ for each $\phi \in \text{Dom}(J)$. Let $(T(t) : t \geq 0)$ be the C_0 -contraction semigroup on H generated by $-J$. Assume moreover that there exists some r_1 such that $(\lambda I + J)^{-r_1}$ is Hilbert-Schmidt. Given these conditions, it is known (see [35], Example 1.3.2) that one can construct a Fréchet nuclear space Φ , which is continuously embedded in H and whose topology is determined by an increasing family of Hilbertian norms $|\cdot|_n, n \geq 0$. This family of norms is such that $(T(t) : t \geq 0)$ restricts to a equicontinuous C_0 -semigroup $(S(t) : t \geq 0)$ on Φ , i.e. $|S(t)\phi|_n \leq |\phi|_n, n \geq 0$. Moreover, the restriction A of $-L$ to Φ is the infinitesimal generator of $(S(t) : t \geq 0)$ on Φ and $A \in \mathcal{L}(\Phi, \Phi)$.

Now, suppose that $Z = (Z_t : t \geq 0)$ is a cylindrical Lévy process in H such that for each $t \geq 0$ the mapping $L_t : H \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous. Then Example 6.2 in [24] shows that there exists a Φ' -valued càdlàg regular Lévy process $L = (L_t : t \geq 0)$ that is a version of $Z = (Z_t : t \geq 0)$.

This way, the Langevin equation on H with (cylindrical) Lévy noise,

$$dY_t = -JY_t dt + dZ_t, \quad Y_0 = y_0,$$

can be reinterpreted as the Langevin equation on Φ' with (genuine) Lévy noise,

$$dY_t = AY_t dt + dL_t, \quad Y_0 = y_0,$$

for which we know from Corollary 5.8 that a unique Φ' -valued càdlàg solution ($Z_t : t \geq 0$) exists and is given by (5.11) (for $B = I$).

The reader interested in the study of Langevin equations driven by a cylindrical Lévy process in a Hilbert space is referred to [40] where sufficient conditions for the existence and uniqueness of solutions are discussed.

The examples given above illustrate applications of Theorem 5.7 in the case where the linear part of the stochastic evolution equation is time-homogeneous, i.e. for $A(t) = A \forall t \geq 0$, where $A \in \mathcal{L}(\Psi, \Psi)$ is the generator of a $(C_0, 1)$ -semigroup.

We now discuss a class of examples in the time-inhomogeneous case under the assumption that each $A(t)$ is the generator of a $(C_0, 1)$ -semigroup. We will restrict ourself to the case where Ψ is a Fréchet nuclear space where can make usage of the theory of Kallianpur and Pérez-Abreu introduced in [32] for the existence of a $(C_0, 1)$ -backward evolution system on Ψ generated by the family $(A(t) : t \geq 0)$.

Example 6.4 Let Ψ be a Fréchet nuclear space. Consider a family $A = (A(t) : t \geq 0) \subseteq \mathcal{L}(\Psi, \Psi)$ wherein each $A(t)$ is the generator of a $(C_0, 1)$ -semigroup $(S_t(s) : s \geq 0)$ on Φ . Assume that for the family A there exists an increasing sequence $(q_n : n \geq 0)$ of norms generating the topology on Ψ such that the following two conditions hold:

- (1) For every $k \geq 0$, there exists $m \geq k$ such that, for each $t \geq 0$, $A(t)$ has a continuous linear extension form Ψ_{q_m} into Ψ_{q_k} (also denoted by $A(t)$) and the mapping $t \mapsto A(t)$ is $\mathcal{L}(\Psi_{q_m}, \Psi_{q_k})$ -continuous.
- (2) The family A is *stable* with respect to $(q_n : n \geq 0)$, i.e. for each $T > 0$ there exists $k_0 \geq 0$ and for $k \geq k_0$ there are constants $M_k = M_k(T) \geq 1$ and $\sigma_k = \sigma_k(T)$ satisfying the condition:

$$\begin{aligned} & q_k (S_{t_1}(s_1)S_{t_2}(s_2) \cdots S_{t_m}(s_m)\psi) \\ & \leq M_k \exp \left(\sigma_k \sum_{j=1}^m s_j \right) q_k(\psi), \quad \forall \psi \in \Psi, s_j \geq 0, \end{aligned} \tag{6.4}$$

whenever $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq T, m \geq 0$.

By Theorem 1.3 in [32] there exists a unique $(C_0, 1)$ -backward evolution system $(U(s, t) : 0 \leq s \leq t < \infty)$ on Ψ which is generated by the family $(A(t) : t \geq 0)$.

Suppose now that we have a family $(D(t) : t \geq 0) \subseteq \mathcal{L}(\Psi, \Psi)$ for which there exists $k_0 \geq 0$ and, for $k \geq k_0$ and $t \geq 0$, $D(t)$ has a continuous linear extension form Ψ_{q_k} into Ψ_{q_k} (also denoted by $D(t)$) and the mapping $t \mapsto D(t)$ is $\mathcal{L}(\Psi_{q_k}, \Psi_{q_k})$ -continuous.

With the same conditions as above for the family $(A(t) : t \geq 0)$, Theorem 1.4 in [32] shows the existence of a unique $(C_0, 1)$ -backward evolution system $(V(s, t) : 0 \leq s \leq t < \infty)$ on Ψ which is generated by the family $(A(t) + D(t) : t \geq 0)$. Moreover $(V(s, t) : 0 \leq s \leq t < \infty)$ satisfies the integral equation

$$V(s, t)\psi = U(s, t)\psi + \int_s^t U(s, r)D(r)V(r, t)\psi dr, \quad \forall t \geq 0, \psi \in \Psi.$$

Now if $L = (L_t : t \geq 0)$ is a Lévy process in a quasi-complete bornological nuclear space Φ , η is a \mathcal{F}_0 -measurable random variable in Ψ , $B \in \mathcal{L}(\Psi, \Phi)$, with families $(A(t) : t \geq 0)$ and $(D(t) : t \geq 0)$ as above, the perturbed stochastic evolution equation

$$dY_t = (A(t)'Y_t + D(t)'Y_t)dt + B'dL_t, \quad t \geq 0, \quad (6.5)$$

with initial condition $Y_0 = \eta$ \mathbb{P} -a.e., has by Theorem 5.7 a unique càdlàg weak solution $(Z_t : t \geq 0)$ given by

$$Z_t = V(t, 0)'\eta + \int_0^t V(t, s)'(A(s) + D(s))'B'L_s ds + B'L_t, \quad \forall t \geq 0.$$

Furthermore, if the initial condition η and the Lévy process L are square integrable, we have by Theorem 5.4 that for every $t > 0$ there exists a continuous Hilbertian seminorm ϱ on Ψ such that $\mathbb{E} \int_0^t \varrho'(Z_s)^2 ds < \infty$.

It is worth to mention that the above result extends to the Lévy noise case the results of Theorems 2.1 and 2.3 in [32], there formulated for square integrable martingale noise under the assumption that the underlying space is nuclear Fréchet.

For an example of families $(A(t) : t \geq 0)$ and $(D(t) : t \geq 0)$ satisfying the conditions given above in this example see Section 3 in [32], p.264–271. The example in [32] points out that perturbed stochastic evolution equations of the form (6.5) arise in the study of the fluctuation limit of a sequence of interacting diffusions (see [26, 45]).

7 Weak convergence of solutions to Stochastic evolution equations

In this section we give sufficient conditions for the weak convergence of a sequence of Ornstein–Uhlenbeck processes. The problem is described as follows.

For each $n = 0, 1, 2, \dots$, let $L^n = (L_t^n : t \geq 0)$ be a Φ' -valued Lévy process with càdlàg paths, $(U^n(s, t) : 0 \leq s \leq t)$ a backward evolution system with family of generators $(A^n(t) : t \geq 0) \subseteq \mathcal{L}(\Psi, \Psi)$ satisfying that the mapping $t \mapsto A^n(t)\psi$ is continuous from $[0, \infty)$ into Ψ for every $\psi \in \Psi$, η^n is a \mathcal{F}_0 -measurable Ψ' -valued regular random variable, and $B^n \in \mathcal{L}(\Psi, \Phi)$.

In view of Theorem 5.7 the Ornstein–Uhlenbeck process $X^n = (X_t^n : t \geq 0)$ corresponding to the generalized Langevin equation

$$dY_t^n = A^n(t)'Y_t^n dt + (B^n)'dL_t^n, \quad Y_0^n = \eta^n, \tag{7.1}$$

has a Ψ' -valued càdlàg regular version which is given by

$$X_t^n = U^n(t, 0)' \eta^n + \int_0^t U^n(t, s)' A^n(s)' (B^n)' L_s^n ds + (B^n)' L_t^n, \quad \forall t \geq 0. \tag{7.2}$$

Observe that because Ψ is ultrabornological, Corollary 4.8 in [23] shows that for each $n = 0, 1, 2, \dots$ the processes $(B^n)'L^n = ((B^n)'L_t^n : t \geq 0)$ and $X^n = (X_t^n : t \geq 0)$ define each a random variable in $D_\infty(\Psi')$ with a Radon distribution.

In the next theorem we give sufficient conditions for the weak convergence of the sequence of Ornstein–Uhlenbeck processes X^n to X^0 in $D_\infty(\Psi')$.

Theorem 7.1 *Assume the following:*

- (1) For $n = 0, 1, 2, \dots$, η^n is independent of L^n .
- (2) $\eta^n \Rightarrow \eta^0$.
- (3) $U^n(0, t)\psi \rightarrow U^0(0, t)\psi$ as $n \rightarrow \infty$ for each $\psi \in \Psi$ and $t \geq 0$, and $A^n(s)U^n(s, t)\psi \rightarrow A^0(s)U^0(s, t)\psi$ as $n \rightarrow \infty$ for each $\psi \in \Psi$ uniformly for (s, t) in bounded intervals of time.
- (4) $(B^n)'L^n \Rightarrow (B^0)'L^0$ in $D_\infty(\Psi')$ as $n \rightarrow \infty$.

Then the sequence $(X^n : n \in \mathbb{N})$ is uniformly tight in $D_\infty(\Psi')$ and $X^n \Rightarrow X^0$ in $D_\infty(\Psi')$ as $n \rightarrow \infty$.

For our proof of Theorem 7.1 we will require the following result which makes usage of the particular form taken by the Ornstein–Uhlenbeck process in (7.2).

Let $F : D_\infty(\Psi') \rightarrow D_\infty(\Psi')$ be defined by

$$F(x)(t) = x_t + \int_0^t U^0(t, s)' A^0(s)' x_s ds, \tag{7.3}$$

Observe that the arguments used in the proof of Lemma 5.6 guarantee that the linear mapping F maps $D_\infty(\Psi')$ into itself.

Lemma 7.2 *The mapping $F : D_\infty(\Psi') \rightarrow D_\infty(\Psi')$ defined by (7.3) is continuous.*

Lemma 7.2 is an extension of Lemma 1 in [19], result originally formulated under the assumption that Ψ is a nuclear Fréchet space (see also Lemma 4.1 in [49]). The proof of Lemma 7.2 can be carried out following line-by-line the arguments used in the proof of Lemma 1 in [19] for the family $(A^0(s)U^0(s, t) : 0 \leq s \leq t)$. Indeed, the only properties required on Ψ is that the Banach-Steinhaus theorem hold and that each $x \in D_T(\Psi')$ belongs to $D_T(\Psi'_p)$ for some continuous seminorm p on Ψ , but since we are assuming that Ψ is barrelled, the aforementioned properties are valid (see Theorem 11.9.1 in [47] and Remark 3.6 in [23]). The details are left to the reader.

Proof of Theorem 7.1 First, observe that since each X_t^n is a regular random variable and Ψ is nuclear, each X_t^n has a Radon distribution (Theorem 2.10 in [20]). In view of Theorem 6.6 in [23] and since Ψ is ultrabornological, it is sufficient to check the following:

- (a) For each $\psi \in \Psi$, the sequence of distributions of $\langle X^n, \psi \rangle$ is uniformly tight on $D_\infty(\mathbb{R})$.
- (b) For all $m \in \mathbb{N}$, $\psi_1, \dots, \psi_m \in \Psi$, and $t_1, \dots, t_m \geq 0$, the probability distribution of $(\langle X_{t_1}^n, \psi_1 \rangle, \dots, \langle X_{t_m}^n, \psi_m \rangle)$ converges in distribution to some probability measure on \mathbb{R}^m .

In effect, for each $n = 1, 2, \dots$, let \tilde{X}^n be defined by

$$\tilde{X}_t^n = U^n(t, 0)' \eta^n + \int_0^t U^0(t, s)' A^0(s)' (B^n)' L_s^n ds + (B^n)' L_t^n, \quad \forall t \geq 0. \quad (7.4)$$

As for X^n we have that each \tilde{X}^n defines a $D_\infty(\Psi')$ -valued random variable. Following the idea of Fernández and Gorostiza in [19] (Theorem 1), observe that in order to prove (a) and (b) it is enough to show

- (c) $\tilde{X}^n \Rightarrow X^0$ in $D_\infty(\Psi')$ as $n \rightarrow \infty$,
- (d) $\int_0^t \langle (B^n)' L_s^n, (A^n U^n(s, t) \psi - A^0 U^0(s, t) \psi) \rangle ds \rightarrow 0$ in probability as $n \rightarrow \infty$ for each $t \geq 0$ and $\psi \in \Psi$,
- (e) $d^\infty(\langle X^n, \psi \rangle, \langle \tilde{X}^n, \psi \rangle) \rightarrow 0$ in probability as $n \rightarrow \infty$ for each $\psi \in \Psi$, where $d^\infty(\cdot, \cdot)$ denotes the Skorokhod metric on $D_\infty(\mathbb{R})$.

In effect, assume that (c), (d) and (e) holds. We start by proving (a). First, (c) shows that for each $\psi \in \Psi$ we have $\langle \tilde{X}^n, \psi \rangle \Rightarrow \langle X^0, \psi \rangle$ in $D_\infty(\mathbb{R})$. Then from (e) we have $\langle X^n, \psi \rangle \Rightarrow \langle X^0, \psi \rangle$ in $D_\infty(\mathbb{R})$ for each $\psi \in \Psi$, which implies (a).

Now we prove (b). Assumption (4) in Theorem 7.1 and (d) shows that as $n \rightarrow \infty$,

$$\left(\langle X_{t_1}^n - \tilde{X}_{t_1}^n, \psi_1 \rangle, \dots, \langle X_{t_m}^n - \tilde{X}_{t_m}^n, \psi_m \rangle \right) \Rightarrow (0, \dots, 0).$$

Hence from (c) we conclude that (b) holds.

Proof of (c): First, observe that $X^0 = U^0(t, 0)' \eta^0 + F((B^0)' L^0)$ and for each $n = 1, 2, \dots$, we have $\tilde{X}^n = U^n(t, 0)' \eta^n + F((B^n)' L^n)$ where F is the mapping defined in (7.3). Assumption (1) in Theorem 7.1 shows that for $\psi \in \Psi$ the real-valued processes $\langle U^n(t, 0)' \eta^n, \psi \rangle = \langle \eta^n, U^n(t, 0) \psi \rangle$ and $\langle F((B^n)' L^n)(t), \psi \rangle = \int_0^t \langle (B^n)' L_s^n, A(s) U(s, t) \psi \rangle ds + \langle (B^n)' L_t^n, \psi \rangle$ are independent. Hence the Ψ' -valued regular processes $U^n(\cdot, 0)' \eta^n$ and $F((B^n)' L^n)$ are independent (see [22], Proposition 2.3). Thus, to prove (c) it suffices to show $U^n(\cdot, 0)' \eta^n \Rightarrow U^0(\cdot, 0)' \eta^0$ and $F((B^n)' L^n) \Rightarrow F((B^0)' L^0)$ with convergence in $D_\infty(\Psi')$.

In effect, for every $\psi \in \Psi$ assumptions (2)-(3) in Theorem 7.1 imply that $\langle U^n(\cdot, 0)' \eta^n, \psi \rangle \Rightarrow \langle U^0(\cdot, 0)' \eta^0, \psi \rangle$ in $D_\infty(\mathbb{R})$ and hence $(\langle U^n(\cdot, 0)' \eta^n, \psi \rangle : n \in \mathbb{N})$ is uniformly tight in $D_\infty(\mathbb{R})$. Likewise, for any choice $\psi_1, \dots, \psi_m \in \Psi$, and

$t_1, \dots, t_m \geq 0$, we have as $n \rightarrow \infty$,

$$\begin{aligned} & \left(\langle U^n(0, t_1)\eta^n, \psi_1 \rangle, \dots, \langle U^n(0, t_m)\eta^n, \psi_m \rangle \right) \\ & \Rightarrow \left(\langle U^0(0, t_1)\eta^0, \psi_1 \rangle, \dots, \langle U^0(0, t_m)\eta^0, \psi_m \rangle \right). \end{aligned}$$

Then $U^n(\cdot, 0)' \eta^n \Rightarrow U^0(\cdot, 0)' \eta^0$ in $D_\infty(\Psi')$ by Theorem 6.6 in [23].

Finally, by assumption (4) in Theorem 7.1 and by Lemma 7.2 we have $F((B^n)'L^n) \Rightarrow F((B^0)'L^0)$ in $D_\infty(\Psi')$. This proves (c) by the arguments given above.

Proof of (d): Assumption (4) in Theorem 7.1 shows that for each $\psi \in \Psi$ the family $(\langle (B^n)'L^n, \psi \rangle : n \in \mathbb{N})$ is uniformly tight in $D_\infty(\mathbb{R})$. Since Ψ is ultrabornological, Theorem 5.10 in [23] shows that the family $((B^n)'L^n : n \in \mathbb{N})$ is uniformly tight in $D_\infty(\Psi')$. Since Ψ is barrelled, Theorem 5.2 in [23] (see the the arguments used in its proof) shows that for each $t > 0$ and $\epsilon \in (0, 1)$ there exists a continuous Hilbertian seminorm q on Ψ and $C > 0$ such that

$$\inf_{n \geq 1} \mathbb{P} \left(\sup_{0 \leq s \leq t} q'(\langle (B^n)'L_s^n \rangle) \leq C \right) > 1 - \epsilon.$$

Then

$$\begin{aligned} & \mathbb{P} \left(\left| \int_0^t \langle (B^n)'L_s^n, (A^n(s)U^n(s, t)\psi - A^0(s)U^0(s, t)\psi) \rangle ds \right| > \epsilon \right) \\ & \leq \mathbb{P} \left(t \sup_{0 \leq s \leq t} q'(\langle (B^n)'L_s^n \rangle) \sup_{0 \leq s \leq t} q \left(A^n(s)U^n(s, t)\psi - A^0(s)U^0(s, t)\psi \right) > \epsilon \right) \\ & \leq \epsilon + \mathbb{P} \left(\sup_{0 \leq s \leq t} q \left(A^n(s)U^n(s, t)\psi - A^0(s)U^0(s, t)\psi \right) > \epsilon/Ct \right). \end{aligned}$$

The last term tends to 0 as $n \rightarrow \infty$ by assumption (3) in Theorem 7.1 and since ϵ is arbitrary we have proved (d).

Proof of (e): Let $\psi \in \Psi$. Since the Skorokhod topology is weaker than the local uniform topology, it suffices to show that for each $T > 0$ and $\psi \in \Psi$,

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left| \langle X_t^n - \tilde{X}_t^n, \psi \rangle \right| \\ & = \sup_{0 \leq t \leq T} \left| \int_0^t \langle (B^n)'L_s^n, (A^n(s)U^n(s, t)\psi - A^0(s)U^0(s, t)\psi) \rangle ds \right|, \end{aligned}$$

converges to 0 in probability as $n \rightarrow \infty$. But this can be proved following the same arguments used in the proof of (d). □

Remark 7.3 Theorem 7.1 constitutes an extension of Theorem 4 in [19] there formulated in the context of Fréchet spaces to our more general context of quasi-complete

bornological nuclear spaces. We point out however that unlike our results in this section in [19] the question of existence and uniqueness of solutions to each equation (7.1) is not addressed.

Theorem 7.1 also extends the results obtained in [33, 49], also within the context of Fréchet spaces. In [33, 49] the assumptions on the family of generators are stronger than our assumptions in Theorem 7.1 and are patterned after those required for A^n to generate a unique U^n (as in Example 6.4).

For each $n = 0, 1, 2, \dots$, observe that since $(B^n)' \in \mathcal{L}(\Phi', \Psi')$ one can easily verify that $(B^n)'L^n$ is a Ψ' -valued Lévy process, hence possesses a family of characteristics $(\mathfrak{m}_n, \mathcal{Q}_n, \nu_n, \rho_n)$ which determine uniquely its distribution as in (3.4). Given the above information, in the next result we reformulate the statement of Theorem 7.1 to replace the assumption of weak convergence of $(B^n)'L^n$ to $(B^0)'L^0$ in $D_\infty(\Psi')$ by convergence of finite-dimensional distributions and some properties of the sequence of characteristics $(\mathfrak{m}_n, \mathcal{Q}_n, \nu_n, \rho_n)$.

Theorem 7.4 *For each $n = 0, 1, 2, \dots$, let $(\mathfrak{m}_n, \mathcal{Q}_n, \nu_n, \rho_n)$ be the characteristics of the Ψ' -valued Lévy process $(B^n)'L^n$. Assume the following:*

- (1) *For $n = 0, 1, 2, \dots$, η^n is independent of L^n .*
- (2) *$\eta^n \Rightarrow \eta^0$.*
- (3) *$U^n(0, t)\psi \rightarrow U^0(0, t)\psi$ as $n \rightarrow \infty$ for each $\psi \in \Psi$ and $t \geq 0$, and $A^n(s)U^n(s, t)\psi \rightarrow A^0(s)U^0(s, t)$ as $n \rightarrow \infty$ for each $\psi \in \Psi$ uniformly for (s, t) in bounded intervals of time.*
- (4) *There exists a continuous Hilbertian seminorm q on Ψ such that $\mathcal{Q}_n \leq q$ and $\rho_n \leq q \forall n \in \mathbb{N}$, and such that the following is satisfied:*
 - (a) *$(\mathfrak{m}_n : n \in \mathbb{N})$ is relatively compact in Ψ' ,*
 - (b) *$\sup_{n \in \mathbb{N}} \|i_{\mathcal{Q}_n, q}\|_{\mathcal{L}_2(\Phi_q, \Phi_{\mathcal{Q}_n})} < \infty$,*
 - (c) *$\sup_{n \in \mathbb{N}} \int_{\Phi'} (q'(f))^2 \wedge 1 \nu_n(df) < \infty$.*
- (5) *$\forall m \in \mathbb{N}, \phi_1, \dots, \phi_m \in \Phi, t_1, \dots, t_m \in [0, T], \langle (B^n)'L^n_{t_1}, \phi_1 \rangle, \dots, \langle (B^n)'L^n_{t_m}, \phi_m \rangle$ converges in distribution to $\langle (B^0)'L^0_{t_1}, \phi_1 \rangle, \dots, \langle (B^0)'L^0_{t_m}, \phi_m \rangle$.*

Then the sequences $((B^n)'L^n : n \in \mathbb{N})$ and $(X^n : n \in \mathbb{N})$ are each uniformly tight in $D_\infty(\Psi')$ and we have $(B^n)'L^n \Rightarrow (B^0)'L^0$ and $X^n \Rightarrow X^0$ in $D_\infty(\Psi')$ as $n \rightarrow \infty$.

Proof By Theorem 7.2 in [23], assumptions (4) and (5) show that $((B^n)'L^n : n \in \mathbb{N})$ is uniformly tight in $D_\infty(\Psi')$ and that $(B^n)'L^n \Rightarrow (B^0)'L^0$ in $D_\infty(\Psi')$ as $n \rightarrow \infty$. Theorem 7.1 then shows that $(X^n : n \in \mathbb{N})$ is uniformly tight in $D_\infty(\Psi')$ and $X^n \Rightarrow X^0$ in $D_\infty(\Psi')$ as $n \rightarrow \infty$. □

8 Remarks and comparisons

In the present paper and in the dual to a quasi-complete bornological nuclear space we introduced stochastic integrals with respect to a Lévy process (Sect. 3), proved

existence and uniqueness of solutions to stochastic evolution equations with additive Lévy noise (Sect. 4) and then considered properties of the solution to these equations (Sect. 5). Our approach for the stochastic integral consists in to define it as a sum of stochastic integrals through the Lévy-Itô decomposition. This is a common practice when dealing with stochastic integration in Hilbert spaces (see e.g. [3, 4, 51]) or in separable Banach spaces (see e.g. [2, 43, 54, 59]) where a Lévy-Itô decomposition is available.

Although our stochastic integrals are defined by means of a Lévy-Itô decomposition some important differences in the construction arises in relation with the Hilbert or Banach space setting.

First, in our dual of nuclear space setting the stochastic integral with respect to the drift term of the decomposition is defined by means of a Radonification argument (also known as regularization in the context of nuclear spaces) and not as a ω -wise Bochner integral as is usually carried out in Banach spaces. Similarly, the stochastic integral with respect to the martingale part of the Lévy-Itô decomposition is defined by means of a Radonification argument (see [21]) and not by mean of an Itô isometry as in the Hilbert space setting (see e.g. [13, 44, 51]), or more generally by an Itô isomorphism as in the Banach space (of martingale type or UMD) setting (see e.g. [16, 17, 58]). The reason for this particular construction is that in duals of nuclear spaces we only consider weak (in the duality sense) moments (Definition 3.2) and hence it is natural to first define our stochastic integrals as a cylindrical process and then use a procedure of Radonification. Observe however that our definition for the stochastic integral with respect to the large jumps part is given as a finite random sum. This is similar to the definition in [4] for the Hilbert space setting.

In Banach spaces there is an increasing interest into consider stochastic integration and stochastic evolution equations driven by cylindrical Lévy noise (see e.g. [3, 29, 38, 40, 41, 53]). In the context of Banach spaces the cylindrical Lévy processes are generalizations of their classical counterparts (see [5]) and is often the case that to define stochastic integrals one requires the integrands possess a Radonification property [3, 39, 58]. In our context of duals of quasi-complete bornological nuclear spaces however there is no gain of generality in considering a cylindrical Lévy process as a noise, this since any cylindrical Lévy process with minimal regularity assumptions can be Radonified (regularized) into a classical Lévy process (see Theorem 3.8 in [22] and Proposition 3.10 in [20]).

In this paper we have considered stochastic evolution equations whose linear part is defined by a family of generators to a $(C_0,1)$ backward evolution system, i.e. its linear part is time-inhomogeneous. As we mentioned in the examples (Sect. 6) these equations arise naturally in modelling in duals of nuclear spaces. Our proof of the existence and uniqueness of weak solutions to (4.15) is a modification of the proof of equivalence of weak and mild solutions in [21] and is based on iterated integration for stochastic convolutions. Existence and uniqueness of weak solutions to stochastic evolution equations with multiplicative Lévy noise where considered in [21] but there the linear part of these equations is defined by an operator which is the generator to a $(C_0,1)$ -semigroup, i.e. the linear part is time-homogeneous.

We have proved that the mild solution to (4.15) has almost surely finite weak square moments (Proposition 4.10). A similar result for stochastic evolution equations driven

by additive cylindrical Lévy processes in a Hilbert space was obtained in ([40], Theorem 5.1). Furthermore, if the noise is square integrable (in our weak sense) then the solution has (strong) square moments in a Hilbert space embedded in the dual space (Theorem 5.4). This is a conclusion which was obtained earlier in the work [32] where the authors considered stochastic evolution equations driven by additive square integrable martingale noise in the dual of a nuclear Fréchet space.

We have shown in Proposition 5.5 that the Ornstein–Uhlenbeck process is a Markov process. This is a standard property which also occurs in the context of Hilbert spaces either for classical or cylindrical Lévy processes (e.g. see [4, 40]).

It is well-known that in the context of Banach spaces there are examples of Ornstein–Uhlenbeck processes with Lévy noise who do not have a càdlàg modification (e.g. see [51], Section 9.4.4). Moreover in the literature there are a variety of sufficient and necessary conditions for the existence of strong and weak (in the duality sense) càdlàg versions for Ornstein–Uhlenbeck processes with either classical or cylindrical Lévy noise (see e.g. [42, 53, 59]). For the context of quasi-complete bornological nuclear spaces in Theorem 5.7 we proved the existence of a unique càdlàg version for the Ornstein–Uhlenbeck process under the assumption that $A(t)$ is a continuous linear operator for every $t \geq 0$. If the backward evolution system is defined by a $(C_0, 1)$ -semigroup this reduces to the assumption that the generator A is linear continuous (Corollary 5.8).

It is shown in [25] that for an Ornstein–Uhlenbeck process with classical Lévy noise in a separable Hilbert space it is always possible to find a larger Hilbert space for which a càdlàg version exists. Observe that in Example 6.3 we provide sufficient conditions on the generator of the C_0 -semigroup so that for a Langevin equation defined on a Hilbert space with cylindrical Lévy noise there exists an appropriate nuclear space so that the Langevin equation is solved in the dual space and the corresponding Ornstein–Uhlenbeck process has a càdlàg version.

In Sect. 7 we introduced sufficient conditions for the weak convergence in the Skorokhod space of a sequence of Ornstein–Uhlenbeck processes in the dual to a quasi-complete bornological nuclear space. The conditions are given in terms of convergence of the initial conditions and of the generators and their corresponding backward evolution systems, and the weak convergence of the Lévy noises (Theorem 7.1), or even convergence of the corresponding sequence of characteristics of the noises (Theorem 7.4). This result generalizes the work by other authors (e.g. see [19, 33, 49]) carried out originally in the context of nuclear Fréchet spaces. In the context of Banach spaces different authors have also studied the Skorokhod convergence of stochastic convolutions and Ornstein–Uhlenbeck processes (e.g. see [41, 48]).

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