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Positive recurrence of a solution of an SDE with variable switching intensities

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Abstract
Positive recurrence of a d-dimensional diffusion with an additive Wiener process, with switching and with one recurrent and one transient regimes and variable switching intensities is established under suitable conditions. The approach is based on embedded Markov chains.

Keywords Diffusion · Switching · Variable switching intensities · Positive recurrence

Mathematics Subject Classification 60H10 · 60J60

1 Introduction

Let us consider the process (X_t, Z_t) with a continuous component X and discrete one Z described by the stochastic differential equation in $\mathbb{R}^d$

$$dX_t = b(X_t, Z_t)\,dt + dW_t, \quad t \ge 0, \quad X_0 = x, \ Z_0 = z, \tag{1}$$

for the component X, while Z_t is a continuous-time conditionally Markov process given X on the state space $S = \{0, 1\}$ with positive intensities of respective transitions $\lambda_{01}(x) =: \lambda_0(x), \ \& \, \lambda_{10}(x) =: \lambda_1(x)$; here the variable x signifies a certain (arbitrary Borel measurable) dependence on the component X; the trajectories of Z are assumed to be càdlàg; the probabilities of jumps for Z are conditionally independent given the

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trajectory of the component X (see the precise description in what follows). Denote

$$b(x,0)=b_-(x),\quad b(x,1)=b_+(x),$$
$$\overline{\lambda}_0 := \sup_{x,z}\lambda_0(x),\quad \underline{\lambda}_0 := \inf_{x,z}\lambda_0(x),\quad \overline{\lambda}_1 := \sup_{x,z}\lambda_1(x),\quad \underline{\lambda}_1 := \inf_{x,z}\lambda_1(x).$$

It is assumed that

$$0<\underline{\lambda}_0\wedge\underline{\lambda}_1\le\overline{\lambda}_0\vee\overline{\lambda}_1<\infty. \tag{2}$$

These conditions along with the boundedness of the function b in x suffice for the process (X_t, Z_t) to be well-defined. A rigorous construction of the system (X, Z) of this type may be given by the SDE system

$$\begin{aligned} dX_t &= b(X_t, Z_t)\,dt + dW_t,\ t\ge 0,\ X_0 = x\in\mathbb{R}^d,\\ dZ_t &= 1(Z_t=0)d\pi^0_t - 1(Z_t=1)d\pi^1_t,\quad Z_0\in\{0,1\}, \end{aligned} \tag{3}$$

where π^i_t, $i = 0, 1$, are two Poisson processes with intensities $\lambda_i(X_t)$, $i = 0, 1$, respectively. More precisely,

$$\pi^i_t = \bar\pi^i_{\phi_i(t)},$$

where $\bar\pi^i_t, i = 0, 1$, are, in turn, two standard Poisson processes with a constant intensity one, independent of the Wiener process (W_t) and of each other, and the time changes

$$t\mapsto\phi_i(t) := \int_0^t \lambda_i(X_s)ds,\quad i=0,1,$$

are applied to each of them, respectively.

By virtue of the assumption (2) the equation between the jumps only concerns the diffusion part of the SDE (3), for which it is well-known since [13] that the equation has a pathwise unique strong solution. The jump moments are stopping times with respect to the filtration $(\mathcal{F}_t = \mathcal{F}^{W,\pi^0,\pi^1}_t,\ t\ge 0)$, and the position of the system after any jump (X_τ, Z_τ) is uniquely determined by the left limiting values $(X_{\tau-}, Z_{\tau-})$:

$$X_\tau = X_{\tau-},\quad Z_\tau = 1(Z_{\tau-}=0).$$

After any such jump, the diffusion part of the SDE is solved starting from the position X_τ until the next jump, say, τ', of the component Z, and the moment of this next jump is determined by the trajectories of π^0_t and (or) of π^1_t and by the intensity $\lambda_{Z_s}(X_s)$, $s<\tau'$. Since there might be only a finite numbers of jumps on any bounded interval of time, then pathwise (and, hence, also weak) uniqueness follows on $[0,\infty)$.

Therefore, the process (X, Z) exists and is markovian. Its (quasi-) generator has a form

$$Lh(x,z) = \frac{1}{2}\Delta_x h(x,z) + b(x,z)\nabla_x h(x,z) + \lambda_z(x)\,(h(x,\bar z) - h(x,z)),$$

where $\bar z := 1(z = 0)$ (that is, $\bar z$ is not a z, the other state from $\{0, 1\}$).

For any $t > 0$ fixed let us define the function

$$v(s,x,z) := \mathbb{E}_{s,x,z} f(X_t, Z_t).$$

The vector-function $v(s,x) = (v(s,x,0), v(s,x,1))$ satisfies the system of PDEs

$$\begin{aligned} &v_s(s,x,0) + L^0 v(s,x,0) + \lambda_0(x)\,(v(s,x,1) - v(s,x,0)) = 0, \quad v(t,x,0) = f(x,0),\\ &v_s(s,x,1) + L^1 v(s,x,1) + \lambda_1(x)\,(v(s,x,0) - v(s,x,1)) = 0, \quad v(t,x,1) = f(x,1), \end{aligned}$$

where

$$L^i = \frac{1}{2}\,\Delta_x + \langle b(x,i), \nabla_x\rangle, \quad i = 0, 1.$$

Due to the results of [12, Theorem 5.5] (see also [7]), its solution is continuous in the variable s for any bounded and continuous f. Hence, the process is Feller's (that is, $\mathbb{E}_{x,z} h(X_t)$ is continuous in x and, of course, bounded for any $h \in C_b$ and any $t > 0$). Since the process is Markov and càdlàg, then it is also strong Markov according to the well-known sufficient condition.

The SDE solution is assumed ergodic under the regime $Z = 0$ and transient under $Z = 1$. We are looking for sufficient conditions for positive recurrence of the strong Markov process (X_t, Z_t). Such a problem was considered in [2] for the exponentially recurrent case; for other references see [1, 5, 8, 11], and the references therein. Under weak ergodic and transient conditions the setting was earlier investigated in [15] for the case of the constant intensities λ_0, λ_1 (i.e., not depending on x). In [9] and [10] other interesting results about the transience and recurrence for diffusions with switching were established; in particular, examples were given of conditions under which the solutions of the SDEs *on the half line with reflection* with two transient "pure" regimes are recurrent, and, vice versa, where the solutions of the SDEs with two recurrent "pure" regimes are transient. Here we tackle the case of a combination of one transient and one recurrent regime. In the case of [15] the lengths of intervals between successive jumps of the discrete component were all independent and independent of the Wiener process. In the general case under the consideration in the present paper they are not independent of W via the component X, and, hence, not independent of each other. This difficulty will be overcome with the help of certain comparison arguments.

The paper consists of the sections: Introduction, Main result, Auxiliaries, and Proof of main result.

2 Main result

Theorem 1 *Let the drift* $b = (b_+, b_-)$ *be bounded and Borel measurable, and let there exist* $r_-, r_+, M > 0$ *such that*

$$0 < \underline{\lambda}_0 \wedge \underline{\lambda}_1 \le \bar{\lambda}_0 \vee \bar{\lambda}_1 < \infty, \tag{4}$$

$$xb_-(x) \le -r_-, \quad xb_+(x) \le +r_+, \quad \forall\, |x| \ge M, \tag{5}$$

and

$$2r_- > d \quad \& \quad \underline{\lambda}_1(2r_- - d) > \overline{\lambda}_0(2r_+ + d). \tag{6}$$

Then the process (X, Z) *is positive recurrent; moreover, there exists* $C > 0$ *such that for all* M_1 *large enough and all* $x \in \mathbb{R}^d$ *and for* $z = 0, 1$

$$\mathbb{E}_{x,z}\tau_{M_1} \le C(x^2 + 1), \tag{7}$$

where

$$\tau_{M_1} := \inf(t \ge 0 : |X_t| \le M_1).$$

Remark 1 On the basis of the theorem 1 it may be proved that the process (X_t, Z_t) has a unique invariant measure (see [4, Sect. 4.4]), and for each nonrandom initial condition x, z there is a convergence to this measure in total variation when $t \to \infty$. We leave a rigorous presentation of this claim till next publications for a wider class of models.

3 Auxiliaries

Denote $\|b\| = \sup_{x,z} |b(x, z)|$. Let $M_1 \gg M$ (the value M_1 will be specified later). Let

$$T_0 := \inf(t \ge 0 : Z_t = 0),$$

and

$$0 \le T_0 < T_1 < T_2 < \dots,$$

where T_n for each $n \ge 1$ is defined by induction as

$$T_n := \inf(t > T_{n-1} : Z_{T_n} - Z_{T_n-} \neq 0).$$

Let

$$\tau := \inf(T_n \ge 0 : |X_{T_n}| \le M_1).$$

To prove the theorem it suffices to evaluate from above the value $\mathbb{E}_{x,z}\tau$ because $\tau_{M_1} \le \tau$. Let $\epsilon > 0$, $q < 1$ be positive values satisfying the equality

$$\overline{\lambda}_0(2r_+ + d + \epsilon) = q\underline{\lambda}_1(2r_- - d - \epsilon) \tag{8}$$

(see (6)). In the proof of the theorem it suffices to assume $|x| > M$.

Lemma 1 *Under the assumptions of the theorem for any $\delta > 0$ there exists M_1 such that*

$$\max\Bigg[\sup_{|x|>M_1} \mathbb{E}_{x,z}\left(\int_0^{T_1} 1(\inf_{0\le s\le t} |X_s| \le M)dt|Z_0=0\right), \\ \sup_{|x|>M_1} \mathbb{E}_{x,z}\left(\int_0^{T_0} 1(\inf_{0\le s\le t} |X_s| \le M)dt|Z_0=1\right)\Bigg] < \delta. \tag{9}$$

Let us denote by X^i_t, $i = 0, 1$ the solutions of the equations

$$dX^i_t = b(X^i_t, i)\,dt + dW_t, \quad t \ge 0, \quad X^i_0 = x. \tag{10}$$

Proof of lemma 1 Let $Z_0 = 0$; then $T_0 = 0$. We have,

$$\mathbb{P}_{x,0}(X_t = X^0_t,\ 0 \le t \le T_1) = 1,$$

due to the uniqueness of solutions of the SDEs (1) (or (3)) and (10) and because of the property of stochastic integrals [6, Theorem 2.8.2] to coincide almost surely (a.s.) on the set where the integrands are equal. Therefore, we estimate for any $|x| > M$ with $z = 0$:

$$\begin{aligned}
&\mathbb{E}_{x,z}\left(\int_0^{T_1} 1(\inf_{0\le s\le t} |X_s| \le M)dt|Z_0 = 0\right) = \mathbb{E}_{x,z}\int_0^{T_1} 1(\inf_{0\le s\le t} |X^0_s| \le M)dt \\
&= \mathbb{E}_{x,z}\int_0^{\infty} 1(t < T_1)1(\inf_{0\le s\le t} |X^0_s| \le M)dt = \int_0^{\infty} \mathbb{E}_{x,z}1(t < T_1)1(\inf_{0\le s\le t} |X^0_s| \le M)dt \\
&\overset{\forall t_0>0}{=} \int_0^{t_0} \mathbb{E}_{x,z}1(t < T_1)1(\inf_{0\le s\le t} |X^0_s| \le M)dt + \int_{t_0}^{\infty} \mathbb{E}_{x,z}1(t < T_1)1(\inf_{0\le s\le t} |X^0_s| \le M)dt \\
&\le \int_0^{t_0} \mathbb{E}_{x,z}1(\inf_{0\le s\le t} |X^0_s| \le M)dt + \int_{t_0}^{\infty} \mathbb{E}_{x,z}1(t < T_1)dt \\
&\le t_0\mathbb{P}_{x,z}(\inf_{0\le s\le t_0} |X^0_s| \le M) + \int_{t_0}^{\infty} \exp(-\underline{\lambda}_0 t)dt.
\end{aligned}$$

Let us fix some t_0, so that

$$t_0 > -\underline{\lambda}_0^{-1} \ln(\underline{\lambda}_0\delta/2).$$

Then

$$\int_{t_0}^{\infty} e^{-\underline{\lambda}_0 s} ds < \delta/2.$$

Now, with this t_0 already fixed, by virtue of the boundedness of b there exists $M_1 > M$ such that for any $|x| \ge M_1$ we get

$$t_0\, \mathbb{P}_{x,z}(\inf_{0\le s\le t_0} |X_s^0| \le M) < \delta/2.$$

Similarly, the bound for the second term in (9) follows if we replace the process X^0 by X^1 and the intensity λ_0 by λ_1. □

Lemma 2 *If M_1 is large enough, then under the assumptions of the theorem for any $|x| > M_1$ for any $k = 0, 1, \ldots$*

$$\mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}}) \le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}})$$
$$- 1(\tau > T_{2k})\mathbb{E}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau|Z_0 = 0, \mathcal{F}_{T_{2k}})((2r_- - d) - \epsilon) \tag{11}$$
$$\le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}}) - 1(\tau > T_{2k})\bar{\lambda}_0^{-1}((2r_- - d) - \epsilon), \tag{12}$$
$$\mathbb{E}_{x,z}(X^2_{T_{2k+2}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) \le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}})$$
$$+ 1(\tau > T_{2k+1})\mathbb{E}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau|Z_0 = 1, \mathcal{F}_{T_{2k+1}})((2r_- + d) + \epsilon) \tag{13}$$
$$\le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) + 1(\tau > T_{2k+1})\underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon). \tag{14}$$

Corollary 1 *If M_1 is large enough, then under the assumptions of the theorem for any $|x| > M_1$ for any $k = 0, 1, \ldots$*

$$\begin{aligned}
&\mathbb{E}_{x,0}X^2_{T_{2k+1}\wedge\tau} - \mathbb{E}_{x,0}X^2_{T_{2k}\wedge\tau} \\
&\quad\le -\mathbb{E}_{x,0}1(\tau > T_{2k})\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau|\mathcal{F}_{T_{2k}})((2r_- - d) - \epsilon) \\
&\quad= -\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)((2r_- - d) - \epsilon) \\
&\quad\le -\mathbb{E}_{x,0}1(\tau > T_{2k})\bar{\lambda}_0^{-1}((2r_- - d) - \epsilon),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{x,1}X^2_{T_{2k+2}\wedge\tau} - \mathbb{E}_{x,1}X^2_{T_{2k+1}\wedge\tau} \\
&\quad\le \mathbb{E}_{x,1}1(\tau > T_{2k+1})(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)((2r_- + d) + \epsilon) \\
&\quad= \mathbb{E}_{x,1}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)((2r_- + d) + \epsilon) \\
&\quad\le \mathbb{E}_{x,1}1(\tau > T_{2k+1})\underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon).
\end{aligned}$$

Proof of lemma 2 **1.** Recall that $T_0 = 0$ under the condition $Z_0 = 0$. We have,

$$T_{2k+1} = \inf(t > T_{2k} : Z_t = 1).$$

In other words, the moment T_{2k+1} may be treated as "T_1 after T_{2k}". Under $Z_0 = 0$ the process X_t coincides with X_t^0 until the moment T_1. Hence, we have on $t \in [0, T_1]$ by Itô's formula

$$dX_t^2 - 2X_t dW_t = (2X_t b_-(X_t) + d)\, dt \le (-2r_- + d)dt,$$

on the set $(|X_t| > M)$ due to the assumptions (5). Further, since $1(|X_t| > M) = 1 - 1(|X_t| \le M)$, we obtain

$$\begin{aligned}
&\int_0^{T_1\wedge\tau} 2X_t b_-(X_t)dt \\
&= \int_0^{T_1\wedge\tau} 2X_t b_-(X_t)1(|X_t| > M)dt + \int_0^{T_1\wedge\tau} 2X_t b_-(X_t)1(|X_t| \le M)dt \\
&\le -2r_- \int_0^{T_1\wedge\tau} 1(|X_t| > M)dt + \int_0^{T_1\wedge\tau} 2M\|b\|1(|X_t| \le M)dt \\
&= -2r_- \int_0^{T_1\wedge\tau} 1dt + \int_0^{T_1\wedge\tau} (2M\|b\| + 2r_-)1(|X_t| \le M)dt \\
&\le -2r_- \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-) \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt.
\end{aligned}$$

Thus, always for $|x| > M_1$,

$$\begin{aligned}
&\mathbb{E}_{x,z} \int_0^{T_1\wedge\tau} 2X_t b_-(X_t)dt \\
&\le -2r_-\mathbb{E}_{x,z} \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\mathbb{E}_{x,z} \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt \\
&= -2r_-\mathbb{E} \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\mathbb{E}_{x,z} \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt \\
&\le -2r_-\mathbb{E} \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\mathbb{E}_{x,z} \int_0^{T_1} 1(|X_t| \le M)dt \\
&\le -2r_-\mathbb{E} \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\delta.
\end{aligned}$$

For a fixed $\epsilon > 0$ let us choose $\delta = \bar{\lambda}_0^{-1}\epsilon/(2M\|b\| + 2r_-)$. Then, since $|x| > M_1$ implies $T_1 \wedge \tau = T_1$ on $(Z_0 = 0)$, and since

$$\bar{\lambda}_0^{-1} \le \mathbb{E}_{x,0}T_1 \le \underline{\lambda}_0^{-1}, \tag{15}$$

we get with $z = 0$

$$\mathbb{E}_{x,z} X^2_{T_1 \wedge \tau} - x^2 \le -(2r_- - d)\mathbb{E}_{x,z} \int_0^{T_1} dt + \overline{\lambda}_0^{-1} \epsilon$$
$$= -(2r_- - d)\mathbb{E}_{x,z} T_1 + \overline{\lambda}_0^{-1} \epsilon \overset{(15)}{\le} -\overline{\lambda}_0^{-1}((2r_- - d) - \epsilon).$$

Substituting here x by $X_{T_{2k}}$ and writing $\mathbb{E}_{x,z}(\cdot|\mathcal{F}_{T_{2k}})$ instead of $\mathbb{E}_{x,z}(\cdot)$, and multiplying by $1(\tau > T_{2k})$, we obtain the bounds (11) and (12), as required.

Note that the bound (15) follows straightforwardly from

$$\mathbb{E}_{x,0} T_1 = \int_0^\infty \mathbb{P}_{x,0}(T_1 \ge t) dt = \int_0^\infty \mathbb{E}_{x,0} \mathbb{P}_{x,0}(T_1 \ge t | \mathcal{F}_t^{X^0}) dt$$
$$= \mathbb{E}_{x,0} \int_0^\infty \exp(-\int_0^t \lambda_0(X_s^0) ds) dt \le \int_0^\infty \exp(-\int_0^t \underline{\lambda}_0 ds) dt$$
$$= \int_0^\infty \exp(-t\underline{\lambda}_0) dt = \underline{\lambda}_0^{-1},$$

and similarly

$$\mathbb{E}_{x,0} T_1 = \int_0^\infty \mathbb{E}_{x,0} \exp\left(-\int_0^t \lambda_0(X_s^0) ds\right) dt$$
$$\ge \int_0^\infty \exp(-\int_0^t \overline{\lambda}_0 ds) dt = \int_0^\infty \exp(-t\overline{\lambda}_0) dt = \overline{\lambda}_0^{-1}.$$

2. The condition $Z_0 = 1$ implies the inequality $T_0 > 0$. We have,

$$T_{2k+2} = \inf(t > T_{2k+1} : Z_t = 0).$$

In other words, the moment T_{2k+2} may be treated as "T_0 after T_{2k+1}". Under the condition $Z_0 = 1$ the process X_t coincides with X_t^1 until the moment T_0. Hence, we have on $[0, T_0]$ by Itô's formula

$$dX_t^2 - 2X_t dW_t = 2X_t b_+(X_t) dt + dt \le (2r_+ + d) dt,$$

on the set $(|X_t| > M)$ due to the assumptions (5). Further, since $1(|X_t| > M) = 1 - 1(|X_t| \le M)$, we obtain

$$\begin{aligned}
&\int_0^{T_0\wedge\tau} 2X_t b_+(X_t)dt \\
&= \int_0^{T_0\wedge\tau} 2X_t b_+(X_t)1(|X_t| > M)dt + \int_0^{T_0\wedge\tau} 2X_t b_+(X_t)1(|X_t| \le M)dt \\
&\le 2r_+ \int_0^{T_0\wedge\tau} 1(|X_t| > M)dt + \int_0^{T_0\wedge\tau} 2M\|b\|1(|X_t| \le M)dt \\
&= 2r_+ \int_0^{T_0\wedge\tau} 1dt + \int_0^{T_1\wedge\tau} (2M\|b\| - 2r_+)1(|X_t| \le M)dt \\
&\le 2r_+ \int_0^{T_0\wedge\tau} 1dt + 2M\|b\| \int_0^{T_0\wedge\tau} 1(|X_t| \le M)dt.
\end{aligned}$$

Thus, for $|x| > M_1$ and with $z = 1$ we have,

$$\begin{aligned}
&\mathbb{E}_{x,z} \int_0^{T_0\wedge\tau} 2X_t b_+(X_t)dt \\
&\le 2r_+\mathbb{E}_{x,z} \int_0^{T_0\wedge\tau} 1dt + 2M\|b\|E_{x,z} \int_0^{T_0\wedge\tau} 1(|X_t| \le M)dt \\
&= 2r_+\mathbb{E}_{x,z} \int_0^{T_0\wedge\tau} 1dt + 2M\|b\|\mathbb{E}_{x,z} \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt \\
&\le 2r_+\mathbb{E}_{x,z} \int_0^{T_0\wedge\tau} 1dt + 2M\|b\|\mathbb{E}_{x,z} \int_0^{T_0} 1(|X_t| \le M)dt \\
&\le 2r_+\mathbb{E}_{x,z} \int_0^{T_0\wedge\tau} 1dt + 2M\|b\|\delta.
\end{aligned}$$

For a fixed $\epsilon > 0$ let us choose $\delta = \underline{\lambda}_1^{-1}\epsilon/(2M\|b\|)$. Then, since $|x| > M_1$ implies $T_0 \wedge \tau = T_0$ on the set $(Z_0 = 1)$, we get (recall that $z = 1$)

$$\begin{aligned}
\mathbb{E}_{x,z} X^2_{T_0\wedge\tau} - x^2 &\le (2r_+ + d)\mathbb{E}_{x,z} \int_0^{T_0} dt + \underline{\lambda}_1^{-1}\epsilon \\
&= (2r_+ + d)\mathbb{E}_{x,z} T_0 + \underline{\lambda}_1^{-1}\epsilon \le \underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon).
\end{aligned}$$

Substituting here $X_{T_{2k+1}}$ instead of x and writing $\mathbb{E}_{x,z}(\cdot|\mathcal{F}_{T_{2k+1}})$ instead of $\mathbb{E}_{x,z}(\cdot)$, and multiplying by $1(\tau > T_{2k+1})$, we obtain the bounds (13) and (14), as required. Lemma 2 is proved. □

Proof of corollary 1 is straightforward by taking expectations.

Lemma 3 *If M_1 is large enough, then under the assumptions of the theorem for any $k = 0, 1, \ldots$*

$$\mathbb{E}_{x,z}(X^2_{T_{2k+2}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}}) \le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}})$$
$$+1(\tau > T_{2k+1})\mathbb{E}_{x,z}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau|Z_0 = 0, \mathcal{F}_{T_{2k+1}}))((2r_+ + 1) + \epsilon)) \tag{16}$$
$$\le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}}) + 1(\tau > T_{2k+1})\underline{\lambda}_1^{-1}((2r_+ + 1) + \epsilon)), \tag{17}$$

and

$$\mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}}) \le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}})$$
$$+1(\tau > T_{2k})\mathbb{E}_{x,z}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau|Z_0 = 0, \mathcal{F}_{T_{2k}})) \tag{18}$$
$$\le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}}) - 1(\tau > T_{2k})\overline{\lambda}_0^{-1}((2r_- - 1) - \epsilon)). \tag{19}$$

Corollary 2 *If M_1 is large enough, then under the assumptions of the theorem for any $k = 0, 1, \ldots$*

$$\begin{aligned}
&\mathbb{E}_{x,0}X^2_{T_{2k+2}\wedge\tau} - \mathbb{E}_{x,0}X^2_{T_{2k+1}\wedge\tau}\\
&\quad\le \mathbb{E}_{x,0}1(\tau > T_{2k+1})(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)((2r_+ + 1) + \epsilon)\\
&\quad= \mathbb{E}_{x,0}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)((2r_+ + 1) + \epsilon)\\
&\quad\le \mathbb{E}_{x,0}1(\tau > T_{2k+1})\underline{\lambda}_1^{-1}((2r_+ + 1) + \epsilon)),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{x,1}X^2_{T_{2k+1}\wedge\tau} - \mathbb{E}_{x,1}X^2_{T_{2k}\wedge\tau}\\
&\quad\le \mathbb{E}_{x,1}1(\tau > T_{2k})(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau)\\
&\quad\le -\mathbb{E}_{x,1}1(\tau > T_{2k})\overline{\lambda}_0^{-1}((2r_- - 1) - \epsilon)).
\end{aligned}$$

Proof of lemma 3 Let $Z_0 = 0$; recall that it implies $T_0 = 0$. If $\tau \le T_{2k+1}$, then (16) is trivial. Let $\tau > T_{2k+1}$. We will substitute x instead of $X_{T_{2k}}$ for a while, and will be using the solution X^1_t of the equation

$$dX^1_t = b(X^1_t, 1)\,dt + dW_t, \quad t \ge T_1, \quad X^1_{T_1} = X_{T_1}.$$

For M_1 large enough, since $|x| \wedge |X_{T_1}| > M_1$ implies $T_2 \le \tau$, and due to the assumptions (5) the double bound

$$\begin{aligned}
&1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1},1}X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau})\\
&\quad\le 1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1},1}(T_2 - T_1)((2r_+ + d) + \epsilon))\\
&\quad\le +1(|X_{T_1}| > M_1)(\underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon))
\end{aligned}$$

is guaranteed in the same way as the bounds (13) and (14) in the previous lemma. In particular, it follows that for $|x| > M_1$

$$
\begin{aligned}
&(\mathbb{E}_{X_{T_1},1} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau}) \le 1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1},1}(T_2\wedge\tau - T_1\wedge\tau)((2r_+ + d) + \epsilon)) \\
&\quad = +1(|X_{T_1}| > M_1)(\underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon)),
\end{aligned}
$$

since $|X_{T_1}| \le M_1$ implies $\tau \le T_1$ and $\mathbb{E}_{X_{T_1},1} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau} = 0$. So, on the set $|x| > M_1$ we have with $z = 0$

$$
\begin{aligned}
&\mathbb{E}_{x,z}(\mathbb{E}_{X_{T_1},1} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau}) \\
&\quad \le \mathbb{E}_{x,z} 1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1},1}(T_2\wedge\tau - T_1\wedge\tau)((2r_+ + d) + \epsilon) \\
&\quad \le \mathbb{E}_{x,z} 1(|X_{T_1}| > M_1)(\underline{\lambda}_1^{-1}((2r_+ + 1) + \epsilon)) \le \underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon).
\end{aligned}
$$

Now substituting back $X_{T_{2k}}$ in place of x and multiplying by $1(\tau > T_{2k+1})$, we obtain the inequalities (16) and (17), as required.
For $Z_0 = 1$ we have $T_0 > 0$, and the bounds (18) and (19) follow in a similar way. Lemma 3 is proved. □

Proof of corollary 2 is straightforward by taking expectations. □

Lemma 4 *Under the assumptions of the theorem for any* $k = 0, 1, \ldots$

$$
1(\tau > T_{2k+1})\mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau) \ge 1(\tau > T_{2k+1})\bar{\lambda}_1^{-1},
$$

and

$$
1(\tau > T_{2k})\mathbb{E}_{X_{T_{2k}},0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau) \le 1(\tau > T_{2k})\underline{\lambda}_0^{-1},
$$

Proof of lemma 4 On the set $\tau > T_{2k+1}$ we have,

$$
\mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau) = \mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2} - T_{2k+1}) \in [\bar{\lambda}_1^{-1}, \underline{\lambda}_1^{-1}].
$$

Similarly, on the set $\tau > T_{2k}$

$$
\mathbb{E}_{X_{T_{2k}},0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau) = \mathbb{E}_{X_{T_{2k}},0}(T_{2k+1} - T_{2k}) \in [\bar{\lambda}_0^{-1}, \underline{\lambda}_0^{-1}].
$$

On the sets $\tau \le T_{2k+1}$ and $\tau \le T_{2k}$, respectively, both sides of the required inequalities equal zero. Lemma 4 follows. □

4 Proof of theorem 1

Consider the case $Z_0 = 0$ where $T_0 = 0$. Since the identity

$$
\tau\wedge T_n = \tau\wedge T_0 + \sum_{m=0}^{n-1}((T_{m+1}\wedge\tau) - (T_m\wedge\tau))
$$

we have,

$$\mathbb{E}_{x,z}(\tau \wedge T_n) = \mathbb{E}_{x,z}\tau \wedge T_0 + \mathbb{E}_{x,z}\sum_{m=0}^{n-1}((T_{m+1} \wedge \tau) - (T_m \wedge \tau)).$$

Due to the convergence $T_n \uparrow \infty$, we get by the monotone convergence theorem

$$\begin{aligned}\mathbb{E}_{x,z}\tau &= \mathbb{E}_{x,z}\tau \wedge T_0 + \sum_{m=0}^{\infty}\mathbb{E}_{x,z}((T_{m+1} \wedge \tau) - (T_m \wedge \tau)) \\ &= \mathbb{E}_{x,z}\tau \wedge T_0 + \sum_{k=0}^{\infty}\mathbb{E}_{x,z}((T_{2k+1} \wedge \tau) - (T_{2k} \wedge \tau)) \\ &\quad + \sum_{k=0}^{\infty}\mathbb{E}_{x,z}((T_{2k+2} \wedge \tau) - (T_{2k+1} \wedge \tau)). \end{aligned} \tag{20}$$

By virtue of the corollary 1, we have

$$\mathbb{E}_{x,z}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \leq ((2r_- - d) - \epsilon)^{-1}\left(\mathbb{E}_{x,z}X^2_{T_{2k+1}\wedge\tau} - \mathbb{E}_{x,z}X^2_{T_{2k}\wedge\tau}\right).$$

Therefore,

$$\begin{aligned}&\mathbb{E}_{x,0}X^2_{T_{2m+2}\wedge\tau} - x^2 \\ &\quad\leq ((2r_+ + d) + \epsilon)\sum_{k=0}^{m}\mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \\ &\quad - ((2r_- - d) - \epsilon)\sum_{k=0}^{m}\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\ &\quad = \sum_{k=0}^{m}\big(-((2r_- - d) - \epsilon)(\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)\ r \\ &\quad +((2r_+ + d) + \epsilon)\mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)\big).\end{aligned}$$

By virtue of Fatou's lemma we get

$$\begin{aligned}x^2 &\geq ((2r_- - d) - \epsilon)\sum_{k=0}^{m}(\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\ &\quad - ((2r_+ + d) + \epsilon)\sum_{k=0}^{m}\mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau).\end{aligned} \tag{21}$$

Note that $1(\tau > T_{2k+1}) \le 1(\tau > T_{2k})$. So, $\mathbb{P}_{x,0}(\tau > T_{2k+1}) \le \mathbb{P}_{x,0}(\tau > T_{2k})$. Hence,

$$\begin{aligned}
&\overline{\lambda}_0 \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) - \underline{\lambda}_1 \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \\
&\quad = \overline{\lambda}_0 \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) 1(\tau \ge T_{2k}) \\
&\quad - \underline{\lambda}_1 \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) 1(\tau \ge T_{2k+1}) \\
&\quad = \overline{\lambda}_0 \mathbb{E}_{x,0} 1(\tau > T_{2k}) \mathbb{E}_{X_{T_{2k}}}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\quad - \underline{\lambda}_1 \mathbb{E}_{x,0} 1(\tau > T_{2k+1}) \mathbb{E}_{X_{T_{2k+1}}}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \\
&\quad \ge \overline{\lambda}_0 \mathbb{E}_{x,0} 1(\tau > T_{2k}) \overline{\lambda}_0^{-1} - \underline{\lambda}_1 \mathbb{E}_{x,0} 1(\tau > T_{2k+1}) \underline{\lambda}_1^{-1} \\
&\quad = \mathbb{E}_{x,0} 1(\tau > T_{2k}) - \mathbb{E}_{x,0} 1(\tau > T_{2k+1}) \ge 0.
\end{aligned}$$

Thus,

$$\mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \le \frac{\overline{\lambda}_0}{\underline{\lambda}_1} \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau).$$

Therefore, we estimate

$$\begin{aligned}
&((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \\
&\quad \le ((2r_+ + d) + \epsilon) \frac{\overline{\lambda}_0}{\underline{\lambda}_1} \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\quad = q((2r_- - d) - \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau).
\end{aligned}$$

So, (21) implies that

$$\begin{aligned}
x^2 &\ge ((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\quad - ((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \\
&\ge (1-q)((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\ge \frac{1-q}{2} ((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\quad + \frac{1-q}{2q} ((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau).
\end{aligned}$$

Denoting $c := \min\left(\frac{1-q}{2q}\left((2r_{+}+d)+\epsilon\right), \frac{1-q}{2}\left((2r_{-}-d)-\epsilon\right)\right)$, we conclude that

$$x^2 \geq c \sum_{k=0}^{2m} \mathbb{E}_{x,0}(T_{k+1} \wedge \tau - T_k \wedge \tau).$$

So, as $m \uparrow \infty$, by the monotone convergence theorem we get the inequality

$$\sum_{k=0}^{\infty} \mathbb{E}_{x,0}(T_{k+1} \wedge \tau - T_k \wedge \tau) \leq c^{-1}x^2.$$

Due to (20), it implies that

$$\mathbb{E}_{x,0}\tau \leq c^{-1}x^2, \tag{22}$$

as required. Recall that this bound is established for $|x| > M_1$, while in the case of $|x| \leq M_1$ the left hand side in this inequality is just zero.
In the case of $Z_0 = 1$ (and, hence, $T_0 > 0$), we have to add the value $\mathbb{E}_{x,z}T_0$ satisfying the bound $\mathbb{E}_{x,1}T_0 \leq \underline{\lambda}_1^{-1}$ to the right hand side of (22), which leads to the bound (7), as required. Theorem 1 is proved. □

Remark 2 The results of the paper may be extended to the equation

$$dX_t = b(X_t, Z_t)\,dt + \sigma(X_t, Z_t)\,dW_t, \quad t \geq 0, \quad X_0 = x, \; Z_0 = z, \tag{23}$$

with a Borel measurable σ under the assumptions of the existence of a strong solutions, or of a weak solution which is weakly unique (because the strong Markov property is needed), in addition to the standing balance type conditions replacing (5) and (6) (while (4) is still valid): $a(x,z) = \sigma\sigma^*(x,z)$ and

$$2xb(x,0) + \mathrm{Tr}\,(a(x,0)) \leq -R_{-}, \quad 2xb(x,1) + \mathrm{Tr}\,(a(x,1)) \leq +R_{+}, \quad \forall\,|x| \geq M, \tag{24}$$

with some $R_{-}, R_{+} > 0$, and

$$\underline{\lambda}_1 R_{-} > \overline{\lambda}_0 R_{+}, \tag{25}$$

where the definitions of $\underline{\lambda}_1$ and $\overline{\lambda}_0$ do not change. The proofs will now involve the diffusion coefficient and will use the assumptions (24) and (25), but otherwise will remain the same as in the case of the unit diffusion matrix.

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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