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Global existence for the stochastic Navier–Stokes equations with small L^p data

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Abstract

We consider the stochastic Navier–Stokes equations in $\mathbb{T}^3$ with multiplicative white noise. We construct a unique local strong solution with initial data in L^p, where $p > 5$. We also address the global existence of the solution when the initial data is small in L^p, with the same range of p.

Keywords Stochastic Navier-Stokes equations · Local existence · Global existence · Strong solutions

1 Introduction

In this paper we address the global well-posedness of the stochastic Navier–Stokes equation (SNSE)

$$du(t,x) = \nu \Delta u(t,x)\,dt - \mathcal{P}\big((u(t,x)\cdot\nabla)u(t,x)\big)\,dt + \sigma(u(t,x))\,d\mathbb{W}(t), \tag{1.1}$$

$$\nabla\cdot u = 0, \qquad (t,x)\in(0,\infty)\times\mathbb{T}^3, \tag{1.2}$$

$$u(0,x) = u_0(x), \qquad x\in\mathbb{T}^3, \tag{1.3}$$

on the 3D torus $\mathbb{T}^3 = [0,1]^3$, where $\nabla\cdot u_0 = 0$ and $\int_{\mathbb{T}^3} u_0 = 0$. Here, u is the velocity field of a stochastic flow, ν is the viscosity, and $\mathcal{P}$ is the Leray (also called Helmholtz-Hodge) orthogonal projection onto the mean zero divergence-free fields. The stochastic term $\sigma(u)d\mathbb{W}(t)$ denotes an infinite-dimensional and possibly degen-

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erate multiplicative white noise which is understood in the Itô sense. Note that in the formulation (1.1) the pressure term has been eliminated by utilizing the projector $\mathcal{P}$.

The stochastic forcing driving the Navier–Stokes equation represents a perturbation during the flow evolution and thus the SNSE may be argued to be a realistic model for fluids. Consequently, much effort has been devoted to studying its well-posedness; cf. [4,9,21,22] for results on mild solutions of the SNSE with Lévy-type jump noise, and [1,5,6,19] for results on mild formulation subject to white noise. The existence of a global L^2 martingale solution to the SNSE with Stratonovich noise in $\mathbb{R}^3$ was proven in [18], and the existence of a martingale solution in $L^{8/d}(0, T; L^4)$, where $d = 2, 3$, was shown in [2] for the SNSE with noise that is colored in space.

There are fewer available results on the existence of strong (pathwise) solutions, and most were established in a Hilbert space setting. For example, within the Hilbert setting, [17] proved the global existence of a strong solution for the 2D SNSE with additive white noise. Also, Flandoli [7] proved the result in the 3D case. For the SNSE with multiplicative noise, Fernando and Sritharan showed in [10] the existence of a global strong solution in a 2D unbounded domain in a Hilbert space, while in [12] Glatt-Holtz and the third author established the existence of a maximal strong solution in a 3D bounded domain by assuming the H^1 regularity for the initial data. In [14], Kim proved the existence with a large probability of a global strong solution to the SNSE with non-degenerate noise, assuming smallness in $H^s(\mathbb{R}^3)$.

Inspired by results on the deterministic Navier–Stokes equation in L^p Banach spaces [8,13], we aim to find a global L^p strong solution to (1.1)–(1.3) in three space dimensions. We show that a unique solution exists for small initial velocity. To be precise, we prove

$$\mathbb{P}(\tau_M = \infty) \geq 1 - C_p^{-1} M^{-p} \mathbb{E}[\|u_0\|_p^p],$$

where $\tau_M = \inf\{t > 0 : \|u(t)\|_p > M\}$ and u evolves in $L^p(\Omega, C([0, \tau_M), L^p))$. Note that we do not impose any structural assumptions on the multiplicative noise, allowing it to be degenerate. In [3], Barbu and Röckner obtained the existence and uniqueness of a global mild solution in L^p $(3/2 < p < 2)$ to the vorticity equation associated with the SNSE. They worked with a convolution-type finite-dimensional noise and small initial vorticity. The convolution structure is needed for obtaining a commutative C_0 noise operator, which is essential for transforming the vorticity equation into a random equation. Also, in [11], Glatt-Holtz and Vicol used multiplicative and linear noise to treat the 3D stochastic Euler equation.

As has been shown in many existing results, a major obstacle when seeking global solutions is a combination of a multiplicative noise and the convective nonlinearity $(u \cdot \nabla)u$. To overcome this difficulty, authors usually introduce stopping times of ascending u-norms to, in a sense, linearize this term in a specified function space. The stopping time argument proves to be a powerful tool for obtaining the local existence, but showing the non-degeneracy of these stopping times is a major problem. In this paper, we truncate the noise and $(u \cdot \nabla)u$ at $\|u\|_p = \delta_0$ at some level $\delta_0 > 0$ (not necessarily small). We first use a stochastic heat equation (SHE) with additive noise (see [20]) to obtain the global solution to the truncated SNSE. Then we establish the existence of a local strong solution of (1.1)–(1.3) by sending $\delta_0 \to \infty$ along integer values and utilizing pathwise uniqueness. Finally, we fix $\delta_0 > 0$ sufficiently small and estimate the probability distribution of $e^{at}\|u\|_p^p$ for a small $a > 0$. We show that (1.1) agrees with a truncated SNSE for all time on a large part of the probability space if the initial velocity is small, obtaining thus a global solution to (1.1)–(1.3).

Working in a function space of low regularity imposes several challenges. First, we need to obtain a global L^p solution to the SHE and then adapt it to the truncated SNSE. Considering that [20] only provides a $W^{m,p}$ solution to the SHE (cf. [20, Chapter 4]) and the $W^{m,p}$ estimate obtained in [20] does not support the L^p convergence of approximating solutions, we extend [20] to obtain an L^p a priori estimate for the SHE. Next, the regularity of the drift function in this L^p estimate must be strictly less than L^p, because the drift corresponds to $(u \cdot \nabla)u$ when one relates the truncated SNSE to the SHE in the fixed point argument, and $(u \cdot \nabla)u$ is less regular than u itself. We utilize the dissipative term to make such an estimate possible. But at a cost, the use of the dissipative term generates a non-linear term $|\nabla(|u|^{p/2})|^2$, which prevents convergence in the strong topology. Hence, we resort to the weak lower-semicontinuity of Hilbert space norms and fulfill the requirement of passing the limit for this term (cf. Lemma 4.4 below). The third difficulty is due to the structure of $(u \cdot \nabla)u$ and the introduced truncation. To overcome this, we apply the fixed point iteration twice. The main trick is to introduce a square of the cut-off, which allows us to treat the difference via a special splitting (cf. (5.19)–(5.20) below). Note that a high-regularity truncation on the SNSE is required by the iteration, while a low-regularity norm is preferable for showing the convergence of the iterated solutions. Overall, we can obtain convergence when $p > 5$. It would be desirable to obtain our theorems in the range $p > 3$ and for $p \geq 3$ for small data (as in [13] in the deterministic case), but this remains open (for the case of additive noise, see however [19]).

We note that all the results also apply to the Stratonovich noise under some modifications on the assumptions on the noise. When interpreted in the Stratonovich sense, (1.1) has an equivalent Itô formulation

$$\begin{aligned} du(t,x) &= \nu\Delta u(t,x)\,dt - \mathcal{P}\big((u(t,x)\cdot\nabla)u(t,x)\big)\,dt \\ &\quad + \frac{1}{2}\operatorname{Tr}(D\sigma(u(t,x))\sigma(u(t,x)))\,dt + \sigma(u(t,x))\,d\mathbb{W}(t). \end{aligned}$$

By assuming that σ and $D\sigma$ are bounded and globally Lipschitz in L^p, all the results and proofs apply without change.

The paper is organized as follows. In Sect. 2, we introduce the notation and preliminaries on stochastic calculus. In Sect. 3, we state our assumptions and the main results. Theorems on the SHE are collected in Sect. 4. The global existence and uniqueness of a strong solution to the truncated SNSE is established in Sect. 5, where we also obtain the local existence of solutions up to a stopping time. The global existence of solutions for small data is obtained in Sect. 6.

2 Notation and preliminaries

2.1 Basic notation

Let $T \in (0, \infty)$. For a scalar function $u = u(t, x)$ on $[0, T) \times \mathbb{T}^3$, we denote its partial derivatives by $\partial_t u = \partial u/\partial t$, and $\partial_i u = \partial u/\partial x_i$. Also, we denote its gradient with respect to x by $\nabla u = (\partial_1 u, \dots, \partial_d u)$.

We use $C^\infty(\mathbb{T}^3)$ for the set of infinitely differentiable functions on $\mathbb{T}^3$ and $\mathcal{D}'(\mathbb{T}^3)$ for the space of distributions $(C^\infty(\mathbb{T}^3))'$. Note that we have $C^\infty(\mathbb{T}^3) \subseteq L^p(\mathbb{T}^3) \subseteq \mathcal{D}'(\mathbb{T}^3)$ for $1 \le p \le \infty$. The usual L^p norms are denoted by $\|\cdot\|_p$.

The m-th Fourier coefficient of an L^1 function f on $\mathbb{T}^3$ is defined as

$$\mathcal{F} f(m) = \hat{f}(m) = \int_{\mathbb{T}^3} f(x) e^{-2\pi i m\cdot x}\, dx, \qquad m \in \mathbb{Z}^3,$$

and the corresponding Fourier series (Fourier inversion) of g at $x \in \mathbb{T}^3$ is

$$(\mathcal{F}^{-1} g)(x) = \sum_{m\in\mathbb{Z}^3} g(m) e^{2\pi i m\cdot x}.$$

Recall that $\mathcal{F}$ can be extended to $\mathcal{D}'(\mathbb{T}^3)$ and $\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id}$ on $\mathcal{D}'(\mathbb{T}^3)$. For $s \in \mathbb{R}$ and $f \in \mathcal{D}'(\mathbb{T}^3)$, we denote

$$J^s f(x) = \sum_{m\in\mathbb{Z}^3} (1 + 4\pi^2|m|^2)^{s/2} \hat{f}(m) e^{2\pi i m\cdot x}, \qquad x \in \mathbb{T}^3$$

and

$$\partial^s f(x) = \sum_{m\in\mathbb{Z}^3} |m|^s \hat{f}(m) e^{2\pi i m\cdot x}, \qquad x \in \mathbb{T}^3.$$

We define $W^{s,p}(\mathbb{T}^3)$ to be the class of functions $f \in \mathcal{D}'(\mathbb{T}^3)$ such that

$$\|f\|_{s,p} = \|J^s f\|_p < \infty, \qquad s \in \mathbb{R}, \qquad p > 1.$$

For the L^2 based spaces, we abbreviate $H^s(\mathbb{T}^3) = W^{s,2}(\mathbb{T}^3)$. Recall that there exists a positive constant C independent of f such that

$$\frac{1}{C}\|f\|_{s,p} \le \|f\|_p + \|\partial^s f\|_p \le C\|f\|_{s,p}, \qquad s \ge 0, \qquad 1 < p < \infty.$$

The Leray orthogonal projection $\mathcal{P}$ is defined by

$$\widehat{(\mathcal{P}u)}_j(m) = \sum_{k=1}^{d} \left(\delta_{jk} - \frac{m_j m_k}{|m|^2}\right) \widehat{u_k}(m), \qquad j = 1, 2, \dots, d. \tag{2.1}$$

Using the Riesz transforms

$$R_j = -\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}, \qquad j = 1, 2, \ldots, d,$$

the Eq. (2.1) for $\mathcal{P}$ may be rewritten as

$$(\mathcal{P}u)_j(x) = \sum_{k=1}^{d}(\delta_{jk} + R_j R_k)u_k(x), \qquad j = 1, 2, \ldots, d,$$

from where

$$\big((I - \mathcal{P})u\big)_j(x) = -\sum_{k=1}^{d} R_j R_k u_k(x), \qquad j = 1, 2, \ldots, d.$$

For convenience, we write

$$W^{s,p}_{\text{sol}} = \{\mathcal{P}f : f \in W^{s,p}\}. \tag{2.2}$$

As usual, C represents a generic positive constant, whose value may increase from line to line, with explicit dependence indicated when necessary. We consider p fixed, so C is allowed to depend on p without an explicit mention.

2.2 Preliminaries on stochastic analysis

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with an augmented filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by a cylindrical Brownian motion $\mathbb{W}$. We assume that $\mathbb{W}$ is an $\mathcal{H}$-valued process for some real separable Hilbert space $\mathcal{H}$, which may be infinite dimensional. Choosing a complete orthonormal basis $\{\mathbf{e}_k\}_{k\geq 1}$ for $\mathcal{H}$, we formally write $\mathbb{W}(t,\omega) = \sum_{k\geq 1} W_k(t,\omega)\mathbf{e}_k$, where $\{W_k : k \in \mathbb{N}\}$ is a collection of mutually independent 1D Brownian motions.

Let $\mathcal{Y}$ be another real separable Hilbert space. Denote by $l^2(\mathcal{H}, \mathcal{Y})$ the set of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{Y}$, i.e., $G \in l^2(\mathcal{H}, \mathcal{Y})$ if and only if G is a linear bounded operator mapping from $\mathcal{H}$ to $\mathcal{Y}$ such that

$$\|G\|^2_{l^2(\mathcal{H},\mathcal{Y})} = \sum_{k=1}^{\dim \mathcal{H}} |G\mathbf{e}_k|^2_{\mathcal{Y}} < \infty.$$

In our context, $\mathcal{Y}$ denotes either $\mathbb{R}$ or $\mathbb{R}^d$, and $\|\cdot\|_{l^2}$ is used interchangeably for $\|\cdot\|_{l^2(\mathcal{H},\mathbb{R})}$ and $\|\cdot\|_{l^2(\mathcal{H},\mathbb{R}^d)}$ when there is no risk of confusion. Note that any operator in $l^2(\mathcal{H}, \mathcal{Y})$ is compact and $l^2(\mathcal{H}, \mathcal{Y})$ is a separable Hilbert space endowed with a scalar product

$$(A, B)_{l^2(\mathcal{H},\mathcal{Y})} = \sum_{k=1}^{\dim \mathcal{H}} (A\mathbf{e}_k, B\mathbf{e}_k)_{\mathcal{Y}}, \qquad A, B \in l^2(\mathcal{H}, \mathcal{Y}).$$

Next, by the Burkholder–Davis–Gundy (BDG) inequality, for $G \in l^2(\mathcal{H}, \mathcal{Y})$ and $1 \le p < \infty$,

$$\mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s G\, d\mathbb{W}_r\right|_{\mathcal{Y}}^p\right] \le C\mathbb{E}\left[\left(\int_0^t \|G\|^2_{l^2(\mathcal{H},\mathcal{Y})}\, dr\right)^{p/2}\right].$$

Using this fact and letting $(J^s f)\mathbf{e}_k = J^s(f\mathbf{e}_k)$, we introduce Banach spaces

$$\mathbb{W}^{s,p} = \Big\{ f : \mathbb{T}^3 \to l^2(\mathcal{H}, \mathcal{Y}) : f\mathbf{e}_k \in W^{s,p}(\mathbb{T}^3) \text{ for each } k, \text{ and } \int_{\mathbb{T}^3} \|J^s f\|^p_{l^2(\mathcal{H},\mathcal{Y})}\, dx < \infty \Big\},$$

with respect to the norm

$$\|f\|_{\mathbb{W}^{s,p}} = \left(\int_{\mathbb{T}^3} \|J^s f\|^p_{l^2(\mathcal{H},\mathcal{Y})}\, dx\right)^{1/p},$$

for $s \ge 0$ and $1 < p < \infty$. Also, $\mathbb{W}^{0,p}$ is abbreviated as $\mathbb{L}^p$. Letting $(\mathcal{P}f)\mathbf{e}_k = \mathcal{P}(f\mathbf{e}_k)$, where $\mathcal{P}$ is the Leray projector, we have $\mathcal{P}f \in \mathbb{W}^{s,p}$ if $f \in \mathbb{W}^{s,p}$. Define

$$\mathbb{W}^{s,p}_{\mathrm{sol}} = \{\mathcal{P}f : f \in \mathbb{W}^{s,p}\}.$$

We assume for (1.1) that σ maps $W^{s,p}_{\mathrm{sol}}$ into $\mathbb{W}^{s,p}_{\mathrm{sol}}$, where $W^{s,p}_{\mathrm{sol}}$ was introduced in (2.2), and that it maps the set of mean zero fields onto itself.

3 Assumptions and main results

We seek a strong (pathwise) solution to (1.1)–(1.3) in $L^p(\mathbb{T}^3)$ for p sufficiently large. Here, we say a solution to a stochastic partial differential equation (SPDE) is strong if, almost surely relative to the given stochastic basis, it satisfies the SPDE in the distributional sense and it evolves in the designated function space (cf. [11,12,15] and references therein). This notion demonstrates a pathwise behavior rather than a law property, which distinguishes it from the martingale solution whose probability law fits the equation.

Suppose σ and g are $(l^2(\mathcal{H}, \mathbb{R}))^d$-valued operators, namely, σ and g have d components and each component is $l^2(\mathcal{H}, \mathbb{R})$-valued. Let A be an operator that is usually unbounded and

$$u(t, x) = u_0(x) + \int_0^t (Au(r, x) + f(r, x))\, dr + \int_0^t \big(\sigma(u(r, x)) + g(r, x)\big)\, d\mathbb{W}(r), \tag{3.1}$$

a d-dimensional stochastic evolution partial differential equation on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Different notions of solutions are defined as follows.

Definition 3.1 (Local Strong Solution) A pair (u, τ) is a local strong L^p solution to (3.1) if τ is a positive stopping time $\mathbb{P}$-almost surely, the stochastic process u is adapted with respect to $\mathcal{F}_t$, it belongs to $L^p(\Omega; C([0, \tau \wedge T], L^p))$, and satisfies

$$\begin{aligned}(u(t), \phi) = (u_0, \phi) + \int_0^t (Au(r) + f(r), \phi)\, dr \\ + \int_0^t \big(\sigma(u(r)) + g(r), \phi\big)\, d\mathbb{W}(r) \text{ a.e. } (t, \omega),\end{aligned} \tag{3.2}$$

for all $\phi \in C^\infty(\mathbb{T}^3)$.

In our applications below, the term $(Au(r), \phi)$ is interpreted using integration by parts.

Definition 3.2 (Maximal Strong Solution) A pair (u, τ) is a maximal strong L^p solution to (3.1) if there exists an increasing sequence of stopping times τ_n with $\tau_n \uparrow \tau$ a.s. such that each pair (u, τ_n) is a local strong solution,

$$\sup_{0\leq t\leq \tau_n} \|u(t)\|_p^p + \int_0^{\tau_n} \int_{\mathbb{T}^3} |\nabla(|u(t)|^{p/2})|^2\, dxdt < \infty,$$

and

$$\sup_{0\leq t\leq \tau} \|u(t)\|_p^p + \int_0^{\tau} \int_{\mathbb{T}^3} |\nabla(|u(t)|^{p/2})|^2\, dxdt = \infty,$$

on the set $\{\tau \leq T\}$.

For the local existence, we assume

$$\sum_{i=1}^3 \|\sigma_i(u)\|_{\mathbb{L}^p} = \sum_{i=1}^3 \left(\int_{\mathbb{T}^3} \|\sigma_i(u)\|_{l^2}^p\, dx\right)^{1/p} \leq C(\|u\|_p + 1) \tag{3.3}$$

and

$$\sum_{i=1}^3 \|\sigma_i(u) - \sigma_i(v)\|_{\mathbb{L}^p} \leq C\|u - v\|_p. \tag{3.4}$$

The following statement is the main result on the local existence of strong solutions.

Theorem 3.1 *(Local strong solution up to a stopping time) Let $p > 5$ and $u_0 \in L^p(\Omega; L^p)$. Then there exists a unique maximal strong solution (u, τ) to (1.1)–(1.3) such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq \tau} \|u(s, \cdot)\|_p^p + \int_0^\tau \sum_j \int_{\mathbb{T}^3} |\partial_j(|u_j(s, x)|^{p/2})|^2\, dxds\right] \leq C\mathbb{E}\big[\|u_0\|_p^p + 1\big],$$

where $C > 0$ is a constant depending on p.

The theorem is proven at the end of Sect. 5.

In the next statement, we address the global existence of solutions, for which we impose, in addition to (3.4), a superlinearity assumption

$$\sum_{i=1}^{3} \|\sigma_i(u)\|_{\mathbb{L}^p} \leq \epsilon_0 \|u\|_p, \tag{3.5}$$

where $\epsilon_0 > 0$.

Theorem 3.2 *(Global strong solution for small data) Suppose that (3.4) and (3.5) hold with $\epsilon_0 \in (0, 1]$ sufficiently small. Let (u, τ) be the solution provided in Theorem 3.1. For every $\epsilon \in (0, 1]$ there exists $\delta > 0$ such that if*

$$\mathbb{E}[\|u_0\|_p^p] \leq \delta, \tag{3.6}$$

then

$$\mathbb{P}(\tau = \infty) \geq 1 - \epsilon.$$

The proof of Theorem 3.2 is given in Sect. 6.

4 Stochastic heat equation on the torus

In this section, we prove the global existence of an L^p solution to the stochastic heat equation

$$du(t, x) = \Delta u(t, x)\, dt + f(t, x)\, dt + g(t, x)\, d\mathbb{W}(t), \tag{4.1}$$

$$u(0, x) = u_0(x) \ \text{ a.s.}, \qquad x \in \mathbb{T}^d \tag{4.2}$$

on $[0, T] \times \mathbb{T}^d$, where $d \in \mathbb{N}$. The functions u, u_0, f, and g are assumed to be scalar valued and have mean zero in x. The white noise $\mathbb{W}$ was introduced above, the drift f is a predictable process evolving in $W^{-1,q}$, where the range of q is stated below, the noise coefficient g takes values in $l^2(\mathcal{H}, \mathbb{R})$, and u_0 is $\mathcal{F}_0$-measurable.

Using the terminology in [20], the Eq. (4.1) is super-parabolic. Also, the a priori estimates for Theorems 4.1.2 and 4.1.4 in [20] remain true on the torus without change. Thus if $u_0 \in L^{p'}(\Omega; W^{m,p'})$, $f \in L^{p'}(\Omega \times [0, T], W^{m,p'})$, and $g \in L^{p'}(\Omega \times [0, T], \mathbb{W}^{m,p'})$ for some $m \in \mathbb{N}$ and $p' \geq 2$, then there exists $u \in L^{p'}(\Omega \times [0, T]; C_{\text{weak}} W^{m,p'})$ satisfying (4.1)–(4.2) in the sense of (3.2). If in addition $(m - k)p' > d$, then u has a version that belongs to $C_b^{0,k}([0, T] \times \mathbb{T}^d)$ $\mathbb{P}$-almost surely. This conclusion of global existence relies on a high regularity of the forcing term f, which needs to be relaxed to apply to the stochastic Navier–Stokes equations.

Theorem 4.1 *Let $2 < p < \infty$ and $0 < T < \infty$. Suppose that $u_0 \in L^p(\Omega, L^p(\mathbb{T}^d))$, $f \in L^p(\Omega \times [0,T], W^{-1,q}(\mathbb{T}^d))$, and $g \in L^p(\Omega \times [0,T], \mathbb{L}^p(\mathbb{T}^d))$ have x-mean zero (ω, t) a.s., with*

$$\frac{dp}{p+d-2} < q \leq p, \tag{4.3}$$

provided $d \geq 2$, or $1 < q \leq p$ if $d = 1$. Then there exists a unique maximal solution $u \in L^p(\Omega; C([0,T], L^p))$ to (4.1)–(4.2) such that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq T} \|u(t,\cdot)\|_p^p + \int_0^T \int_{\mathbb{T}^d} |\nabla(|u(t,x)|^{p/2})|^2\, dxdt\right] \\ &\leq C\mathbb{E}\left[\|u_0\|_p^p + \int_0^T \|f(s,\cdot)\|_{-1,q}^p\, ds + \int_0^T \|g(s)\|_{\mathbb{L}^p}^p\, ds\right], \end{aligned} \tag{4.4}$$

where $C > 0$ depends on T and p.

Recall that we use the notation

$$\|g(t)\|_{\mathbb{L}^p}^p = \int_{\mathbb{T}^d} \|g(t,x)\|_{l^2(\mathcal{H},\mathbb{R})}^p\, dx.$$

Introduce the standard convolution function $\rho \in C_0^\infty(\mathbb{R}^d)$ such that supp $\rho \subseteq \{x \in \mathbb{R}^d : |x| \leq 1\}$ and $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. Assume also that ρ is nonnegative and radial. Set $\rho_\epsilon = \epsilon^{-d}\rho(\cdot/\epsilon)$.

The next lemma is needed when approximating the forcing term in (4.1).

Lemma 4.2 *Let $q \in (1, \infty)$. If $f \in W^{-1,q}(\mathbb{T}^d)$, then $f * \rho_\epsilon \to f$ in $W^{-1,q}(\mathbb{T}^d)$ as $\epsilon \to 0$.*

Proof of Lemma 4.2 The mapping $S = -\Delta + I$ is a Banach space isomorphism $S\colon W^{1,q}(\mathbb{T}^d) \to W^{-1,q}(\mathbb{T}^d)$, which commutes with the convolution operator. Thus the statement follows by applying S to $(S^{-1}f) * \rho_\epsilon \to S^{-1}f$ in $W^{1,q}$.

Remark 4.3 Note that the above proof implies that if $f \in L^p(\Omega \times [0,T], W^{-1,q})$, then $f * \rho_\epsilon \to f$ in $L^p(\Omega \times [0,T], W^{-1,q})$.

The following lemma is essential when passing to the limit in the inequality (4.4).

Lemma 4.4 *Let $p \geq 2$. If*

$$u_n \to u \text{ in } L^p(\Omega; L^\infty([0,T], L^p)) \text{ as } n \to \infty$$

and

$$\nabla(|u_n(\omega,t,x)|^{p/2}) \text{ are uniformly bounded in } L^2(\Omega \times [0,T], L^2), \tag{4.5}$$

then

$$\liminf_{n\to\infty} \mathbb{E}\left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_n(\omega,t,x)|^{p/2})|^2\, dxdt\right] \geq \mathbb{E}\left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u(\omega,t,x)|^{p/2})|^2\, dxdt\right]. \tag{4.6}$$

Proof of Lemma 4.4 First, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\lim_k \mathbb{E}\left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_{n_k}(\omega,t,x)|^{p/2})|^2\, dxdt\right] = \liminf_n \mathbb{E}\left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_n(\omega,t,x)|^{p/2})|^2\, dxdt\right]. \tag{4.7}$$

Observe that, by (4.7), it suffices to prove (4.6) for a subsequence of $\{u_{n_k}\}_k$. For simplicity of notation, relabel $\{u_{n_k}\}_k$ as $\{u_n\}_n$. Passing to a subsequence, we may assume that $|u_n|^{p/2} \to |u|^{p/2}$ a.e. in $\Omega \times \mathbb{T}^d \times (0,T)$, and thus, by the Dominated Convergence Theorem, we get

$$|u_n|^{p/2} \to |u|^{p/2} \text{ in } L^2(\Omega \times [0,T], L^2).$$

By (4.5), we may pass to a subsequence and assume that

$$\nabla(|u_n(\omega,t,x)|^{p/2}) \to g \text{ weakly in } L^2(\Omega\times[0,T],L^2) \text{ as } n\to\infty,$$

for some $g \in L^2(\Omega \times [0,T], L^2)$, which also implies

$$\liminf_{n\to\infty} \mathbb{E}\left[\int_0^T \int_{\mathbb{T}^d} |\nabla(|u_n(\omega,t,x)|^{p/2})|^2\, dxdt\right] \geq \mathbb{E}\left[\int_0^T \int_{\mathbb{T}^d} |g|^2\, dxdt\right],$$

by the weak lower-semicontinuity of the Hilbert space norm. In order to obtain (4.6), we only need to prove that g and $\nabla(|u|^{p/2})$ agree as elements in $L^2(\Omega\times[0,T],L^2)$. To establish this, let $\varphi \in C^\infty(\mathbb{T}^d)$ be arbitrary. Then, for all $j = 1,\ldots,d$, we have

$$\big(g_j,\varphi\big) = \lim_n\big(\partial_j(|u_n|^{p/2}),\varphi\big) = -\lim_n\big(|u_n|^{p/2},\partial_j\varphi\big) = -\big(|u|^{p/2},\partial_j\varphi\big) = \big(\partial_j(|u|^{p/2}),\varphi\big),$$

where $(\cdot,\cdot)$ represents the inner product on $L^2(\Omega\times[0,T],L^2)$. Thus we obtain that $g(t,\omega)$ and $\nabla(|u(t,\omega)|^{p/2})$ agree in $L^2(\mathbb{T}^d)$ (t,ω)-a.e.

Proof of Theorem 4.1 Denote $u_0^\epsilon = u_0 * \rho_\epsilon$, $f^\epsilon = f * \rho_\epsilon$, and $g^\epsilon = g * \rho_\epsilon$. By Young's inequality, we have $u_0^\epsilon \in L^{p'}(\Omega; W^{m,p'})$, $f^\epsilon \in L^{p'}(\Omega\times[0,T],W^{m,p'})$, and $g^\epsilon \in L^{p'}(\Omega\times[0,T],\mathbb{W}^{m,p'})$ for $m\in\mathbb{N}_0$ and $2 \le p' < \infty$. Note that $u_0^\epsilon \to u_0$ in $L^p(\Omega,L^p)$, $f^\epsilon(t,\cdot)\to f(t,\cdot)$ in $L^p(\Omega\times[0,T],W^{-1,q})$, and $g^\epsilon(t,\cdot)\to g(t,\cdot)$ in $L^p(\Omega\times[0,T],\mathbb{L}^p)$ as $\epsilon\to 0$. Now, consider

$$du^\epsilon(t,x) = \Delta u^\epsilon(t,x)\,dt + f^\epsilon(t,x)\,dt + g^\epsilon(t,x)\,d\mathbb{W}_t, \tag{4.8}$$

$$u^\epsilon(0,x) = u_0^\epsilon(x) \ \text{ a.s.} \tag{4.9}$$

Clearly, assumptions of Theorem 4.1.4 in [20] are fulfilled. Therefore, there exists $u^\epsilon \in L^p(\Omega \times [0,T], W^{m,p})$ satisfying (4.8)–(4.9) in the sense of Definition 3.1. By Corollary 4.1.4 in [20], u^ϵ has a modification that belongs to $C_b^{0,n}([0,T] \times \mathbb{T}^d)$ $\mathbb{P}$-a.s. if $m > n + d/p$. We shall choose m sufficiently large and use the continuously differentiable modification of u^ϵ.

Applying the Itô formula to $h(y) = |y|^p$ with $y = u^\epsilon(t,x)$, we get

$$
\begin{aligned}
|u^\epsilon(t)|^p = |u_0^\epsilon|^p &+ p\int_0^t |u^\epsilon(r)|^{p-2}u^\epsilon(r)\big(\Delta u^\epsilon(r) + f^\epsilon(r)\big)\,dr \\
&+ p\int_0^t |u^\epsilon(r)|^{p-2}u^\epsilon(r)g^\epsilon(r)\,d\mathbb{W}_r \\
&+ \frac{p(p-1)}{2}\int_0^t |u^\epsilon(r)|^{p-2}\|g^\epsilon(r)\|^2_{l^2(\mathcal{H},\mathbb{R})}\,dr.
\end{aligned}
$$

We integrate both sides of the equation in x and apply the stochastic Fubini theorem obtaining

$$
\begin{aligned}
\|u^\epsilon(t)\|_p^p = \|u_0^\epsilon\|_p^p &+ p\int_0^t\int_{\mathbb{T}^d} |u^\epsilon(r)|^{p-2}u^\epsilon(r)\big(\Delta u^\epsilon(r) + f^\epsilon(r)\big)\,dxdr \\
&+ p\int_0^t\int_{\mathbb{T}^d} |u^\epsilon(r)|^{p-2}u^\epsilon(r)g^\epsilon(r)\,dxd\mathbb{W}_r \\
&+ \frac{p(p-1)}{2}\int_0^t\int_{\mathbb{T}^d} |u^\epsilon(r)|^{p-2}\|g_i^\epsilon(r)\|^2_{l^2}\,dxdr.
\end{aligned}
\tag{4.10}
$$

For the dissipative term, we have

$$
p\int_{\mathbb{T}^d} |u^\epsilon|^{p-2}u^\epsilon\Delta u^\epsilon\,dx = -p(p-1)\int_{\mathbb{T}^d}|u^\epsilon|^{p-2}|\nabla u^\epsilon|^2\,dx = -\frac{4(p-1)}{p}\int_{\mathbb{T}^d}|\nabla|u^\epsilon|^{p/2}|^2\,dx.
\tag{4.11}
$$

It then follows from (4.10) and (4.11) that

$$
\begin{aligned}
&\|u^\epsilon(t)\|_p^p + \frac{4(p-1)}{p}\int_0^t\int_{\mathbb{T}^d}|\nabla(|u^\epsilon(r)|^{p/2})|^2\,dxdr \\
&\le \|u_0^\epsilon\|_p^p + p\int_0^t\left|\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}u^\epsilon(r)f^\epsilon(r)\,dx\right|dr \\
&\quad + \frac{p(p-1)}{2}\int_0^t\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}\|g^\epsilon(r)\|^2_{l^2}\,dxdr \\
&\quad + p\left|\int_0^t\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}u^\epsilon(r)g^\epsilon(r)\,dxd\mathbb{W}_r\right| \\
&= \|u_0^\epsilon\|_p^p + I_1 + I_2 + I_3.
\end{aligned}
\tag{4.12}
$$

With $q' = q/(q-1)$, we have

$$I_1 \le C \int_0^t \|f^\epsilon\|_{-1,q} \||u^\epsilon|^{p-2}u^\epsilon\|_{1,q'}\, dr \le C \int_0^t \|f^\epsilon(r)\|_{-1,q} (\||u^\epsilon|^{p-2}u^\epsilon\|_{q'} + \|\nabla(|u^\epsilon|^{p-2}u^\epsilon)\|_{q'})\, dr, \tag{4.13}$$

where, recall, we allow C to depend on p throughout. Since $\int_{\mathbb{T}^d} u^\epsilon = 0$, we have, as in [16], a Poincaré type inequality

$$\||u^\epsilon|^{p-2}u^\epsilon\|_{q'} \le C\|\nabla(|u^\epsilon|^{p-2}u^\epsilon)\|_{q'} \tag{4.14}$$

when $p, q' \in (1, \infty)$. In [16, Lemma 3] the inequality

$$\||v|^{p-1}\|_{q'} \le C\|\nabla(|v|^{p-1})\|_{q'} \tag{4.15}$$

was proven for v such that $\int_{\mathbb{T}^3} v = 0$, but the same proof works (by means of a contradiction argument) for (4.14) as well. By (4.13) and (4.14), we get

$$I_1 \le C \int_0^t \|f^\epsilon\|_{-1,q} \|\nabla(|u^\epsilon|^{p-2}u^\epsilon)\|_{q'}\, dr. \tag{4.16}$$

Now, note that

$$\begin{aligned} \|\nabla(|u^\epsilon|^{p-2}u^\epsilon)\|_{q'} &\le C\left\||u^\epsilon|^{p/2-1}\nabla(|u^\epsilon|^{p/2})\right\|_{q'} \le C\left\||u^\epsilon|^{p/2-1}\right\|_{\bar{r}}\left\|\nabla(|u^\epsilon|^{p/2})\right\|_2 \\ &= C\|u^\epsilon\|_{\bar{r}(p-2)/2}^{(p-2)/2}\|\nabla(|u^\epsilon|^{p/2})\|_2 = C\||u^\epsilon|^{p/2}\|_{\bar{r}(p-2)/p}^{(p-2)/p}\|\nabla(|u^\epsilon|^{p/2})\|_2, \end{aligned} \tag{4.17}$$

where $1/\bar{r} + 1/2 = 1/q'$, i.e.,

$$\frac{1}{\bar{r}} + \frac{1}{q} = \frac{1}{2}. \tag{4.18}$$

(The assumptions on the exponents p and q imply $q > 2$.) It is easy to check that the condition (4.3) gives

$$2 \le \frac{\bar{r}(p-2)}{p} < \frac{2d}{d-2},$$

when $d \ge 2$. By the Gagliardo-Nirenberg inequality and (4.15), with $p-1$ and q' replaced by $p/2$ and 2 respectively, we have with $w = |u^\epsilon|^{p/2}$ the inequality

$$\|w\|_{\bar{r}(p-2)/p} \le C\|w\|_2^{1-\alpha}\|\nabla w\|_2^{\alpha},$$

where $\alpha = d(1/2 - p/\bar{r}(p-2))$, and thus using (4.17), we get

$$\|\nabla(|u^\epsilon|^{p-2}u^\epsilon)\|_{q'} \le C\||u^\epsilon|^{p/2}\|_2^{(1-\alpha)(p-2)/p}\|\nabla(|u^\epsilon|^{p/2})\|_2^{1+\alpha(p-2)/p}. \tag{4.19}$$

From (4.16)–(4.19), we thus obtain

$$
\begin{aligned}
I_1 &\le C\int_0^t \|f^\epsilon\|_{-1,q}\||u^\epsilon|^{p/2}\|_2^{(1-\alpha)(p-2)/p}\|\nabla(|u^\epsilon|^{p/2})\|_2^{1+\alpha(p-2)/p}\,dr \\
&\le \delta\int_0^t \|\nabla(|u^\epsilon|^{p/2})\|_2^2\,dr + \delta t\sup_{0\le r\le t}\|u(r,\cdot)\|_p^p + C_\delta\int_0^t \|f^\epsilon\|_{-1,q}^p\,dr
\end{aligned}
\tag{4.20}
$$

with $\delta > 0$ arbitrarily small, where we applied Young's inequality in the last step. Next, for the term I_2 in (4.12), we write

$$
I_2 = \frac{p(p-1)}{2}\int_0^t\int_{\mathbb{T}^d} |u^\epsilon(r)|^{p-2}\|g^\epsilon(r)\|_{l^2}^2\,dxdr \le \delta\int_0^t \|u^\epsilon(r)\|_p^p\,dr + C_\delta\int_0^t \|g^\epsilon(r)\|_{\mathbb{L}^p}^p\,dr\,.
$$

Finally, we consider the last term in (4.12). Using Minkowski's integral inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left\|\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}u^\epsilon(r)g^\epsilon(r)\,dx\right\|_{l^2}^2 dr\right)^{1/2}\right] \\
&\quad\le \mathbb{E}\left[\left(\int_0^T\left(\int_{\mathbb{T}^d}\left\||u^\epsilon(r)|^{p-2}u^\epsilon(r)g^\epsilon(r)\right\|_{l^2}dx\right)^2 dr\right)^{1/2}\right] \\
&\quad= \mathbb{E}\left[\left(\int_0^T\left(\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-1}\|g^\epsilon(r)\|_{l^2}\,dx\right)^2 dr\right)^{1/2}\right] \\
&\quad\le \mathbb{E}\left[\sup_{r\in[0,T]}\|u^\epsilon(r)\|_p^{p/2}\left(\int_0^T\left(\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}\|g^\epsilon(r)\|_{l^2}^2\,dx\right)dr\right)^{1/2}\right],
\end{aligned}
\tag{4.21}
$$

where we abbreviated $l^2 = l^2(\mathcal{H},\mathbb{R})$. Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left\|\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}u^\epsilon(r)g^\epsilon(r)\,dx\right\|_{l^2}^2 dr\right)^{1/2}\right] \\
&\quad\le \mathbb{E}\left[\sup_{r\in[0,T]}\|u^\epsilon(r)\|_p^{p/2}\left(\int_0^T\left(\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}\|g^\epsilon(r)\|_{l^2}^2\,dx\right)dr\right)^{1/2}\right] \\
&\quad\le \frac{1}{8p}\mathbb{E}\left[\sup_{r\in[0,T]}\|u^\epsilon(r)\|_p^p\right] + C\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}\|g^\epsilon(r,x)\|_{l^2}^2\,dxdr\right] \\
&\quad\le \frac{1}{4p}\mathbb{E}\left[\sup_{r\in[0,T]}\|u^\epsilon(r)\|_p^p\right] + C_T\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}\|g^\epsilon(r,x)\|_{l^2}^p\,dxdr\right],
\end{aligned}
$$

where we used Young's inequality in the last step. Note that the far right side is finite by (4.1.21) in [20]. Thus, from the BDG inequality, we get

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{T}^d}|u^\epsilon(r)|^{p-2}u^\epsilon(r)g^\epsilon(r)\,dxd\mathbb{W}_r\right|\right] \\ &\quad\leq \frac{1}{4p}\mathbb{E}\left[\sup_{r\in[0,T]}\|u^\epsilon(r)\|_p^p\right]+C\mathbb{E}\left[\int_0^T\|g^\epsilon(r)\|_{\mathbb{L}^p}^p\,dr\right]. \end{aligned} \tag{4.22}$$

Now, setting δ in (4.20) to be sufficiently small, taking the supremum over $t\in[0,T]$ on both sides of (4.12), and then computing the expectation, we obtain

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|u^\epsilon(t)\|_p^p+\frac{1}{p}\int_0^t\int_{\mathbb{T}^d}|\nabla(|u^\epsilon(r)|^{p/2})|^2\,dxdr\right)\right] \\ &\quad\leq \frac{1}{2}\mathbb{E}\left[\sup_{r\in[0,T]}\|u^\epsilon(r)\|_p^p\right]+\mathbb{E}[\|u_0^\epsilon\|_p^p]+C\mathbb{E}\left[\int_0^T(\|f^\epsilon(r)\|_{-1,q}^p+\|g^\epsilon(r)\|_{\mathbb{L}^p}^p)\,dr\right], \end{aligned}$$

which implies

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\epsilon(t)\|_p^p\right]\leq 2\mathbb{E}[\|u_0^\epsilon\|_p^p]+C\mathbb{E}\left[\int_0^T(\|f^\epsilon(r)\|_{-1,q}^p+\|g^\epsilon(r)\|_{\mathbb{L}^p}^p)\,dr\right] \tag{4.23}$$

and

$$\begin{aligned} &\frac{1}{2p}\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}|\nabla(|u^\epsilon(r)|^{p/2})|^2\,dxdr\right] \\ &\quad\leq \frac{1}{4}\mathbb{E}\left[\sup_{r\in[0,T]}\|u^\epsilon(r)\|_p^p\right]+\frac{1}{2}\mathbb{E}[\|u_0^\epsilon\|_p^p]+C\mathbb{E}\left[\int_0^T(\|f^\epsilon(r)\|_{-1,q}^p+\|g^\epsilon(r)\|_{\mathbb{L}^p}^p)\,dr\right]. \end{aligned}$$

In summary,

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\epsilon(t)\|_p^p+\int_0^T\int_{\mathbb{T}^d}|\nabla(|u^\epsilon(r)|^{p/2})|^2\,dxdr\right] \\ &\quad\leq C\mathbb{E}\left[\|u_0^\epsilon\|_p^p+\int_0^T(\|f^\epsilon(r)\|_{-1,q}^p+\|g^\epsilon(r)\|_{\mathbb{L}^p}^p)\,dr\right]. \end{aligned} \tag{4.24}$$

Note that the derivation of (4.23) does not depend on ϵ. Thus, we may apply the same procedure to $u^\epsilon - u^{\epsilon'}$ and obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u^\epsilon(t) - u^{\epsilon'}(t)\|_p^p\right] \leq C\mathbb{E}\left[\|u_0^\epsilon - u_0^{\epsilon'}\|_p^p + \int_0^T (\|f^\epsilon(r) - f^{\epsilon'}(r)\|_{-1,q}^p + \|g^\epsilon(r) - g^{\epsilon'}(r)\|_{\mathbb{L}^p}^p)\, dr\right].$$

Since each u^ϵ belongs to $L^p(\Omega; C([0,T], L^p))$ and they converge in $L^p(\Omega; L^\infty([0,T], L^p))$, they have a limit in $L^p(\Omega; C([0,T], L^p))$, and there exists a subsequence u^{ϵ_n} which converges to that limit in $L^\infty([0,T], L^p)$ almost surely. We denote this limit by u and we now prove that it is a strong L^p solution to (4.1)–(4.2). Since

$$(u^\epsilon(t), \phi) = (u_0^\epsilon, \phi) + \int_0^t ((\Delta u^\epsilon(r) + f^\epsilon(r)), \phi)\, dr + \int_0^t (g^\epsilon(r), \phi)\, d\mathbb{W}_r, \qquad (t, \omega)\text{-a.e.},$$

for all $\phi \in C^\infty(\mathbb{T}^d)$ and all $\epsilon > 0$, by the Hölder inequality and the dominated convergence theorem, we have

$$(u^{\epsilon_n}(t), \phi) - (u_0^{\epsilon_n}, \phi) \to (u(t), \phi) - (u_0, \phi)$$

and

$$\int_0^t \big((u^{\epsilon_n}(r), \Delta\phi) + (f^{\epsilon_n}(r), \phi)\big)\, dr \to \int_0^t \big((u(r), \Delta\phi) + (f(r), \phi)\big)\, dr$$

for a.e. (t, ω) as $n \to \infty$. By the BDG inequality,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t (g^{\epsilon_n}(r) - g(r), \phi)\, d\mathbb{W}_r\right|\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^T \|(g^{\epsilon_n}(r) - g(r), \phi)\|_{l^2}^2\, dr\right)^{1/2}\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^T \left(\int_{\mathbb{T}^d} \|g^{\epsilon_n}(r) - g(r)\|_{l^2}^2\, dx\right) \|\phi\|_2^2\, dr\right)^{1/2}\right] \\
&\leq C\|\phi\|_2 \mathbb{E}\left[\left(\int_0^T \left(\int_{\mathbb{T}^d} \|g^{\epsilon_n}(r) - g(r)\|_{l^2}^p\, dx\right)^{2/p} dr\right)^{1/2}\right] \\
&\leq C\|\phi\|_2 \mathbb{E}\left[\int_0^T \|g^{\epsilon_n}(r) - g(r)\|_{\mathbb{L}^p}^p\, dr\right],
\end{aligned}$$

which converges to 0 as $n \to \infty$. This implies that for a further subsequence, which we still denote by u^{ϵ_n}, we have

$$\int_0^t (g^{\epsilon_n}(r), \phi)\, d\mathbb{W}_r \xrightarrow{n\to\infty} \int_0^t (g(r), \phi)\, d\mathbb{W}_r, \quad (t, \omega)\text{-a.e.}$$

Using Lemma 4.4 and letting $n \to \infty$ in (4.24), we obtain (4.4).

Suppose u_1, u_2 are two strong L^p solutions to (4.1)–(4.2). Then $v = u_1 - u_2$ satisfies

$$\begin{aligned} dv(t, x) &= \Delta v(t, x)\, dt, \\ v(0, x) &= 0 \ \text{ a.s.} \end{aligned}$$

on $[0, T] \times \mathbb{T}^d$. Then $v \equiv 0$ $\mathbb{P}$-a.s.

For convenience we also state the vector-valued version of the previous theorem. Thus, consider (4.1)–(4.2) on $\mathbb{T}^d$ but with u, f, g, and u_0 being $\mathbb{R}^D$-valued, where $D \in \mathbb{N}$. Then, under the assumptions of Theorem 4.1, we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\le t\le T} \|u(t, \cdot)\|_p^p + \sum_{j=1}^D \int_0^T \int_{\mathbb{T}^d} |\nabla(|u_j(t, x)|^{p/2})|^2\, dxdt\right] \\ &\quad \le C\mathbb{E}\left[\|u_0\|_p^p + \int_0^T \|f(t, \cdot)\|_{-1,q}^p\, dt + \sum_{j=1}^D \int_0^T \int_{\mathbb{T}^d} \|g_j(t, x)\|_{l^2(\mathcal{H},\mathbb{R})}^p\, dxdt\right]. \end{aligned} \tag{4.25}$$

5 Stochastic truncated Navier–Stokes equation

From here on, we restrict our considerations to the space dimension 3, although all the statements can be adjusted to any dimension $d \ge 2$. Also, with a constant $\delta_0 > 0$ which is not necessarily small, denote by $\varphi\colon [0, \infty) \to [0, 1]$ a decreasing smooth function such that $\varphi \equiv 1$ on $[0, \delta_0/2]$ and $\varphi \equiv 0$ on $[\delta_0, \infty)$. In addition, we assume

$$|\varphi(t_1) - \varphi(t_2)| \le \frac{C}{\delta_0}|t_1 - t_2|, \qquad t_1, t_2 \ge 0.$$

We consider a stochastic Navier–Stokes equations on $[0, T] \times \mathbb{T}^3$, truncated by this function, which reads

$$\begin{aligned} du(t, x) &= \Delta u(t, x)\, dt - \varphi(\|u(t)\|_p)^2 \mathcal{P}\big((u(t, x) \cdot \nabla)u(t, x)\big)\, dt \\ &\qquad + \varphi(\|u(t)\|_p)^2 \sigma(u(t, x))\, d\mathbb{W}_t, \\ \nabla \cdot u(t, x) &= 0, \\ u(0, x) &= u_0(x) \text{ a.s.}, \qquad x \in \mathbb{T}^3, \end{aligned} \tag{5.1}$$

where σ is $(l^2(\mathcal{H},\mathbb{R}))^3$-valued, $u_0 \in L^p(\Omega; L^p)$ is $\mathcal{F}_0$-measurable with $p > 5$, and $\nabla \cdot u_0 = 0$ with $\int_{\mathbb{T}^3} u_0\,dx = 0$ a.s. assumed throughout. Our goal in this section is to find the unique global solution for (5.1) by applying a fixed point argument.

We note that the reason for the square in the two factors containing $\varphi(\|u(t)\|_p)$ in (5.1) is the splitting (5.19)–(5.20) (and similarly (5.26)–(5.27)), which assures that every term is linearized either by matching $u^{(n)}$ with $\varphi^{(n)}$ or $u^{(n-1)}$ with $\varphi^{(n-1)}$.

Theorem 5.1 *Let $p > 5$ and $u_0 \in L^p(\Omega; L^p)$. For every $T > 0$, there exists a unique strong solution $u \in L^p(\Omega; C([0,T], L^p))$ to (5.1) such that*

$$\mathbb{E}\left[\sup_{0\le s\le T}\|u(s,\cdot)\|_p^p + \sum_j \int_0^T\int_{\mathbb{T}^3} |\nabla(|u_j(s,x)|^{p/2})|^2\,dxds\right] \le C\mathbb{E}\big[\|u_0\|_p^p\big] + C_T. \tag{5.2}$$

In order to solve (5.1), we use the iteration

$$\begin{aligned} du^{(n)} - \Delta u^{(n)}\,dt &= -\varphi(\|u^{(n)}\|_p)\varphi(\|u^{(n-1)}\|_p)\mathcal{P}\big((u^{(n-1)}\cdot\nabla)u^{(n-1)}\big)\,dt\\ &\quad + \varphi(\|u^{(n)}\|_p)\varphi(\|u^{(n-1)}\|_p)\sigma(u^{(n-1)})\,d\mathbb{W}_t,\\ \nabla\cdot u^{(n)} &= 0,\\ u^{(n)}(0) &= u_0 \text{ a.s.,} \qquad x\in\mathbb{T}^3, \end{aligned} \tag{5.3}$$

where $u^{(0)}$ is the strong solution to

$$\begin{aligned} &du^{(0)}(t,x) - \Delta u^{(0)}(t,x)\,dt = 0,\\ &\nabla\cdot u^{(0)}(t,x) = 0,\\ &u^{(0)}(0,x) = u_0(x) \text{ a.s.,} \qquad x\in\mathbb{T}^3. \end{aligned}$$

Utilizing the results from the previous section, we conclude that $u^{(0)} \in L^p(\Omega; C([0,T], L^p))$ and

$$\mathbb{E}\left[\sup_{0\le t\le T}\|u^{(0)}(t,\cdot)\|_p^p + \sum_j \int_0^T\int_{\mathbb{T}^3} |\nabla(|u_j^{(0)}(t,x)|^{p/2})|^2\,dxdt\right] \le C\mathbb{E}[\|u_0\|_p^p]. \tag{5.4}$$

We need to prove that at each step n, there exists a unique solution $u^{(n)} \in L^p(\Omega; C([0,T], L^p))$ to (5.3), which is uniformly bounded in a manner consistent with (5.4). Thus we first consider the equation

$$\begin{aligned} &du - \Delta u\,dt = -\varphi(\|u\|_p)\varphi(\|v\|_p)\mathcal{P}\big((v\cdot\nabla)v\big)\,dt + \varphi(\|u\|_p)\varphi(\|v\|_p)\sigma(v)\,d\mathbb{W}_t,\\ &\nabla\cdot u = 0,\\ &u(0,x) = u_0, \qquad \text{a.s.,} \qquad x\in\mathbb{T}^3, \end{aligned} \tag{5.5}$$

where v is divergence-free and satisfies

$$\mathbb{E}\left[\sup_{0\le t\le T}\|v_j(t,\cdot)\|_p^p+\sum_j\int_0^T\int_{\mathbb{T}^3}|\nabla(|v(t,x)|^{p/2})|^2\,dxdt\right]\le C\mathbb{E}[\|u_0\|_p^p]+C_T. \tag{5.6}$$

In order to solve (5.5), we employ the iteration procedure

$$\begin{aligned} du^{(n)}-\Delta u^{(n)}\,dt &= -\varphi(\|u^{(n-1)}\|_p)\varphi(\|v\|_p)\mathcal{P}\big((v\cdot\nabla)v\big)\,dt\\ &\quad+\varphi(\|u^{(n-1)}\|_p)\varphi(\|v\|_p)\sigma(v)\,d\mathbb{W}_t,\\ \nabla\cdot u^{(n)}&=0,\\ u^{(n)}(0)&=u_0, \qquad \text{a.s.}, \qquad x\in\mathbb{T}^3, \end{aligned} \tag{5.7}$$

for v which is divergence-free and satisfies (5.6). Note that $u^{(n)}$ in (5.7) is not the same as in (5.3).

We shall prove the existence by obtaining an exponential rate of convergence for the fixed point iteration, for both (5.7) and (5.3), and then claiming that a sequence of random variables converges to zero a.s. if their expectation approaches zero rapidly. For this purpose, the following auxiliary result is essential.

Lemma 5.2 *Let ξ_n be a sequence of nonnegative random variables such that $\mathbb{E}[\xi_n]\le\eta^n$, for $n\in\mathbb{N}$, where $\eta\in(0,1)$. Then, $\xi_n\to0$ almost surely.*

Proof of Lemma 5.2 Denote the probability event $\{\omega\in\Omega:\xi_n(\omega)\ge1/m\}$ by A_n^m. If $\xi_n(\omega)$ does not converge to zero as $n\to\infty$, then $\omega\in\cup_{m=1}^\infty\cap_{n=1}^\infty\cup_{k=n}^\infty A_k^m$. For each fixed $m\in\mathbb{N}$,

$$\sum_n^\infty\mathbb{P}(A_n^m)\le m\sum_n^\infty\mathbb{E}[\xi_n]<\infty,$$

and thus $\mathbb{P}(\limsup_{n\to\infty}A_n^m)=0$ by the Borel-Cantelli Lemma. Hence,

$$\mathbb{P}(\cup_{m=1}^\infty\cap_{n=1}^\infty\cup_{k=n}^\infty A_k^m)=\lim_{m\to\infty}\mathbb{P}(\limsup_{n\to\infty}A_n^m)=0,$$

completing the proof.

Remark 5.3 This conclusion can be extended from expectation and the probability measure to integration with respect to any finite measure. In particular, we could use the integration on $\Omega\times[0,T]$ with respect to the product measure.

For convenience, we abbreviate

$$\begin{aligned} \varphi^{(n)}&=\varphi(\|u^{(n)}\|_p), \qquad n\in\mathbb{N},\\ \varphi_v&=\varphi(\|v\|_p), \end{aligned}$$

for the rest of the section. The next lemma asserts uniform boundedness of $u^{(n)}$, which is needed in the fixed point argument.

Lemma 5.4 *Let $p > 5$, $n \in \mathbb{N}$, and $T > 0$. Suppose $u_0 \in L^p(\Omega; L^p)$ and assume that for each $n \in \{1, 2, \ldots, k-1\}$, there exists a unique solution $u^{(n)} \in L^p(\Omega; C([0,T], L^p))$ to the initial value problem (5.7), where v and $u^{(n)}$ satisfy (5.6). Then for $n = k$, the initial value problem (5.7) also has a unique solution $u^{(k)} \in L^p(\Omega; C([0,T], L^p))$, and moreover,*

$$\mathbb{E}\left[\sup_{0\le t\le T} \|u^{(k)}(t,\cdot)\|_p^p + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla(|u_j^{(k)}(t,x)|^{p/2})|^2\, dx dt\right] \le C\mathbb{E}\big[\|u_0\|_p^p\big] + C_T. \tag{5.8}$$

Proof Let $n = k$. We apply Theorem 4.1 (cf. the inequality (4.25)) to the equation

$$\begin{aligned} du_j^{(n)} - \Delta u_j^{(n)}\, dt &= -\varphi^{(n-1)}\varphi_v\big(\mathcal{P}\big((v\cdot\nabla)v\big)\big)_j\, dt \\ &\quad + \varphi^{(n-1)}\varphi_v\sigma_j(v)\, d\mathbb{W}_t, \qquad j = 1, 2, 3. \end{aligned} \tag{5.9}$$

We write the first term on the right side of (5.9) as

$$-\sum_i \varphi^{(n-1)}\varphi_v\partial_i\big(\mathcal{P}\big(v_i v\big)\big)_j\, dt.$$

In order to apply (4.25), we need to estimate

$$\begin{aligned} C\mathbb{E}\left[\int_0^T \|\varphi^{(n-1)}\varphi_v v_i v\|_q^p\, ds\right] &\le C\mathbb{E}\left[\int_0^T \varphi^{(n-1)}\varphi_v\|v_i\|_r^p\|v\|_l^p\, ds\right] \\ &\le C\mathbb{E}\left[\int_0^T \varphi^{(n-1)}\varphi_v\|v_i\|_r^p\|v\|_p^p\, ds\right] \le C\delta_0^p\mathbb{E}\left[\int_0^T \varphi^{(n-1)}\varphi_v\|v_i\|_r^p\, ds\right], \end{aligned} \tag{5.10}$$

where

$$\frac{3p}{p+1} < q \le p \tag{5.11}$$

and

$$\frac{1}{r} + \frac{1}{l} = \frac{1}{q}.$$

For the last inequality in (5.10) we require

$$l \le p \tag{5.12}$$

and then use $\varphi_v\|v\|_p^p \le C\delta_0^p\varphi_v$. In order to bound the last expression in (5.10), we also need $r < 3p$. When we consider below the differences of iterates (cf. (5.21)–(5.23) below), we however need a stronger inequality

$$r \le p. \tag{5.13}$$

For the sake of exposition, we fix the exponents at this point as

$$q = (3p + \eta_0)/(p + 1) \text{ and } r = l = 2q.$$

The parameter $\eta_0 > 0$ is chosen so that

$$\frac{3p + \eta_0}{p + 1} < \frac{p}{2}, \tag{5.14}$$

which is possible when $p > 5$. It remains to estimate the last term in (5.9) (cf. (4.25)), i.e.,

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{T}^d} \|\varphi^{(n-1)}\varphi_v \sigma(v)\|_{l^2(\mathcal{H},\mathbb{R}^d)}^p \, dxds\right] \le C\mathbb{E}\left[\int_0^T \varphi_v(\|v\|_p^p + 1)\, ds\right] \le CT,$$

using sub-linear growth of the noise (3.3), and we obtain (5.8).

Lemma 5.5 *Let $p > 5$ and suppose that $u_0 \in L^p(\Omega; L^p)$. Then there exists $t \in (0, T]$ such that the initial value problem (5.5), where v satisfies (5.6), has a unique strong solution $u \in L^p(\Omega; C([0, t], L^p))$, which satisfies*

$$\mathbb{E}\left[\sup_{0\le s\le t} \|u(s,\cdot)\|_p^p + \sum_j \int_0^t \int_{\mathbb{T}^3} |\nabla(|u_j(s,x)|^{p/2})|^2 \, dxds\right] \le C\mathbb{E}\big[\|u_0\|_p^p\big] + C_t. \tag{5.15}$$

Proof of Lemma 5.5 We employ the fixed point argument on the iteration (5.7). The difference $z^{(n)} = u^{(n+1)} - u^{(n)}$ satisfies

$$dz_j^{(n)} - \Delta z_j^{(n)}\, dt = \sum_i \partial_i f_{ij}\, dt + g_j\, d\mathbb{W}_t, \qquad j = 1, 2, 3, \tag{5.16}$$

where

$$f_{ij} = -(\varphi^{(n)} - \varphi^{(n-1)})\varphi_v(\mathcal{P}(v_i v))_j$$

and

$$g_j = \big(\varphi^{(n)} - \varphi^{(n-1)}\big)\varphi_v \sigma_j(v)\, d\mathbb{W}_t.$$

In addition to (5.16), we have

$$\begin{aligned} &\nabla \cdot z^{(n)} = 0, \\ &z^{(n)}(0) = 0 \ \text{ a.s.}, \quad x \in \mathbb{T}^3. \end{aligned}$$

Note that

$$|\varphi^{(n)} - \varphi^{(n-1)}| \le \frac{C}{\delta_0}\Big|\|u^{(n)}\|_p - \|u^{(n-1)}\|_p\Big| \le \frac{C}{\delta_0}\|u^{(n)} - u^{(n-1)}\|_p = \frac{C}{\delta_0}\|z^{(n-1)}\|_p. \tag{5.17}$$

Now, we apply (4.25). The second term on the right side of (4.25) is estimated as

$$\begin{aligned} C\sum_i \mathbb{E}\left[\int_0^t \|(\varphi^{(n)} - \varphi^{(n-1)})\varphi_v v_i v\|_q^p\, ds\right] &\le \frac{C}{\delta_0^p}\mathbb{E}\left[\int_0^t \varphi_v^p \|z^{(n-1)}\|_p^p \|v\|_r^p \|v\|_l^p\, ds\right] \\ &\le C\mathbb{E}\left[\int_0^t \|z^{(n-1)}\|_p^p\, ds\right] \le C\, t\, \mathbb{E}\left[\sup_{s\in[0,t]} \|z^{(n-1)}\|_p^p\right], \end{aligned}$$

where we used (5.12) and (5.13) in the second inequality. For the last term in (4.25), we estimate

$$\begin{aligned} C\mathbb{E}\left[\int_0^t \int_{\mathbb{T}^d} \|g(s,x)\|^p_{l^2(\mathcal{H},\mathbb{R}^d)}\, dxds\right] &\le C_{\delta_0}\mathbb{E}\left[\int_0^t \|z^{(n-1)}\|_p^p\, ds\right] \\ &\le C_{\delta_0}\mathbb{E}\left[\int_0^t \|z^{(n-1)}\|_p^p ds\right] \le C\, t\, \mathbb{E}\left[\sup_{s\in[0,t]} \|z^{(n-1)}\|_p^p ds\right]. \end{aligned}$$

This concludes the proof of existence of a fixed point for (5.5) on $[0,t]$ in $L^p_\omega L^\infty_t L^p_x$ if $t > 0$ is sufficiently small. It is standard to adapt the contraction argument above to the proof of uniqueness, and we omit the details. We denote this unique fixed point by u. Observing the exponential rate of convergence, we apply Lemma 5.2 and obtain $\varphi(\|u^{(n)}(t)\|_p) \to \varphi(\|u(t)\|_p)$ for a.e.-(ω, t). Then, by applying the dominated convergence theorem, we obtain that u is indeed a solution to (5.7). Thus, (5.15) holds by Lemma 4.4.

Proof of Theorem 5.1 Consider the iteration (5.3), i.e.,

$$\begin{aligned} du^{(n)} - \Delta u^{(n)}\, dt &= -\varphi^{(n)}\varphi^{(n-1)}\mathcal{P}\big((u^{(n-1)}\cdot\nabla)u^{(n-1)}\big)\, dt \\ &\quad + \varphi^{(n)}\varphi^{(n-1)}\sigma(u^{(n-1)})\, d\mathbb{W}_t, \\ \nabla\cdot u^{(n)} &= 0, \\ u^{(n)}(0) &= u_0, \qquad \text{a.s.}, \qquad x\in\mathbb{T}^3 \end{aligned}$$

on $(0,T]\times\mathbb{T}^3$. First assume that T is a sufficiently small constant as determined in Lemma 5.5 above; at the end of the proof, we extend the solution to the full range by the pathwise uniqueness. Lemma 5.5 implies the existence of a unique maximal solution $u^{(n)}$, which satisfies

$$\mathbb{E}\left[\sup_{0\le t\le T}\|u^{(n)}(t,\cdot)\|_p^p + \sum_j \int_0^T \int_{\mathbb{T}^3} |\nabla(|u_j^{(n)}(t,x)|^{p/2})|^2\, dxdt\right] \le C_{T,\delta_0} + C\mathbb{E}\big[\|u_0\|_p^p\big]. \tag{5.18}$$

In order to apply the fixed point technique, we consider the difference

$$v^{(n)} = u^{(n+1)} - u^{(n)},$$

for which

$$\begin{aligned}
&dv^{(n)} - \Delta v^{(n)}\,ds + \Big(\varphi^{(n+1)}\varphi^{(n)}\mathcal{P}\big((u^{(n)}\cdot\nabla)u^{(n)}\big) - \varphi^{(n)}\varphi^{(n-1)}\mathcal{P}\big((u^{(n-1)}\cdot\nabla)u^{(n-1)}\big)\Big)\,ds\\
&\quad = \Big(\varphi^{(n+1)}\varphi^{(n)}\sigma(u^{(n)}) - \varphi^{(n)}\varphi^{(n-1)}\sigma(u^{(n-1)})\Big)\,d\mathbb{W}_s,\\
&\nabla\cdot v^{(n)} = 0,\\
&v^{(n)}(0) = 0 \ \text{ a.s.}
\end{aligned}$$

We rewrite the first equation as

$$dv_j^{(n)} - \Delta v_j^{(n)}\,dt = \sum_i \partial_i f_{ij}\,dt + g_j\,d\mathbb{W}_t, \qquad j = 1,2,3,$$

where

$$\begin{aligned}
f_{ij} &= -\varphi^{(n+1)}\varphi^{(n)}\big(\mathcal{P}(u_i^{(n)}u^{(n)})\big)_j + \varphi^{(n)}\varphi^{(n-1)}\big(\mathcal{P}(u_i^{(n-1)}u^{(n-1)})\big)_j\\
&= -\varphi^{(n)}(\varphi^{(n+1)} - \varphi^{(n)})(\mathcal{P}(u_i^{(n)}u^{(n)}))_j - \varphi^{(n)}(\varphi^{(n)} - \varphi^{(n-1)})(\mathcal{P}(u_i^{(n)}u^{(n)}))_j\\
&\quad - \varphi^{(n)}\varphi^{(n-1)}(\mathcal{P}(v_i^{(n-1)}u^{(n)}))_j - \varphi^{(n)}\varphi^{(n-1)}(\mathcal{P}(u_i^{(n-1)}v^{(n-1)}))_j\\
&= f_{ij}^{(1)} + f_{ij}^{(2)} + f_{ij}^{(3)} + f_{ij}^{(4)}
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
g_j &= \varphi^{(n+1)}\varphi^{(n)}\sigma_j(u^{(n)}) - \varphi^{(n)}\varphi^{(n-1)}\sigma_j(u^{(n-1)})\\
&= \varphi^{(n)}(\varphi^{(n+1)} - \varphi^{(n)})\sigma_j(u^{(n)}) + \varphi^{(n)}(\varphi^{(n)} - \varphi^{(n-1)})\sigma_j(u^{(n)})\\
&\qquad + \varphi^{(n)}\varphi^{(n-1)}\big(\sigma_j(u^{(n)}) - \sigma_j(u^{(n-1)})\big)\\
&= g_j^{(1)} + g_j^{(2)} + g_j^{(3)}.
\end{aligned} \tag{5.20}$$

We first apply (4.25) to all the terms on the far right side of (5.19). Now, choose the exponents q, r, and l as in (5.11)–(5.14). Regarding the first term in (5.19), we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T \|f^{(1)}\|_q^p\,ds\right] &\le C\sum_i \mathbb{E}\left[\int_0^T (\varphi^{(n+1)}-\varphi^{(n)})^p(\varphi^{(n)})^p\|u_i^{(n)}u^{(n)}\|_q^p\,ds\right]\\ &\le \frac{C}{\delta_0^p}\mathbb{E}\left[\int_0^T \|v^{(n)}\|_p^p(\varphi^{(n)})^p\|u^{(n)}\|_r^p\|u^{(n)}\|_l^p\,ds\right]\\ &\le C\delta_0^p\mathbb{E}\left[\int_0^T \|v^{(n)}\|_p^p\,ds\right]\end{aligned} \tag{5.21}$$

by

$$|\varphi^{(n+1)}-\varphi^{(n)}| \le \frac{C}{\delta_0}\Big|\|u^{(n+1)}\|_p - \|u^{(n)}\|_p\Big| \le \frac{C}{\delta_0}\|u^{(n+1)}-u^{(n)}\|_p = \frac{C}{\delta_0}\|v^{(n)}\|_p,$$

as in (5.17), and where we also used (5.12) and (5.13) in the last inequality in (5.21). As in (5.21), we have

$$\mathbb{E}\left[\int_0^T \|f^{(2)}\|_q^p\,ds\right] \le C\delta_0^p\mathbb{E}\left[\int_0^T \|v^{(n-1)}\|_p^p\,ds\right]. \tag{5.22}$$

Similarly,

$$\mathbb{E}\left[\int_0^T \|f^{(3)}\|_q^p\,ds\right] + \mathbb{E}\left[\int_0^T \|f^{(4)}\|_q^p\,ds\right] \le C\delta_0^p\mathbb{E}\left[\int_0^T \|v^{(n-1)}\|_p^p\,ds\right]. \tag{5.23}$$

Summarizing (5.21), (5.22), and (5.23), we get

$$\mathbb{E}\left[\int_0^T \|f\|_q^p\,ds\right] \le C_{\delta_0}\,T\,\mathbb{E}\left[\sup_{s\in[0,T]}\|v^{(n-1)}\|_p^p\right] + C_{\delta_0}\,T\,\mathbb{E}\left[\sup_{s\in[0,T]}\|v^{(n)}\|_p^p\right].$$

Next, we turn to the three terms in (5.20). For the first one, we have

$$\begin{aligned}&C\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}\|g^{(1)}(s,x)\|_{l^2(\mathcal{H},\mathbb{R}^d)}^p\,dxds\right]\\ &\quad\le C\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}(\varphi^{(n)})^p(\varphi^{(n+1)}-\varphi^{(n)})^p\|\sigma(u^{(n)})\|_{l^2(\mathcal{H},\mathbb{R}^d)}^p\,dxds\right]\\ &\quad\le \frac{C}{\delta_0^p}\mathbb{E}\left[\int_0^T(\varphi^{(n)})^p\|v^{(n)}\|_p^p(\|u^{(n)}\|_p^p+1)\,ds\right]\\ &\quad\le C_{\delta_0}\mathbb{E}\left[\int_0^T\|v^{(n)}\|_p^p\,ds\right],\end{aligned} \tag{5.24}$$

and similarly,

$$
\begin{aligned}
&C\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}\|g^{(2)}(s,x)\|^p_{l^2(\mathcal{H},\mathbb{R}^d)}\,dxds\right]+C\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}\|g^{(3)}(s,x)\|^p_{l^2(\mathcal{H},\mathbb{R}^d)}\,dxds\right]\\
&\quad\le C_{\delta_0}\mathbb{E}\left[\int_0^T\|v^{(n-1)}\|_p^p\,ds\right].
\end{aligned}
\tag{5.25}
$$

We may summarize (5.24) and (5.25) as

$$
C\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^d}\|g(s,x)\|^p_{l^2(\mathcal{H},\mathbb{R}^d)}\,dxds\right]\le C_{\delta_0}\,T\,\mathbb{E}\left[\sup_{s\in[0,T]}\|v^{(n-1)}\|_p^p\right]+C_{\delta_0}\,T\,\mathbb{E}\left[\sup_{s\in[0,T]}\|v^{(n)}\|_p^p\right].
$$

Therefore, we obtain the existence of a fixed point u of (5.1) in $L^p_\omega L^\infty_t L^p_x$ on $[0,t^*]$, where $t^*\in(0,T)$ is a sufficiently small constant. Since each $u^{(n)}\in L^p(\Omega;C([0,t^*],L^p))$, so is u. By Lemma 5.5,

$$
\begin{aligned}
(u^{(n)}(s),\phi)&=(u_0,\phi)+\int_0^s(u^{(n)}(r),\Delta\phi)\,dr\\
&\quad+\sum_j\int_0^s\big(\varphi^{(n)}\varphi^{(n-1)}\mathcal{P}\big(u_j^{(n-1)}u^{(n-1)}\big),\partial_j\phi\big)\,dr\\
&\quad+\int_0^s(\varphi^{(n)}\varphi^{(n-1)}\sigma(u^{(n-1)}),\phi)\,d\mathbb{W}_r,\qquad(s,\omega)\text{-a.e.},
\end{aligned}
$$

for all $\phi\in C^\infty(\mathbb{T}^3)$. The exponential convergence rate and Remark 5.3 imply that $\varphi(\|u^{(n)}(s)\|_p)$, $\varphi(\|u^{(n-1)}(s)\|_p)\to\varphi(\|u(s)\|_p)$ for a.e. (s,ω). Together with the divergence free condition, the Hölder inequality, and the dominated convergence theorem, we get

$$
\begin{aligned}
&\int_0^s(u^{(n)}(r),\Delta\phi)\,dr+\sum_j\int_0^s(\varphi^{(n)}\varphi^{(n-1)}\mathcal{P}(u_j^{(n-1)}u^{(n-1)}),\partial_j\phi)\,dr\\
&\quad\to\int_0^s\big((u(r),\Delta\phi)+(\varphi^2\mathcal{P}(uu_j),\phi)\big)\,dr
\end{aligned}
$$

for a.e. (s,ω) as $n\to\infty$. Also, by the BDG inequality and assumptions on σ,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t^*)}\left|\int_0^s(\varphi^{(n)}\varphi^{(n-1)}\sigma(u^{(n-1)})-\varphi^2\sigma(u),\phi)\,d\mathbb{W}_r\right|\right]\\
&\quad\le\mathbb{E}\left[\left(\int_0^{t^*}\|(\varphi^{(n)}\varphi^{(n-1)}\sigma(u^{(n-1)})-\varphi^2\sigma(u),\phi)\|_{l^2}^2\,dr\right)^{1/2}\right].
\end{aligned}
$$

Moreover, the right side goes to zero exponentially fast as $n \to \infty$. This implies

$$\int_0^s (\varphi^{(n)}\varphi^{(n-1)}\sigma(u^{(n-1)}), \phi)\, d\mathbb{W}_r \xrightarrow{n\to\infty} \int_0^s (\varphi^2\sigma(u), \phi)\, d\mathbb{W}_r, \qquad (s,\omega)\text{-a.e.}$$

Letting $n \to \infty$, we obtain that u solves (5.1). On the other hand, the inequality (5.2) follows by using Lemma 4.4 on (5.18). This completes the existence.

To prove the uniqueness, suppose that (5.1) has two strong solutions $u, v \in L^p(\Omega; C([0,t^*], L^p))$. Then $w = u - v$ satisfies

$$\begin{aligned}
&dw - \Delta w\, dt = -\big(\varphi_u^2 \mathcal{P}\big((u\cdot\nabla)u\big) - \varphi_v^2 \mathcal{P}\big((v\cdot\nabla)v\big)\big)\, dt + \big(\varphi_u^2\sigma(u) - \varphi_v^2\sigma(v)\big)\, d\mathbb{W}_t,\\
&\nabla\cdot w = 0,\\
&w(0) = 0, \qquad \text{a.s.}
\end{aligned}$$

on $(0,t^*] \times \mathbb{T}^3$, where $\varphi_v = \varphi(\|v\|_p)$ and $\varphi_u = \varphi(\|u\|_p)$. As before, we write the first equation component-wise as

$$dw_j - \Delta w_j\, dt = \sum_i \partial_i f_{ij}\, dt + g_j\, d\mathbb{W}_t, \qquad j = 1,2,3,$$

where

$$\begin{aligned}
f_{ij} &= -\varphi_u^2\big(\mathcal{P}(u_i u)\big)_j + \varphi_v^2\big(\mathcal{P}(v_i v)\big)_j\\
&= -\varphi_u(\varphi_u - \varphi_v)(\mathcal{P}(u_i u))_j - \varphi_u\varphi_v(\mathcal{P}(w_i u))_j\\
&\quad - \varphi_u\varphi_v(\mathcal{P}(v_i w))_j - \varphi_v(\varphi_u - \varphi_v)(\mathcal{P}(v_i v))_j
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
g_j &= \varphi_u^2\sigma_j(u) - \varphi_v^2\sigma_j(v)\\
&= \varphi_u(\varphi_u - \varphi_v)\sigma_j(u) + \varphi_v(\varphi_u - \varphi_v)\sigma_j(v) + \varphi_u\varphi_v\big(\sigma_j(u) - \sigma_j(v)\big).
\end{aligned} \tag{5.27}$$

We now show similarly as above that

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t^*]} \|w\|_p^p\right] &\le C\mathbb{E}\left[\int_0^{t^*} \|f\|_q^p\, ds\right] + C\mathbb{E}\left[\int_0^{t^*}\int_{\mathbb{T}^d} \|g(s,x)\|^p_{l^2(\mathcal{H},\mathbb{R}^d)}\, dxds\right]\\
&\le C_{\delta_0}\, t^*\, \mathbb{E}\left[\sup_{s\in[0,t^*]} \|w\|_p^p\right],
\end{aligned}$$

and obtain the pathwise uniqueness by setting t^* sufficiently small. Thus, we have obtained a unique strong solution of (5.1) in $L^p(\Omega; C([0,t^*], L^p))$.

Now, we turn to the global existence. First, note that the deterministic time $t^* > 0$ from above does not depend on the initial data. Now, let $T > 0$ be arbitrary and let n^* be a positive integer such that $T/n^* < t^*$. Denote $t_i = iT/n^*$ for

$i \in \{0, 1, \ldots, n^*\}$. Applying the existence and pathwise uniqueness inductively on $[t_i, t_{i+1}]$, $i \in \{0, 1, \ldots, n^*\}$, we obtain a unique strong solution to (5.1) on $[0, T]$ and (5.2) holds.

Proof of Theorem 3.1 For $n = 1, 2, \ldots$, denote by $u^{(n)}$ the solution of the truncated SNSE (5.1) with $\delta_0 = n$. Also, introduce the stopping times

$$\tau_n(\omega) = \begin{cases} \inf\left\{t > 0 : \|u^{(n)}(t, \omega)\|_p \geq n/2\right\}, & \text{if } \|u^{(n)}(0, \omega)\|_p < n/2, \\ 0, & \text{if } \|u^{(n)}(0, \omega)\|_p \geq n/2. \end{cases}$$

By uniqueness, the sequence is non-decreasing a.s. and $u^{(m)} = u^{(n)}$ on $[0, \tau_m \wedge \tau_n]$. Let $\tau = \lim_n \tau_n \wedge T$. Then, $\mathbb{P}(\tau > 0) = 1$. Also, for any integer $n \in \mathbb{N}$, define $u = u^{(n)}$ on $[0, \tau_n \wedge T]$. It is easy to check that (u, τ) satisfies all the required properties.

6 Global solutions and energy decay

The truncated stochastic Navier–Stokes equations reads

$$\begin{aligned} du(t,x) &= \Delta u(t,x)\,dt - \varphi(\|u(t)\|_p)^2 \mathcal{P}\big((u(t,x)\cdot\nabla)u(t,x)\big)\,dt \\ &\qquad + \varphi(\|u(t)\|_p)^2 \sigma(u(t,x))d\mathbb{W}_t, \\ \nabla\cdot u(t,x) &= 0, \\ u(0,x) &= u_0(x) \text{ a.s.}, \qquad x \in \mathbb{T}^3 \end{aligned} \tag{6.1}$$

on $[0, \infty) \times \mathbb{T}^3$ with $\operatorname{div} u_0 = 0$ and $\int_{\mathbb{T}^3} u_0\,dx = 0$ a.s. Note that in the previous section, we have proved the global well-posedness of this initial value problem. Recall that $\delta_0 > 0$ and that $\varphi\colon [0, \infty) \to [0, 1]$ is a decreasing smooth function with $\varphi \equiv 1$ on $[0, \delta_0/2]$ and $\varphi \equiv 0$ on $[\delta_0, \infty)$. In addition, we assumed

$$|\varphi(t_1) - \varphi(t_2)| \leq \frac{C}{\delta_0}|t_1 - t_2|, \qquad t_1, t_2 \geq 0.$$

We shall set $\delta_0 > 0$ sufficiently small. Note that when $\|u\|_p$ is below $\delta_0/2$, the initial value problem (1.1)–(1.3) coincides with this cut-off model. Hence, an estimate of the likelihood that $\|u\|_p$ exceeds $\delta_0/2$ determines the time of existence for the solution to (1.1)–(1.3). The next result is essential for estimating that likelihood.

Theorem 6.1 *Let $p > 5$. Then the global solution $u \in L^p(\Omega; C([0, \infty), L^p))$ to (6.1) satisfies*

$$\mathbb{E}\left[\sup_{s\in[0,\infty)} e^{as}\|u(s)\|_p^p + \int_0^\infty e^{as}\sum_i \|\nabla(|u_i(s)|^{p/2})\|_2^2\,ds\right] \leq C\mathbb{E}[\|u_0\|_p^p], \tag{6.2}$$

provided $a, \delta_0, \epsilon_0 > 0$ are sufficiently small constants.

Recall that the constant $\epsilon_0 > 0$ is in the condition (3.5).

Proof Let $T > 0$. Applying the Itô-Wentzel formula to $F_i(t) = e^{at}\|u_i(t)\|_p^p$, for a fixed $i \in \{1, 2, 3\}$, we obtain

$$d(e^{at}\|u_i(t)\|_p^p) = ae^{at}\|u_i(t)\|_p^p\,dt + e^{at}d(\|u_i(t)\|_p^p). \tag{6.3}$$

Utilizing the Itô expansion in the proof of Theorem 4.1 (cf. (4.12)) and (6.3), we have

$$\begin{aligned} &e^{at}\|u_i(t)\|_p^p + \frac{4(p-1)}{p}\int_0^t e^{as}\int_{\mathbb{T}^3}|\nabla(|u_i(s)|^{p/2})|^2\,dxds \\ &\quad = \|u_{0i}\|_p^p - p\int_0^t e^{as}\varphi^2\int_{\mathbb{T}^3}|u_i|^{p-2}u_i(\mathcal{P}(u\cdot\nabla)u)_i\,dxds \\ &\qquad + p\int_0^t e^{as}\varphi^2\int_{\mathbb{T}^3}|u_i|^{p-2}u_i\sigma_i(u)\,dxd\mathbb{W}_s \\ &\qquad + \frac{p(p-1)}{2}\int_0^t e^{as}\varphi^4\int_{\mathbb{T}^3}|u_i|^{p-2}\|\sigma_i(u)\|_{l^2}^2\,dxds + a\int_0^t e^{as}\|u_i(s)\|_p^p\,ds. \end{aligned} \tag{6.4}$$

Now, choose q, r, and l as in (5.11)–(5.14) and $\bar{r}$ as in (4.18). By integration by parts, we have

$$\begin{aligned} &pe^{as}\varphi^2\left|\int_{\mathbb{T}^3}|u_i(s)|^{p-2}u_i(s)(\mathcal{P}(u\cdot\nabla)u)_i\,dx\right| \\ &\quad = pe^{as}\varphi^2\left|\sum_j\int_{\mathbb{T}^3}\partial_j(|u_i(s)|^{p-2}u_i(s))(\mathcal{P}(u_ju))_i\,dx\right| \\ &\quad \le Ce^{as}\varphi^2\|\nabla(|u_i(s)|^{p/2})\|_2\||u_i|^{(p-2)/2}\|_{\bar{r}}\|u_j\|_r\|u\|_l \\ &\quad \le C\delta_0 e^{as}\varphi\|\nabla(|u_i(s)|^{p/2})\|_2\||u_i|^{(p-2)/2}\|_{\bar{r}}\|u_j\|_p, \end{aligned}$$

using $\varphi\|u\|_p \le \delta_0$ in the last step. As in the proof of Theorem 4.1 above, we get

$$\begin{aligned} &\left|\int_0^t pe^{as}\varphi^2\int_{\mathbb{T}^3}|u_i(s)|^{p-2}u_i(s)(\mathcal{P}(u\cdot\nabla)u)_i\,dxds\right| \\ &\quad \le \delta\int_0^t e^{as}\sum_i\|\nabla(|u_i(t)|^{p/2})\|_2^2\,ds + C_\delta\delta_0^\kappa\int_0^t e^{as}\|u(s)\|_p^p\,ds, \end{aligned} \tag{6.5}$$

where $\delta > 0$ is arbitrary and where $\kappa > 0$ is a constant depending on p. Note that the first term on the right side may be absorbed in the dissipative term if $\delta > 0$ is sufficiently small. Also, by using the Poincaré type inequality

$$\||v|^{p/2}\|_2 \le C\|\nabla(|v|^{p/2})\|_2, \tag{6.6}$$

for v such that $\int_{\mathbb{T}^d} v\,dx = 0$, as in (4.15), the second term in (6.5) may also be absorbed if $\delta_0 > 0$ is sufficiently small. Regarding the fourth term in (6.4), we use the superlinearity assumption on the noise (3.5) and obtain

$$\frac{p(p-1)}{2} e^{as} \varphi^4 \int_{\mathbb{T}^3} |u_i(s)|^{p-2} \|\sigma_i(u)\|_{l^2}^2\, dx \leq C\epsilon_0^2 e^{as} \varphi^4 \|u(s)\|_p^p.$$

This term can be controlled in the same way as the last term in (6.5). Likewise, the last term in (6.4) may also be absorbed in the dissipative part if $a > 0$ is sufficiently small constant (independent of p). Combining the estimates above and absorbing the second, fourth, and fifth terms on the right-hand side of (6.4), we arrive at

$$\begin{aligned}
&\frac{1}{2} e^{at} \|u_i(t)\|_p^p + \frac{1}{2} \int_0^t e^{as} \|\nabla(|u_i(s)|^{p/2})\|_2^2\, ds \\
&\quad \leq \|u_i(0)\|_p^p + p \left| \int_0^t \int_{\mathbb{T}^3} e^{as} \varphi^2 |u_i(s)|^{p-2} u_i(s) \sigma_i(u)\, dx d\mathbb{W}_s \right|
\end{aligned}$$

since $4(p-1)/p \geq 1/2$. Hence,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} e^{at} \|u_i(t)\|_p^p \right] + \mathbb{E}\left[\int_0^T e^{as} \|\nabla(|u_i(s)|^{p/2})\|_2^2\, ds \right] \\
&\quad \leq 2\mathbb{E}\left[\|u_i(0)\|_p^p \right] + C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t e^{as} \varphi^2 \int_{\mathbb{T}^3} |u_i(s)|^{p-2} u_i(s) \sigma_i(u)\, dx d\mathbb{W}_s \right| \right].
\end{aligned} \tag{6.7}$$

For the last term in (6.7), we apply the same approach as in (4.21)–(4.22), except that we use the assumption (3.5). We thus obtain

$$C_p \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t e^{as} \varphi^2 \int_{\mathbb{T}^3} |u_i(s)|^{p-2} u_i(s) \sigma_i(u)\, dx d\mathbb{W}_s \right| \right] \leq C_\delta \epsilon_0 \mathbb{E}\left[\int_0^T e^{as} \varphi^2 \|u(s)\|_p^p ds \right].$$

Using also (6.6), by taking $\epsilon_0 > 0$ sufficiently small, the right-hand side may be absorbed in the left side of (6.7). Therefore,

$$\mathbb{E}\left[\sup_{s\in[0,T]} e^{as} \|u(s)\|_p^p + \int_0^T e^{as} \sum_i \|\nabla(|u_i(s)|^{p/2})\|_2^2\, ds \right] \leq C\mathbb{E}[\|u_0\|_p^p],$$

and (6.2) follows upon sending $T \to \infty$.

Now, we are ready to prove the main theorem on the global existence of solutions for small data.

Proof of Theorem 3.2 Let $\epsilon_0, \delta_0, a > 0$ be as in Theorem 6.1. Assume that (3.6) holds for some $\delta > 0$. By Markov's inequality, we have

$$\mathbb{P}\left(\sup_{t\in[0,\infty)} e^{at}\|u(t)\|_p^p \geq \frac{\delta_0}{2}\right) \leq \frac{C}{\delta_0}\mathbb{E}[\|u_0\|_p^p] \leq \frac{C\delta}{\delta_0}.$$

The assertion is then obtained by choosing $\delta > 0$ sufficiently small.

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