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On the energy transfer to high frequencies in the damped/driven nonlinear Schrödinger equation

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Abstract
We consider a damped/driven nonlinear Schrödinger equation in $\mathbb{R}^n$, where n is arbitrary, $\mathbb{E}u_t - \nu\Delta u + i|u|^2u = \sqrt{\nu}\eta(t,x), \quad \nu > 0$, under odd periodic boundary conditions. Here $\eta(t,x)$ is a random force which is white in time and smooth in space. It is known that the Sobolev norms of solutions satisfy $\|u(t)\|_m^2 \le C\nu^{-m}$, uniformly in $t \ge 0$ and $\nu > 0$. In this work we prove that for small $\nu > 0$ and any initial data, with large probability the Sobolev norms $\|u(t,\cdot)\|_m$ with $m > 2$ become large at least to the order of $\nu^{-\kappa_{n,m}}$ with $\kappa_{n,m} > 0$, on time intervals of order $\mathcal{O}(\frac{1}{\nu})$. It proves that solutions of the equation develop short space-scale of order ν to a positive degree, and rigorously establishes the (direct) cascade of energy for the equation.

Keywords NLS · Sobolev norms · Energy cascading

1 Introduction

In this work we study a damped/driven nonlinear Schrödinger equation

$$u_t - \nu\Delta u + i|u|^2u = \sqrt{\nu}\eta(t,x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

i.e. a CGL equation without linear dispersion, with cubic Hamiltonian nonlinearity and a random forcing. The dimension n is any, $0 < \nu \le 1$ is the viscosity constant

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and the random force η is white in time t and regular in x. The equation is considered under the odd periodic boundary conditions,

$$u(t,\ldots,x_j,\ldots)=u(t,\ldots,x_j+2\pi,\ldots)=-u(t,\ldots,x_j+\pi,\ldots),\quad j=1,\ldots,n.$$

The latter implies that u vanishes on the boundary of the cube of half-periods $K^n = [0,\pi]^n$,

$$u\mid_{\partial K^n}=0.$$

We denote by $\{\varphi_d(\cdot),\ d=(d_1,\ldots,d_n)\in\mathbb{N}^n\}$ the trigonometric basis in the space of odd periodic functions,

$$\varphi_d(x)=(\tfrac{2}{\pi})^{\frac{n}{2}}\sin(d_1x_1)\cdots\sin(d_nx_n).$$

The basis is orthonormal with respect to the normalised scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ in $L_2(K^n,\pi^{-n}dx)$,

$$\langle\!\langle u,v\rangle\!\rangle=\int_{K^n}\langle u(x),v(x)\rangle\pi^{-n}dx, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ is the real scalar product in $\mathbb{C}$, $\langle u,v\rangle=\Re u\bar{v}$. It is formed by eigenfunctions of the Laplacian:

$$(-\Delta)\varphi_d=|d|^2\varphi_d.$$

The force $\eta(t,x)$ is a random field of the form

$$\eta(t,x)=\frac{\partial}{\partial t}\xi(t,x),\quad \xi(t,x)=\sum_{d\in\mathbb{N}^n}b_d\beta_d(t)\varphi_d(x). \tag{1.3}$$

Here $\beta_d(t)=\beta_d^R(t)+i\beta_d^I(t)$, where $\beta_d^R(t),\beta_d^I(t)$ are independent real-valued standard Brownian motions, defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $\{\mathcal{F}_t;\, t\geq 0\}$. The set of real numbers $\{b_d,\ d\in\mathbb{N}^n\}$ is assumed to form a non-zero sequence, satisfying

$$0<B_{m_*}<\infty,\quad m_*=\min\{m\in\mathbb{Z}: m>n/2\}, \tag{1.4}$$

where for a real number k we set

$$B_k:=\sum_{d\in\mathbb{N}^n}|d|^{2k}|b_d|^2\leq\infty.$$

For $m\geq 0$ we denote by H^m the Sobolev space of order m, formed by complex odd periodic functions, equipped with the homogeneous norm,

$$\|u\|_m = \|(-\Delta)^{\frac{m}{2}} u\|_0,$$

where $\|\cdot\|_0$ is the L^2-norm on K^n, $\|u\|_0^2 = \langle\langle u, u\rangle\rangle$ (see (1.2)). If we write $u \in H^m$ as Fourier series, $u(x) = \sum_{d\in\mathbb{N}^n} u_d\varphi_d(x)$, then $\|u\|_m^2 = \sum_{d\in\mathbb{N}^n} |d|^{2m}|u_d|^2$.

Equation (1.1) with small ν belongs to a group of equations, describing turbulence in the CGL equations. These equations have got quite a lot of attention in physical literature as models for turbulence in various media, e.g. see [3, Chapter 5]. In particular – as a natural model for hydrodynamical turbulence since Eq. (1.1) is obtained from the Navier-Stokes system by replacing the Euler term $(u \cdot \nabla)u$, which is a quadratic Hamiltonian nonlinearity, by $i|u|^2u$, which is a cubic Hamiltonian nonlinearity, see [13].

The global solvability of Eq. (1.1) for any space dimension n is established in [8,10]. It is proved there that if

$$u(0, x) = u_0(x), \tag{1.5}$$

where $u_0 \in H^m \cap C(K^n)$, $m \in \mathbb{N}$, and if $B_m < \infty$, then the problem (1.1), (1.5) has a unique strong solution $u(t, x)$ in H^m which we write as $u(t, x; u_0)$, or $u(t; u_0)$, or $u_\nu(t; u_0)$. Its norm satisfies

$$\mathbb{E}\|u(t; u_0)\|_m^2 \leq C_m\nu^{-m}, \quad t \geq 0,$$

where C_m depends on $\|u_0\|_m$, $|u_0|_\infty$ and B_m, B_{m_*}. Furthermore, denoting by $C_0(K^n)$ the space of continuous complex functions on K^n, vanishing at ∂K^n, we have that the solutions $u(t, x)$ define a Markov process in $C_0(K^n)$. Moreover, if the noise $\eta(t, \cdot)$ is non-degenerate in the sense that in (1.3) all coefficients b_d are non-zero, then this process is mixing.[1]

Our goal is to study the growth of higher Sobolev norms for solutions of Eq. (1.1) as $\nu \to 0$ on time intervals of order $\mathcal{O}(\frac{1}{\nu})$. The main result of this work is the following.

Theorem 1 *For any real number $m > 2$, in addition to (1.4), assume that $B_m < \infty$. Then there exists $\kappa_{n,m} > 0$ such that for every fixed quadruple $(\delta, \kappa, \mathscr{K}, T_0)$, where*

$$\kappa \in (0, \kappa_{n,m}), \quad \delta \in (0, \tfrac{1}{8}), \quad \mathscr{K}, T_0 > 0,$$

there exists a $\nu_0 > 0$ with the property that if $0 < \nu \leq \nu_0$, then for every $u_0 \in H^m \cap C_0(K^n)$, satisfying

$$|u_0|_\infty \leq \mathscr{K}, \quad \|u_0\|_m \leq \nu^{-\kappa m}, \tag{1.6}$$

the solution $u(t, x; u_0)$ is such that

[1] We note that solutions of eqs. (1.1) with complex ν behave differently, and solubility of those equations with large n is unknown.

(1)
$$\mathbb{P}\Big\{\sup_{t\in[t_0,t_0+T_0\nu^{-1}]}\|u_\nu^\omega(t)\|_m > \nu^{-m\kappa}\Big\}\ge 1-\delta,\quad \forall\, t_0\ge 0.$$

(2) If m is an integer, $m\ge 3$, then a possible choice of $\kappa_{n,m}$ is $\kappa_{n,m}=\frac{1}{35}$, and there exists $C\ge 1$, depending on $\kappa<\frac{1}{35}$, $\mathscr{K}$, m, B_{m_*} and B_m, such that

$$C^{-1}\nu^{-2m\kappa+1}\le \mathbb{E}\left(\nu\int_{t_0}^{t_0+\nu^{-1}}\|u_\nu(s)\|_m^2 ds\right)\le C\nu^{-m},\quad \forall\, t_0\ge 0. \tag{1.7}$$

A similar result holds for the classical C^k-norms of solutions:

Proposition 2 *For any integer $m\ge 2$ in addition to (1.4) assume that $B_m<\infty$. Then for every fixed triplet $K, \mathcal{K}, T_0>0$ and any $0<\kappa<1/16$ we have*

$$\mathbb{P}\left\{\sup_{t\in[t_0,t_0+T_0\nu^{-1}]}|u_\nu^\omega(t;u_0)|_{C^m} > K\nu^{-m\kappa}\right\}\to 1 \quad as \quad \nu\to 0, \tag{1.8}$$

for each $t_0\ge 0$, if u_0 satisfies $|u_0|_\infty\le\mathcal{K}$, $|u_0|_{C^m}\le\nu^{-\kappa m}$. The rate of convergence depends only on the triplet and κ.

For a proof of this result see the extended version of our work [6]. Due to (1.8), for any $m>2+n/2$ we have

$$\mathbb{P}\left\{\sup_{T_0\le t\le t_0+T_0\nu^{-1}}\|u(t)\|_m\ge K\nu^{-\lfloor m-\frac{n}{2}\rfloor\kappa}\right\}\to 1 \quad \text{as} \quad \nu\to 0,$$

for every $K>0$ and $0<\kappa<1/16$, where for $a\in\mathbb{R}$ we denote $\lfloor a\rfloor=\max\{n\in\mathbb{Z}: n<a\}$. This improves the first assertion of Theorem 1 for large m.

We have the following two corollaries from Theorem 1, valid if the Markov process defined by the Eq. (1.1) is mixing:

Corollary 3 *Assume that $B_m<\infty$ for all m and $b_d\neq 0$ for all d. Then Eq. (1.1) is mixing and for any $\kappa<1/35$ and $0<\nu\le\nu_0$ its unique stationary measure μ_ν satisfies*

$$C^{-1}\nu^{-2m\kappa+1}\le\int\|u\|_m^2\mu_\nu(du)\le C\nu^{-m},\quad 3\le m\in\mathbb{N}. \tag{1.9}$$

Here C and ν_0 are as in Theorem 1.

Corollary 4 *Under the assumptions of Corollary 3, for any $u_0\in C^\infty$ we have*

$$\tfrac{1}{2}C^{-1}\nu^{-2m\kappa+1}\le\mathbb{E}\|u(s;u_0)\|_m^2\le 2C\nu^{-m},\quad 3\le m\in\mathbb{N},$$

if $s\ge T(\nu,u_0,\kappa,B_m,B_{m_*})$, where C is the same as in (1.9).

Theorem 1 rigorously establishes the energy cascade to high frequencies for solutions of Eq. (1.1) with small ν. Indeed, if $u_0(x)$ and $\eta(t,x)$ are smooth functions of x (or even trigonometric polynomials of x), then in view of (1.7) for $0<\nu\ll 1$ and $t\gtrsim \nu^{-1}$ a substantial part of the energy $\frac{1}{2}\sum|u_d(t)|^2$ of a solution $u(t,x;u_0)$ is carried by high modes u_d with $|d|\gg 1$. Relation (1.7) (valid for all integer $m\geq 3$) also means that the averaged in time space-scale l_x of solutions for (1.1) satisfies $l_x\in[\nu^{1/2},\nu^{1/35}]$, and goes to zero with ν (see [1,9]). We recall that the energy cascade to high frequencies and formation of short space-scale is the driving force of the Kolmogorov theory of turbulence, see [5].

We mention that in the work [12] the stochastic CGL equation

$$u_t-(\nu+i)\Delta u+i|u|^2u=\sqrt{\nu}\eta(t,x),\quad 0<\nu\leq 1, \tag{1.10}$$

with linear dispersion and white in time random force η as in (1.3) was considered under the odd periodic boundary conditions, and the inviscid limiting dynamics as $\nu\to 0$ was examined. However, since the limiting Eq. $(1.10)_{\nu=0}$ is a regular PDE in difference with the Eq. $(1.1)_{\nu=0}$, the results on the inviscid limit in [12] differ in spirit from those in our work, and we do not discuss them now.

Deterministic versions of the result of Theorem 1 for Eq. (1.1) with $\eta=0$, where ν is a small non-zero complex number such that $\Re\nu\geq 0$ and $\Im\nu\leq 0$ are known, see [9]. In particular, if ν is a positive real number and u_0 is a smooth function of order one, then for any integer $m\geq 4$ a solution $u_\nu(t,x;u_0)$ satisfies estimates (1.7) with the averaging $\nu\mathbb{E}\int_t^{t+\nu^{-1}}\dots ds$ replaced by $\nu^{1/3}\int_0^{\nu^{-1/3}}\dots ds$, with the same upper bound and with the lower bound $C_m\nu^{-\kappa_m m}$, where $\kappa_m\to 1/3$ as $m\to\infty$. Moreover, it was then shown in [2] that the lower bounds remain true with $\kappa=1/3$, and that the estimates $\sup_{t\in[0,|\nu|^{-1/3}]}\|u(t)\|_{C^m}\geq C_m|\nu|^{-m/3}$, $m\geq 2$, hold for smooth solutions of Eq. (1.1) with $\eta=0$ and any non-zero complex "viscosity" ν.

The better quality of the lower bounds for solutions of the deterministic equations is due to an extra difficulty which occurs in the stochastic case: when time grows, simultaneously with increasing of high Sobolev norms of a solution, its L_2-norm may decrease, which accordingly would weaken the mechanism, responding for the energy transfer to high modes. Significant part of the proof of Theorem 1 is devoted to demonstration that the L_2-norm of a solution cannot go down without sending up the second Sobolev norm.

If $\eta=0$ and $\nu=i\delta\in i\mathbb{R}$, then (1.1) is a Hamiltonian PDE (the defocusing Schrödinger equation), and the L_2-norm is its integral of motion. If this integral is of order one, then the results of [9] (see there Appendix 3) imply that at some point of each time-interval of order $\delta^{-1/3}$ the C^m-norm of a corresponding solution will become $\gtrsim\delta^{-m\kappa}$ if $m\geq 2$, for any $\kappa<1/3$. Furthermore, if $n=2$ and $\delta=1$, then due to [4] for $m>1$ and any $M>1$ there exists a $T=T(m,M)$ and a smooth $u_0(x)$ such that $\|u_0\|_m<M^{-1}$ and $\|u(T;u_0)\|_m>M$.

The paper is organized as follows. In Sect. 2, we recall the results from [8,10] on solutions of the Eq. (1.1). Next we show in Sect. 3 that if the noise η is non-degenerate, the L^2-norm of a solution of Eq. (1.1) cannot stay too small on time intervals of order $\mathcal{O}(\frac{1}{\nu})$ with high probability, unless its H^2-norm gets very large (see Lemma 12). Then in Sect. 4 we derive from this fact the assertion (1) of Theorem 1. We prove assertion (2) and both corollaries in Sect. 5.

Constants in estimates never depend on ν, unless otherwise stated. For a metric space M we denote by $\mathcal{B}(M)$ the Borel σ-algebra on M, and by $\mathcal{P}(M)$ – the space of probability Borel measures on M. By $\mathcal{D}(\xi)$ we denote the law of a r.v. ξ, and by $|\cdot|_p$ – the norm in $L_p(K^n)$.

2 Solutions and estimates

Strong solutions for the Eq. (1.1) are defined in the usual way:

Definition 5 Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be the filtered probability space as in the introduction. Let u_0 in (1.5) be a r.v., measurable in $\mathcal{F}_0$ and independent from the Wiener process ξ (e.g., $u_0(x)$ may be a non-random function). Then a random process $u(t) = u(t; u_0) \in C_0(K^n)$, $t \in [0, T]$, adapted to the filtration, is called a strong solution of (1.1), (1.5), if

(1) a.s. its trajectories $u(t)$ belong to the space

$$\mathcal{H}([0,T]) := C([0,T], C_0(K^n)) \cap L^2([0,T], H^1);$$

(2) we have

$$u(t) = u_0 + \int_0^t (\nu\Delta u - i|u|^2 u)ds + \sqrt{\nu}\,\xi(t), \quad \forall t \in [0,T], \quad a.s.,$$

where both sides are regarded as elements of H^{-1}.

If (1)-(2) hold for every $T < \infty$, then $u(t)$ is a strong solution for $t \in [0, \infty)$. In this case a.s. $u \in C([0,\infty), C_0(K^n)) \cap L^2_{loc}([0,\infty), H^1)$.

Everywhere below when we talk about solutions for the problem (1.1), (1.5) we assume that the r.v. u_0 is as in the definition above.

The global well-posedness of Eq. (1.1) was established in [8,10]:

Theorem 6 *For any $u_0 \in C_0(K^n)$ the problem (1.1), (1.5) has a unique strong solution $u^\omega(t, x; u_0)$, $t \geq 0$. The family of solutions $\{u^\omega(t; u_0)\}$ defines in the space $C_0(K^n)$ a Fellerian Markov process.*

In [8,10] the theorem above is proved when (1.4) is replaced by the weaker assumption $B_* < \infty$, where $B_* = \sum |b_d|$ (note that $B_* \leq C_n B_{m_*}^{1/2}$).

The transition probability for the obtained Markov process in $C_0(K^n)$ is

$$P_t(u, \Gamma) = \mathbb{P}\{u(t; u) \in \Gamma\}, \quad u \in C_0(K^n),\ \Gamma \in \mathscr{B}(C_0(K^n)),$$

and the corresponding Markov semigroup in the space $\mathscr{P}(C_0(K^n))$ of Borel measures on $C_0(K^n)$ is formed by the operators $\{\mathcal{B}_t^*, t \geq 0\}$,

$$\mathcal{B}_t^* \mu(\Gamma) = \int_{C_0(K^n)} P_t(u, \Gamma)\mu(du), \quad t \in \mathbb{R}.$$

Then $\mathcal{B}_t^* \mu = \mathcal{D}u(t; u_0)$ if u_0 is a r.v., independent from ξ and such that $\mathcal{D}(u_0) = \mu$.

Introducing the slow time $\tau = \nu t$ and denoting $v(\tau, x) = u(\frac{\tau}{\nu}, x)$, we rewrite Eq. (1.1) in the following form, more convenient for some calculations:

$$\frac{\partial v}{\partial \tau} - \Delta v + i\nu^{-1}|v|^2 v = \tilde{\eta}(\tau, x), \tag{2.1}$$

where

$$\tilde{\eta}(\tau, x) = \frac{\partial}{\partial \tau}\tilde{\xi}(\tau, x), \quad \tilde{\xi}(\tau, x) = \sum_{d \in \mathbb{N}^n} b_d \tilde{\beta}_d(\tau)\varphi_d(x),$$

and $\tilde{\beta}_d(\tau) := \nu^{1/2}\beta_d(\tau\nu^{-1})$, $d \in \mathbb{N}^d$, is another set of independent standard complex Brownian motions.

Let $\Upsilon \in C^\infty(\mathbb{R})$ be any smooth function such

$$\Upsilon(r) = \begin{cases} 0, & \text{for } r \leq \frac{1}{4}; \\ r, & \text{for } r \geq \frac{1}{2}. \end{cases}$$

Writing $v \in \mathbb{C}$ in the polar form $v = re^{i\Phi}$, where $r = |v|$, and recalling that $\langle \cdot, \cdot \rangle$ stands for the real scalar product in $\mathbb{C}$, we apply Itô's formula to $\Upsilon(|v|)$ and obtain that the process $\Upsilon(\tau) := \Upsilon(|v(\tau)|)$ satisfies

$$\begin{aligned} \Upsilon(\tau) = \Upsilon_0 + \int_0^\tau \Big[&\Upsilon'(r)(\nabla r - r|\nabla\Phi|^2) \\ &+ \frac{1}{2}\sum_{d \in \mathbb{N}^n} b_d^2 \Big(\Upsilon''(r)\langle e^{i\Phi}, \varphi_d\rangle^2 + \Upsilon'(r)\frac{1}{r}(|\varphi_d|^2 - \langle e^{i\Phi}, \varphi_d\rangle^2)\Big)\Big]ds + \mathbb{W}(\tau), \end{aligned} \tag{2.2}$$

where $\Upsilon_0 = \Upsilon(|v(0)|)$ and $\mathbb{W}(\tau)$ is the stochastic integral

$$\mathbb{W}(\tau) = \sum_{d \in \mathbb{N}^n} \int_0^\tau \Upsilon'(r) b_d \varphi_d \langle e^{i\Phi}, d\tilde{\beta}_d(s)\rangle.$$

In [10] Eq. (2.1) is considered with $\nu = 1$ and, following [8], the norm $|v(t)|_\infty$ of a solution v is estimated via $\Upsilon(t)$ (since $|v| \leq \Upsilon + 1/2$). But the nonlinear term $i\nu^{-1}|v|^2 v$ does not contribute to Eq. (2.2), which is the same as the Υ-equation (2.3) in [10] (and as the corresponding equation in [8, Section 3.1]). So the estimates on $|\Upsilon(t)|_\infty$ and the resulting estimates on $|v(t)|_\infty$, obtained in [10], remain true for solutions of (2.1) with any ν. Thus we get the following upper bound for quadratic exponential moments of the L_∞-norms of solutions:[2]

Theorem 7 *For any $T > 0$ there are constants $c_* > 0$ and $C > 0$, depending only on B_* and T, such that for any r.v. $v_0^\omega \in C_0(K^n)$ as in Definition 5, any $\tau \geq 0$ and any $c \in (0, c_*]$, a solution $v(\tau; v_0)$ of Eq. (2.1) satisfies*

$$\mathbb{E} \exp(c \sup_{\tau \leq s \leq \tau+T} |v(s)|_\infty^2) \leq C\, \mathbb{E} \exp(5c\, |v_0|_\infty^2) \leq \infty. \tag{2.3}$$

In [10] the result above is proved for a deterministic initial data v_0. The theorem's assertion follows by averaging the result of [10] in v_0^ω.

The estimate (2.3) is crucial for derivation of further properties of solutions, including the given below upper bounds for their Sobolev norms, obtained in the work [8]. Since the scaling of the equation in [8] differs from that in (2.1) and the result there is a bit less general than in the theorem below, a sketch of the proof is given in Appendix B.

Theorem 8 *Assume that $B_m < \infty$ for some $m \in \mathbb{N}$, and $v_0 = v_0^\nu \in H^m \cap C_0(K^n)$ satisfies*

$$|v_0|_\infty \leq M, \quad \|v_0\|_m \leq M_m \nu^{-m}, \quad 0 < \nu \leq 1.$$

Then

$$\mathbb{E}\|v(\tau; v_0)\|_m^2 \leq C_m \nu^{-m}, \quad \forall \tau \in [0, \infty), \tag{2.4}$$

where $C_{M,m}$ also depends on M, M_m and B_m, B_{m_*}.

Neglecting the dependence on ν, we have that if $B_m < \infty$, $m \in \mathbb{N}$, and a r.v. $v_0^\omega \in H^m \cap C_0(K^n)$ satisfies $\mathbb{E}\|v_0\|_m^2 < \infty$ and $\mathbb{E}\exp(c\,|v_0|_\infty^2) < \infty$ for some $c > 0$, then Eq. (2.1) has a solution, equal v_0 at $t = 0$, such that

$$\mathbb{E}\|v(\tau; v_0)\|_m^2 \leq e^{-t}\mathbb{E}\|v_0\|_m^2 + C, \quad \tau \geq 0, \tag{2.5}$$

$$\mathbb{E} \sup_{0 \leq \tau \leq T} \|v(\tau; v_0)\|_m^2 \leq C', \tag{2.6}$$

where $C > 0$ depend on c, ν, $\mathbb{E}\exp(c\,|v_0|_\infty^2)$, B_{m_*} and B_m, while C' also depends on $\mathbb{E}\|v_0\|_m^2 < \infty$ and T. See Appendix B.

As it is shown in [10], the estimate (2.3) jointly with an abstract theorem from [11], imply that under a mild nondegeneracy assumption on the random force the Markov process in the space $C_0(K^n)$, constructed in Theorem 6, is mixing:

Theorem 9 *For each $\nu > 0$, there is an integer $N = N(B_*, \nu) > 0$ such that if $b_d \neq 0$ for $|d| \leq N$, then the Eq. (1.1) is mixing. I.e. it has a unique stationary measure $\mu_\nu \in \mathscr{P}(C_0(K^n))$, and for any probability measure $\lambda \in \mathscr{P}(C_0(K^n))$ we have $\mathcal{B}_t^*\lambda \rightharpoonup \mu_\nu$ as $t \to \infty$.*

[2] In [8] polynomial moments of the random variables $\sup_{\tau \leq s \leq \tau+T} |v(s)|_\infty^2)$ are estimated, and in [10] these results are strengthened to the exponential bounds (2.3).

Under the assumption of Theorem 8, for any $u_0 \in H^m$ the law $\mathcal{D}u(t; u_0)$ of a solution $u(t; u_0)$ is a measure in H^m. The mixing property in Theorem 9 and (2.4) easily imply

Corollary 10 *If under the assumptions of Theorem 9 $B_m < \infty$ for some $m \in \mathbb{N}$ and $u_0 \in H^m$, then $\mathcal{D}(u(t; u_0)) \rightharpoonup \mu_\nu$ in $\mathcal{P}(H^m)$.*

In view of Theorems 7, 8 with $v_0 = 0$ and the established mixing, we have:

Corollary 11 *Under the assumptions of Theorem 9, if $v^{st}(\tau)$ is the stationary solution of the equation, then*

$$\mathbb{E} \exp(c_* \sup_{\tau \le s \le \tau + T} |v^{st}(s)|_\infty^2) \le \mathcal{C},$$

where the constant $\mathcal{C} > 0$ depends only on T and B_*. If in addition $B_m < \infty$ for some $m \in \mathbb{N} \cup \{0\}$, then $\mathbb{E}\|v^{st}(\tau)\|_m^2 \le C_m \nu^{-m}$, where C_m depends on B_* and B_m.

Finally we note that applying Itô's formula to $\|v^{st}(\tau)\|_0^2$, where v^{st} is a stationary solution of (2.1), and taking the expectation we get the balance relation

$$\mathbb{E}\|v^{st}(\tau)\|_1^2 = B_0. \tag{2.7}$$

We cannot prove that $\mathbb{E}\|v^{st}(\tau)\|_0^2 \ge B' > 0$ for some ν-independent constant B', and cannot bound from below the energy $\frac{1}{2}\mathbb{E}\|v(\tau; v_0)\|_0^2$ of a solution v by a positive ν-independent quantity. Instead in next section we get a weaker conditional lower bound on the energies of solutions.

3 Conditional lower bound for the L^2-norm of solutions

In this section we prove the following result:

Lemma 12 *Let $B_2 < \infty$ and $u(\tau; u_0)$, where $u_0 \in H^2 \cap C_0(K^n)$ is non-random, be a solution of Eq. (2.1). Take any constants $\chi > 0, \Gamma \ge 1, \tau_0 \ge 0$, and define the stopping time*

$$\tau_\Gamma := \inf\{\tau \ge \tau_0 : \|u(\tau)\|_2 \ge \Gamma\}$$

(as usual, $\tau_\Gamma = \infty$ if the set under the inf*-sign is empty). Then*

$$\mathbb{E} \int_{\tau_0}^{\tau \wedge \tau_\Gamma} \mathbb{I}_{[0,\chi]}(\|u(s)\|_0) ds \le 2(1 + \tau - \tau_0) B_0^{-1} \chi \Gamma, \tag{3.1}$$

for any $\tau > \tau_0$.

Proof We establish the result by adapting the proof from [16] (also see [11, Theorem 5.2.12]) to non-stationary solutions. The argument relies on the concept of local time for semi-martingales (see e.g. [14, Chapter VI.1] for details of the concept). By $[\cdot]_b$ we denote the quasinorm $[u]_b^2 = \sum_d |u_d|^2 b_d^2$.

Without loss of generality we assume $\tau_0 = 0$. Otherwise we just need to replace $u(\tau, x)$ by the process $\tilde{u}(\tau, x) := u(\tau + \tau_0, x)$, apply the lemma with $\tau_0 = 0$ and with u_0 replaced by the initial data $\tilde{u}_0^\omega = u^\omega(\tau_0; u_0)$, and then average the estimate in the random $\tilde{u}_0^\omega$.

Let us write the solution $u(\tau; u_0)$ as $u(\tau) = \sum_{d\in\mathbb{N}^n} u_d(\tau)\varphi_d$. For any fixed function $g \in C^2(\mathbb{R})$, consider the process

$$f(\tau) = g(\|u(\tau \wedge \tau_\Gamma)\|_0^2).$$

Since

$$\partial_u g(\|u\|_0^2) = 2g'(\|u\|_0^2)\langle\langle u, \cdot\rangle\rangle, \quad \partial_{uu} g(\|u\|_0^2) = 4g''(\|u\|_0^2)\langle\langle u, \cdot\rangle\rangle\langle\langle u, \cdot\rangle\rangle$$
$$+2g'(\|u\|_0^2)\langle\langle \cdot, \cdot\rangle\rangle,$$

then by Itô's formula we have

$$f(\tau) = f(0) + \int_0^{\tau\wedge\tau_\Gamma} A(s)ds + \sum_{d\in\mathbb{N}^n} b_d \int_0^{\tau\wedge\tau_\Gamma} 2g'(\|u(s)\|_0^2)\langle u_d(s), d\beta_d(s)\rangle, \quad (3.2)$$

$$\begin{aligned} A(s) &= 2g'(\|u\|_0^2)\langle\langle u, \Delta u - \frac{1}{\nu} i|u|^2 u\rangle\rangle + 2\sum_d b_d^2\big(g''(\|u\|_0^2)|u_d|^2 + g'(\|u\|_0^2)\big) \\ &= -2g'(\|u\|_0^2)\|u\|_1^2 + 2g''(\|u\|_0^2)[u]_b^2 + 2g'(\|u\|_0^2)B_0, \quad u = u(s). \end{aligned} \quad (3.3)$$

Step 1: We firstly show that for any bounded measurable set $G \subset \mathbb{R}$, denoting by $\mathbb{I}_G$ its indicator function, we have the following equality

$$\begin{aligned} 2\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_G(f(s))\left(g'(\|u(s)\|_0^2)\right)^2[u(s)]_b^2 ds = \int_{-\infty}^{\infty} \mathbb{I}_G(a) \\ \Big[\mathbb{E}(f(\tau) - a)_+ - \mathbb{E}(f(0) - a)_+ - \mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a+\infty)}(f(s))A(s)ds\Big]da. \end{aligned} \quad (3.4)$$

Let $L(\tau, a)$, $(\tau, a) \in [0, \infty) \times \mathbb{R}$, be the local time for the semi martingale $f(\tau)$ (see e.g. [14, Chapter VI.1]). Since in view of (3.2) the quadratic variation of the process $f(\tau)$ is

$$d\langle f, f\rangle_s = \sum_d (2g'(\|u\|_0^2)|u_d|b_d)^2 = 4\big(g'(\|u\|_0^2)\big)^2[u]_b^2,$$

then for any bounded measurable set $G \subset \mathbb{R}$, we have the following equality (known as the occupation time formula, see [14, Corollary VI.1.6]),

$$\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_G(f(s))4\big(g'(\|u(s)\|_0^2)\big)^2 [u(s)]_b^2 ds = \int_{-\infty}^{\infty} \mathbb{I}_G(a)L(\tau,a)da. \tag{3.5}$$

For the local time $L(\tau, a)$, due to Tanaka's formula (see [14, Theorem VI.1.2]) we have

$$\begin{aligned}(f(\tau)-a)_+ =&(f(0)-a)_+ \\ &+ \sum_{d\in\mathbb{N}^n} b_d \int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,+\infty)}\big(f(s)\big)2g'(\|u(s)\|_0^2)\langle u_d(s), d\beta_d(s)\rangle \\ &+ \int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,+\infty)}(f(s))A(s)ds + \frac{1}{2}L(\tau,a).\end{aligned} \tag{3.6}$$

Taking expectation of both sides of (3.5) and (3.6) we obtain the required equality (3.4).
Step 2: Let us choose $G = [\rho_0, \rho_1]$ with $\rho_1 > \rho_0 > 0$, and $g(x) = g_{\rho_0}(x) \in C^2(\mathbb{R})$ such that $g'(x) \geq 0$, $g(x) = \sqrt{x}$ for $x \geq \rho_0$ and $g(x) = 0$ for $x \leq 0$. Then due to the factors $\mathbb{I}_G(f)$ and $\mathbb{I}_G(a)$ in (3.4), we may there replace $g(x)$ by $\sqrt{x}$, and accordingly replace $g(\|u\|_0^2)$, $g'(\|u\|_0^2)$ and $g''(\|u\|_0^2)$ by $\|u\|_0$, $\frac{1}{2}\|u\|_0^{-1}$ and $-\frac{1}{4}\|u\|_0^{-3}$. So the relation (3.4) takes the form

$$\begin{aligned}&\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_G(f(s))\|u(s)\|_0^{-2}[u(s)]_b^2 = 2\int_{\rho_0}^{\rho_1}\Big[\mathbb{E}(f(\tau)-a)_+ - \mathbb{E}(f(0)-a)_+\Big]da \\ &-2\int_{\rho_0}^{\rho_1}\Big\{\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,+\infty)}\Big(f(s)\Big)\Big[\frac{2}{2\|u(s)\|_0}(B_0 - \|u(s)\|_1^2) \\ &- \frac{2}{4\|u(s)\|_0^3}[u(s)]_b^2\Big]ds\Big\}da.\end{aligned}$$

Since the l.h.s. of the above equality is non-negative, we have

$$\begin{aligned}&\int_{\rho_0}^{\rho_1}\Big[\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,+\infty)}\Big(f(s)\Big)\frac{1}{\|u(s)\|_0^3}\Big(B_0\|u(s)\|_0^2 - \tfrac{1}{2}[u(s)]_b^2\Big)ds\Big]da \\ &\leq \int_{\rho_0}^{\rho_1}\mathbb{E}\Big[\big((f(\tau)-a)_+ - (f(0)-a)_+\big) + \int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,+\infty)}\big(f(s)\big)\frac{\|u(s)\|_1^2}{\|u(s)\|_0}ds\Big]da.\end{aligned} \tag{3.7}$$

Noting that

$$B_0\|u\|_0^2 - \tfrac{1}{2}[u(s)]_b^2 = \sum_{d\in\mathbb{N}^n}(B_0 - \frac{1}{2}b_d^2)|u_d|^2 \geq \frac{B_0}{2}\|u\|_0^2,$$

that by the definition of the stopping time τ_Γ

$$(f(\tau)-a)_+ - (f(0)-a)_+ \leq \Gamma,$$

and that by interpolation,

$$\int_0^{\tau\wedge\tau_\Gamma} \frac{\|u(s)\|_1^2}{\|u(s)\|_0} ds \le \int_0^{\tau\wedge\tau_\Gamma} \|u(s)\|_2 ds \le (\tau\wedge\tau_\Gamma)\Gamma,$$

we derive from (3.7) the relation

$$\frac{B_0}{2}\int_{\rho_0}^{\rho_1} \big(\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,+\infty)}\big(f(s)\big)\|u(s)\|_0^{-1}ds\big)da \le (\rho_1-\rho_0)\Gamma\big(1+\tau\big).$$

When $\rho_0 \to 0$, we have $g(x) \to \sqrt{x}$ and $f(\tau) \to \|u(\tau\wedge\tau_\Gamma)\|_0$. So sending ρ_0 to 0 and using Fatou's lemma we get from the last estimate that

$$\int_0^{\rho_1} \mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,\infty)}\Big(\|u(s)\|_0\Big)\|u(s)\|_0^{-1}dsda \le 2\rho_1(1+\tau)B_0^{-1}\Gamma.$$

As the l.h.s. above is not smaller than

$$\frac{1}{\chi}\int_0^{\rho_1} \mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,\chi]}(\|u(s)\|_0)dsda,$$

then

$$\frac{1}{\rho_1}\int_0^{\rho_1} \mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,\chi]}(\|u(s)\|_0)dsda \le 2(1+\tau)B_0^{-1}\Gamma\,\chi. \tag{3.8}$$

By the monotone convergence theorem

$$\lim_{a\to 0}\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(a,\chi]}(\|u(s)\|_0)ds = \mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(0,\chi]}(\|u(s)\|_0)ds,$$

so we get from (3.8) that

$$\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{(0,\chi]}(\|u(s)\|_0)ds \le 2(1+\tau)B_0^{-1}\Gamma\chi. \tag{3.9}$$

Step 3: We continue to verify that

$$\mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{\{0\}}(\|u(s)\|_0)ds = 0. \tag{3.10}$$

To do this let us fix any index $d \in \mathbb{N}^n$ such that $b_d \neq 0$. The process $u_d(\tau)$ is a semimartingale, $du_d = v_d ds + b_d d\beta_d$, where $v_d(s)$ is the d-th Fourier coefficient of $\Delta u + \frac{1}{\nu}i|u|^2u$ for the solution $u(\tau) = \sum_d u_d(\tau)\varphi_d$ which we discuss. Consider the stopping time

$$\tau_R = \inf\{s \le \tau\wedge\tau_\Gamma : |u(s)|_\infty \ge R\}.$$

Due to (2.3) and (2.6), $\mathbb{P}(\tau_R = \tau \wedge \tau_\Gamma) \to 1$ as $R \to \infty$. Let us denote $u_d^R(\tau) = u_d(\tau \wedge \tau_R)$. To prove (3.10) it suffices to verify that

$$\pi(\delta) := \mathbb{E}\int_0^{\tau\wedge\tau_\Gamma} \mathbb{I}_{\{|u_d(s)|<\delta\}} ds \to 0 \quad \text{as} \quad \delta \to 0.$$

If we replace above u_d by u_d^R, then the obtained new quantity $\pi^R(\delta)$ differs from $\pi(\delta)$ at most by $\mathbb{P}(\tau_R < \tau \wedge \tau_\Gamma)$. The process u_d^R is an Ito process with a bounded drift. So by [7, Theorem 2.2.2, p. 52], $\pi^R(\delta)$ goes to zero with δ. Thus, given any $\varepsilon > 0$, we firstly choose R sufficiently big and then δ sufficiently small to achieve $\pi(\delta) < \varepsilon$, for a suitable $\delta(\varepsilon) > 0$. So (3.10) is verified. Jointly with (3.9) this proves (3.1). □

4 Lower bounds for Sobolev norms of solutions

In this section we work with Eq. (1.1) in the original time scale t and provide lower bounds for the H^m-norms of its solutions with $m > 2$. This will prove the assertion (1) of Theorem 1. As always, the constants do not depend on ν, unless otherwise stated.

Theorem 13 *For any integer $m \geq 3$, if $B_m < \infty$ and*

$$0 < \kappa < \tfrac{1}{35}, \quad T_0 \geq 0, \quad T_1 > 0,$$

then for any r.v. $u_0(x) \in H^m \cap C_0(K^n)$, satisfying

$$\mathbb{E}\|u_0\|_m^2 < \infty, \quad \mathbb{E}\exp(c\,|u_0|_\infty^2) \leq C < \infty \tag{4.1}$$

for some $c, C > 0$, we have

$$\mathbb{P}\left\{\sup_{T_0 \leq t \leq T_0 + T_1\nu^{-1}} \|u(t; u_0)\|_m \geq K\nu^{-m\kappa}\right\} \to 1 \quad \text{as} \quad \nu \to 0, \tag{4.2}$$

for every $K > 0$.

Proof Consider the complement to the event in (4.2):

$$Q^\nu = \left\{\sup_{T_0 \leq t \leq T_0 + \frac{T_1}{\nu}} \|u(t)\|_m < K\nu^{-m\kappa}\right\}.$$

We will prove the assertion (4.2) by contradiction. Namely, we assume that there exists a $\gamma > 0$ and a sequence $\nu_j \to 0$ such that

$$\mathbb{P}(Q^{\nu_j}) \geq 5\gamma \quad \text{for} \quad j = 1, 2, \ldots, \tag{4.3}$$

and will derive a contradiction. Below we write Q^{ν_j} as Q and always suppose that

$$\nu \in \{\nu_1, \nu_2, \dots\}.$$

The constants in the proof may depend on $\mathcal{K}$, K, γ, $B_{m\vee m_*}$, but not on ν.

Without lost of generality we assume that $T_1 = 1$. For any $T_0 > 0$, due to (2.5) and (2.3) the r.v. $\tilde{u}_0 := u(T_1)$ satisfies (4.1) with c replaced by $c/5$. So considering $\tilde{u}(t,x) = u(t+T_0, x)$ we may assume that $T_0 = 0$.

Let us denote $J_1 = [0, \frac{1}{\nu}]$. Due to Theorem 7,

$$\mathbb{P}(Q_1) \geq 1-\gamma, \quad Q_1 = \{\sup_{t\in J_1} |u(t)|_\infty \leq C_1(\gamma)\},$$

uniformly in ν, for a suitable $C_1(\gamma)$. Then, by the definition of Q and Sobolev's interpolation,

$$\|u^\omega(t)\|_l \leq C_{l,\gamma}\nu^{-l\kappa}, \quad \omega \in Q\cap Q_1,\ t\in J_1, \tag{4.4}$$

for $l \in [0, m]$ (and any $\nu \in \{\nu_1, \nu_2, \dots\}$).

Denote $J_2 = [0, \frac{1}{2\nu}]$ and consider the stopping time

$$\tau_1 = \inf\{t \in J_2 : \|u(t)\|_2 \geq C_{2,\gamma}\nu^{-2\kappa}\} \leq \tfrac{1}{2\nu}.$$

Then $\tau_1 = \frac{1}{2\nu}$ for $\omega \in Q\cap Q_1$. So due to (3.1) with $\Gamma = C_{2,\gamma}\nu^{-2\kappa}$, for any $\chi > 0$, we have

$$\begin{aligned}\mathbb{E}\Big(\nu\int_{J_2}\mathbb{I}_{[0,\chi]}(\|u(s)\|_0)ds\,\mathbb{I}_{Q\cap Q_1}(\omega)\Big) &= \mathbb{E}\Big(\nu\int_0^{\frac{1}{2\nu}\wedge\tau_1}\mathbb{I}_{[0,\chi]}(\|u(s)\|_0)ds\,\mathbb{I}_{Q\cap Q_1}(\omega)\Big)\\ &\leq \mathbb{E}\Big(\nu\int_0^{\frac{1}{2\nu}\wedge\tau_1}\mathbb{I}_{[0,\chi]}(\|u(s)\|_0)ds\Big) \leq C\nu^{-2\kappa}\chi.\end{aligned}$$

Consider the event

$$\Lambda = \{\omega \in Q\cap Q_1 : \|u(s)\|_0 \leq \chi,\ \forall s \in J_2\}.$$

Due to the above, we have,

$$\mathbb{P}(\Lambda) \leq 2\mathbb{E}\Big(\nu\int_{J_2}\mathbb{I}_{[0,\chi]}(\|u(s)\|_0)ds\,\mathbb{I}_{Q\cap Q_1}(\omega)\Big) \leq 2C\nu^{-2\kappa}\chi.$$

So $\mathbb{P}(\Lambda) \leq \gamma$ if we choose

$$\chi = c_3(\gamma)\nu^{2\kappa}, \quad c_3(\gamma) = \gamma(2C)^{-1}. \tag{4.5}$$

Let us set

$$Q_2 = (Q \cap Q_1) \setminus \Lambda, \quad \mathbb{P}(Q_2) \geq 3\gamma, \tag{4.6}$$

and for χ as in (4.5), consider the stopping time

$$\tilde{\tau}_1 = \inf\{t \in J_2 : \|u(t)\|_0 \geq \chi\}.$$

Then $\tilde{\tau}_1 \leq \frac{1}{2\nu}$ for all $\omega \in Q_2$. Consider the function

$$v(t,x) := u(\tilde{\tau}_1 + t, x), \quad t \in [0, \tfrac{1}{2\nu}].$$

It solves Eq. (1.1) with modified Wiener processes and with initial data $v_0(x) = u^\omega(\tilde{\tau}_1, x)$, satisfying

$$\|v_0^\omega\|_0 \geq \chi = c\nu^{2\kappa} \quad \text{if} \quad \omega \in Q_2. \tag{4.7}$$

Now we introduce another stopping time, in terms of $v(t,x)$:

$$\tau_2 = \inf\{t \in [0, \tfrac{1}{2\nu}] : \|v(t)\|_m \geq K\nu^{-m\kappa}\} \leq \tfrac{1}{2\nu}.$$

For $\omega \in Q_2$, $\tau_2 = \frac{1}{2\nu}$ and in view of (4.4)

$$\|v^\omega(t)\|_l \leq C_3(\gamma)\nu^{-l\kappa}, \quad t \in [0, \tfrac{1}{2\nu}],\ l \in [0,m], \quad \forall\, \omega \in Q_2. \tag{4.8}$$

Step 1: Let us estimate from above the increment $\mathscr{E}(t,x) = |v(t \wedge \tau_2, x)|^2 - |v_0(x)|^2$. Due to Itô's formula, we have that

$$\mathscr{E}(t,x) = 2\nu \int_0^{t\wedge\tau_2} \Big(\langle v(s,x), \Delta v(s,x)\rangle + \sum_{d\in\mathbb{N}^n} b_d^2 \varphi_d^2(x)\Big) ds + \sqrt{\nu}\, M(t,x),$$
$$M(t,x) = \int_0^{t\wedge\tau_2} \sum_{d\in\mathbb{N}^n} b_d \varphi_d(x) \langle v(s,x), d\beta_d(s)\rangle.$$

We treat M as a martingale $M(t)$ in the space H^1. Since in view of (A.3) for $0 \leq s < \tau_2$ we have

$$\|v(s)\varphi_d\|_1 \leq C\big(|v(s)|_\infty \|\varphi_d\|_1 + |v(s)\|_1 |\varphi_d|_\infty\big) \leq C(\zeta d + \zeta^{(m-1)/m}\nu^{-\kappa}),$$

where $\zeta = \sup_{0\leq s\leq\frac{1}{\nu}} |u(s)|_\infty$ (the assertion is empty if $\tau_2 = 0$), then for any $0 < T_* \leq \frac{1}{2\nu}$

$$\mathbb{E}\|M(T_*)\|_1^2 \leq \int_0^{T_*} \mathbb{E} \sum_d b_d^2 \|\varphi_d v(s)\|_1^2 ds \leq CT_*\nu^{-2\kappa}, \tag{4.9}$$

where we used that $B_1 < \infty$. So by Doob's inequality

$$\mathbb{P}\Big(\sup_{0\le s\le T_*} \|M(s)\|_1^2 \ge r^2\Big) \le C T_* r^{-2}\nu^{-2\kappa}, \quad \forall r > 0. \tag{4.10}$$

Let us choose

$$T_* = \nu^{-b}, \quad b \in (0,1),$$

where b will be specified later. Then $1 \le T_* \le \frac{1}{2\nu}$ if ν is sufficiently small, so due to (4.10)

$$\mathbb{P}(Q_3) \ge 1-\gamma, \quad Q_3 = \{\sup_{0\le\tau\le T_*} \|M(\tau)\|_1 \le C_4(\gamma)\nu^{-\kappa}\sqrt{T_*}\},$$

for a suitable $C_4(\gamma)$ (and for $\nu \ll 1$); thus $\mathbb{P}(Q_2\cap Q_3) \ge 2\gamma$. Since $\|\langle v, \Delta v\rangle\|_1 \le C|v|_\infty\|v\|_3$ by (A.2) and $\|\sum_d b_d\varphi_d\|_1 \le C$, then in view of (4.8) and the definition of Q_3,

$$\|\mathscr{E}^\omega(\tau)\|_1 \le C(\gamma)(\nu^{1-3\kappa}T_* + \nu^{\frac12-\kappa}T_*^{1/2}), \qquad \forall\tau\in[0,T_*], \ \ \forall\omega\in Q_2\cap Q_3. \tag{4.11}$$

Step 2: For any $x \in K^n$, denoting $R(t) = |v(t,x)|^2$, $a(t) = \Delta v(t,x)$ and $\xi(t) = \xi(t,x)$, we write the equation for $v(t) := v(t,x)$ as an Itô process:

$$dv(t) = (-iRv + \nu a)dt + \sqrt{\nu}\, d\xi(t). \tag{4.12}$$

Setting $w(t) = e^{i\int_0^t R(s)ds}v(t)$, we observe that w also is an Itô process, $w(0) = v_0$ and $dv = e^{-i\int_0^t R(s)ds}dw - iRv\,dt$. From here and (4.12),

$$w(t) = v_0 + \nu\int_0^t e^{i\int_0^s R(s')ds'}a(s)ds + \sqrt{\nu}\int_0^t e^{i\int_0^s R(s')ds'}d\xi(s).$$

So $v(t\wedge\tau_2) = v(t\wedge\tau_2, x)$ can be written as

$$v(t\wedge\tau_2,x) = I_1(t\wedge\tau_2,x) + I_2(t\wedge\tau_2,x) + I_3(t\wedge\tau_2,x), \tag{4.13}$$

where

$$I_1(t,x) = e^{-i\int_0^t |v(s,x)|^2ds}v_0, \quad I_2(t,x) = \nu\int_0^t e^{-i\int_s^t |v(s',x)|^2ds'}\Delta v(s,x)ds,$$
$$I_3(t,x) = \sqrt{\nu}e^{-i\int_0^t |v(s',x)|^2ds'}\int_0^t e^{i\int_0^s |v(s',x)|^2ds'}d\xi(s,x).$$

Our next goal is to obtain a lower bound for $\|v(T_*)\|_1$ when $\omega\in Q_2\cap Q_3$, using the above decomposition (4.13).

Step 3: We first deal with the stochastic term $I_3(t)$. For $0 \le s \le s_1 \le T_* \wedge \tau_2$ we set

$$W(s, s_1, x) := \exp(i \int_s^{s_1} |v(s', x)|^2 ds'), \quad F(s, s_1, x) := \int_s^{s_1} |v(s', x)|^2 ds'; \quad (4.14)$$

then $W(s, s_1, x) = \exp\big(iF(s, s_1, x)\big)$. The functions F and W are periodic in x, but not odd. Speaking about them we understand $\|\cdot\|_m$ as the non-homogeneous Sobolev norm, so $\|F\|_m^2 = \|F\|_0^2 + \|(-\Delta)^{m/2} F\|_0^2$, etc. We write I_3 as

$$I_3(t) = \sqrt{\nu}\, \overline{W}(0, t \wedge \tau_2, x) \int_0^{t \wedge \tau_2} W(0, s, x) d\xi(s, x). \quad (4.15)$$

In view of (A.1),

$$\|\exp(iF(s, s_1 \cdot))\|_k \le C_k (1 + |F(s, s_1, \cdot)|_\infty)^{k-1} \|F(s, s_1, \cdot)\|_k, \quad k \in \mathbb{N}. \quad (4.16)$$

For any $s \in J = [0, T_* \wedge \tau_2)$, by (A.3) and the definition of τ_2, we have that $v := v(s)$ satisfies

$$\||v|^2\|_1 \le C|v|_\infty \|v\|_1 \le C|v|_\infty \|v\|_0^{1-1/m} \|v\|_m^{1/m} \le C' |v|_\infty^{2-1/m} \nu^{-\kappa} \quad (4.17)$$

(this assertion is empty if $\tau_2 = 0$ since then $J = \emptyset$). So for $s, s_1 \in J$,

$$|F(s, s_1, \cdot)|_\infty \le |s_1 - s| \sup_{s' \in J} |v(s')|_\infty^2, \quad \|F(s, s_1, \cdot)\|_k \le C\nu^{-\kappa k} |s_1 - s| \big(\sup_{s' \in J} |v(s')|_\infty\big)^{2-k/m}$$

for $k \le m$. Then, due to (4.16),

$$\|W(0, s \wedge \tau_2, \cdot)\|_1 \le C' T_* \nu^{-\kappa} (1 + \sup_{s \in J} |v(s)|_\infty^2). \quad (4.18)$$

Consider the stochastic integral in (4.15),

$$N(t, x) = \int_0^t W(0, s, x) d\xi(s, x).$$

The process $t \mapsto W(0, t, x)$ is adapted to the filtration $\{\mathcal{F}_t\}$, and

$$dW(0, t, x) = i|v(t, x)|^2 W(0, t, x) dt.$$

So integrating by parts (see, e.g., [14, Proposition IV.3.1]) we re-write N as

$$N(t, x) = W(0, t, x)\xi(t, x) - i \int_0^t \xi(s, x)|v(s, x)|^2 W(0, s, x) ds,$$

and we see from (4.15) that

$$I_3(t)=\sqrt{\nu}\xi(t\wedge\tau_2,x)+i\sqrt{\nu}\int_0^{t\wedge\tau_2}\xi(s,x)|v(s,x)|^2W(s,t\wedge\tau_2,x)ds. \tag{4.19}$$

Due to (1.4) and since $B_m<\infty$, the Wiener process $\xi(t,x)$ satisfies

$$\mathbb{E}\|\xi(T_*,x)\|_1^2\le CB_1T_*,$$

and

$$\mathbb{E}\sup_{0\le t\le T_*}|\xi(t,\cdot)|_\infty\le\sum_{d\in\mathbb{N}^n}b_d(\mathbb{E}\sup_{0\le t\le T_*}|\beta_d(t)\varphi_d|_\infty)\le CB_*\sqrt{T_*},$$

(we recall that $B_*=\sum_{d\in\mathbb{N}^n}|b_d|<\infty$). Therefore,

$$\mathbb{P}(Q_4)\ge 1-\gamma,\quad Q_4=\{\sup_{0\le t\le T_*}(\|\xi(t)\|_1\vee|\xi(t)|_\infty)\le CT_*^{1/2}\},$$

with a suitable $C=C(\gamma)$. Let

$$\tilde{Q}=\bigcap_{i=1}^4 Q_i,$$

then $\mathbb{P}(\tilde{Q})\ge\gamma$. As $\tau_2=T_*$ for $\omega\in\tilde{Q}$, then due to (4.17), (4.18), (4.19) and (A.3), for $\omega\in\tilde{Q}$ we have

$$\begin{aligned}\sup_{0\le t\le T_*}\|I_3^\omega(t)\|_1&\le\sqrt{\nu}\sup_{0\le t\le T_*}\Big(\|\xi^\omega(t)\|_1+\int_0^t\|\xi^\omega(s)|v^\omega(s)|^2W^\omega(s,t)\|_1ds\Big)\\&\le CT_*^{5/2}\nu^{\frac{1}{2}-\kappa}.\end{aligned} \tag{4.20}$$

Setp 4: We then consider the term $I_2=\nu\int_0^{t\wedge\tau_2}\bar{W}(s,t\wedge\tau_2,x)\Delta v(s,x)ds$. To bound its H^1-norm we need to estimate $\|W\Delta v\|_1$. Since

$$\|\partial_x^aW\partial_x^bv\|_0\le C\|W\|_3^{1/3}\|v\|_3^{2/3}|v|_\infty^{1/3}\quad\text{if }\ |a|=1,|b|=2,$$

(see [17, Proposition 3.6]), we have

$$\|W\Delta v\|_1\le C(\|v\|_3+\|W\|_3^{1/3}\|v\|_3^{2/3}|v|_\infty^{1/3}).$$

Then in view of (4.16) and (4.8), for $\omega\in\tilde{Q}$

$$\|W\Delta v\|_1\le C\big(\nu^{-3\kappa}+(T_*^3\nu^{-3\kappa})^{1/3}\nu^{-2\kappa}\big)\le C\nu^{-3\kappa}T_*,$$

and accordingly

$$\sup_{0\le t\le T_*}\|I_2^\omega(t)\|_1 \le \nu \sup_{0\le t\le T_*}\int_0^t \|W^\omega(s,T_*)\Delta v^\omega(s)\|_1 ds \le C\nu^{1-3\kappa}T_*^2, \quad \forall\omega\in\tilde{Q}. \tag{4.21}$$

Step 5: Now we estimate from below the H^1-norm of the term $I_1^\omega(T_*,x)$, $\omega\in\tilde{Q}$. Writing it as $I_1^\omega(T_*,x)=e^{-iT_*|v_0(x)|^2}e^{-i\int_0^{T_*}\mathscr{E}(s,x)ds}v_0(x)$ wee see that

$$\|I_1^\omega(T_*)\|_1 \ge \|\nabla(\exp(-iT_*|v_0|^2)v_0\|_0 - \|\nabla(\exp(-i\int_0^{T_*}\mathscr{E}(s)ds))v_0\|_0 - \|v_0\|_1.$$

This first term on the r.h.s is

$$T_*\|v_0\nabla(|v_0|^2)\|_0 = T_*\tfrac{2}{3}\|\nabla|v_0|^3\|_0 \ge CT_*\||v_0|^3\|_0 \ge CT_*\|v_0\|_0^3 \ge CT_*\nu^{6\kappa}, \quad C>0,$$

where we have used the fact that $u|_{\partial K^n}=0$, Poincaré's inequality and (4.7).

For $\omega\in\tilde{Q}$ and $0\le s\le T_*$, in view of (4.11), the second term is bounded by

$$\left\|\left(\int_0^{T_*}\nabla\mathscr{E}(s)ds\right)v_0\right\|_0 \le CT_*|v_0|_\infty \sup_{0\le s\le T_*}\|\mathscr{E}(s)\|_1 \le CT_*(\nu^{1-3\kappa}T_*+\nu^{\frac{1}{2}-\kappa}T_*^{1/2}).$$

Therefore, using (4.11), we get for the term $I_1^\omega(T_*)$ the following lower bound:

$$\|I_1^\omega(T_*)\|_1 \ge C\Big(\nu^{6\kappa}T_* - T_*\big(\nu^{1-3\kappa}T_*+\nu^{\frac{1}{2}-\kappa}T_*^{1/2}\big)-\nu^{-\kappa}\Big).$$

Recalling $T_*=\nu^{-b}$ we see that if we assume that

$$\begin{cases} 6\kappa-b<-\kappa, \\ 6\kappa-b<1-3\kappa-2b, \\ 6\kappa-b<1/2-\kappa-\frac{3}{2}b, \end{cases} \tag{4.22}$$

then for $\omega\in\tilde{Q}$,

$$\|I_1^\omega(T_*)\|_1 \ge C\nu^{6\kappa}T_*, \quad C>0, \tag{4.23}$$

provided that ν is sufficiently small.

Step 6: Finally, remembering that $\tau_2=T_*$ for $\omega\in\tilde{Q}$ and combining the relations (4.20), (4.21) and (4.23) to estimate the terms of (4.13), we see that for $\omega\in\tilde{Q}$ we have

$$\|v^\omega(T_*)\|_1 \ge \|I_1^\omega(T_*)\|_1 - \|I_2^\omega(T_*)\|_1 - \|I_3^\omega(\tau_*)\|_1 \ge \tfrac{1}{2}C_1\nu^{6\kappa-b}, \quad C_1>0, \tag{4.24}$$

if we assume in addition to (4.22) that

$$6\kappa - b < \frac{1}{2} - \kappa - \frac{5}{2}b, \tag{4.25}$$

and ν is small. Note that this relation implies the last two in (4.22).

Combining (4.8) and (4.24) we get that

$$\nu^{-b+7\kappa} \leq C_2^{-1}, \tag{4.26}$$

for all sufficiently small ν. Thus we have obtained a contradiction with the existence of the sets $Q^{\nu j}$ as at the beginning of the proof if (for a chosen κ) we can find a $b \in (0, 1)$ which meets (4.22), (4.25) and

$$-b + 7\kappa < 0.$$

Noting that this is nothing but the first relation in (4.22), we see that we have obtained a contradiction if

$$\kappa < \tfrac{1}{7}b, \quad \kappa < \tfrac{1}{14} - \tfrac{3}{14}b,$$

for some $b \in (0, 1)$. We see immediately that such a b exists if and only if $\kappa < \frac{1}{35}$. □

Amplification If we replace the condition $m \geq 3$ with the weaker assumption

$$\mathbb{R} \ni m > 2,$$

then the statement (4.2) remains true for $0 < \kappa < \kappa(n, m)$ with a suitable (less explicit) constant $\kappa(n, m) > 0$. In this case we obtain a contradiction with the assumption (4.3) by deriving a lower bound for $\|v(T_*)\|_\alpha$, where $\alpha = \min\{1, m - 2\} \in (0, 1]$, using the decomposition (4.13). The proof remains almost identical except that now, firstly, we bound $\|I_2\|_\alpha$ $(\alpha < 1)$ from above using the following estimate from [15, Theorem 5, p. 206] (also see there p. 14):

$$\|W\Delta u\|_\alpha \leq C\|u\|_{2+\alpha}(|W|_\infty + |W|_\infty^{1-\frac{2\alpha}{n}} \|W\|_2^{\frac{2\alpha}{n}});$$

and, secondly, estimate $\|I_1^\omega(T_*)\|_\alpha$ $(\alpha < 1)$ from below as

$$\|I_1^\omega(T_*)\|_\alpha \geq \|I_1^\omega(T_*)\|_1^{2-\alpha} \|I_2^\omega(T_*)\|_2^{-1+\alpha},$$

which directly follows from Sobolev's interpolation. See [6] for more details.

5 Lower bounds for time-averaged Sobolev norms

In this section we prove the assertion (2) of Theorem 1. We provide each space H^r, $r \geq 0$, with the scalar product

$$\langle\langle u, v\rangle\rangle_r := \langle\langle(-\Delta)^{\frac{r}{2}} u, (-\Delta)^{\frac{r}{2}} v\rangle\rangle,$$

corresponding to the norm $\|u\|_r$. Let $u(t) = \sum u_d(t)\varphi_d$ be a solution of Eq. (1.1). Applying Itô's formula to the functional $\|u\|_m^2$, we have for any $0 \leq t < t' < \infty$ the relation

$$\begin{aligned} \|u(t')\|_m^2 =& \|u(t)\|_m^2 + 2\int_t^{t'} \langle\langle u(s), \nu\Delta u(s) - i|u(s)|^2 u(s)\rangle\rangle_m ds \\ &+ 2\nu B_m(t'-t) + 2\sqrt{\nu} M(t,t'), \end{aligned} \tag{5.1}$$

where M stands for the real scalar productis the stochastic integral

$$M(t,t') := \int_t^{t'} \sum_{d\in\mathbb{N}^n} b_d |d|^{2m} \langle u_d(s), d\beta_d(s)\rangle.$$

Let us fix a $\gamma \in (0, \frac{1}{8})$. Due to Theorems 7 and 13, for small enough ν there exists an event $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) \geq 1 - \gamma/2$, such that for all $\omega \in \Omega_1$ we have:

a) $\sup_{0\leq t\leq \frac{1}{\nu}} |u^\omega(t)|_\infty \leq C(\gamma)$, for a suitable $C(\gamma) > 0$;
b) there exist $t_\omega \in [0, \frac{1}{3\nu}]$ and $t'_\omega \in [\frac{2}{3\nu}, \frac{1}{\nu}]$ satisfying

$$\|u^\omega(t_\omega)\|_m,\ \|u^\omega(t'_\omega)\|_m \geq \nu^{-m\kappa}. \tag{5.2}$$

Since for the martingale $M(0,t)$ we have that

$$\mathbb{E}|M(0, \tfrac{1}{\nu})|^2 \leq B_m \mathbb{E}\int_0^{\frac{1}{\nu}} \|u(s)\|_m^2 ds =: X_m,$$

then by Doob's inequality

$$\mathbb{P}(\Omega_2) \geq 1 - \frac{\gamma}{2}, \qquad \Omega_2 = \Big\{ \sup_{0\leq t\leq \frac{1}{\nu}} |M(0,t)| \leq c(\gamma) X_m^{1/2} \Big\}.$$

Now let us set $\hat{\Omega} = \Omega_1 \cap \Omega_2$. Then $\mathbb{P}(\hat{\Omega}) \geq 1 - \gamma$ for small enough ν, and for any $\omega \in \hat{\Omega}$ there are two alternatives:

(i) there exists a $t^0_\omega \in [0, \frac{1}{3\nu}]$ such that $\|u^\omega(t^0_\omega)\|_m = \frac{1}{3}\nu^{-\kappa m}$. Then from (5.1) and (5.2) in view of (A.4) we get

$$\frac{8}{9}\nu^{-2m\kappa} + 2\nu \int_{t^0_\omega}^{t'_\omega} \|u^\omega(s)\|^2_{m+1} ds \le C(m,\gamma) \int_0^{\frac{1}{\nu}} \|u^\omega(s)\|^2_m ds + 2B_m$$
$$+ 2\sqrt{\nu} c(\gamma) X_m^{1/2}.$$

(ii) There exists no $t \in [0, \frac{1}{3\nu}]$ with $\|u^\omega(t)\|_m = \frac{1}{3}\nu^{-\kappa m}$. In this case, since $\|u^\omega(t)\|_m$ is continuous with respect to t, then due to (5.2) $\|u^\omega(t)\|_m > \frac{1}{3}\nu^{-m\kappa}$ for all $t \in [0, \frac{1}{3\nu}]$. This leads to the relation

$$\tfrac{1}{27}\nu^{-2m\kappa-1} \le \int_0^{\frac{1}{\nu}} \|u^\omega(s)\|^2_m ds.$$

In both cases for $\omega \in \hat{\Omega}$ we have:

$$\frac{1}{27}\nu^{-2m\kappa} \le C'(m,\gamma) \int_0^{\frac{1}{\nu}} \|u(s)\|^2_m ds + 2B_m + \nu c(\gamma)^2 + X_m.$$

It implies that

$$\mathbb{E}\nu \int_0^{\frac{1}{\nu}} \|u(\tau)\|^2_m d\tau \ge C\nu^{-2m\kappa+1}$$

(for small enough ν), and gives the lower bound in (1.7).

The upper bound follows directly from Theorem 8.

Proof of Corollaries 3 and 4 Since $B_k < \infty$ for each k and all coefficients b_d are non-zero, then Eq. (1.1) is mixing in the spaces H^m, $m \in \mathbb{N}$, see Corollary 10. As the stationary solution v^{st} satisfies Corollary 11 with any m, then for each $\mu \in \mathbb{N}$ and $M > 0$, interpolating the norm $\|u\|_\mu$ via $\|u\|_0$ and $\|u\|_m$ with m sufficiently large we get that the stationary measure μ_ν satisfies

$$\int \|u\|^M_\mu \mu_\nu(du) < \infty \quad \forall \mu \in \mathbb{N}, \; \forall M > 0. \tag{5.3}$$

Similar, in view of (2.5) and Theorem 7,

$$\mathbb{E}\|u(t;u_0)\|^M_\mu \le C_\nu(u_0) \quad \forall t \ge 0, \tag{5.4}$$

for each $u_0 \in C^\infty$ and every μ and M as in (5.3). Now let us consider the integral in (1.7) and write it as

$$J_t := \nu \int_t^{t+\nu^{-1}} \mathbb{E}\|u(s)\|^2_m ds.$$

Replacing the integrand in J_t with $\mathbb{E}(\|u_\nu(s)\|_m \wedge N)^2$, $N \geq 1$, using the convergence

$$\mathbb{E}\big(\|u(s; v_0)\|_m \wedge N\big)^2 \to \int \big(\|u\|_m \wedge N\big)^2 \mu_\nu(du) \quad \text{as} \quad s \to \infty \quad \forall N, \tag{5.5}$$

which follows from Corollary 10, and the estimates (5.3), (5.4) we get that

$$J_t \to \int \|u\|_m^2 \, \mu_\nu(du) \quad \text{as} \quad t \to \infty. \tag{5.6}$$

This convergence and (1.7) imply the assertion of Corollary 3.

Now the convergence (5.5) jointly with estimates (5.3), (5.4) and (1.8) imply Corollary 4. □

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Appendix A. Some estimates

For any integer $l \in \mathbb{N}$ and $F \in H^l$ we have that

$$\| \exp(iF(x))\|_l \leq C_l(1 + |F|_\infty)^{l-1}\|F\|_l. \tag{A.1}$$

Indeed, to verify (A.1) it suffices to check that for any non-zero multi-indices $\beta_1, \ldots, \beta_{l'}$, where $1 \leq l' \leq l$ and $|\beta_1| + \cdots + |\beta_{l'}| = l$, we have

$$\|\partial_x^{\beta_1} F \cdots \partial_x^{\beta_{l'}} F\|_0 \leq C|F|_\infty^{l'-1}\|F\|_l. \tag{A.2}$$

But this is the assertion of Lemma 3.10 in [17]. Similarly,

$$\|FG\|_r \leq C_r(|F|_\infty\|G\|_r + |G|_\infty\|F\|_r), \quad F, G \in H^r, \quad r \in \mathbb{N}, \tag{A.3}$$

see [17, Proposition 3.7] (this relation is known as Moser's estimate). Finally, since for $|\beta| \leq m$ we have $|\partial_x^\beta v|_{2m/\beta|} \leq C|v|_\infty^{1-|\beta|/m}\|v\|_m^{|\beta|/m}$ (see relation (3.17) in [17]), then

$$|\langle\langle |v|^2 v, v\rangle\rangle_m| \leq C_m\|v\|_m^2|v|_\infty^2, \qquad |\langle\langle |v|^2 v, v\rangle\rangle_m| \leq C'_m\|v\|_{m+1}^{\frac{2m}{m+1}}|v|_\infty^{\frac{2m+4}{m+1}}. \tag{A.4}$$

Appendix B. Proof of Theorem 8

Applying Ito's formula to a solution $v(\tau)$ of Eq. (2.1) we get a slow time version of the relation (5.1):

$$\|v(\tau)\|_m^2 = \|v_0\|_m^2 + 2\int_0^\tau \big(-\|v\|_{m+1}^2 - \nu^{-1}\langle\langle i|v|^2 v, v\rangle\rangle_m\big)ds + 2B_m\tau + 2M(\tau), \tag{B.1}$$

where $M(\tau)=\int_0^\tau\sum_d b_d|d|^{2m}\langle v_d(s), d\beta_d(s)\rangle$. Since in view of (A.4)

$$\mathbb{E}\big|\langle\!\langle |v|^2v, v\rangle\!\rangle_m\big| \le C_m\big(\mathbb{E}\|v\|_{m+1}^2\big)^{\frac{m}{m+1}}\mathbb{E}\big(|v|_\infty^{2m+4}\big)^{\frac{1}{m+1}},$$

then denoting $\mathbb{E}\|v(\tau)\|_r^2 =: g_r(\tau)$, $r\in\mathbb{N}\cup\{0\}$, taking the expectation of (B.1), differentiating the result and using (2.3), we get that

$$\frac{d}{d\tau}g_m \le -2g_{m+1}+C_m\nu^{-1}g_{m+1}^{\frac{m}{m+1}}+2B_m \le -2g_{m+1}\big(1-C'_m\nu^{-1}g_m^{-\frac{1}{m}}+2B_m\big), \quad \text{(B.2)}$$

since $g_m\le g_0^{1/(m+1)}g_{m+1}^{m/(m+1)}\le C_m g_{m+1}^{m/(m+1)}$. We see that if $g_m\ge(2\nu^{-1}C'_m)^m$, then the r.h.s. of (B.2) is

$$\le -g_{m+1}+2B_m\le -C_m^{-1}g_m^{(m+1)/m}+2B_m\le -\bar{C}_m\nu^{-m-1}+2B_m, \quad \text{(B.3)}$$

which is negative if $\nu\ll 1$. So if

$$g_m(\tau)<(2\nu^{-1}C'_m)^m \quad \text{(B.4)}$$

at $\tau=0$, then (B.4) holds for all $\tau\ge 0$ and (2.4) follows. If $g_m(0)$ violates (B.4), then in view of (B.2) and (B.3), for $\tau\ge 0$, while (B.4) is false, we have that

$$\frac{d}{d\tau}g_m\le -C_m g_m^{(m+1)/m}+2B_m,$$

which again implies (2.4). Besides, in view of (B.2),

$$\frac{d}{d\tau}g_m\le -g_m+C_m(\nu,|v_0|_\infty,B_{m_*},B_m).$$

This relation immediately implies (2.5).

Now let us return to Eq. (B.1). Using Doob's inequality and (2.4) we find that

$$\mathbb{E}\big(\sup_{0\le\tau\le T}|M(\tau)|^2\big)\le C<\infty.$$

Next, applying (A.4) and Young's inequality we get

$$\int_0^\tau\big(-\|v\|_{m+1}^2-\nu^{-1}\langle\!\langle i|v|^2v, v\rangle\!\rangle_m\big)ds\le C_m\int_0^\tau|v(s)|_\infty^{2m+3}ds, \quad \forall\, 0\le\tau\le T.$$

Finally, using in (B.1) the last two displayed formulas jointly with (2.3) we obtain (2.6).

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