



Gaussian fluctuations for the stochastic heat equation with colored noise

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Abstract

In this paper, we present a *quantitative* central limit theorem for the d -dimensional stochastic heat equation driven by a *Gaussian multiplicative noise*, which is *white* in time and has a spatial covariance given by the *Riesz kernel*. We show that the spatial average of the solution over an Euclidean ball is close to a Gaussian distribution, when the radius of the ball tends to infinity. Our central limit theorem is described in the *total variation distance*, using Malliavin calculus and Stein's method. We also provide a functional central limit theorem.

Keywords Stochastic heat equation · Central limit theorem · Malliavin calculus · Stein's method

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1 Introduction

We consider the stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \sigma(u) \dot{W}, \tag{1.1}$$

on $\mathbb{R}_+ \times \mathbb{R}^d$ with initial condition $u(0, x) = 1$, where $\dot{W}(t, x)$ is a centered Gaussian noise with covariance $\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)|x - y|^{-\beta}$, $0 < \beta < \min(d, 2)$. In this paper, $|\cdot|$ denotes the Euclidean norm and we assume the nonlinear term σ is a Lipschitz function with $\sigma(1) \neq 0$; see point (iii) in Remark 3.

Our first result is the following quantitative central limit theorem concerning the spatial average of the solution $u(t, x)$ over $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$.

Theorem 1.1 *For all $t > 0$, there exists a constant $C = C(t, \beta)$, depending on t and β , such that*

$$d_{\text{TV}} \left(\frac{1}{\sigma_R} \int_{B_R} [u(t, x) - 1] dx, Z \right) \leq CR^{-\beta/2},$$

where d_{TV} denotes total variation distance (see (1.2)), $Z \sim N(0, 1)$ is a standard normal random variable and $\sigma_R^2 = \text{Var}(\int_{B_R} [u(t, x) - 1] dx)$. Moreover, the normalization σ_R is of order $R^{d-\frac{\beta}{2}}$, as $R \rightarrow +\infty$ [(see (1.8)].

Remark 1 In the above result we have assumed $u(0, x) = 1$ for the sake of simplicity. However, one can easily extend the result to cover more general initial deterministic condition $u(0, x)$. In fact, this is the topic of Corollary 3.3.

We will mainly rely on the methodology of Malliavin-Stein approach to prove the above result. Such an approach was introduced by Nourdin and Peccati in [9] to, among other things, quantify Nualart and Peccati’s fourth moment theorem in [14]. Notably, if F is a Malliavin differentiable (that means, F belongs to the Sobolev space $\mathbb{D}^{1,2}$), centered Gaussian functional with unit variance, the well-known Malliavin-Stein bound implies

$$d_{\text{TV}}(F, Z) := \sup \left\{ P(F \in A) - P(Z \in A) : A \subset \mathbb{R} \text{ Borel sets} \right\} \tag{1.2}$$

$$\leq 2\sqrt{\text{Var}(\langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})} \quad \text{with } Z \sim N(0, 1), \tag{1.3}$$

where D is the Malliavin derivative, L^{-1} is the pseudo-inverse of the Ornstein-Uhlenbeck operator and $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ denotes the inner product in the Hilbert space \mathfrak{H} associated with the covariance of W ; see also the monograph [10].

It was observed in [16] that instead of $\langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}$, one can work with the term $\langle DF, v \rangle_{\mathfrak{H}}$ once F can be represented as a Skorohod integral $\delta(v)$; see the following result from [16, Proposition 3.1] (see also [8,12]).

Proposition 1.2 *If F is a centered random variable in the Sobolev space $\mathbb{D}^{1,2}$ with unit variance such that $F = \delta(v)$ for some \mathfrak{H} -valued random variable v , then, with $Z \sim N(0, 1)$,*

$$d_{TV}(F, Z) \leq 2\sqrt{\text{Var}(\langle DF, v \rangle_{\mathfrak{H}})}. \tag{1.4}$$

This estimate enables us to bring in tools from stochastic analysis.

Remark 2 Throughout this paper we only work with the total variation distance. However, we point out that we can get the same rate in other frequently used distances. Indeed, if \bar{d} denotes either the Kolmogorov distance, the Wasserstein distance or the Fortet-Mourier distance, we have as well

$$\bar{d}(F, Z) \leq 2\sqrt{\text{Var}(\langle DF, v \rangle_{\mathfrak{H}})},$$

where F, Z are given as in (1.4); see e.g. [10, Chapter 3] for the properties of Stein’s solution. Thus, proceeding in the exact same lines as in this paper, we will get the same rate in these distances.

It is known (see e.g. [4,18]) that Eq. (1.1) has a unique *mild solution* $\{u(t, x) : t \geq 0, x \in \mathbb{R}^d\}$, in the sense that it is adapted to the filtration generated by W , uniformly bounded in $L^2(\Omega)$ over $[0, T] \times \mathbb{R}^d$ (for any finite T) and satisfies the following integral equation in the sense of Dalang-Walsh

$$u(t, x) = \int_{\mathbb{R}^d} p_t(x - y)u(0, y)dy + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y)\sigma(u(s, y))W(ds, dy), \tag{1.5}$$

where $p_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$ denotes the heat kernel and is the *fundamental solution* to the corresponding *deterministic* heat equation.

Let us introduce some handy notation. For fixed $t > 0$, we define

$$\varphi_R(s, y) := \int_{B_R} p_{t-s}(x - y)dx \quad \text{and} \quad G_R(t) := \int_{B_R} [u(t, x) - 1]dx. \tag{1.6}$$

If we put $F_R(t) := G_R(t)/\sigma_R$, then we write, due to the Fubini’s theorem,

$$F_R(t) = \frac{1}{\sigma_R} \int_0^t \int_{\mathbb{R}^d} \varphi_R(s, y)\sigma(u(s, y))W(ds, dy) = \delta(v_R), \tag{1.7}$$

with $v_R(s, y) = \sigma_R^{-1} \mathbf{1}_{[0,t]}(s)\varphi_R(s, y)\sigma(u(s, y))$ taking into account that the Dalang-Walsh integral (1.5) is a particular case of the Skorohod integral; see [11].

By Proposition 1.2, the proof of Theorem 1.1 reduces to estimating the variance of $\langle DF_R(t), v_R \rangle_{\mathfrak{H}}$. One of the key steps, our Proposition 3.2, provides the exact asymptotic behavior of the normalization σ_R^2 :

$$\sigma_R^2 = \text{Var}(G_R(t)) \sim \left(k_\beta \int_0^t \eta^2(s) ds \right) R^{2d-\beta}, \quad \text{as } R \rightarrow +\infty; \tag{1.8}$$

and throughout this paper, we will reserve the notation k_β and $\eta(\cdot)$ for

$$k_\beta := \int_{B_1^2} |x_1 - x_2|^{-\beta} dx_1 dx_2 \quad \text{and} \quad \eta(s) := \mathbb{E}[\sigma(u(s, y))]. \tag{1.9}$$

- Remark 3** (i) The definition of η does not depend on the spatial variable, due to the *strict stationarity* of the solution, meaning that the finite-dimensional distributions of the process $\{u(t, x + y), x \in \mathbb{R}^d\}$ do not depend on y ; see [4].
 (ii) Another consequence of the *strict stationarity* is that the quantity

$$K_p(t) := \mathbb{E}[|u(t, x)|^p] \tag{1.10}$$

does not depend on x . Moreover, $K_p(\cdot)$ is uniformly bounded on compact sets.

- (iii) Under our constant initial condition (i.e. $u(0, x) \equiv 1$), the assumption $\sigma(1) \neq 0$ is *necessary and sufficient* in our paper. It is necessary, in view of the usual *Picard iteration*, to exclude the situation where $u(t, x) \equiv 1$ is the unique solution, and it is sufficient to guarantee that the integral in (1.8) is nonzero. Moreover, we have the following equivalence

$$\sigma(1) = 0 \Leftrightarrow \sigma_R = 0, \forall R > 0 \Leftrightarrow \sigma_R = 0 \text{ for some } R > 0 \Leftrightarrow \lim_{R \rightarrow \infty} \sigma_R^2 R^{\beta-2d} = 0;$$

whose verification can be done in the same way as in [5, Lemma 3.4], so we leave it as an easy exercise for interested readers.

- (iv) Due to the stationarity again, we can see that Theorem 1.1 still holds true when we replace B_R by $\{x \in \mathbb{R}^d : |x - a_R| \leq R\}$, with a_R possibly varying as $R \rightarrow +\infty$. Moreover, the normalization σ_R in this translated version remains unchanged. To see this translational “invariance”, one can alternatively go through the exactly same arguments as in the proof of Theorem 1.1. Note that this invariance under translation justifies the statement in our abstract.
 (v) By going through the exactly same arguments as in the proof of Theorem 1.1, we can obtain the following result: If we replace the ball B_R by the box $\Lambda_R := \{(x_1, \dots, x_d) \in \mathbb{R}^d : |x_i| \leq R, i = 1, \dots, d\}$, Theorem 1.1 still holds true with a slightly different normalization σ'_R given by

$$\sigma'_R \sim \left(\int_0^t \eta^2(s) \int_{\Lambda_1^2} |z - y|^{-\beta} dz dy ds \right)^{1/2} R^{d-\frac{\beta}{2}}, \quad \text{as } R \rightarrow +\infty.$$

Our second main result is the following functional version of Theorem 1.1.

Theorem 1.3 Fix any finite $T > 0$ and recall (1.9). Then, as $R \rightarrow +\infty$, we have

$$\left\{ R^{\frac{\beta}{2}-d} \int_{B_R} [u(t, x) - 1] dx \right\}_{t \in [0, T]} \Rightarrow \left\{ \sqrt{k_\beta} \int_0^t \eta(s) dY_s \right\}_{t \in [0, T]},$$

where Y is a standard Brownian motion and the above weak convergence takes place on the space of continuous functions $C([0, T])$.

A similar problem for the stochastic heat equation on \mathbb{R} has been recently considered in [8], but only in the case of a space-time white noise, where $\eta^2(s)$ appearing in the limiting variance (1.8) is replaced by $\mathbb{E}[\sigma^2(u(s, y))]$. Such a phenomenon also appeared in the case of the one-dimensional wave equation, see the recent paper [5], whose authors also considered the Riesz kernel for the spatial fractional noise therein, which corresponds to the case $\beta = 2 - 2H$.

Remark 4 When $\sigma(x) = x$, the random variable $G_R(t)$ defined in (1.6) has an explicit Wiener chaos expansion:

$$G_R(t) = \int_0^t \int_{\mathbb{R}^d} \varphi_R(s, y) W(ds, dy) + \text{higher-order chaoses.}$$

In this linear case, the asymptotic result (1.8) reduces to $\sigma_R^2 \sim tk_\beta R^{2d-\beta}$, while the first chaotic component of $G_R(t)$ is centered Gaussian with variance equal to

$$\int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, y)\varphi_R(s, z)|y - z|^{-\beta} dydz \sim tk_\beta R^{2d-\beta}, \quad \text{as } R \rightarrow +\infty;$$

see (3.9). Therefore, due to the orthogonality of Wiener chaoses with different orders, we can conclude that $F_R(t) = G_R(t)/\sigma_R$ is asymptotically Gaussian. It is worth pointing out that, unlike in our case, for the linear stochastic heat equation driven by space-time white noise as considered in [8], the central limit is *chaotic*, meaning that each projection on the Wiener chaos contributes to the Gaussian limit. In this case, the proof of asymptotic normality could be based on the chaotic central limit theorem from [7, Theorem 3] (see also [10, Section 6.3]).

The rest of the paper is organized as follows. We present the proofs in Sects. 3 and 4, after we recall briefly some preliminaries in Sect. 2.

Along the paper, we will denote by C a generic constant that may depend on the fixed time T , the parameter β and σ , and it can vary from line to line. We also denote by $\|\cdot\|_p$ the $L^p(\Omega)$ -norm and by L the Lipschitz constant of σ .

2 Preliminaries

Let us build an isonormal Gaussian process from the colored noise W as follows. We begin with a centered Gaussian family of random variables $\{W(\psi) : \psi \in C_c^\infty([0, \infty) \times \mathbb{R}^d)\}$, defined on some complete probability space, such that

$$\mathbb{E}[W(\psi)W(\phi)] = \int_0^\infty \int_{\mathbb{R}^{2d}} \psi(s, x)\phi(s, y)|x - y|^{-\beta} dx dy ds =: \langle \psi, \phi \rangle_{\mathfrak{H}},$$

where $C_c^\infty([0, \infty) \times \mathbb{R}^d)$ is the space of infinitely differentiable functions with compact support on $[0, \infty) \times \mathbb{R}^d$. Let \mathfrak{H} be the Hilbert space defined as the completion of $C_c^\infty([0, \infty) \times \mathbb{R}^d)$ with respect to the above inner product. By a density argument,

we obtain an isonormal Gaussian process $W := \{W(\psi) : \psi \in \mathfrak{H}\}$ meaning that for any $\psi, \phi \in \mathfrak{H}$,

$$\mathbb{E}[W(\psi)W(\phi)] = \langle \psi, \phi \rangle_{\mathfrak{H}}.$$

Let $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ be the natural filtration of W , with \mathcal{F}_t generated by $\{W(\phi) : \phi \text{ continuous and with compact support in } [0, t] \times \mathbb{R}^d\}$. Then for any \mathbb{F} -adapted, jointly measurable random field X such that

$$\mathbb{E}(\|X\|_{\mathfrak{H}}^2) = \mathbb{E}\left(\int_0^\infty ds \int_{\mathbb{R}^{2d}} dx dy X(s, y)X(s, x)|x - y|^{-\beta}\right) < \infty, \tag{2.1}$$

the stochastic integral

$$\int_0^\infty \int_{\mathbb{R}^d} X(s, y)W(ds, dy)$$

is well-defined in the sense of Dalang-Walsh and the following isometry property is satisfied

$$\mathbb{E}\left(\left|\int_0^\infty \int_{\mathbb{R}^d} X(s, y)W(ds, dy)\right|^2\right) = \mathbb{E}(\|X\|_{\mathfrak{H}}^2).$$

In the sequel, we recall some basic facts on Malliavin calculus and we refer readers to the books [11,12] for any unexplained notation and result.

Denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let \mathcal{S} be the space of simple functionals of the form $F = f(W(h_1), \dots, W(h_n))$ for $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in \mathfrak{H}, 1 \leq i \leq n, n \in \mathbb{N}$. Then, DF is the \mathfrak{H} -valued random variable defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))h_i.$$

The derivative operator D is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathfrak{H})$ for any $p \geq 1$ and we let $\mathbb{D}^{1,p}$ be the completion of \mathcal{S} with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathfrak{H}}^p]\right)^{1/p}.$$

We denote by δ the adjoint of D characterized by the integration-by-parts formula

$$\mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF \rangle_{\mathfrak{H}}]$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \text{Dom } \delta \subset L^2(\Omega; \mathfrak{H})$, the domain of δ . The operator δ is also called the Skorohod integral, because in the case of the Brownian motion, it coincides with Skorohod’s extension of the Itô integral (see e.g. [6,13]). In our context, any

adapted random field X satisfying (2.1) belongs to $\text{Dom}\delta$, and $\delta(X)$ coincides with the Dalang-Walsh stochastic integral. As a consequence, the mild formulation (1.5) can be rewritten as

$$u(t, x) = 1 + \delta\left(p_{t-\cdot}(x - *)\sigma(u(\cdot, *))\right).$$

It is known that for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $u(t, x) \in \mathbb{D}^{1,p}$ for any $p \geq 2$ and the derivative satisfies, for $t \geq s$, the linear equation

$$D_{s,y}u(t, x) = p_{t-s}(x - y)\sigma(u(s, y)) + \int_s^t \int_{\mathbb{R}^d} p_{t-r}(x - z)\Sigma(r, z)D_{s,y}u(r, z)W(dr, dz), \tag{2.2}$$

where $\Sigma(r, z)$ is an adapted process, bounded by L . If in addition $\sigma \in C^1(\mathbb{R})$, then $\Sigma(r, z) = \sigma'(u(r, z))$. This result is proved in [11, Proposition 2.4.4] in the case of the stochastic heat equation with Dirichlet boundary conditions on $[0, 1]$ driven by a space-time white noise. Its proof can be easily adapted to our case, see also [1,15].

In the end of this section, we record a technical lemma, whose proof can be found in [3, Lemma 3.11].

Lemma 2.1 *For any $p \in [2, +\infty)$, $0 \leq s \leq t \leq T$ and $x, y \in \mathbb{R}^d$, we have for every $(s, y) \in [0, T] \times \mathbb{R}^d$,*

$$\|D_{s,y}u(t, x)\|_p \leq Cp_{t-s}(x - y) \tag{2.3}$$

for some constant $C = C_{T,p}$ that may depend on T and p .

Note that in [5], the p -norm of the Malliavin derivative $D_{s,y}u(t, x)$ can be bounded by a multiple of the fundamental solution of the wave equation. So it is natural to expect that a similar estimate like (2.3) may hold for a large family of stochastic partial differential equations.

3 Proof of Theorem 1.1

We first state a lemma, which will be used below.

Lemma 3.1 *Let Z be a standard Gaussian vector on \mathbb{R}^d and $\beta \in (0, d)$. Then*

$$\sup_{s>0} \int_{\mathbb{R}^d} p_s(x - y)|x|^{-\beta} dx = \sup_{s>0} \mathbb{E}\left[|y + \sqrt{s}Z|^{-\beta}\right] \leq C|y|^{-\beta}, \tag{3.1}$$

for some constant C that only depends on d and β .

Proof We want to show that

$$\int_{\mathbb{R}^d} p_s(x - y)|x|^{-\beta} dx \leq C|y|^{-\beta},$$

where C does not depend on s . We first prove this for $s = 1$.

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2}} |x|^{-\beta} dx &= \int_{|x| \leq \frac{|y|}{2}} e^{-\frac{|x-y|^2}{2}} |x|^{-\beta} dx + \int_{|x| > \frac{|y|}{2}} e^{-\frac{|x-y|^2}{2}} |x|^{-\beta} dx \\ &\leq \int_{|x| \leq \frac{|y|}{2}} e^{-\frac{|y|^2}{8}} |x|^{-\beta} dx + 2^\beta \int_{|x| \geq \frac{|y|}{2}} e^{-\frac{|x-y|^2}{2}} |y|^{-\beta} dx \\ &\leq C_1 e^{-\frac{|y|^2}{8}} |y|^{-\beta+d} + C_2 |y|^{-\beta}, \end{aligned}$$

where the constants C_1 and C_2 only depend on d, β . So we have proved (3.1) for $s = 1$.

Therefore, we have for general $s > 0$,

$$\mathbb{E}[|y + \sqrt{s}Z|^{-\beta}] = \mathbb{E}\left[|(ys^{-1/2} + Z)\sqrt{s}|^{-\beta}\right] \leq C|ys^{-1/2}|^{-\beta}(\sqrt{s})^{-\beta} = C|y|^{-\beta}.$$

That is, we have proved (3.1) for general $s > 0$.

Now we begin with the estimate (1.8) recalled below.

Proposition 3.2 *With the notation (1.6) and (1.9), we have*

$$\lim_{R \rightarrow +\infty} R^{\beta-2d} \mathbb{E}[G_R^2(t)] = k_\beta \int_0^t \eta^2(s) ds.$$

Proof We define $\Psi(s, y) = \mathbb{E}[\sigma(u(s, 0))\sigma(u(s, y))]$. Then

$$\begin{aligned} \mathbb{E}[G_R^2(t)] &= \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, y)\varphi_R(s, z)\mathbb{E}[\sigma(u(s, y))\sigma(u(s, z))]|y - z|^{-\beta} dydzds \\ &= \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, \xi + z)\varphi_R(s, z)\Psi(s, \xi)|\xi|^{-\beta} d\xi dz ds, \quad \text{by stationarity.} \end{aligned}$$

We claim that

$$\lim_{|\xi| \rightarrow +\infty} \sup_{0 \leq s \leq t} |\Psi(s, \xi) - \eta^2(s)| = 0. \tag{3.2}$$

To see (3.2), we apply a version of the Clark-Ocone formula for square integrable functionals of the noise W and we can write

$$\sigma(u(s, y)) = \mathbb{E}[\sigma(u(s, y))] + \int_0^s \int_{\mathbb{R}^d} \mathbb{E}\left[D_{r,\gamma}(\sigma(u(s, y)))|\mathcal{F}_r\right] W(dr, d\gamma).$$

As a consequence, $\mathbb{E}[\sigma(u(s, y))\sigma(u(s, z))] = \eta^2(s) + T(s, y, z)$, where

$$T(s, y, z) = \int_0^s \int_{\mathbb{R}^{2d}} \mathbb{E}\left\{\mathbb{E}\left[D_{r,\gamma}(\sigma(u(s, y)))|\mathcal{F}_r\right]\mathbb{E}\left[D_{r,\lambda}(\sigma(u(s, z)))|\mathcal{F}_r\right]\right\}$$

$$\times |\gamma - \lambda|^{-\beta} d\gamma d\lambda dr. \tag{3.3}$$

By the chain-rule (see [11]), $D_{r,\gamma}(\sigma(u(s, y))) = \Sigma(s, y)D_{r,\gamma}u(s, y)$, with $\Sigma(\cdot, \ast)$ \mathbb{F} -adapted and bounded by L (in particular, $\Sigma(s, y) = \sigma'(u(s, y))$, when σ is differentiable.) This implies, using (2.3),

$$\begin{aligned} & \left| \mathbb{E} \left\{ \mathbb{E} \left[D_{r,\gamma}(\sigma(u(s, y))) | \mathcal{F}_r \right] \mathbb{E} \left[D_{r,\lambda}(\sigma(u(s, z))) | \mathcal{F}_r \right] \right\} \right| \\ & \leq L^2 \|D_{r,\gamma}u(s, y)\|_2 \|D_{r,\lambda}u(s, z)\|_2 \leq Cp_{s-r}(\gamma - y)p_{s-r}(\lambda - z), \end{aligned} \tag{3.4}$$

for some constant C . Therefore, substituting (3.4) into (3.3) and using the semigroup property in (3.5), we can write

$$\begin{aligned} |T(s, y, z)| & \leq C \int_0^s \int_{\mathbb{R}^{2d}} p_{s-r}(\gamma - y)p_{s-r}(\lambda - z)|\gamma - \lambda|^{-\beta} d\gamma d\lambda dr \\ & = C \int_0^s \int_{\mathbb{R}^d} p_{2s-2r}(\gamma + z - y)|\gamma|^{-\beta} d\gamma dr \end{aligned} \tag{3.5}$$

$$\leq C|y - z|^{-\beta} \xrightarrow{|y-z| \rightarrow +\infty} 0, \tag{3.6}$$

by Lemma 3.1. This completes the verification of (3.2).

Let us continue our proof of Proposition 3.2 by proving that

$$R^{\beta-2d} \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, \xi + z)\varphi_R(s, z)[\Psi(s, \xi) - \eta^2(s)]|\xi|^{-\beta} d\xi dz ds \rightarrow 0, \tag{3.7}$$

as $R \rightarrow +\infty$. Notice that

$$\int_{\mathbb{R}^d} \varphi_R(s, \xi + z)\varphi_R(s, z) dz = \int_{B_R^2} p_{2(t-s)}(x_1 - x_2 - \xi) dx_1 dx_2 \leq CR^d. \tag{3.8}$$

From (3.2) we deduce that given any $\varepsilon > 0$, we find $K = K_\varepsilon > 0$ such that $|\Psi(s, \xi) - \eta^2(s)| < \varepsilon$ for all $s \in [0, t]$ and $|\xi| \geq K$. Now we divide the integration domain in (3.7) into two parts $|\xi| \leq K$ and $|\xi| > K$.

Case (i): On the region $|\xi| \leq K$, since $\Psi(s, y - z) - \eta^2(s) = T(s, y, z)$ is uniformly bounded, using (3.8), we can write

$$\begin{aligned} & R^{\beta-2d} \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, \xi + z)\varphi_R(s, z)|\Psi(s, \xi) - \eta^2(s)| \mathbf{1}_{\{|\xi| \leq K\}} |\xi|^{-\beta} d\xi dz ds \\ & \leq CR^{\beta-2d} \int_0^t \int_{|\xi| \leq K} \left(\int_{\mathbb{R}^d} \varphi_R(s, \xi + z)\varphi_R(s, z) dz \right) |\xi|^{-\beta} d\xi ds \\ & \leq CR^{\beta-d} \int_0^t \int_{|\xi| \leq K} |\xi|^{-\beta} d\xi ds \xrightarrow{R \rightarrow +\infty} 0, \text{ because of } \beta < d. \end{aligned}$$

Case (ii): On the region $[0, t] \times B_K^c$, $|\Psi(s, \xi) - \eta^2(s)| < \varepsilon$. Thus,

$$\begin{aligned} & R^{\beta-2d} \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, \xi + z) \varphi_R(s, z) |\Psi(s, \xi) - \eta^2(s)| \mathbf{1}_{\{|\xi| > K\}} |\xi|^{-\beta} d\xi dz ds \\ & \leq \varepsilon R^{\beta-2d} \int_0^t \int_{\mathbb{R}^d} \int_{B_R^2} p_{2(t-s)}(x_1 - x_2 - \xi) |\xi|^{-\beta} dx_1 dx_2 d\xi ds \\ & = \varepsilon R^{\beta-2d} \int_0^t \int_{B_R^2} \mathbb{E} \left[|x_1 - x_2 - \sqrt{2s}Z|^{-\beta} \right] dx_1 dx_2 ds \\ & = \varepsilon \int_0^t \int_{B_1^2} \mathbb{E} \left[|x_1 - x_2 - R^{-1}\sqrt{2s}Z|^{-\beta} \right] dx_1 dx_2 ds \leq C \varepsilon t k_\beta, \text{ by Lemma 3.1,} \end{aligned}$$

with Z a standard Gaussian random vector on \mathbb{R}^d . This establishes the limit (3.7), since $\varepsilon > 0$ is arbitrary.

Now, it suffices to show that

$$R^{\beta-2d} \int_0^t \eta^2(s) \int_{\mathbb{R}^2} \varphi_R(s, \xi + z) \varphi_R(s, z) |\xi|^{-\beta} d\xi dz ds \rightarrow k_\beta \int_0^t \eta^2(s) ds. \tag{3.9}$$

as $R \rightarrow +\infty$. This follows from arguments similar to those used above. In fact, previous computations imply that the left-hand side of (3.9) is equal to

$$\int_0^t \eta^2(s) \left(\int_{B_1^2} \mathbb{E} \left[|x_1 - x_2 - R^{-1}\sqrt{2s}Z|^{-\beta} \right] dx_1 dx_2 \right) ds.$$

In view of Lemma 3.1 and dominated convergence theorem, we obtain the limit in (3.9) and hence the proof of the proposition is completed. \square

Remark 5 By using the same argument as in the proof of Proposition 3.2, we obtain an asymptotic formula for $\mathbb{E}[G_R(t_i)G_R(t_j)]$ with $t_i, t_j \in \mathbb{R}_+$, which is a useful ingredient for our proof of Theorem 1.3. Suppose $t_i, t_j \in \mathbb{R}_+$, we write

$$\mathbb{E}[G_R(t_i)G_R(t_j)] = \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^{2d}} \varphi_R^{(i)}(s, y) \varphi_R^{(j)}(s, z) \Psi(s, y - z) |y - z|^{-\beta} dy dz ds,$$

with $\varphi_R^{(i)}(s, y) = \int_{B_R} p_{t_i-s}(x - y) dx$, and we obtain

$$\begin{aligned} & \lim_{R \rightarrow +\infty} R^{\beta-2d} \mathbb{E}[G_R(t_i)G_R(t_j)] \\ & = \lim_{R \rightarrow +\infty} R^{\beta-2d} \int_0^{t_i \wedge t_j} ds \eta^2(s) \int_{\mathbb{R}^{2d}} \varphi_R^{(i)}(s, y) \varphi_R^{(j)}(s, z) |y - z|^{-\beta} dy dz \\ & = k_\beta \int_0^{t_i \wedge t_j} \eta^2(s) ds. \end{aligned}$$

We are now ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1 Recall from (1.7) that

$$F_R := F_R(t) = \delta(v_R), \text{ with } v_R(s, y) = \sigma_R^{-1} \mathbf{1}_{[0,t]}(s) \varphi_R(s, y) \sigma(u(s, y)).$$

Moreover,

$$D_{s,y} F_R = \mathbf{1}_{[0,t]}(s) \frac{1}{\sigma_R} \int_{B_R} D_{s,y} u(t, x) dx,$$

and by (2.2) and Fubini’s theorem, we can write

$$\begin{aligned} \int_{B_R} D_{s,y} u(t, x) dx &= \varphi_R(s, y) \sigma(u(s, y)) \\ &+ \int_s^t \int_{\mathbb{R}^d} \varphi_R(r, z) \Sigma(r, z) D_{s,y} u(r, z) W(dr, dz). \end{aligned}$$

Therefore, we have the following decomposition $\langle DF_R, v_R \rangle_{\mathfrak{H}} = B_1 + B_2$, with

$$B_1 = \sigma_R^{-2} \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, y) \varphi_R(s, y') \sigma(u(s, y)) \sigma(u(s, y')) |y - y'|^{-\beta} dy dy' ds$$

and

$$\begin{aligned} B_2 &= \sigma_R^{-2} \int_0^t \int_{\mathbb{R}^{2d}} \left(\int_s^t \int_{\mathbb{R}^d} \varphi_R(r, z) \Sigma(r, z) D_{s,y} u(r, z) W(dr, dz) \right) \\ &\times \varphi_R(s, y') \sigma(u(s, y')) |y - y'|^{-\beta} dy dy' ds. \end{aligned}$$

By Proposition 1.2, $d_{TV}(F_R, Z) \leq 2\sqrt{\text{Var}[\langle DF_R, v_R \rangle_{\mathfrak{H}}]} \leq 2\sqrt{2}(A_1 + A_2)$, with

$$\begin{aligned} A_1 &= \sigma_R^{-2} \int_0^t ds \left(\int_{\mathbb{R}^{4d}} \varphi_R(s, y) \varphi_R(s, y') \varphi_R(s, \tilde{y}) \varphi_R(s, \tilde{y}') |y - y'|^{-\beta} \right. \\ &\times |\tilde{y} - \tilde{y}'|^{-\beta} \text{Cov}[\sigma(u(s, y)) \sigma(u(s, y')), \sigma(u(s, \tilde{y})) \sigma(u(s, \tilde{y}'))] dy dy' d\tilde{y} d\tilde{y}' \Big)^{1/2} \end{aligned}$$

and

$$\begin{aligned} A_2 &= \sigma_R^{-2} \int_0^t \left(\int_{\mathbb{R}^{6d}} \int_s^t \varphi_R(r, z) \varphi_R(r, \tilde{z}) \varphi_R(s, y') \varphi_R(s, \tilde{y}') \right. \\ &\times \mathbb{E} \left\{ \Sigma(r, z) D_{s,y} u(r, z) \Sigma(r, \tilde{z}) D_{s,\tilde{y}} u(r, \tilde{z}) \sigma(u(s, \tilde{y}')) \sigma(u(s, y')) \right\} \\ &\times |y - y'|^{-\beta} |\tilde{y} - \tilde{y}'|^{-\beta} |z - \tilde{z}|^{-\beta} dy dy' d\tilde{y} d\tilde{y}' dz d\tilde{z} dr \Big)^{1/2} ds. \end{aligned}$$

The proof will be done in two steps:

Step 1: Let us first estimate the term A_2 . By Lemma 2.1,

$$\begin{aligned} & \left| \mathbb{E} \left\{ \Sigma(r, z) D_{s,y} u(r, z) \Sigma(r, \tilde{z}) D_{s,\tilde{y}} u(r, \tilde{z}) \sigma(u(s, \tilde{y}')) \sigma(u(s, y')) \right\} \right| \\ & \leq K_4^2(t) L^2 \|D_{s,y} u(r, z)\|_4 \|D_{s,\tilde{y}} u(r, \tilde{z})\|_4 \leq C p_{r-s}(y - z) p_{r-s}(\tilde{y} - \tilde{z}), \end{aligned}$$

where $K_4(t)$ is introduced in (1.10). By Proposition 3.2, we have

$$\begin{aligned} A_2 & \leq C R^{\beta-2d} \int_0^t \left(\int_{\mathbb{R}^{6d}} \int_s^t \varphi_R(r, z) \varphi_R(r, \tilde{z}) \varphi_R(s, y') \varphi_R(s, \tilde{y}') p_{r-s}(y - z) \right. \\ & \quad \left. \times p_{r-s}(\tilde{y} - \tilde{z}) |y - y'|^{-\beta} |\tilde{y} - \tilde{y}'|^{-\beta} |z - \tilde{z}|^{-\beta} dy dy' d\tilde{y} d\tilde{y}' dz d\tilde{z} dr \right)^{1/2} ds, \end{aligned}$$

where the integral in the spatial variables can be rewritten as

$$\begin{aligned} \mathbf{I}_R & := \int_{B_R^4} \int_{\mathbb{R}^{6d}} p_{t-r}(x - z) p_{t-r}(\tilde{x} - \tilde{z}) p_{t-s}(x' - y') p_{t-s}(\tilde{x}' - \tilde{y}') p_{r-s}(y - z) \\ & \quad \times p_{r-s}(\tilde{y} - \tilde{z}) |y - y'|^{-\beta} |\tilde{y} - \tilde{y}'|^{-\beta} |z - \tilde{z}|^{-\beta} dx d\tilde{x} dx' d\tilde{x}' dy dy' d\tilde{y} d\tilde{y}' dz d\tilde{z}. \end{aligned}$$

Making the change of variables ($\theta = x - z, \tilde{\theta} = \tilde{x} - \tilde{z}, \eta' = x' - y', \tilde{\eta}' = \tilde{x}' - \tilde{y}', \eta = y - z$ and $\tilde{\eta} = \tilde{y} - \tilde{z}$) yields,

$$\begin{aligned} \mathbf{I}_R & = \int_{B_R^4} \left(\int_{\mathbb{R}^{6d}} p_{t-r}(\theta) p_{t-r}(\tilde{\theta}) p_{t-s}(\eta') p_{t-s}(\tilde{\eta}') p_{r-s}(\eta) p_{r-s}(\tilde{\eta}) \right. \\ & \quad \times |x - x' - \theta + \eta + \eta'|^{-\beta} |\tilde{x} - \tilde{x}' - \tilde{\theta} + \tilde{\eta} + \tilde{\eta}'|^{-\beta} \\ & \quad \left. \times |x - \tilde{x} - \theta + \tilde{\theta}|^{-\beta} d\eta d\eta' d\tilde{\eta} d\tilde{\eta}' d\theta d\tilde{\theta} \right) dx d\tilde{x} dx' d\tilde{x}'. \end{aligned}$$

This can be written as

$$\begin{aligned} \mathbf{I}_R & = \int_{B_R^4} dx d\tilde{x} dx' d\tilde{x}' \mathbb{E} \left\{ |x - x' + \sqrt{t-r} Z_1 + \sqrt{r-s} Z_5 + \sqrt{t-s} Z_3|^{-\beta} \right. \\ & \quad \left. \times |\tilde{x} - \tilde{x}' + \sqrt{t-r} Z_2 + \sqrt{r-s} Z_6 + \sqrt{t-s} Z_4|^{-\beta} |x - \tilde{x} + \sqrt{t-r}(Z_1 + Z_2)|^{-\beta} \right\} \\ & \leq C \int_{B_R^4} dx d\tilde{x} dx' d\tilde{x}' \mathbb{E} \left\{ |x - x' + \sqrt{t-r} Z_1|^{-\beta} |\tilde{x} - \tilde{x}' - \sqrt{t-r} Z_2|^{-\beta} \right. \\ & \quad \left. \times |x - \tilde{x} + \sqrt{t-r}(Z_1 + Z_2)|^{-\beta} \right\}, \end{aligned}$$

with Z_1, \dots, Z_6 i.i.d. standard Gaussian vectors on \mathbb{R}^d , here we have used Lemma 3.1 repeatedly for Z_5, Z_3, Z_6, Z_4 to obtain the last inequality. Making the change of variables $x \rightarrow Rx, \tilde{x} \rightarrow R\tilde{x}, x' \rightarrow Rx'$ and $\tilde{x}' \rightarrow R\tilde{x}'$, we can write

$$\begin{aligned} \mathbf{I}_R &= CR^{4d-3\beta} \mathbb{E} \int_{B_1^4} dx d\tilde{x} dx' d\tilde{x}' |x - x' + R^{-1}\sqrt{t-r}Z_1|^{-\beta} \\ &\quad \times |\tilde{x} - \tilde{x}' - R^{-1}\sqrt{t-r}Z_2|^{-\beta} |x - \tilde{x} + R^{-1}\sqrt{t-r}(Z_1 + Z_2)|^{-\beta} \\ &\leq CR^{4d-3\beta} \mathbb{E} \int_{B_2^3} dy_1 dy_2 dy_3 |y_1 + R^{-1}\sqrt{t-r}Z_1|^{-\beta} |y_2 - R^{-1}\sqrt{t-r}Z_2|^{-\beta} \\ &\quad \times |y_3 + R^{-1}\sqrt{t-r}(Z_1 + Z_2)|^{-\beta}, \end{aligned}$$

where the second inequality follows from the change of variables $x - x' = y_1, \tilde{x} - \tilde{x}' = y_2, x - \tilde{x} = y_3$. Taking into account the fact that

$$\sup_{z \in \mathbb{R}^d} \int_{B_2} |y + z|^{-\beta} dy < +\infty, \tag{3.10}$$

we obtain

$$\mathbf{I}_R \leq CR^{4d-3\beta} \left(\sup_{z \in \mathbb{R}^d} \int_{B_2} |y + z|^{-\beta} dy \right)^3 \leq CR^{4d-3\beta},$$

and it follows immediately that $A_2 \leq CR^{-\beta/2}$.

Step 2: We now estimate A_1 . We begin by estimating the covariance

$$\text{Cov} \left[\sigma(u(s, y))\sigma(u(s, y')), \sigma(u(s, \tilde{y}))\sigma(u(s, \tilde{y}')) \right]. \tag{3.11}$$

Using a version of Clark–Ocone formula for square integrable functionals of the noise W , we can write

$$\begin{aligned} \sigma(u(s, y))\sigma(u(s, y')) &= \mathbb{E}[\sigma(u(s, y))\sigma(u(s, y'))] \\ &\quad + \int_0^s \int_{\mathbb{R}^d} \mathbb{E} \left[D_{r,z}(\sigma(u(s, y))\sigma(u(s, y'))) | \mathcal{F}_r \right] W(dr, dz). \end{aligned}$$

Then, we represent the covariance (3.11) as

$$\begin{aligned} &\int_0^s \int_{\mathbb{R}^{2d}} \mathbb{E} \left\{ \mathbb{E} \left[D_{r,z}(\sigma(u(s, y))\sigma(u(s, y'))) | \mathcal{F}_r \right] \right. \\ &\quad \times \left. \mathbb{E} \left[D_{r,z'}(\sigma(u(s, \tilde{y}))\sigma(u(s, \tilde{y}'))) | \mathcal{F}_r \right] \right\} |z - z'|^{-\beta} dz dz' dr. \end{aligned}$$

By the chain rule, $D_{r,z}(\sigma(u(s, y))\sigma(u(s, y'))) = \sigma(u(s, y))\Sigma(s, y')D_{r,z}u(s, y') + \sigma(u(s, y'))\Sigma(s, y)D_{r,z}u(s, y)$. Therefore, $\|\mathbb{E}[D_{r,z}(\sigma(u(s, y))\sigma(u(s, y'))) | \mathcal{F}_r]\|_2$ is bounded by $2K_4(t)L\{\|D_{r,z}u(s, y)\|_4 + \|D_{r,z}u(s, y')\|_4\}$. Using Lemma 2.1 again, we see that the covariance (3.11) is bounded by

$$4L^2K_4^2(t) \int_0^s \int_{\mathbb{R}^{2d}} \left(\|D_{r,z}u(s, y)\|_4 + \|D_{r,z}u(s, y')\|_4 \right)$$

$$\begin{aligned} & \times \left(\|D_{r,z'}u(s, \tilde{y})\|_4 + \|D_{r,z'}u(s, \tilde{y}')\|_4 \right) |z - z'|^{-\beta} dz dz' dr \\ & \leq C \int_0^s \int_{\mathbb{R}^{2d}} (p_{s-r}(y - z) + p_{s-r}(y' - z)) \\ & \quad \times (p_{s-r}(\tilde{y} - z') + p_{s-r}(\tilde{y}' - z')) |z - z'|^{-\beta} dz dz' dr. \end{aligned}$$

Consequently, the spatial integral in the expression of A_1 can be bounded by

$$\begin{aligned} \mathbf{J}_R & := C \int_0^s \int_{\mathbb{R}^{6d}} \varphi_R(s, y) \varphi_R(s, y') \varphi_R(s, \tilde{y}) \varphi_R(s, \tilde{y}') \\ & \quad \times |y - y'|^{-\beta} |\tilde{y} - \tilde{y}'|^{-\beta} |z - z'|^{-\beta} (p_{s-r}(y - z) + p_{s-r}(y' - z)) \\ & \quad \times (p_{s-r}(\tilde{y} - z') + p_{s-r}(\tilde{y}' - z')) dy dy' d\tilde{y} d\tilde{y}' dz dz' dr \\ & = 4C \int_0^s dr \int_{\mathbb{R}^{6d}} dy dy' d\tilde{y} d\tilde{y}' dz dz' \varphi_R(s, y) \varphi_R(s, y') \varphi_R(s, \tilde{y}) \varphi_R(s, \tilde{y}') \\ & \quad \times |y - y'|^{-\beta} |\tilde{y} - \tilde{y}'|^{-\beta} |z - z'|^{-\beta} p_{s-r}(y - z) p_{s-r}(\tilde{y} - z'), \quad \text{by symmetry.} \end{aligned}$$

Then, it follows from exactly the same argument as in the estimation of \mathbf{I}_R in the previous step that $\mathbf{J}_R \leq CR^{4d-3\beta}$. Indeed, we have

$$\begin{aligned} & \int_0^s \int_{B_R^4} \int_{\mathbb{R}^{6d}} p_{t-s}(x - y) p_{t-s}(x' - y') p_{t-s}(\tilde{x} - \tilde{y}) p_{t-s}(\tilde{x}' - \tilde{y}') \\ & \quad \times |y - y'|^{-\beta} |\tilde{y} - \tilde{y}'|^{-\beta} |z - z'|^{-\beta} p_{s-r}(y - z) p_{s-r}(\tilde{y} - z') \\ & \quad \quad \quad dx dx' d\tilde{x} d\tilde{x}' dy dy' d\tilde{y} d\tilde{y}' dz dz' dr \\ & = \int_0^s dr \int_{B_R^4} dx dx' d\tilde{x} d\tilde{x}' \int_{\mathbb{R}^{6d}} p_{t-s}(\theta) p_{t-s}(\theta') p_{t-s}(\tilde{\theta}) p_{t-s}(\tilde{\theta}') p_{s-r}(\eta) \\ & \quad \times p_{s-r}(\tilde{\eta}) |x - x' - \theta + \theta'|^{-\beta} |\tilde{x} - \tilde{x}' - \tilde{\theta} + \tilde{\theta}'|^{-\beta} \\ & \quad \times |x - \tilde{x} - \eta + \theta + \tilde{\theta} + \tilde{\eta}|^{-\beta} d\theta d\theta' d\tilde{\theta} d\tilde{\theta}' d\eta d\tilde{\eta} \\ & = \int_0^s dr \int_{B_R^4} \mathbb{E} \left\{ |x - x' + \sqrt{t-s}(Z_2 - Z_1)|^{-\beta} |\tilde{x} - \tilde{x}' + \sqrt{t-s}(Z_4 - Z_3)|^{-\beta} \right. \\ & \quad \times \left. |x - \tilde{x} + \sqrt{t-s}(Z_1 + Z_3) + \sqrt{2s-2r}Z_5|^{-\beta} \right\} dx dx' d\tilde{x} d\tilde{x}' \\ & \leq C \int_0^s dr \int_{B_R^4} \mathbb{E} \left\{ |x - x' - \sqrt{t-s}Z_1|^{-\beta} |\tilde{x} - \tilde{x}' - \sqrt{t-s}Z_3|^{-\beta} \right. \\ & \quad \times \left. |x - \tilde{x} + \sqrt{t-s}(Z_1 + Z_3)|^{-\beta} \right\} dx dx' d\tilde{x} d\tilde{x}' \tag{3.12} \\ & \leq CR^{4d-3\beta} \mathbb{E} \int_{B_1^4} dx dx' d\tilde{x} d\tilde{x}' |x - x' - R^{-1}\sqrt{t-s}Z_1|^{-\beta} \\ & \quad \times |\tilde{x} - \tilde{x}' - R^{-1}\sqrt{t-s}Z_3|^{-\beta} \times |x - \tilde{x} + R^{-1}\sqrt{t-s}(Z_1 + Z_3)|^{-\beta} \\ & \leq CR^{4d-3\beta} \mathbb{E} \int_{B_2^3} |y_1 - R^{-1}\sqrt{t-s}Z_1|^{-\beta} \end{aligned}$$

$$\begin{aligned} & \times |y_2 - R^{-1}\sqrt{t-s}Z_3|^{-\beta} \times |y_3 + R^{-1}\sqrt{t-s}(Z_1 + Z_3)|^{-\beta} dy_1 dy_2 dy_3 \\ & \leq CR^{4d-3\beta} \left(\sup_{z \in \mathbb{R}^d} \int_{B_2} |y+z|^{-\beta} dy \right)^3 \leq CR^{4d-3\beta}, \quad \text{by (3.10),} \end{aligned} \tag{3.13}$$

where we have used Lemma 3.1 for Z_5, Z_2, Z_4 to obtain (3.12). This gives us the desired estimate for A_1 and finishes our proof. \square

With a slight modification of the above proof, we can extend Theorem 1.1 to more general initial conditions.

Corollary 3.3 *Assume that there are two positive constants c_1, c_2 such that $c_1 \leq u(0, x) \leq c_2$, and assume one of the following two conditions:*

- (1) σ is a non-negative, nondecreasing Lipschitz function with $\sigma(c_1) > 0$.
- (2) $-\sigma$ is a non-negative, nondecreasing Lipschitz function with $\sigma(c_1) < 0$.

Define

$$\tilde{F}_R(t) = \frac{1}{\tilde{\sigma}_R} \int_{B_R} [u(t, x) - \mathbb{E}u(t, x)] dx, \tag{3.14}$$

where

$$\tilde{\sigma}_R^2 = \text{Var} \left(\int_{B_R} [u(t, x) - \mathbb{E}u(t, x)] dx \right) \in (0, \infty)$$

for every $R > 0$. Then, there exists a constant $C = C(t, \beta)$, depending on t and β , such that

$$d_{TV}(\tilde{F}_R(t), N(0, 1)) \leq CR^{-\beta/2}.$$

Proof We write $\tilde{F}_R(t)$ as $\tilde{G}_R(t)/\tilde{\sigma}_R$. Let $u_1(t, x)$ be the solution to Eq. (1.1) with initial condition $u(0, x) = c_1$. According to the weak comparison principle (see [2, Theorem 1.1]), $u(t, x) \geq u_1(t, x)$ almost surely for every $t \geq 0$ and $x \in \mathbb{R}^d$, which immediately implies

$$\sigma(u(s, y))\sigma(u(s, z)) \geq \sigma(u_1(s, y))\sigma(u_1(s, z)) \geq 0 \quad \text{almost surely,}$$

so that

$$\begin{aligned} \mathbb{E}[\tilde{G}_R^2(t)] &= \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, y)\varphi_R(s, z) \mathbb{E}[\sigma(u(s, y))\sigma(u(s, z))] |y-z|^{-\beta} dy dz ds \\ &\geq \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, y)\varphi_R(s, z) \mathbb{E}[\sigma(u_1(s, y))\sigma(u_1(s, z))] |y-z|^{-\beta} dy dz ds. \end{aligned} \tag{3.15}$$

In view of point (iii) from Remark 3, our assumption $\sigma(c_1) \neq 0$ guarantees that the last integral (3.15) has the exact order $R^{2d-\beta}$: More precisely,

$$(3.15) \sim \kappa_\beta \left(\int_0^t \eta_1(s)^2 ds \right) R^{2d-\beta} \quad \text{as } R \rightarrow +\infty,$$

where $\eta_1(s) := \mathbb{E}[\sigma(u_1(s, y))]$ does not depend on y and $\int_0^t \eta_1(s)^2 ds \in (0, \infty)$; see also (1.9).

Now let $u_2(t, x)$ be the solution to the Eq. (1.1) with initial condition $u(0, x) = c_2$. Notice that by our assumption on σ , we get $\sigma(c_2) \neq 0$. And by applying the same weak comparison principle, we deduce that

$$\begin{aligned} \mathbb{E}[\tilde{G}_R^2(t)] &\leq \int_0^t \int_{\mathbb{R}^{2d}} \varphi_R(s, y)\varphi_R(s, z)\mathbb{E}[\sigma(u_2(s, y))\sigma(u_2(s, z))]|y - z|^{-\beta} dydz ds \\ &\sim \kappa_\beta \left(\int_0^t \eta_2(s)^2 ds \right) R^{2d-\beta} \quad \text{as } R \rightarrow +\infty, \end{aligned}$$

where $\eta_2(s) := \mathbb{E}[\sigma(u_2(s, y))]$ does not depend on y and $\int_0^t \eta_2(s)^2 ds \in (0, \infty)$. This implies that

$$0 < \liminf_{R \rightarrow +\infty} \tilde{\sigma}_R^2 R^{\beta-2d} \leq \limsup_{R \rightarrow +\infty} \tilde{\sigma}_R^2 R^{\beta-2d} < +\infty.$$

The rest of the proof follows the same lines as the proof of Theorem 1.1. □

4 Proof of Theorem 1.3

In order to prove Theorem 1.3, we need to establish the convergence of the finite-dimensional distributions as well as the tightness.

Convergence of finite-dimensional distributions. Fix $0 \leq t_1 < \dots < t_m \leq T$ and consider

$$F_R(t_i) := R^{\frac{\beta}{2}-d} \int_{B_R} [u(t_i, x) - 1] dx = \delta(v_R^{(i)}) \text{ for } i = 1, \dots, m,$$

where

$$v_R^{(i)}(s, y) = \mathbf{1}_{[0, t_i]}(s) R^{\frac{\beta}{2}-d} \sigma(u(s, y)) \varphi_R^{(i)}(s, y)$$

with $\varphi_R^{(i)}(s, y) = \int_{B_R} p_{t_i-s}(x - y) dx$. Set $\mathbf{F}_R = (F_R(t_1), \dots, F_R(t_m))$ and let Z be a centered Gaussian vector on \mathbb{R}^m with covariance $(C_{i,j})_{1 \leq i, j \leq m}$ given by

$$C_{i,j} := \kappa_\beta \int_0^{t_i \wedge t_j} \eta^2(r) dr, \text{ with } \eta(r) := \mathbb{E}[\sigma(u(r, y))].$$

To proceed, we need the following generalization of [10, Theorem 6.1.2]; see [8, Proposition 2.3] for a proof.

Lemma 4.1 *Let $F = (F^{(1)}, \dots, F^{(m)})$ be a random vector such that $F^{(i)} = \delta(v^{(i)})$ for $v^{(i)} \in \text{Dom } \delta$ and $F^{(i)} \in \mathbb{D}^{1,2}$, $i = 1, \dots, m$. Let Z be an m -dimensional centered Gaussian vector with covariance $(C_{i,j})_{1 \leq i,j \leq m}$. For any C^2 function $h : \mathbb{R}^m \rightarrow \mathbb{R}$ with bounded second partial derivatives, we have*

$$|\mathbb{E}[h(F)] - \mathbb{E}[h(Z)]| \leq \frac{m}{2} \|h''\|_\infty \sqrt{\sum_{i,j=1}^m \mathbb{E}[(C_{i,j} - \langle DF^{(i)}, v^{(j)} \rangle_{\mathfrak{H}})^2]},$$

where $\|h''\|_\infty := \sup \left\{ \left| \frac{\partial^2}{\partial x_i \partial x_j} h(x) \right| : x \in \mathbb{R}^d, i, j = 1, \dots, m \right\}$.

In view of Lemma 4.1, the proof of $F_R \xrightarrow{\text{law}} Z$ boils down to the $L^2(\Omega)$ -convergence of $\langle DF_R(t_i), v_R^{(j)} \rangle_{\mathfrak{H}}$ to $C_{i,j}$ for each i, j , as $R \rightarrow +\infty$. The case $i = j$ has been covered in the proof of Theorem 1.1 and for the other cases, we need to show

$$\mathbb{E}[F_R(t_i)F_R(t_j)] \rightarrow C_{i,j} \tag{4.1}$$

and

$$\text{Var}(\langle DF_R(t_i), v_R^{(j)} \rangle_{\mathfrak{H}}) \rightarrow 0, \tag{4.2}$$

as $R \rightarrow +\infty$. The point (4.1) has been established in Remark 5. To see point (4.2), we put

$$B_1(i, j) = R^{\beta-2d} \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^d} \varphi_R^{(i)}(s, y) \varphi_R^{(j)}(s, y') \sigma(u(s, y)) \sigma(u(s, y')) \times |y - y'|^{-\beta} dy dy' ds$$

and

$$B_2(i, j) = R^{\beta-2d} \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^{2d}} \varphi_R^{(j)}(s, y') \sigma(u(s, y')) |y - y'|^{-\beta} \times \left(\int_s^{t_i} \int_{\mathbb{R}^d} \varphi_R^{(i)}(r, z) \Sigma(r, z) D_{s,y} u(r, z) W(dr, dz) \right) dy dy' ds,$$

so that $\langle DF_R(t_i), v_R^{(j)} \rangle_{\mathfrak{H}} = B_1(i, j) + B_2(i, j)$. Therefore,

$$\begin{aligned} & \mathbb{E} \left[\left(C_{i,j} - \langle DF_R(t_i), v_R^{(j)} \rangle_{\mathfrak{H}} \right)^2 \right] \\ & \leq 3(C_{i,j} - \mathbb{E}[B_1(i, j)])^2 + 3\text{Var}[B_1(i, j)] + 3\text{Var}[B_2(i, j)]. \end{aligned}$$

Going through the same lines as for the estimation of A_1, A_2 , one can verify easily that both $\text{Var}[B_1(i, j)]$ and $\text{Var}[B_2(i, j)]$ vanish asymptotically, as R tends to infinity. By recalling that $C_{ij} = \lim_{R \rightarrow +\infty} \mathbb{E}[B_1(i, j)]$, the convergence of finite-dimensional distributions is thus established. \square

Tightness The following proposition together with Kolmogorov’s criterion ensures the tightness of the processes $\{R^{\frac{\beta}{2}-d} \int_{B_R} [u(t, x) - 1] dx, t \in [0, T]\}, R > 0$.

Proposition 4.2 *Recall the notation (1.6). For any $0 \leq s < t \leq T$, for any $p \in [2, \infty)$ and any $\alpha \in (0, 1)$, it holds that*

$$\mathbb{E}[|G_R(t) - G_R(s)|^p] \leq CR^{p(d-\frac{\beta}{2})}(t-s)^{\alpha p/2}, \tag{4.3}$$

where the constant $C = C_{p,T,\alpha}$ may depend on T, p and α .

Notice that the p th moment of an increment of the solution $u(t, x) - u(s, x)$ is bounded by a constant times $|t - s|^{\frac{\alpha p}{2}(1-\frac{\beta}{2})}$, for any $\alpha \in (0, 1)$, see [17]; and here the spatial averaging has improved the Hölder continuity.

Now we present the proof of Proposition 4.2. Let $0 \leq s < t \leq T$ and set

$$\Theta_{x,t,s}(r, y) = p_{t-r}(x - y)\mathbf{1}_{\{r \leq t\}} - p_{s-r}(x - y)\mathbf{1}_{\{r \leq s\}}.$$

Then,

$$G_R(t) - G_R(s) = \int_0^T \int_{\mathbb{R}^d} \left(\int_{B_R} \Theta_{x,t,s}(r, y) dx \right) \sigma(u(r, y)) W(dr, dy),$$

so that

$$\begin{aligned} & \mathbb{E}[|G_R(t) - G_R(s)|^p] \\ & \leq C \left\| \int_0^T dr \int_{\mathbb{R}^{2d}} dy dy' |y - y'|^{-\beta} \left(\int_{B_R} \Theta_{x,t,s}(r, y) dx \right) \sigma(u(r, y)) \right. \\ & \quad \times \left. \left[\int_{B_R} \Theta_{\tilde{x},t,s}(r, y) d\tilde{x} \right] \sigma(u(r, y')) \right\|_{p/2}^{p/2} \text{ by Burkholder's inequality} \\ & \leq C \left\{ \int_0^T dr \int_{\mathbb{R}^{2d}} dy dy' |y - y'|^{-\beta} \left| \int_{B_R} \Theta_{x,t,s}(r, y) dx \right| \times \left| \int_{B_R} \Theta_{\tilde{x},t,s}(r, y) d\tilde{x} \right| \right. \\ & \quad \times \left. \left\| \sigma(u(r, y)) \sigma(u(r, y')) \right\|_{p/2} \right\}^{p/2} \text{ by Minkowski's inequality.} \end{aligned}$$

Therefore, taking into account that $\|\sigma(u(r, y))\sigma(u(r, y'))\|_{p/2}$ is uniformly bounded, we obtain

$$\begin{aligned} & \mathbb{E}[|G_R(t) - G_R(s)|^p] \\ & \leq C \left\{ \int_0^T \int_{\mathbb{R}^{2d}} \left| \int_{B_R} \Theta_{x,t,s}(r, y) dx \right| \left| \int_{B_R} \Theta_{\tilde{x},t,s}(r, y) d\tilde{x} \right| |y - y'|^{-\beta} dy dy' dr \right\}^{p/2} \end{aligned}$$

$$\begin{aligned}
 &\leq C \left\{ \int_0^s \int_{\mathbb{R}^{2d}} \left| \int_{B_R} (p_{t-r}(x-y) - p_{s-r}(x-y)) dx \right| \right. \\
 &\quad \times \left. \left| \int_{B_R} (p_{t-r}(\tilde{x}-y') - p_{s-r}(\tilde{x}-y')) d\tilde{x} \right| |y-y'|^{-\beta} dy dy' dr \right\}^{p/2} \\
 &\quad + C \left[\int_s^t \int_{\mathbb{R}^{2d}} \left(\int_{B_R^2} p_{t-r}(x-y) p_{t-r}(\tilde{x}-y') dx d\tilde{x} \right) |y-y'|^{-\beta} dy dy' dr \right]^{p/2} \\
 &=: C(\mathbf{T}_1^{p/2} + \mathbf{T}_2^{p/2}).
 \end{aligned}$$

Estimation of the term \mathbf{T}_1 . We need Lemma 3.1 in [2]: For all $\alpha \in (0, 1)$, $x, y \in \mathbb{R}^d$ and $t' \geq t > 0$,

$$|p_t(x) - p_{t'}(x)| \leq C t^{-\alpha/2} (t' - t)^{\alpha/2} p_{4t'}(x).$$

Thus,

$$\begin{aligned}
 &(t-s)^{-\alpha} \mathbf{T}_1 \\
 &\leq \int_0^s dr (s-r)^{-\alpha} \int_{B_R^2} dx d\tilde{x} \int_{\mathbb{R}^{2d}} dy dy' p_{4(t-r)}(x-y) p_{4(t-r)}(\tilde{x}-y') |y-y'|^{-\beta} \\
 &= \int_0^s dr (s-r)^{-\alpha} \int_{B_R^2} dx d\tilde{x} \int_{\mathbb{R}^{2d}} dy dy' p_{4(t-r)}(\theta) p_{4(t-r)}(\tilde{\theta}) |x-\tilde{x}-\theta+\tilde{\theta}|^{-\beta},
 \end{aligned}$$

by the change of variable $\theta = x - y$, $\tilde{\theta} = \tilde{x} - y'$. Consequently, with Z a standard Gaussian random vector on \mathbb{R}^d , we continue to write

$$\begin{aligned}
 (t-s)^{-\alpha} \mathbf{T}_1 &\leq \int_0^s dr (s-r)^{-\alpha} \int_{B_R^2} dx d\tilde{x} \mathbb{E} \left[|x-\tilde{x} + \sqrt{8t-8r} Z|^{-\beta} \right] \\
 &= R^{2d-\beta} \int_0^s dr (s-r)^{-\alpha} \int_{B_1^2} dx d\tilde{x} \mathbb{E} \left[|x-\tilde{x} + R^{-1} \sqrt{8t-8r} Z|^{-\beta} \right] \\
 &\leq C R^{2d-\beta} \int_0^s dr (s-r)^{-\alpha} \int_{B_1^2} dx d\tilde{x} |x-\tilde{x}|^{-\beta} \quad \text{using Lemma 3.1} \\
 &\leq C R^{2d-\beta}.
 \end{aligned}$$

Estimation of the term \mathbf{T}_2 . We use a similar change of variable as before:

$$\begin{aligned}
 \mathbf{T}_2 &= \int_s^t dr \int_{B_R^2} dx d\tilde{x} \int_{\mathbb{R}^{2d}} p_{t-r}(\theta) p_{t-r}(\tilde{\theta}) |x-\tilde{x} + \tilde{\theta} - \theta|^{-\beta} d\theta d\tilde{\theta} \\
 &= \int_s^t dr \int_{B_R^2} dx d\tilde{x} \mathbb{E} \left\{ |x-\tilde{x} + \sqrt{2t-2r} Z|^{-\beta} \right\}
 \end{aligned}$$

$$\begin{aligned} &\leq CR^{2d-\beta} \int_s^t dr \int_{B_1^2} dx d\tilde{x} |x - \tilde{x}|^{-\beta} \quad \text{using Lemma 3.1} \\ &= CR^{2d-\beta} (t-s) k_\beta. \end{aligned}$$

Combining the above two estimates, we obtain (4.3). \square

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