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On a stochastic evolution equation with random growth conditions

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Abstract A stochastic forcing of a non-linear singular/degenerated parabolic problem with random growth conditions is proposed in the framework of Orlicz Lebesgue and Sobolev spaces with variable random exponents. We give a result of existence and uniqueness of the solution, for additive and multiplicative problems.

Keywords w-Operator · Random variable exponent · Stochastic forcing

Mathematics Subject Classification 35K92 · 46E30 · 60H15

1 Introduction

Problems in variable exponent Lebesgue and Sobolev spaces (i.e. when the classical Lebesgue exponent p depends on the time–space arguments) have been intensively studied since the years 2000. One can find now in the literature, since the founding work of Zhikov [24], many references concerning the theoretical mathematical point of view, but also many applications in physics and image restoration.

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In addition to the important scientific contribution of Zhikov let us mention the monograph [11] and we invite the reader to consult the references of this book for more information on general Orlicz-type spaces.

The main physical motivation for the study of Lebesgue and Sobolev spaces with variable exponent was induced by the modelling of electrorheological fluids and we refer to [21] and the monograph [20].

Another classical application concerns image restoration, as in [18] for example.

Following the general remarks in [1–3,24] for the elliptic case with $p(x)$ and [4,12] in the parabolic one with $p(t, x)$ (and the important literature of these authors), each model is subject to certain variation of the nonlinear terms: parameters that determine a model, that are constant in certain ranges, have to change when some threshold values are reached. This can be done for example by varying the exponents which are describing the growth conditions of the nonlinear terms.

This is e.g. the case in transformations of thermo-rheological fluids, since these fluids strongly depend on the temperature and the temperature can be given by another equation. In this way, one has to consider models given by systems of type $u_t + A(u, v) = f,\ v_t + Bv = g$ where A and B are nonlinear operators and the growth of A depends on $p(v)$; for example when $A(u, v) = -\text{div}\,[|\nabla u|^{p(v)-2}\nabla u]$.

Since reality is complex, one always considers flawed models and/or data. This is why it is of interest to consider random or stochastic problems.

In the case of random variable exponents, let us mention extensions of [15] and of the properties of the maximal function to the case of a random exponent $p(\omega)$ in [5,17] for martingales and to $p(x, \omega)$ in [22]. This corresponds for example to the case of a system of type $u_t + A(u, v) = f,\ v_t + B(\omega, v) = g(\omega)$ where v gives A a random behavior.

In the case of a stochastic forcing, if the system is of type $du + A(u, v)dt = f dw,\ v_t + B(v) = g$ where w denotes a Wiener process, one can find in the literature the existence of a solution with values in general Orlicz-spaces [19] that corresponds to the $-\Delta_{p(x)}$ case, and [7] for $-\Delta_{p(t,x)}$ stochastic problems.

Thinking about a system, it seems then more natural to consider a stochastic perturbation acting on both equations, i.e., considering systems of type $du + A(u, v)dt = f dw,\ dv + B(v)dt = gdw$. Hence our interest in this paper is the study of problems with growth conditions described by a variable exponent p which may depend on t, x and ω with suitable measurability assumptions with respect to a given filtration. Let us remark that the properties of It's integral will be formally compatible with the technical assumptions on p and on the operator used in the sequel: the predictability of the solution to It's problem with Hölder-continuous paths. This last property is of importance since one needs, for technical reasons, to consider log-Hölder continuous[1] exponents p with respect to the variables t and x.

[1] A function f is Log-Hölder continuous if, for a constant $c \geq 0$, $|f(x) - f(y)| \leq c/\ln[e + 1/|x - y|]$. If f is Hölder continuous with Hölder exponent α, then it is also log-Hölder continuous since $|f(x) - f(y)|\ln[e + 1/|x - y|] \leq c|x - y|^\theta \ln[e + 1/|x - y|]$ and since $\alpha \mapsto \alpha^\theta \ln[e + 1/\alpha]$ is continuous on $[0, M]$ for any positive M.

In this paper, our aim is to study existence and uniqueness of the solution to

$$(P,h)\begin{cases} du - \operatorname{div} \partial j(\omega,t,x,\nabla u) = h(\cdot,u)\,dw & \text{in } \Omega\times(0,T)\times D,\\ u=0 & \text{on } \Omega\times(0,T)\times\partial D,\\ u(0,\cdot)=u_0 & \text{in } L^2(D). \end{cases} \tag{1}$$

where

- $T>0$, $D\subset\mathbb{R}^d$ is a bounded Lipschitz domain, $Q:=(0,T)\times D$,
- $w=\{w_t,\mathscr{F}_t; 0\le t\le T\}$ is a Wiener process on the classical Wiener space $(\Omega,\mathscr{F},P)$.
- $h:(\omega,t,x,\lambda)\in\Omega\times Q\times\mathbb{R}\mapsto h(\omega,t,x,\lambda)\in\mathbb{R}$ is a Carathéodory function, uniformly Lipschitz continuous with respect to λ, such that the mapping $(\omega,t,x)\mapsto h(\omega,t,x,\lambda)$ is in $N^2_W(0,T;L^2(D))$ for any $\lambda\in\mathbb{R}$.
- $j:(\omega,t,x,\xi)\in\Omega\times Q\times\mathbb{R}^d\mapsto j(\omega,t,x,\xi)\in\mathbb{R}^+$ is a Carathéodory function (continuous with respect to ξ, measurable with respect to (ω,t,x)) which is convex and Gâteaux differentiable with respect to ξ, for a.e. (ω,t,x). ∂ denotes this G-differentiation.
- $p:\Omega\times Q\to(1,\infty)$ is a variable exponent such that

$$1<p^-:=\operatorname{ess}\inf_{(\omega,t,x)} p(\omega,t,x)\le p^+:=\operatorname{ess}\sup_{(\omega,t,x)} p(\omega,t,x)<\infty.$$

For the precise assumptions on j and p we refer to Sects. 2 and 4.

2 Function spaces

Let us define

$$N^2_W(0,T;L^2(D)):=L^2(\Omega\times(0,T);L^2(D))$$

endowed with $dt\otimes dP$ and the predictable σ-field $\mathscr{P}_T$ generated by

$$]s,t]\times A,\quad 0\le s<t\le T,\quad A\in\mathscr{F}_s,$$

which is the natural space of Itô integrable stochastic processes. Let $S^2_W(0,T;H^k_0(D))$ be the subset of simple, predictible processes with values in $H^k_0(D)$ for sufficiently large values of k. Note that $S^2_W(0,T;H^k_0(D))$ is densely imbedded into $N^2_W(0,T;L^2(D))$.

If $(X,\mathscr{A},\mu)$ is a σ-finite measure space and $p:X\to\mathbb{R}$ is a measurable function with values in $[p^-,p^+]\subset(1,+\infty)$, one denotes by $L^{p(\cdot)}(X,d\mu)$ the variable exponent Lebesgue space of measurable functions f such that $\int_X|f(x)|^{p(x)}d\mu(x)<+\infty$. This space is endowed with the Luxemburg norm defined by

$$\|f\|=\inf\left\{\lambda>0\;\middle|\;\int_X|\lambda^{-1}f(x)|^{p(x)}d\mu(x)\le 1\right\}$$

and we refer to [11] for the basic definitions and properties of variable exponent Lebesgue and Sobolev spaces.
In this paper X is $\Omega \times Q$, $d\mu = d(t,x) \otimes dP$ and we are interested in measurable variable exponents $p : \Omega \times Q \to \mathbb{R}$ such that

$$1 < \operatorname{ess} \inf_{(\omega,t,x)} p(\omega,t,x) =: p^- \leq p(\omega,t,x) \leq p^+ := \operatorname{ess} \sup_{(\omega,t,x)} p(\omega,t,x) < \infty.$$

Moreover we assume that ω a.s. in Ω, $(t,x) \mapsto p(\omega,t,x)$ is log-Hölder continuous [11, Definition 4.1.1, p. 100] and that for all $t \geq 0$, $(\omega,s,x) \mapsto p(\omega,s,x)$ is $\mathscr{F}_t \times \mathscr{B}(0,t) \times \mathscr{B}(D)$-measurable. For this kind of variable exponents we introduce the spaces

$$\mathscr{E}_{\omega,t} := L^2(D) \cap W_0^{1,p(\omega,t,\cdot)}(D)$$

endowed with the norm $\|u\| = \|u\|_{L^2(D)} + \|\nabla u\|_{p(\omega,t,\cdot)}$.

The following function space serves as the variable exponent version of the classical Bochner space setting:

$$X_\omega(Q) := \{u \in L^2(Q) \cap L^1(0,T; W_0^{1,1}(D)) \mid \nabla u \in (L^{p(\omega,\cdot)}(Q))^d\}$$

which is a reflexive Banach space with respect to the norm

$$\|u\|_{X_\omega(Q)} = \|u\|_{L^2(Q)} + \|\nabla u\|_{L^{p(\omega,\cdot)}(Q)}.$$

$X_\omega(Q)$ is a generalization of the space

$$X(Q) := \{u \in L^2(Q) \cap L^1(0,T; W_0^{1,1}(D)) \mid \nabla u \in (L^{p(t,x)}(Q))^d\}$$

which has been introduced in [12] for the case of a variable exponent that is not depending on ω. For the basic properties of $X(Q)$, we refer to [12]. For $u \in X_\omega(Q)$, it follows directly from the definition that $u(t) \in L^2(D) \cap W_0^{1,1}(D)$ for almost every $t \in (0,T)$. Moreover, from $\nabla u \in (L^{p(\omega,\cdot)}(Q))^d$ and the theorem of Fubini it follows that $\nabla u(t,\cdot)$ is in $(L^{p(\omega,t,\cdot)}(D))^d$ a.e. in $\Omega \times (0,T)$.

Let us introduce the space

$$\mathscr{E} := \{u \in L^2(\Omega \times Q) \cap L^{p^-}(\Omega \times (0,T); W_0^{1,p^-}(D)) \mid \nabla u \in (L^{p(\cdot)}(\Omega \times Q))^d\}$$

which is a reflexive Banach space with respect to the norm

$$\|u\|_{\mathscr{E}} = \|u\|_{L^2(\Omega \times Q)} + \|\nabla u\|_{p(\cdot)}, \quad u \in \mathscr{E}.$$

Thanks to Fubini's theorem and since the inequality of Poincaré is available with respect to (t,x), $u \in \mathscr{E}$ implies that $u(\omega) \in X_\omega(Q)$ a.s. in Ω and $u(\omega,t) \in L^2(D) \cap W_0^{1,p(\omega,t,\cdot)}(D)$ for almost all $(\omega,t) \in \Omega \times (0,T)$.

3 Main result

Definition 1 A solution to (P, h) is a function $u \in \mathscr{E} \cap L^2(\Omega; C([0,T]; L^2(D))) \cap N^2_W(0,T; L^2(D))$ such that

$$u(t) - u_0 - \int_0^t \operatorname{div} \partial j(\omega, s, x, \nabla u) ds = \int_0^t h(\cdot, u) dw$$

holds a.e. in $\Omega \times D$ and for all $t \in [0, T]$.
Or, equivalently, such that $u(0, \cdot) = u_0$ and

$$\partial_t \Big[u(t) - \int_0^t h(\cdot, u) dw \Big] - \operatorname{div} \partial j(\omega, t, x, \nabla u) = 0$$

holds a.e. in $X'_\omega(Q)$.

Remark 3.1 The equivalence pointed out in the definition is argued in Sect. 6.2.

Our main result is the following:

Theorem 1 *Under assumptions (J1) to (J3), there exists a unique solution to* (P, h). *Moreover, if* u_1, u_2 *are solutions to* (P, h_1) *and* (P, h_2) *respectively, then:*

$$\begin{aligned} & E\left(\sup_{t\in[0,T]} \|(u_1 - u_2)(t)\|^2_{L^2(D)} \right) \\ & \quad + E\left(\int_Q \partial j(\omega, s, x, \nabla u_1) - \partial j(\omega, s, x, \nabla u_2) \cdot \nabla(u_1 - u_2) d(s,x) \right) \\ & \quad \le CE \int_Q |h_1(\cdot, u_1) - h_2(\cdot, u_2)|^2 \, d(s,x). \end{aligned} \tag{2}$$

Remark 3.2 Of course, our result can be immediately extended to the case of a multi dimensional noise given by a linear combination of independent real-valued Brownian motions.

4 Assumptions

Let

$$j : \Omega \times (0,T) \times D \times \mathbb{R}^d \to \mathbb{R}^+, \ (\omega, t, x, \xi) \mapsto j(\omega, t, x, \xi)$$

be a Carathéodory function (continuous with respect to ξ, measurable with respect to (ω, t, x)) which is convex and Gâteaux differentiable with respect to ξ, for a.e. (ω, t, x). We will denote its Gâteaux derivative by ∂j. Moreover, we assume

(J1) There exist $C_1 > 0$, $C_2 \geq 0$ and $g_1, g_2 \in L^1(\Omega \times Q)$ such that

$$j(\omega, t, x, \xi) \geq C_1 |\xi|^{p(\omega,t,x)} - g_1(\omega, t, x), \tag{3}$$

$$j(\omega, t, x, \xi) \leq C_2 |\xi|^{p(\omega,t,x)} + g_2(\omega, t, x) \tag{4}$$

a.e. in (ω, t, x) for all $\xi \in \mathbb{R}^d$.

(J2) For all $t \in [0, T]$

$$j : \Omega \times (0, t) \times D \times \mathbb{R}^d \to \mathbb{R}, \quad (\omega, s, x, \xi) \mapsto j(\omega, s, x, \xi)$$

is $\mathscr{F}_t \times \mathscr{B}(0, t) \times \mathscr{B}(D) \times \mathscr{L}^d$-measurable.

(J3) Almost surely, there exist two continuous functions $d_\omega : [0, \infty) \to (0, \infty)$ and $w_\omega : [0, \infty) \to [0, \infty)$ with $w_\omega(r) = 0$ if and only if $r = 0$ satisfying

$$d_\omega \left(\|\nabla u\|_{L^{p(\omega,\cdot)}(Q)} + \|\nabla v\|_{L^{p(\omega,\cdot)}(Q)} \right) w_\omega \left(\|\nabla u - \nabla v\|_{L^{p(\omega,\cdot)}(Q)} \right) - \nu_\omega(u, v) \tag{5}$$

$$\leq \int_0^T \int_D (\partial j(\omega, t, x, \nabla u) - \partial j(\omega, t, x, \nabla v)) \cdot \nabla(u - v) \, dx \, dt$$

for all $u, v \in X_\omega(Q)$ a.s. in Ω where $\nu_\omega(u, v) \to 0$ if

$$\int_0^T \int_D (\partial j(\omega, t, x, \nabla u) - \partial j(\omega, t, x, \nabla v)) \cdot \nabla(u - v) \, dx \, dt \to 0.$$

Some additional information and examples are detailed in the Appendix of the paper concerning such operators we have called (weak) w-operators.

Remark 4.1 Thanks to $(J2)$, the mapping $(\omega, s, x, \xi) \mapsto \partial j(\omega, s, x, \xi)$ is $\mathscr{F}_t \times \mathscr{B}(0, t) \times \mathscr{B}(D) \times \mathscr{L}_d$-measurable for every $t \in [0, T]$.

Lemma 1 *The convex functional*

$$J : \mathscr{E} \to \mathbb{R}, \; u \mapsto \int_{\Omega \times Q} j(\omega, t, x, \nabla u) \, d(t, x) \otimes dP$$

is continuous and Gâteaux differentiable with

$$\langle \partial G J(u), v \rangle = \int_{\Omega \times Q} \partial j(\omega, t, x, \nabla u) \cdot \nabla v \, d(t, x) \otimes dP$$

for all $u, v \in \mathscr{E}$. In particular, ∂J is maximal monotone

Proof J is continuous because of $(J1)$ and since it is a Nemytskii operator induced by j. For $u, v \in \mathscr{E}$ we have

$$\lim_{h\to 0^+} \frac{J(u+hv)-J(u)}{h} = \lim_{h\to 0^+} \int_{\Omega\times Q} \frac{j(\omega,t,x,\nabla u + h\nabla v) - j(\omega,t,x,\nabla u)}{h}\, d(t,x)\otimes dP \tag{6}$$

Thanks to the properties of j we have a.e. in $\Omega \times Q$

$$\lim_{h\to 0^+} \frac{j(\omega,t,x,\nabla u + h\nabla v) - j(\omega,t,x,\nabla u)}{h} = \partial j(\omega,t,x,\nabla u)\cdot\nabla v \tag{7}$$

moreover, since

$$h \mapsto \frac{j(\omega,t,x,\nabla u + h\nabla v) - j(\omega,t,x,\nabla u)}{h}$$

is nondecreasing, it follows from the Beppo–Levi theorem that

$$\lim_{h\to 0^+} \frac{J(u+hv)-J(u)}{h} = \int_{\Omega\times Q} \partial j(\omega,t,x,\nabla u)\cdot\nabla v\, d(t,x)\otimes dP. \tag{8}$$

It is left to prove that the integral on the right hand side of (8) is finite. Since

$$\begin{aligned} &-j(\omega,t,x,\nabla(u-v)) + j(\omega,t,x,\nabla u) \\ &\quad\le \partial G j(\omega,t,x,\nabla u)\cdot\nabla v \\ &\quad\le j(\omega,t,x,\nabla(u+v)) - j(\omega,t,x,\nabla u), \end{aligned} \tag{9}$$

a.e. in (ω,t,x), it follows from $(J1)$ that

$$\begin{aligned} &|\partial j(\omega,t,x,\nabla u)\cdot\nabla v| \\ &\quad\le \max\{j(\omega,t,x,\nabla(u+v)) - j(\omega,t,x,\nabla u), j(\omega,t,x,\nabla(u-v)) \\ &\qquad -j(\omega,t,x,\nabla u)\} \\ &\quad\le |j(\omega,t,x,\nabla(u+v))| + |j(\omega,t,x,\nabla(u-v))| + 2|j(\omega,t,x,\nabla u)| \\ &\quad\le C_2 2^{p^+ +1}(|\nabla u|^{p(\omega,t,x)} + |\nabla v|^{p(\omega,t,x)}) + 2(C_2|\nabla u|^{p(\omega,t,x)} + 2g_2). \end{aligned} \tag{10}$$

Using (10) and writing $d\mu := d(t,x)\otimes dP$ we arrive at

$$\begin{aligned} &|\langle \partial J(u), v\rangle| \\ &\quad\le \int_{\Omega\times Q} |\partial j(\omega,t,x,\nabla u)\cdot\nabla v|\, d\mu \\ &\quad\le \int_{\Omega\times Q} C_2 2^{p^+ +1}\left(|\nabla u|^{p(\omega,t,x)} + |\nabla v|^{p(\omega,t,x)}\right) + 2(C_2|\nabla u|^{p(\omega,t,x)} + 2g_2)\, d\mu \end{aligned}$$

and from (11) it follows that $\partial J(u) \in \mathscr{E}'$. Since J is a convex, continuous and Gâteaux-differentiable functional, its Gâteaux derivative is a maximal monotone operator (see [6, Theorem 2.8., p. 47]). □

Remark 4.2 With similar arguments as in the proof of Lemma 1 one shows that

(i) For a.e. $(\omega, t) \in \Omega \times (0, T)$ the convex functional

$$J_D : W_0^{1,p(\omega,t,\cdot)}(D) \to \mathbb{R},\ u \mapsto \int_D j(\omega, t, x, \nabla u)\, dx$$

is continuous and Gâteaux differentiable with respect to u: for all v in $W_0^{1,p(\omega,t,\cdot)}(D)$,

$$\langle \partial J_D(u), v\rangle = \int_D \partial j(\omega, t, x, \nabla u) \cdot \nabla v\, dx.$$

(ii) For a.e. $\omega \in \Omega$, the convex functional

$$J_Q : X_\omega(Q) \to \mathbb{R},\ u \mapsto \int_0^T \int_D j(\omega, t, x, \nabla u)\, dx\, dt = \int_0^T J_D(u)\, dx\, dt$$

is continuous, convex and Gâteaux differentiable with

$$\begin{aligned} \langle \partial J_Q(u), v\rangle_{X'_\omega(Q), X_\omega(Q)} &= \int_0^T \int_D \partial j(\omega, t, x, \nabla u) \cdot \nabla v\, dx\, dt \\ &= \int_0^T \langle \partial G J_D(u), v\rangle_{W^{-1,p'(\cdot)}(D), W_0^{1,p(\cdot)}(D)}\, dt \end{aligned} \tag{11}$$

for all $u, v \in X_\omega(Q)$.

In particular, as an immediate consequence of Lemma 1 we have

$$\begin{aligned} \langle \partial J(u), v\rangle_{\mathscr{E}', \mathscr{E}} &= \int_{\Omega \times Q} \partial j(\omega, t, x, \nabla u) \cdot \nabla v\, d\mu \\ &= \int_\Omega \langle \partial J_Q(u), v\rangle_{X'_\omega(Q), X_\omega(Q)}\, dP \\ &= \int_\Omega \int_0^T \langle \partial J_D(u), v\rangle_{W^{-1,p'(\cdot)}(D), W_0^{1,p(\cdot)}(D)}\, dt\, dP. \end{aligned} \tag{12}$$

5 The additive case for $h \in S^2_W(0, T; H^k_0(D))$

Assume, in this section, that $h \in S^2_W(0, T; H^k_0(D))$ for a big enough value of k. Since $W^{-1,q'}(D)$ is a separable Banach space, the notion of weak-measurability and Pettis measurability theorem yield the following proposition.

Proposition 1 *For $q \geq \max(2, p^+)$ and $\varepsilon > 0$, the operator*

$$A : \Omega \times (0, T) \times W_0^{1,q}(D) \to W^{-1,q'}(D),\qquad (\omega, t, u) \mapsto A(\omega, t, u) = -\varepsilon \Delta_q(u) + \partial J_D(\omega, t, u),$$

satisfies the following properties:

- *A is monotone for a.e. $(\omega, t) \in \Omega \times (0, T)$.*
- *A is progressively measurable, i.e. for every $t \in [0, T]$ the mapping*

$$A : \Omega \times (0, t) \times W_0^{1,q}(D) \to W^{-1,q'}(D), \quad (\omega, s, u) \mapsto A(\omega, s, u)$$

is $\mathscr{F}_t \times \mathscr{B}(0, t) \times \mathscr{B}(W_0^{1,q}(D))$-measurable.

It is then a consequence of [16, Theorem 2.1, p. 1253][2] that:

Proposition 2 *Let $h \in S_W^2(0, T; H_0^k(D))$ for $k > 0$ large enough. The operator $-A$ satisfies the hypotheses of [16, Theorem 2.1, p. 1253], therefore for any $\varepsilon > 0$ there exists a unique*

$$u^\varepsilon \in L^2(\Omega; C([0, T]; L^2(D))) \cap N_W^2(0, T; L^2(D)) \cap L^q(\Omega; L^q(0, T; W_0^{1,q}(D)))$$

that solves

$$u^\varepsilon(t) - u_0 + \int_0^t \partial J_D(u^\varepsilon) - \varepsilon \Delta_q(u^\varepsilon)\, dt = \int_0^t h\, dw \tag{13}$$

in $W^{-1,q'}(D)$ for all $t > 0$ a.s. in Ω.

Remark 5.1 In particular, it follows that u^ε such that $u^\varepsilon(0) = u_0$ satisfies (13) if and only if

$$v^\varepsilon := u^\varepsilon - \int_0^{\cdot} h\, dw$$

satisfies the random equation

$$\partial_t v^\varepsilon - \varepsilon \Delta_q \left(v^\varepsilon + \int_0^{\cdot} h\, dw \right) + \partial J_Q \left(v^\varepsilon + \int_0^{\cdot} h\, dw \right) = 0 \tag{14}$$

in $L^{q'}(0, T; W^{-1,q'}(D))$ a.s. in Ω. Using the regularity of u^ε and that the function h is in $S_W^2(0, T; H_0^k(D))$ we find $v^\varepsilon \in L^q(\Omega; L^q(0, T; W_0^{1,q}(D))$. Now, from (14) we get $\partial_t v^\varepsilon \in L^{q'}(0, T; W^{-1,q'}(D))$ a.s. in Ω. Therefore we can use v^ε as a test function in (14).

[2] Rmk: [9, Proposition 3.17, p. 84] and [16, Theorem 2.3, p. 1254] yield $u^\varepsilon \in L^2(\Omega, C([0, T]; L^2(D)))$.

Lemma 2 *There exists $G \in L^1(\Omega)$ such that for all $t \in [0, T]$*

$$\|v^\varepsilon(t)\|^2_{L^2(D)} + J^*_{Q_t}(\partial J_{Q_t}(u^\varepsilon)) + 2J_{Q_t}(u^\varepsilon) + \frac{\varepsilon}{q}\int_0^t \int_D |\nabla u^\varepsilon|^q \, dx \, ds \le G(\omega) + \|u_0\|^2_{L^2(D)} \tag{15}$$

a.s. in Ω, where $Q_t := (0, t) \times D$.

Proof We fix $t \in [0, T]$ and write $Q_t := (0, t) \times D$. Using v^ε as a test function in (14) and integration by parts, we obtain

$$\frac{1}{2}\|v^\varepsilon(t)\|^2_{L^2(D)} - \frac{1}{2}\|u_0\|^2_{L^2(D)} + \varepsilon\langle -\Delta_q u^\varepsilon, u^\varepsilon\rangle + \langle \partial J_{Q_t}(u^\varepsilon), u^\varepsilon\rangle = \varepsilon\left\langle -\Delta_q u^\varepsilon, \int_0^\cdot h \, dw\right\rangle + \left\langle \partial J_{Q_t}(u^\varepsilon), \int_0^\cdot h \, dw\right\rangle \tag{16}$$

Note that $-\Delta_q u = \partial J_1(u)$ in Q_t where

$$J_1(u) = \int_0^t \int_D \frac{1}{q}|\nabla u|^q \, dx.$$

Using the Fenchel inequality we get from (16)

$$\frac{1}{2}\|v^\varepsilon(t)\|^2_{L^2(D)} - \frac{1}{2}\|u_0\|^2_{L^2(D)} + \varepsilon J_1(u^\varepsilon) + \varepsilon (J_1)^*\left(\partial J_1(u^\varepsilon)\right) + J_{Q_t}(u^\varepsilon) + J^*_{Q_t}\left(\partial J_{Q_t}(u^\varepsilon)\right) = \varepsilon\left\langle \partial J_1(u^\varepsilon), \int_0^\cdot h \, dw\right\rangle + \left\langle \partial J_{Q_t}(u^\varepsilon), \int_0^\cdot h \, dw\right\rangle$$

For all $\alpha > 0$ we have

$$\begin{aligned}\left\langle \partial J_{Q_t}(u^\varepsilon), \int_0^\cdot h \, dw\right\rangle &= \left\langle \alpha\partial J_{Q_t}(u^\varepsilon), \frac{1}{\alpha}\int_0^\cdot h \, dw\right\rangle \\ &= \alpha\left\langle \partial J_{Q_t}(u^\varepsilon), \frac{1}{\alpha}\int_0^\cdot h \, dw\right\rangle \\ &\le \alpha J^*_{Q_t}(\partial J_{Q_t}(u^\varepsilon)) + \alpha J_{Q_t}\left(\frac{1}{\alpha}\int_0^\cdot h \, dw\right).\end{aligned}$$

Plugging (17) in (17) and using the Fenchel–Young inequality for J_1 we get

$$\begin{aligned}
&\frac{1}{2}\|v^\varepsilon(t)\|^2_{L^2(D)} - \frac{1}{2}\|u_0\|^2_{L^2(D)} \\
&\quad + J^*_{Q_t}(\partial J_{Q_t}(u^\varepsilon)) + J_{Q_t}(u^\varepsilon) + \varepsilon \int_0^t \int_D \frac{1}{q}|\nabla u^\varepsilon(t)|^q \, dx\, ds \\
&\quad \le \varepsilon \left(\int_0^t \int_D \frac{q-1}{q}|\nabla u^\varepsilon|^q + \frac{1}{q}\left|\nabla \int_0^s h\, dw\right|^q \, dx\, ds \right) + \alpha J^*_{Q_t}(\partial J_{Q_t}(u^\varepsilon)) \\
&\qquad + \alpha J_{Q_t}\left(\frac{1}{\alpha}\int_0^\cdot h\, dw\right).
\end{aligned} \tag{17}$$

For $\alpha = \frac{1}{2}$ and for all $t \in [0, T]$

$$\begin{aligned}
&\|v^\varepsilon(t)\|^2_{L^2(D)} + J^*_{Q_t}(\partial J_{Q_t}(u^\varepsilon)) + 2J_{Q_t}(u^\varepsilon) + 2\varepsilon \int_0^t \int_D |\nabla u^\varepsilon|^q \, dx\, ds \\
&\quad \le 2\int_0^t \int_D |\nabla \int_0^s h\, dw|^q \, dx\, ds + J_{Q_t}\left(2\int_0^\cdot h\, dw\right) ds + \|u_0\|^2_{L^2(D)}.
\end{aligned} \tag{18}$$

Since ∂_{x_i} is a continuous linear operator from $H^k_0(D)$ to $L^2(D)$, we have

$$\nabla \int_0^t h\, dw = \int_0^t \nabla h\, dw$$

for all $t \in [0, T]$ and a.s. in Ω. From $h \in S^2_W(0, T; H^k_0(D))$ for $k > 0$ large enough it follows that $\nabla h \in L^\infty(\Omega \times Q)^d$ and

$$t \mapsto \int_0^t \nabla h\, dw \in C([0, T]; L^\infty(\Omega \times D)^d).$$

Therefore, using $(J1)$, we get

$$\begin{aligned}
&J_{Q_t}\left(2\int_0^\cdot h\, dw\right) ds \\
&\quad \le C_2 \int_Q \left|\int_0^t \nabla h\, dw\right|^{p(\omega,\cdot)} d(t,x) \int_Q g_2(\omega, t, x)\, d(t,x)
\end{aligned} \tag{19}$$

Thanks to the regularity of ∇h in particular it follows that

$$\left|\int_0^\cdot \nabla h\, dw\right| \in L^r(\Omega \times Q)$$

for any $1 \le r < \infty$ and therefore by Fubini's Theorem

$$\omega \mapsto G_1(\omega) := \int_Q \left| \int_0^t \nabla h \, dw \right|^{p(\omega,\cdot)} + \left| \int_0^t \nabla h \, dw \right|^q d(t,x)$$

is in $L^1(\Omega)$. Moreover,

$$\omega \mapsto G_2(\omega) := \int_Q g_2(\omega,t,x) \, d(t,x)$$

is in $L^1(\Omega)$. Writing $G = G_1 + G_2$, plugging (19) into (18) and rearranging the terms we arrive at (15). □

Lemma 3 *There exists a full measure set $\tilde{\Omega} \subset \Omega$ such that for any $\omega \in \tilde{\Omega}$,*

(i) *$\varepsilon \nabla u^\varepsilon$ is bounded in $L^q(0,T;(L^q(D))^d)$,*
(ii) *v^ε is bounded in $C([0,T];L^2(D))$ and in $L^{p^-}(0,T;W_0^{1,p^-}(D))$, in particular, $v^\varepsilon(t)$ in bounded in $L^2(D)$ for all $t \in (0,T]$.*
(iii) *$\nabla u^\varepsilon(\omega)$ is bounded in $L^{p(\omega,\cdot)}(Q)$ and therefore $v^\varepsilon(\omega)$ is bounded in the space $X_\omega(Q)$.*

Proof By $(J1)$ we have a.s. in Ω

$$\begin{aligned} J_Q^*(\partial J_Q(u^\varepsilon)) + 2J_Q(u^\varepsilon) &= \langle \partial J_Q(u^\varepsilon), u^\varepsilon \rangle + J_Q(u^\varepsilon) \\ &\ge 2J_Q(u^\varepsilon) - J_Q(0) \\ &= \int_Q j(\omega,s,x,\nabla u^\varepsilon) - j(\omega,s,x,0) \, d(s,x) \\ &\ge C_1 \int_Q |\nabla u^\varepsilon|^{p(\cdot)} - g_1(\omega,s,x) - g_2(\omega,s,x) \, d(s,x) \end{aligned} \tag{20}$$

Combining (20) with (15) we arrive at

$$\|v^\varepsilon(t)\|^2_{L^2(D)} + C_1 \int_Q |\nabla u^\varepsilon|^{p(\cdot)} \, d(t,x) \le \tilde{G}(\omega) + \|u_0\|^2_{L^2(D)}, \tag{21}$$

where $\tilde{G} = G + \int_Q g_1(\omega,s,x) + g_2(\omega,s,x) \, d(s,x) \in L^1(\Omega)$. □

Lemma 4 *For $\omega \in \tilde{\Omega}$ fixed, $\partial J_Q(u^\varepsilon)$ is bounded in $X'_\omega(Q)$.*

Proof Using $(J1)$ and (15) it follows that

$$J_Q^*(\partial J_Q(u^\varepsilon)) \le G(\omega) + \|u_0\|^2_{L^2(D)} + \int_Q g_1 \, d(t,x) =: K(\omega,u_0). \tag{22}$$

From (22), the Fenchel–Young inequality and $(J1)$ for any $v \in X_\omega(Q)$ it follows that

$$
\begin{aligned}
|\langle \partial J_Q(u^\varepsilon), v\rangle| &\le J_Q^*(\partial J_Q(u^\varepsilon)) + J_Q(v) \\
&\le K(\omega, u_0) + C_2 \int_Q |\nabla v|^{p(\omega,\cdot)} + g_2 \, d(t,x).
\end{aligned} \tag{23}
$$

□

The following Lemma is a direct consequence of Lemma 3 and Lemma 4:

Lemma 5 *For any $\omega \in \tilde{\Omega}$ there exists a (not relabeled) subsequence of $v^\varepsilon(\omega)$ and $v \in X_\omega(Q) \cap L^\infty(0,T; L^2(D))$ such that, for $\varepsilon \downarrow 0$,*

(i) $v^\varepsilon \overset{*}{\rightharpoonup} v$ *in* $L^\infty(0,T; L^2(D))$,
(ii) $\nabla v^\varepsilon \rightharpoonup \nabla v$ *in* $(L^{p(\omega,\cdot)}(Q))^d$,
(iii) $v^\varepsilon \rightharpoonup v$ *in* $X_\omega(Q)$
(iv) *There exists* $\alpha(T) \in L^2(D)$ *such that* $v^\varepsilon(T) \rightharpoonup \alpha(T)$ *in* $L^2(D)$.
(v) *Moreover, there exists* $\mathrm{B} \in X'_\omega(Q)$, $\mathrm{B} = b - \operatorname{div} G$ *with* $b \in L^2(Q)$ *and* $G \in (L^{p'(\omega,\cdot)}(Q))^d$ *such that*

$$
\partial J_Q(u^\varepsilon) \rightharpoonup b - \operatorname{div} G \text{ in } X'_\omega(Q),
$$

we recall that $u^\varepsilon = v^\varepsilon + \int_0^t h \, dw$.

We take $\varphi = \rho\zeta$ such that $\rho \in \mathscr{D}([0,T])$ and $\zeta \in \mathscr{D}(D)$ as a test function and we have

$$
\begin{aligned}
\int_0^T \int_D -v^\varepsilon \partial_t \varphi \, dx ds - \varepsilon \langle \Delta_q(u^\varepsilon), \varphi\rangle + \langle \partial J_Q(u^\varepsilon), \varphi\rangle \\
= \int_D u_0 \varphi(0,x) - v^\varepsilon(T,x)\varphi(T,x) \, dx
\end{aligned} \tag{24}
$$

Since $\varepsilon \nabla u^\varepsilon$ is bounded in $L^q(0,T; (L^q(D))^d)$, it follows that

$$
\langle -\varepsilon \Delta_q(u^\varepsilon), \varphi\rangle \to 0
$$

for $\varepsilon \downarrow 0$. We can pass to the limit in all the other terms in (24) and arrive at

$$
-\int_0^T \int_D v \partial_t \varphi \, dx \, ds + \int_D \zeta(\alpha(T)\rho(T) - u_0 \rho(0)) \, dx + \langle \mathrm{B}, \varphi\rangle = 0 \tag{25}
$$

and therefore

$$
v_t + B = 0 \tag{26}
$$

in $\mathscr{D}'(Q)$. From (26) we get $v_t \in X'_\omega(Q)$ and therefore v is in

$$
W_\omega(Q) := \{v \in X_\omega(Q) \mid v_t \in X'_\omega(Q)\} \hookrightarrow C([0,T]; L^2(D)).
$$

In particular, since $\mathscr{D}(Q)$ is dense in $X_\omega(Q)$, (26) holds also in $X'_\omega(Q)$. Now, using the integration by parts formula in $W_\omega(Q)$ (see [12]) it follows that

$$\langle v_t, \varphi \rangle = -\int_0^T \int_D v \partial_t \varphi + \int_D \zeta (v(T)\rho(T) - u_0 \rho(0))\, dx \tag{27}$$

Now, we can identify $\alpha(T)$ with $v(T)$: indeed, plugging (27) in (25) we can apply (26) to get

$$\int_D \zeta \rho(T)(\alpha(T) - v(T))\, dx = 0. \tag{28}$$

Moreover, we find that the whole sequence $v^\varepsilon(T)$ converges weakly to $v(T)$. As the argumentation also holds true for any $t \in [0, T]$, we get that $v^\varepsilon(t) \rightharpoonup v(t)$ in $L^2(D)$ for all $t \in [0, T]$.

Lemma 6 *In addition to Lemma 5, $B = \partial J_Q(u)$ in $X'_\omega(Q)$, $\langle \partial J_Q(u^\varepsilon), u^\varepsilon \rangle \to \langle \partial J(u), u \rangle$ for $\varepsilon \downarrow 0$ where $u = v + \int_0^t h\, dw$, $\nabla u^\varepsilon \to \nabla u$ in $L^{p(\omega,\cdot)}(Q)$ and $\nabla v^\varepsilon \to \nabla v$ in $L^{p(\omega,\cdot)}(Q)$ as well.*

Proof Using v as a test function in (26), from integration by parts in $W_\omega(Q)$ we obtain

$$\frac{1}{2}\|v(T)\|^2 - \frac{1}{2}\|u_0\|^2 + \langle \mathrm{B}, v \rangle = 0. \tag{29}$$

On the other hand, using v^ε as a test function in (24) and applying integration by parts we obtain

$$\begin{aligned} &\frac{1}{2}\|v^\varepsilon(T)\|^2 - \frac{1}{2}\|u_0\|^2 - \varepsilon \langle \Delta_q u^\varepsilon, u^\varepsilon \rangle + \langle \partial J_Q(u^\varepsilon), u^\varepsilon \rangle \\ &\quad = -\varepsilon \left\langle \Delta_q u^\varepsilon, \int_0^\cdot h\, dw \right\rangle + \left\langle \partial J_Q(u^\varepsilon), \int_0^\cdot h\, dw \right\rangle \end{aligned} \tag{30}$$

discarding nonnegative terms for $\varepsilon \downarrow 0$ in the limit of (30) we get

$$\frac{1}{2}\|v(T)\|^2 - \frac{1}{2}\|u_0\|^2 + \limsup_{\varepsilon \downarrow 0} \langle \partial J_Q(u^\varepsilon), u^\varepsilon \rangle \le \left\langle \mathrm{B}, \int_0^\cdot h\, dw \right\rangle. \tag{31}$$

Now, from (26) and (27) we obtain

$$\limsup_{\varepsilon \downarrow 0} \langle \partial J_Q(u^\varepsilon), u^\varepsilon \rangle \le \langle \mathrm{B}, u \rangle. \tag{32}$$

Since $X_\omega(Q)$ is reflexive and ∂J_Q is the Gâteaux derivative of the convex and lower semicontinuous functional J_Q, from [21, Theorem 3.32] it follows that ∂J_Q is maximal monotone and therefore it follows from [6, Lemma 2.3, p. 38] and (32) that $B = \partial J_Q(u)$ in $X'_\omega(Q)$ and $\langle \partial J_Q(u^\varepsilon), u^\varepsilon \rangle \to \langle \partial J(u), u \rangle$.
As a consequence, $\lim_{\varepsilon \downarrow 0} \langle \partial J_Q(u^\varepsilon) - \partial J_Q(u), u^\varepsilon - u \rangle = 0$ and Assumption (J3) with Appendix 1 yield the strong convergence claimed at the end of the Lemma. ◻

From Lemma 5 and (25) it follows that

$$\partial_t v + \partial J_Q(u) = 0 \tag{33}$$

and $\partial_t v$ is in $X'_\omega(Q)$ a.s. in Ω. If $v_1 = u_1 - \int_0^t h\,dw$ and $v_2 = u_2 - \int h\,dw$ are both satisfying (33), then subtracting the equations we arrive at

$$\partial_t(u_1 - u_2) + (\partial J_Q(u_1) - \partial J_Q(u_2)) = 0 \tag{34}$$

and from (34) it follows that $(u_1 - u_2) \in W_\omega(Q)$ a.s. in Ω. Therefore we can use $(u_1 - u_2)$ as a test function in (34) and from integration by parts in $W_\omega(Q)$ it follows that $u_1 = u_2$ a.e. in Q for a.e. $\omega \in \Omega$. Therefore, one may conclude by the following proposition:

Proposition 3 *The convergences pointed out in Lemmata 5 and 6 hold for the whole sequences v^ε and u^ε.*

Lemma 7 *We have: $v \in L^2(\Omega; C([0,T]; L^2(D)))$, $v^\varepsilon(\omega, t, \cdot) \to v(\omega, t, \cdot)$ in $L^2(D)$, ω a.s. and for any t, and $\nabla v^\varepsilon \to \nabla v$ in $L^{p(\cdot)}(\Omega \times Q)$.*

Proof We know already that $v^\varepsilon(\omega, t) \rightharpoonup v(\omega, t)$ in $L^2(D)$ for almost every $\omega \in \Omega$ and all $t \in [0, T]$ as $\varepsilon \downarrow 0$. As mentioned above, since T can be replaced by any t, using (29) and (30) with $T = t$ and that $B = \partial J_Q(u)$ we get

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{2}\|v^\varepsilon(t)\|^2_{L^2(D)} \le \frac{1}{2}\|v(t)\|^2_{L^2(D)} \tag{35}$$

and from (35) it follows that

$$\lim_{\varepsilon\downarrow 0} \|v^\varepsilon(t)\|_{L^2(D)} = \|v(t)\|_{L^2(D)}, \tag{36}$$

and (36) together with the weak convergence in $L^2(D)$ yields $v^\varepsilon(\omega, t) \to v(\omega, t)$ in $L^2(D)$ for almost every $\omega \in \Omega$, for all $t \in [0, T]$.

From Lemma 2 and (20) it follows that for all $t \in [0, T]$, a.s. in Ω

$$\|v^\varepsilon(t)\|^2_{L^2(D)} + \int_Q |\nabla u^\varepsilon|^{p(\omega,\cdot)}\,dx\,ds \le G_1 + G_2 + \|u_0\|^2_{L^2(D)} \tag{37}$$

with $G_1, G_2 \in L^1(\Omega)$.

From Lebesgue's dominated convergence theorem and the uniform convexity of $L^2(\Omega \times Q)$ and $L^{p(\cdot)}(\Omega \times Q)$ with similar arguments as in [14], it now follows that $v^\varepsilon \to v$ in $L^2(\Omega \times (0,T); L^2(D))$ and $\nabla u^\varepsilon \to \nabla u$ in $L^{p(\cdot)}(\Omega \times Q)$. In particular, we get that $u^\varepsilon \to u = v + \int_0^t h\,dw$ in $L^2(\Omega \times (0,T); L^2(D))$ as well. Now we need to prove that $v \in L^2(\Omega; C([0,T]; L^2(D)))$. We already know that $v : \Omega \times (0,T) \to L^2(D)$ is a (predictible) stochastic process. Since $v(\omega, \cdot) \in W_\omega(Q) \hookrightarrow C([0,T]; L^2(D))$ for a.e. $\omega \in \Omega$ the measurability follows from [9, Proposition 3.17,

p. 84] with arguments as in [13, Corollary 1.1.2, p. 8]. From (37) it now follows that v is in $L^2(\Omega; C([0, T]; L^2(D)))$. □

Summarizing all previous results we are able to pass to the limit with $\varepsilon \downarrow 0$ in (14). For the limit function u we have shown the following result:

Proposition 4 *For $h \in S^2_W(0, T; H^k_0(D))$ there exists a full-measure set $\tilde{\Omega}$ and $u \in \mathscr{E} \cap L^2(\Omega; C([0, T]; L^2(D))) \cap N^2_W(0, T; L^2(D))$ such that for all $\omega \in \tilde{\Omega}$*

$$u(t) - u_0 - \int_0^t \partial J_D(u(s))\, ds = \int_0^t h\, dw \tag{38}$$

a.e. in D for all $t \in [0, T]$.

6 The additive case for general *h*

6.1 Uniform estimates

Now we want to derive existence for arbitrary $h \in N^2_W(0, T; H^k_0(D))$ from the previous results. From the density of $S^2_W(0, T; H^k_0(D))$ in $N^2_W(0, T; H^k_0(D))$ it follows that there exists $(h_n) \subset S^2_W(0, T; H^k_0(D))$ such that $h_n \to h$ in $N^2_W(0, T; H^k_0(D))$. Let us remark that since $N^2_W(0, T; H^k_0(D))$ is a separable set there exists a countable set $\Lambda \subset S^2_W(0, T; H^k_0(D))$ such that $(h_n) \subset \Lambda$ (irrespective of $h \in N^2_W(0, T; H^k_0(D))$). Thus, the full-measure set $\tilde{\Omega}$ introduced in the above proposition can be shared by all the elements of Λ.

Lemma 8 *For $h_n, h_m \in \Lambda$, let u_n, u_m be solutions to (38) with right-hand side h_n, and h_m respectively. There exists a constant $K_1 \geq 0$ not depending on $m, n \in \mathbb{N}$, such that*

$$E\left(\sup_{t\in[0,T]} \|u_n(t)\|^2_{L^2(D)}\right) + J^*(\partial J(u_n)) + J(u_n) \leq K_1 \left(\|h_n\|^2_{L^2(\Omega\times Q)} + \|u_0\|^2_{L^2(D)}\right) \tag{39}$$

for all $n \in \mathbb{N}$,

$$\begin{aligned} &E\left(\sup_{t\in[0,T]} \|(u_n - u_m)(t)\|^2_{L^2(D)}\right) + \langle \partial J_Q(u_n) - \partial J_Q(u_m), u_n - u_m\rangle \\ &\leq K_1 \|h_n - h_m\|^2_{L^2(\Omega\times Q)} \end{aligned} \tag{40}$$

for all $n, m \in \mathbb{N}$.

Proof Let u_n be a solution to (38) with right-hand side h_n and u_m be a solution to (38) with right-hand side h_m. Denoting u^ε_n and u^ε_m the corresponding approximate solutions to (13), using the Itô formula and discarding the nonnegative term it follows that for all $t \in [0, T]$ a.s. in Ω we have

$$\frac{1}{2}\|u_n^\varepsilon(t) - u_m^\varepsilon(t)\|_{L^2(D)}^2 + \langle \partial J_{Q_t}\left(u_n^\varepsilon\right) - \partial J_{Q_t}\left(u_m^\varepsilon\right), u_n^\varepsilon - u_m^\varepsilon \rangle \tag{41}$$
$$\leq \int_D \int_0^t (h_n - h_m)\left(u_n^\varepsilon - u_m^\varepsilon\right)\, dw\, dx + \frac{1}{2}\int_0^t \int_D (h_n - h_m)^2\, dx\, ds.$$

Using the convergence results of Lemmata 5 to 7 (see Proposition 3), it follows that, for a.e. $\omega \in \Omega$, $u_n^\varepsilon \to u_n$ in $L^2(Q)$, $u_n^\varepsilon(t) \to u_n(t)$ in $L^2(D)$ for all $t \in [0, T]$, $u_n^\varepsilon \to u_n$ in $X_\omega(Q)$, $\partial J_{Q_t}(u_n^\varepsilon) \rightharpoonup \partial J_{Q_t}(u_n)$ in $X'_\omega(Q)$ and $\langle \partial J_{Q_t}(u_n^\varepsilon), u_n^\varepsilon\rangle \to \langle \partial J_{Q_t}(u_n), u_n\rangle$ for $\varepsilon \downarrow 0$ (and resp. with m):

$$\lim_{\varepsilon\downarrow 0}\left\langle \partial J_{Q_t}\left(u_n^\varepsilon\right) - \partial J_{Q_t}\left(u_m^\varepsilon\right), u_n^\varepsilon - u_m^\varepsilon\right\rangle = \left\langle \partial J_{Q_t}(u_n) - \partial J_{Q_t}(u_m), u_n - u_m\right\rangle. \tag{42}$$

Moreover, by Itô isometry we have that

$$\int_0^t (h_n - h_m)(u_n^\varepsilon - u_m^\varepsilon)\, dw \to \int_0^t (h_n - h_m)(u_n - u_m)\, dw \tag{43}$$

in $L^2(\Omega; C([0, T]; L^2(D)))$ for $\varepsilon \downarrow 0$, hence passing to a (not relabeled) subsequence if necessary, it follows that (43) holds a.s. in Ω and for all $t \in [0, T]$. Taking the supremum over $[0, T]$ and then taking expectation, we arrive at

$$E\left(\sup_{t\in[0,T]}\|u_n(t)-u_m(t)\|_{L^2(D)}^2\right)+2E(\langle\partial J_Q(u_n)-\partial J_Q(u_m), u_n-u_m\rangle) \tag{44}$$
$$\leq E\left(\|u_{0,n} - u_{0,m}\|_{L^2(D)}^2\right) + \|h_n - h_m\|_{L^2(\Omega\times Q)}^2$$
$$+2E\left(\sup_{t\in[0,T]}\int_0^t \int_D (h_n - h_m)(u_n - u_m)\, dx\, dw\right).$$

For the last term on the right-hand side of (44), for any $\gamma > 0$ we use Burkholder, Hölder and Young inequality to estimate

$$E\left(\sup_{t\in[0,T]}\int_0^t \int_D (h_n - h_m)(u_n - u_m)\, dx\, dw\right) \tag{45}$$
$$\leq 3E\left(\int_0^T \left(\int_D (h_n - h_m)(u_n - u_m)\, dx\right)^2 ds\right)^{1/2}$$
$$\leq 3E\left(\int_0^T \|h_n - h_m\|_{L^2(D)}^2\|u_n - u_m\|_{L^2(D)}^2\, dt\right)^{1/2}$$

$$
\leq 3E\left[\left(\sup_{t\in[0,T]}\|u_n-u_m\|^2_{L^2(D)}\right)^{1/2}\left(\int_0^T\|h_n-h_m\|^2_{L^2(D)}\right)^{1/2}\right]
$$
$$
\leq 3\gamma E\left(\sup_{t\in[0,T]}\|u_n-u_m\|^2_{L^2(D)}\right)+\frac{3}{\gamma}\|h_n-h_m\|^2_{L^2(\Omega\times Q)}
$$

Plugging (45) into (44), and choosing $\gamma > 0$ small enough and $u_{0,n} = u_{0,m}$ we find $K_1 \geq 0$ such that (40) holds.

Again, using the Itô formula and discarding the nonnegative term it follows that for all $t \in [0, T]$ a.s. in Ω,

$$
\frac{1}{2}\|u_n^\varepsilon(t)\|^2_{L^2(D)}+\left\langle\partial J_{Q_t}\left(u_n^\varepsilon\right),u_n^\varepsilon\right\rangle
$$
$$
\leq\frac{1}{2}\|u_{0,n}\|^2_{L^2(D)}+\int_D\int_0^t h_n u_n^\varepsilon\,dw\,dx+\frac{1}{2}\int_0^t\int_D|h_n|^2\,dx\,ds.
$$

Passing to the limit as above, yields

$$
\frac{1}{2}\|u_n(t)\|^2_{L^2(D)}+\langle\partial J_{Q_t}(u_n),u_n\rangle
$$
$$
\leq\frac{1}{2}\|u_{0,n}\|^2_{L^2(D)}+\int_D\int_0^t h_n u_n\,dw\,dx+\frac{1}{2}\int_0^t\int_D|h_n|^2\,dx\,ds.
$$

And then, as above, we arrive at (39) since by Fenchel–Young inequality it follows that $E(\langle\partial J_Q(u_n),u_n\rangle)=\langle\partial J(u_n),u_n\rangle=J^*(\partial J(u_n))+J(u_n)$. □

Let us fix an arbitrary $h \in N^2_W(0,T;L^2(D))$ and let $(h_n) \subset \Lambda$ be a sequence of simple functions such that $h_n \to h$ in $N^2_W(0,T;L^2(D))$. Let u_n be the solution to (38) with right-hand side h_n for $n \in \mathbb{N}$. From Lemma 8, (40) it follows that for $m, n \to \infty$

$$
E\left(\|(u_n-u_m)(t)\|^2_{C([0,T];L^2(D))}\right)\to 0. \tag{46}
$$

In particular, (46) implies that (u_n) is a Cauchy sequence in $L^2(\Omega;C([0,T];L^2(D)))$ and in $N^2_W(0,T;L^2(D))$, hence $u_n \to u \in L^2(\Omega;C([0,T];L^2(D)))\cap N^2_W(0,T;L^2(D))$ for $n\to\infty$.

Moreover, we have the following

Lemma 9 $\partial J(u_n) \rightharpoonup \partial J(u)$ *in* $\mathscr{E}'$ *and* $\langle\partial J(u_n),u_n\rangle\to\langle\partial J(u),u\rangle$ *for* $n\to\infty$ *for a non-relabeled subsequence.*

Proof Since (h_n) is bounded in $N^2_W(0,T;L^2(D))$, for any $v\in\mathscr{E}$ by Fenchel–Young inequality and thanks to Lemma 8, (39) and $(J1)$ it follows that there exists a constant $K_3 \geq 0$ such that

$$\begin{aligned}|\langle \partial J(u_n), v\rangle| &\le J(v) + J^*(\partial J(u_n)) \\ &\le J(v) + K \\ &\le C_2 \int_{\Omega\times Q} |\nabla v|^{p(\cdot)} d\mu + K_3.\end{aligned} \tag{47}$$

From (47) it follows that there exists a constant $K_4 > 0$ not depending on $n \in \mathbb{N}$ such that

$$\|\partial J(u_n)\|_{\mathscr{E}'} = \sup_{\|v\|_{\mathscr{E}}\le 1} |\langle \partial J(u_n), v\rangle| \le K_4. \tag{48}$$

Since $\mathscr{E}'$ is reflexive, from (48) it follows that there exists a subsequence, still denoted $(\partial J(u_n))$, and $B \in \mathscr{E}'$ such that $\partial J(u_n) \rightharpoonup B$ in $\mathscr{E}'$.
From Lemma 8-(39) and $(J1)$ it follows that there exists a constant $K_5 \ge 0$ not depending on $n \in \mathbb{N}$ such that

$$\|\nabla u_n\|_{p(\cdot)} \le K_5 \tag{49}$$

and since (u_n) is bounded in $N^2_W(0, T; L^2(D))$ (see (39)), it follows that (u_n) is bounded in the reflexive space $\mathscr{E}$. Therefore, passing again to a (not relabeled) subsequence if necessary, there exists $u \in \mathscr{E}$ such that $u_n \rightharpoonup u$ in $\mathscr{E}$ for $n \to \infty$. Since $\partial J : \mathscr{E} \to \mathscr{E}'$ is maximal monotone (see Lemma 1), the assertion follows from [6, Lemma 2.3, p. 38] and (40). □

6.2 Passage to the limit

Proposition 5 *Theorem 1 holds in the additive case: for any $h \in N^2_W(0, T; L^2(D))$, there exists a unique $u \in \mathscr{E} \cap L^2(\Omega; C([0, T]; L^2(D))) \cap N^2_W(0, T; L^2(D))$ and a full measure set $\tilde{\Omega} \in \mathscr{F}$ such that for every $\omega \in \tilde{\Omega}$ and for all $t \in [0, T]$*

$$u(t) - u_0 + \int_0^t \partial J_D(u)\, ds = \int_0^t h\, dw$$

holds a.e. in D. Moreover, (2) holds for two given $h_1, h_2 \in N^2_W(0, T; L^2(D))$.

Proof Let us fix an arbitrary $h \in N^2_W(0, T; L^2(D))$ and let $(h_n) \subset S^2_W(0, T; H^k_0(D))$ be a sequence of simple functions such that $h_n \to h$ in $N^2_W(0, T; L^2(D))$. Let u_n be the solution to (38) with right-hand side h_n for $n \in \mathbb{N}$. According to the results of the previous subsections, there exists a (not relabeled) subsequence of (u_n) with the following convergence results for $n \to \infty$:

(a) $u_n \to u$ in $L^2(\Omega; C([0, T]; L^2(D)))$, in $N^2_W(0, T; L^2(D))$ and a.s. in $C([0, T]; L^2(D))$ for a subsequence if needed. In particular, $u(0, \cdot) = u_0\, dP \otimes dx$-a.e. in $\Omega \times D$
(b) $\nabla u_n \rightharpoonup \nabla u$ in $L^{p(\cdot)}(\Omega \times Q)$
(c) $\partial J(u_n) \rightharpoonup \partial J(u)$ in $\mathscr{E}'$.

We fix $A \in \mathscr{F}$, $\rho \in \mathscr{D}([0,T) \times D)$ and $\phi = \chi_A \rho$. Note that thanks to the regularity of h_n we have

$$v_n := u_n - \int_0^t h_n \, dw \in \mathscr{E}.$$

Therefore, using Lemma 1 it follows that for all $n \in \mathbb{N}$

$$-\left(\int_{\Omega \times Q} v_n \partial_t \phi \, d\mu + \int_{\Omega \times D} u_0 \phi(\omega, 0, x) \, dP \, dx \right) + \langle \partial J(u_n), \phi \rangle = 0 \quad (50)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket for $\mathscr{E}'$, $\mathscr{E}$. Thanks to the Itô isometry

$$\int_{A \times Q} \int_0^t h_n \, dw \, d\mu \to \int_{A \times Q} \int_0^t h \, dw \, d\mu,$$

for $n \to \infty$. Therefore , we can pass to the limit with $n \to \infty$ and obtain

$$-\int_{A \times Q} \left(u - \int_0^t h \, dw \right) \partial_t \rho \, d\mu - \int_{A \times D} u_0 \rho(0, x) \, dP \, dx + \langle \partial J_Q(u), \chi_A \rho \rangle = 0. \quad (51)$$

Thanks to the monotonicity of ∂J, by an argument similar to the one pointed out after (34), from (51) we get that u is unique, hence the whole sequence u_n has the convergence properties $a.$)-$c.$). With a separability argument from (51) and from Lemma 1 it follows that there exists a full-measure set $\tilde{\Omega} \subset \Omega$ not depending on ρ, such that

$$\int_Q \partial_t \left(u - \int_0^t h \, dw \right) \rho \, d\mu + \langle \partial J_Q(u), \rho \rangle = 0 \quad (52)$$

for all $\omega \in \tilde{\Omega}$ and for all $\rho \in \mathscr{D}(Q)$. Moreover, a.s. in Ω

$$u - \int_0^t h \, dw \in C([0,T]; L^2(D))$$

and from (52) it follows that

$$\partial_t \left(u - \int_0^t h \, dw \right) \in X'_\omega(Q) \hookrightarrow L^{q'}(0,T; W^{-1,q'}(D))$$

for $q \geq p^+ + 2$. Thus we can integrate (52) and use Lemma 1 to obtain a.s.

$$u(t) - u_0 + \int_0^t \partial J_D(u) \, ds = \int_0^t h \, dw. \quad (53)$$

To conclude the proof, let us mention that the uniqueness of the solution is based on the argument following (34) and that Lemma 8, (40) and Lemma 9 yield the stability result. □

7 The multiplicative case: the main result

We consider now the general case where $h : (\omega, t, x, \lambda) \in \Omega \times Q \times \mathbb{R} \mapsto h(\omega, t, x, \lambda) \in \mathbb{R}$ is a Carathéodory function, uniformly Lipschitz continuous with respect to λ, such that the mapping $(\omega, t, x) \mapsto h(\omega, t, x, \lambda)$ is in $N^2_W(0, T; L^2(D))$ for any $\lambda \in \mathbb{R}$. Thus, by classical arguments based on Nemytskii operators, one has that $h(\cdot, v) \in N^2_W(0, T; L^2(D))$ when $v \in N^2_W(0, T; L^2(D))$.
Thus, the proof of the main result is based on the remark that u is a solution of

$$\partial_t\Big[u(t) - \int_0^t h(\cdot, u)dw\Big] - \operatorname{div} \partial j(\omega, t, x, \nabla u) = 0$$

and initial condition u_0 if and only if u is a fixed-point of the application

$$\mathscr{T} : N^2_W(0, T, L^2(D)) \to N^2_W(0, T, L^2(D)),\ S \mapsto u_S$$

where u_S is the solution, for the same initial condition, to

$$\partial_t\Big[u(t) - \int_0^t h(\cdot, S)dw\Big] - \operatorname{div} \partial j(\omega, t, x, \nabla u) = 0.$$

From Proposition 5, Application $\mathscr{T}$ is well-defined.
Moreover, if S_1 and S_2 are given in $N^2_W(0, T, L^2(D))$ and u_{S_1}, u_{S_2} are the corresponding solutions, then for all $t \in (0, T)$

$$\begin{aligned} E\|(u_{S_1} - u_{S_2})(t)\|^2_{L^2(D)} &\le CE\int_0^t \|h(\cdot, S_1) - h(\cdot, S_2)\|^2_{L^2(D)} ds \\ &\le CL\int_0^t E\|S_1 - S_2\|^2_{L^2(D)} ds, \end{aligned} \tag{54}$$

where L is the Lipschitz constant of h. We fix $\alpha > 0$. Multiplying (54) by $e^{-\alpha t}$ and integrating over $(0, T)$ we find

$$\begin{aligned} &\int_0^T E\|(u_{S_1} - u_{S_2})(t)\|^2_{L^2(D)} e^{-\alpha t}\, dt \\ &\quad \le CL\int_0^T \frac{d}{dt}\left(-\frac{1}{\alpha}e^{-\alpha t}\right)\int_0^t E\|S_1 - S_2\|^2_{L^2(D)}\, ds\, dt \end{aligned} \tag{55}$$

Using integration by parts on the right-hand side of (55) we obtain

$$\int_0^T E\|(u_{S_1}-u_{S_2})(t)\|^2_{L^2(D)}e^{-\alpha t}\,dt \le \frac{CL}{\alpha}(1-e^{-\alpha T})\int_0^T E\|S_1-S_2\|^2_{L^2(D)}e^{-\alpha t}\,dt \tag{56}$$

Choosing $\alpha > 0$ such that $\frac{CL}{\alpha} < 1$ the Banach fixed point theorem and the equivalence of the weighted norm with the L^2-norm yields the proof of Theorem 1.

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Appendix 1: w-Operators

Definition 2 Let X be a Banach space and $A : X \to X'$ an operator. A is a w-operator if there exist continuous functions $d : [0,+\infty) \to (0,+\infty)$ and $w : [0,+\infty) \to [0,+\infty)$ with $w(r) = 0$ if and only if $r = 0$ such that

$$\forall u, v \in X,\ d(\|u\| + \|v\|)w(\|u - v\|) \le \langle A(u) - A(v), u - v\rangle.$$

A is a weak-w operator if

$$\forall u, v \in X,\ d(\|u\| + \|v\|)w(\|u - v\|) - \nu(u, v) \le \langle A(u) - A(v), u - v\rangle.$$

where $\nu(u, v) \to 0$ if $\langle A(u) - A(v), u - v\rangle \to 0$.

Let us remark that, of course, a w-operator is a strictly monotone operator and that for a given weak w-operator A, if (u_n) is a bounded sequence such that $\langle A(u_n) - A(u), u_n - u\rangle \to 0$ then u_n converges to u (strongly). Indeed, $\nu(u_n, u) \to 0$ and since d is uniformly strictly positive on bounded sets of $[0,+\infty[$, the above assumption yields the convergence of $w(\|u_n - u\|)$ to 0 when n goes to infinity. Denote by $a_n = \|u_n - u\|$. It is a bounded sequence and there exists a subsequence (a_{n_k}) that converges to $a = \limsup_n a_n$. Since w is a continuous function, $w(a_{n_k}) \to w(a)$. But $w(a_{n_k})$ has to converge to 0, so $w(a) = 0$ and $a = \limsup_n \|u_n - u\|$. This yields the result.

An example of a w-operator is given by $Au = -\mathrm{div}\,[a(t, x)|\nabla u|^{p(t,x)-2}\nabla u]$ for a measurable function $a : Q \to \mathbb{R}$ such that $0 < \alpha \le a(t, x) \le \beta < +\infty$ for almost every $(t, x) \in Q$ and where $1 \le p^- \le p(t, x) \le p^+ < +\infty$ on the space

$$X = \{u \in L^1(0, T, W_0^{1,1}(D)),\ \nabla u \in L^{p(t,x)}(Q)\}.$$

The presence of the function d is mainly due to possible values of $p(t, x)$ less than 2 (see Appendix 2).

Then, an example of a weak w-operator is given in Appendix 3 by the operator $\partial J : X \to X'$ where ∂J is the Gâteaux derivative of the convex function

$$J : u \in X \mapsto \int_Q \frac{1}{p(t, x)}|\nabla u|^{p(t,x)} - \delta\cos(|\nabla u|)\, d(t, x) \in \mathbb{R}$$

for $2 \leq p(t,x) \leq p^+ < +\infty$ and $\delta \in (0,1)$.

Let us remark that Assumption (J3) means that, a.s. $A_\omega = \partial J_Q : X_\omega(Q) \to X'_\omega(Q)$ is an operator of type weak w-operator. Indeed, the coefficients a, p and the set X can be ω-dependent.

Appendix 2

An example of a w-operator is given by

$$Au = -\mathrm{div}\,[a(t,x)|\nabla u|^{p(t,x)-2}\nabla u]$$

for a measurable function $a : Q \to \mathbb{R}$ such that $0 < \alpha \leq a(t,x) \leq \beta < +\infty$ for almost every $(t,x) \in Q$ and where $1 \leq p^- \leq p(t,x) \leq p^+ < +\infty$ on the space

$$X = \{u \in L^1(0,T,W_0^{1,1}(D)),\ \nabla u \in L^{p(t,x)}(Q)\}.$$

Indeed, note first that for any $u, v \in X$,

$$\begin{aligned}
&\langle A(u) - A(v), u - v\rangle \\
&\quad = \int_Q a(t,x)\Big[(|\nabla u|^{p(t,x)-2}\nabla u - |\nabla v|^{p(t,x)-2}\nabla v)\cdot\nabla(u-v)\Big]\,d(t,x) \\
&\quad = \int_{Q^+} a(t,x)\Big[(|\nabla u|^{p(t,x)-2}\nabla u - |\nabla v|^{p(t,x)-2}\nabla v)\cdot\nabla(u-v)\Big]\,d(t,x) \\
&\qquad + \int_{Q^-} a(t,x)\Big[(|\nabla u|^{p(t,x)-2}\nabla u - |\nabla v|^{p(t,x)-2}\nabla v)\cdot\nabla(u-v)\Big]d(t,x)
\end{aligned}$$

where

$$Q^+ = \{(t,x) \in Q \mid p(t,x) \geq 2\},\quad Q^- = \{(t,x) \in Q \mid p(t,x) < 2\}.$$

We recall that [10, Lemma 4.4, p. 13] yields

$$\Big(|\nabla u|^{p(t,x)-2}\nabla u - |\nabla v|^{p(t,x)-2}\nabla v\Big)\cdot\nabla(u-v) \geq 2^{2-p(t,x)}|\nabla(u-v)|^{p(t,x)}$$

a.e. in Q^+ and therefore,

$$\begin{aligned}
&\int_{Q^+}\Big(|\nabla u|^{p(t,x)-2}\nabla u - |\nabla v|^{p(t,x)-2}\nabla v\Big)\cdot\nabla(u-v)\,d(t,x) \\
&\quad \geq 2^{2-p^+}\int_{Q^+}|\nabla(u-v)|^{p(t,x)}\,d(t,x),
\end{aligned}$$

and,

$$\int_{Q^+}|\nabla(u-v)|^{p(t,x)}\,d(t,x) \leq \frac{2^{p^+-2}}{\alpha}\langle A(u)-A(v), u-v\rangle$$

For almost every $(t,x)\in Q^-$, [8, Proposition 17.3, p. 235] yields

$$\begin{aligned}&\Big(|\nabla u|^{p(t,x)-2}\nabla u-|\nabla v|^{p(t,x)-2}\nabla v\Big)\cdot\nabla(u-v)\\&\quad\ge(p(t,x)-1)|\nabla(u-v)|^2\Big(1+|\nabla u|^2+|\nabla v|^2\Big)^{\frac{p(t,x)-2}{2}}.\end{aligned}$$

Thanks to the generalized Young inequality, for any $0<\varepsilon\le 1$, it follows

$$\begin{aligned}&\int_{Q^-}|\nabla u-\nabla v|^{p(\cdot)}\,d(t,x)\\&=\int_{Q^-}\frac{|\nabla u-\nabla v|^{p(t,x)}}{(1+|\nabla u|^2+|\nabla v|^2)^{p(t,x)\frac{2-p(t,x)}{4}}}(1+|\nabla u|^2+|\nabla v|^2)^{p(t,x)\frac{2-p(t,x)}{4}}\,d(t,x)\\&\le\int_{Q^-}\varepsilon^{\frac{p(t,x)-2}{p(t,x)}}\frac{|\nabla u-\nabla v|^2}{(1+|\nabla u|^2+|\nabla v|^2)^{\frac{2-p(t,x)}{2}}}\,d(t,x)\\&\quad+\varepsilon\int_{Q^-}(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p(t,x)}{2}}\,d(t,x)\\&\le\frac{1}{\varepsilon(p^--1)\alpha}\int_{Q^-}a(t,x)(|\nabla u|^{p(t,x)-2}\nabla u-|\nabla v|^{p(t,x)-2}\nabla v)\cdot\nabla(u-v)\,d(t,x)\\&\quad+\varepsilon\int_{Q^-}(1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\,d(t,x)\\&\le\frac{1}{\alpha\varepsilon(p^--1)}\langle A(u)-A(v),u-v\rangle+\varepsilon\int_{Q^-}(1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\,d(t,x).\end{aligned}$$

By denoting $M=\max(\frac{1}{\alpha(p^--1)},\frac{2^{p^+-2}}{\alpha})$, one gets that, for any $\varepsilon\in(0,1)$,

$$\begin{aligned}&\int_Q|\nabla u-\nabla v|^{p(\cdot)}\,d(t,x)\\&\quad\le\frac{M}{\varepsilon}\langle A(u)-A(v),u-v\rangle+\varepsilon\int_Q(1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\,d(t,x).\end{aligned}$$

Now consider the two possible cases:
If, on the one hand, $\int_Q(1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\,d(t,x)\le M\langle A(u)-A(v),u-v\rangle$, then

$$\begin{aligned}&\int_Q|\nabla u-\nabla v|^{p(\cdot)}\,d(t,x)\le 2M\langle A(u)-A(v),u-v\rangle\\&\quad\le\frac{2M}{|Q|}\langle A(u)-A(v),u-v\rangle\int_Q(1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\,d(t,x);\end{aligned}$$

if, on the other hand, $\int_Q (1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\, d(t,x) > M\langle A(u)-A(v), u-v\rangle$, then, for $\varepsilon^2 = \frac{M\langle A(u)-A(v),u-v\rangle}{\int_Q (1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\, d(t,x)}$, one has

$$\int_Q |\nabla u - \nabla v|^{p(\cdot)}\, d(t,x) \\ \le 2\sqrt{M\langle A(u)-A(v), u-v\rangle (1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)})\, d(t,x)}.$$

Thus, denoting by $\psi(x) = \min(x, x^2)$ for nonnegative x, there exists a constant K such that

$$\psi\left(\int_Q |\nabla u - \nabla v|^{p(\cdot)}\, d(t,x)\right) \\ \le K\langle A(u)-A(v), u-v\rangle \int_Q \left(1+|\nabla u|^{p(t,x)}+|\nabla v|^{p(t,x)}\right) d(t,x).$$

Since, for any U, by definition of the Luxemburg norm,

$$\min[\|\nabla U\|^{p^-}, \|\nabla U\|^{p^+}] \le \int_Q |\nabla U|^{p(\cdot)}\, d(t,x) \le \max[\|\nabla U\|^{p^-}, \|\nabla U\|^{p^+}],$$

one has that

$$d(\|\nabla u\| + \|\nabla v\|) w(\|\nabla(u-v)\|) \le \langle A(u)-A(v), u-v\rangle$$

where, for nonnegative x,

$$w(x) = \frac{1}{K}\min\left(x^{p^-}, x^{2p^+}\right) \text{ and } d^{-1}(x) = |Q| + 2\max\left(x^{p^+}, x^{p^-}\right).$$

The conclusion is then a consequence of Poincaré's inequality.

Appendix 3

Let us also give an example of a weak w-operator:
denote by $X = \{u \in L^1(0,T, W_0^{1,1}(D)),\ \nabla u \in L^{p(\cdot)}(Q)\}$, where $2 \le p(t,x) \le p^+ < +\infty$, and for any $\delta \in (0,1)$, consider

$$J : u \in X \mapsto \int_Q \frac{1}{p(t,x)} |\nabla u|^{p(t,x)} - \delta \cos(|\nabla u|)\, d(t,x) \in \mathbb{R}.$$

If we define $j : Q \times [0, +\infty) \to \mathbb{R}$ by $j(t, x, s) = \dfrac{s^{p(t,x)}}{p(t, x)} - \delta \cos(s)$, then $J(u) = \int_Q j(t, x, |\nabla u|)\, d(t, x)$. Moreover, for fixed $(t, x) \in Q$, and $s \geq 0$

$$\partial_s j(t, x, s) = s^{p(t,x)-1} + \delta \sin(s) \text{ and } \partial_s^2 j(t, x, s) = (p(t, x) - 1)s^{p(t,x)-2} \\ +\delta \cos(s).$$

For $s \in [0, 1]$, $\partial_s^2 j(t, x, s) \geq \delta \cos(1)$ and for $s > 1$, $\partial_s^2 j(t, x, s) \geq 1 - \delta$. Therefore

$$\partial_s^2 j(t, x, s) \geq \min(\delta \cos 1, 1 - \delta) := \bar{\alpha} > 0$$

for all $(t, x) \in Q$ and j is a convex function of the variable s for any fixed $(t, x) \in Q$, thus J is a convex function and $\partial J : X \to X'$, $u \mapsto \partial J(u)$ where

$$\langle \partial J(u), v \rangle = \int_Q \frac{\partial_s j(t, x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dxdt$$

is a maximal monotone operator. For $(t, x) \in Q$ fixed let us set

$$\alpha : Q \times [0, \infty) \to \mathbb{R}, \quad (t, x, s) \mapsto s^{\frac{p(t,x)-2}{2}} + \delta \frac{\sin(\sqrt{s})}{\sqrt{s}}, \tag{57}$$

then,

$$\frac{1}{2} \int_0^{s^2} \alpha(t, x, \sigma) d\sigma = \int_0^s \sigma \alpha(t, x, \sigma^2) d\sigma = j(t, x, s) \tag{58}$$

and for any $(t, x) \in Q$, $\alpha(t, x, \cdot)$ is a continuous function. Thus, [23] Lemma 25.26 b), p. 524 yields for all $u, v \in X$, a.e. in Q

$$(\alpha(t, x, |\nabla u|^2)\nabla u - \alpha(t, x, |\nabla v|^2)\nabla v) \cdot \nabla(u - v) \geq \bar{\alpha}|\nabla u - \nabla v|^2, \tag{59}$$

and from (58) and (59) it follows that

$$\left(\frac{\partial_s j(t, x, |\nabla u|)}{|\nabla u|} \nabla u - \frac{\partial_s j(t, x, |\nabla v|)}{|\nabla u|} \nabla v \right) \cdot \nabla(u - v) \geq \bar{\alpha}|\nabla u - \nabla v|^2. \tag{60}$$

for all $u, v \in X$ a.e. in Q. By integration over Q, we obtain

$$\forall u, v \in X, \ \ \langle \partial J(u) - \partial J(v), u - v \rangle \geq \bar{\alpha} \int_Q |\nabla(u - v)|^2 \, d(t, x). \tag{61}$$

Note that for every $u \in X$

$$J(u) = \int_Q j_0(|\nabla u|) + j_1(t, x, |\nabla u|)$$

with $j_1 : Q \times [0, \infty) \to \mathbb{R}$ defined by $j_1(t, x, s) = \frac{s^{p(t,x)}}{p(t,x)}$ and $j_0 : [0, +\infty) \to \mathbb{R}$ defined by $j_0(s) = -\delta \cos(s)$. If we define

$$\alpha_0 : (0, \infty) \to \mathbb{R}, \quad \alpha_0(s) := \delta \frac{\sin \sqrt{s}}{\sqrt{s}},$$

then

$$\frac{1}{2} \int_0^{s^2} \alpha_0(\sigma) d\sigma = \int_0^s \sigma \alpha_0(\sigma^2) d\sigma = j_0(s) \tag{62}$$

Thus, $j_0'(s) = \delta \sin(s)$ is a δ-Lipschitz function and with the same arguments as in [23], proof of Lemma 25.26 d), p. 550 we get

$$|\alpha_0(|\nabla u|^2)\nabla u - \alpha_0(|\nabla v|^2)\nabla v| \leq 3\delta |\nabla(u - v)|. \tag{63}$$

From (63) it follows that for all $u, v \in X$, a.e. in Q,

$$\left| \frac{j_0'(|\nabla u|)}{|\nabla u|} \nabla u - \frac{j_0'(|\nabla v|)}{|\nabla v|} \nabla v \right| \leq 3\delta |\nabla(u - v)|. \tag{64}$$

Thus, for $p(t, x) \geq 2$ we arrive at

$$\begin{aligned}
&\langle \partial J(u) - \partial J(v), u - v \rangle \\
&= \int_Q \Big(|\nabla u|^{p(t,x)-2} \nabla u - |\nabla v|^{p(t,x)-2} \nabla v \\
&\quad + \frac{j_0'(|\nabla u|)}{|\nabla u|} \nabla u - \frac{j_0'(|\nabla v|)}{|\nabla v|} \nabla v \Big) \cdot \nabla(u - v) \, d(t, x) \\
&\geq 2^{2-p^+} \int_Q |\nabla(u - v)|^{p(t,x)} dx dt - 3\delta \int_Q |\nabla(u - v)|^2 \, d(t, x)
\end{aligned}$$

and ∂J is a weak w-operator thanks to (61).

Remark 7.1 The previous example holds also true for

$$j_1(t, x, s) = \frac{1}{p(t, x)} (1 + s)^{p(t,x)} - \frac{1}{p(t, x) - 1} (1 + s)^{p(t,x)-1}$$

with $2 \leq p(t, x) \leq p^+ < +\infty$.

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